Test Theories for Lorentz Invariance

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Abstract. After a very short review of the principles underlying Special Relativity, their meaning, and their consequences, we first describe the basic experiments testing SR in a model–independent way which is the most basic way to describe experiments testing the foundations of SR. In order to be able to give quantitative estimates of the validation of SR and, even more important, in order to be able to compare conceptually different experiments, one introduces test theories. We give a review of test theories needed for a consistent description of tests of Lorentz Invariance. The main emphasize is on kinematical test theories of Robertson and Mansouri–Sexl type. Though these test theories were very important in reaching a new understanding of the experimental foundation of SR, an extensive discussion shows that are kinematical test theories are incomplete and, thus, dynamical test theories like the Standard Model Extension are superior.

1 Introduction

1.1 Postulates of Special Relativity

Special Relativity (SR) is one of the rare examples where essentially everything, the formalism as well as all physical consequences, can be bases on two postulates only. These two postulates are

Postulate 1: The speed of light c is constant.
Postulate 2: The relativity principle.

The first postulate may be replaced by a perhaps even more simple one, namely by the statement that light is a unique phenomenon, that is, between an event and a worldline there are two and only two light rays. The light ray, in particular, does not depend on the trajectory the event of the emission point lies on. Otherwise there will be more than two light rays. The second postulate then makes sure that the measured velocity of light does not depend on its direction and on the velocity of the observer.

The two postulates have some immediate consequences which all can be tested in experiments:

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- The velocity of light, c, does not depend on
  - the velocity of the source (what is a statement of the uniqueness of the phenomenon)
  - the velocity of the observer,
  - the direction of propagation,
  - the polarization or frequency of the light ray.
- The relativity principle implies that
  - the limiting velocity of all particles is the speed of light

$$c = c_+ = c_- = c_\nu = v_p^{\max} = v_e^{\max} = v_{\text{grav}}$$

    (otherwise there is a preferred frame in contradiction to the second postulate), with the consequence that
  - c is universal and, thus, can be interpreted as *geometry*,
  - that *all* physics is the same in *all* inertial systems, that is, experimental results do not depend on the
    - · orientation of the laboratory and
    - · on the velocity of the laboratory.

The experimental status of the foundations of SR has been reviewed recently in [1–3] and a description of technological applications of SR can be found in [4].

1.2 The Consequences

From the above postulates one can derive the Lorentz transformations

$$t' = \frac{1}{\sqrt{1-v^2}}(t - \boldsymbol{x}\cdot\boldsymbol{v}) \tag{1}$$

$$\boldsymbol{x}' = \boldsymbol{x}_\perp + \frac{1}{\sqrt{1-v^2}}\left(\boldsymbol{v}_\parallel - \boldsymbol{v}t\right) , \tag{2}$$

where $\boldsymbol{x}_\parallel = \boldsymbol{x}\cdot\boldsymbol{v}/v^2$ and $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$. These transformations lead to the following effects

1. time dilation,
2. twin paradox,
3. Doppler effect,
4. length contraction,
5. addition of velocities,
6. Sagnac effect, and
7. Thomas precession.

All these effects except length contraction have been confirmed in experiments with high accuracy.

1.3 The Ether

One consequence of the Galilei–transformations is the addition of velocities: If a body moves with velocity $\boldsymbol{u}$ with respect to an inertial system S, then another inertial system S' moving with $-\boldsymbol{v}$ with respect to S observes the body with a velocity

$$\boldsymbol{u}' = \boldsymbol{u} + \boldsymbol{v}\,. \tag{3}$$

This applies to all velocities, in particular to the speed of light. That frame in which the speed of light is isotropic and is what appears in Maxwell's equations, is called the ether frame.

If $\boldsymbol{c}$ is the speed of light in the ether frame, then in a frame moving with respect to the ether the velocity of light is $\boldsymbol{c}' = \boldsymbol{c} + \boldsymbol{v}$ with an orientation and velocity–dependent modulus

$$\begin{aligned} c'(\theta, v) &= \sqrt{c^2 + v^2 + 2cv\cos\theta} \\ &\approx c\left(1 + \frac{v}{c}\cos\theta + \frac{1}{2}\frac{v^2}{c^2}\left(1 + 3\cos\theta\right)\right) + \mathcal{O}(v^4/c^4)\,, \end{aligned} \tag{4}$$

where $\theta = \measuredangle(\boldsymbol{c}, \boldsymbol{v})$. This orientation dependence was looked for in the Michelson–Morley experiments.

2 Test Theories

2.1 What are Test Theories?

Test theories are parametrized generalizations or "violations" of theories under consideration. Calculations of experiments using these generalized theories lead to a variety of effects which are absent in the ordinary theory. A comparison of the calculated effects with experimental results leads to estimates of the parameters characterizing the violation of the theory. One main aspect is that only one generalized theory is taken in order to describe all possible effects.

Consequently, tests theories have the following advantages and tasks

1. Parametrization and identification of possible violation.
2. Quantification of degree of validity.
3. Different (!) experiments can be compared.

In particular the last point is important since in principle different tests may need different theories. For example, while the Michelson–Morley experiment examines the outcome of an interference experiment during a change of the orientation of the apparatus, the Kennedy–Thorndike experiments examines the same for a change in the velocity of the apparatus. Both situations are different and have nothing to do with one another. Only within tests theories both possible results can be described by a (different) combination of one set of parameters.

Different test theories contain a different number of parameters characterizing the deviation from the standard theory. Accordingly, for different test theories one needs a different number of independent tests in order to verify within the experimental limits the theory under consideration. Each test theory defines itself the experiments needed for that.

The quality or richness of a test theory depends on the number of parameters. More parameters can describe a wider range of hypothetical effects and therefore a more complete characterization or verification of the theory under consideration is possible. However, in some cases it is preferable to restrict to a small set of parameters in order to focus on distinguished features. Examples of this are the Robertson test theory or the c^2 formalism [5, 6] which is a special case of the $TH\epsilon\mu$–formalism [7] and which is equivalent a one–parameter subset of the Standard Model Extension, see Table 1. The fully parametrized Standard Model Extension is rather cumbersome to treat. It is obvious that one needs as many independent tests as there are parameters which have to be determined. In the c^2 formalism only one test is needed; for the RMS theory we need three tests and for the SME one needs at the end more than 100 tests. One of the theoretical tasks is to find out that tests which may yield the best estimate for the parameters under consideration.

Beside the reasons mentioned above, test theories also play the role to mediate between the experimental results and a full theory of, e.g., quantum gravity, see Fig. 1.

There are two classes of test theories for SR, a kinematical and a dynamical. Kinematical test theories have been worked out by Robertson [8] and by Mansouri and Sexl [9], dynamical test theories are the $TH\epsilon\mu$–Formalism [7, 10], the Extended Standard Model [11] or even more general setups [12].

2.2 Kinematical Test Theories

Kinematical tests theories discuss the transformation between inertial frames moving with different velocities. At first, these transformation possess the general structure $x'^a = f^a{}_b(v)x^b$, where v is the relative velocity between the two frames. Each kinematical test theory considers a certain class of these general transformations. These transformations are used in order to describe experiments in different frames which, in general, may depend on v. The kinematical test theory of Robertson and Mansouri and Sexl were a very important step for the understanding of the structure of SR. Within this test theory the three famous classical experiments of Michelson–Morley, Kennedy–Thorndike, and Ives

Table 1. Test theories and their number of parameters. In the SME n = number of different elementary particles like electrons, protons, neutrons, etc.

Test Theory	Number of Parameters
c^2–formalism	1 parameter
Robertson–Mansouri–Sexl	3 parameters
$\chi - g$–formalism	19 parameter
Extended Standard Model	$19 + n48$ parameter

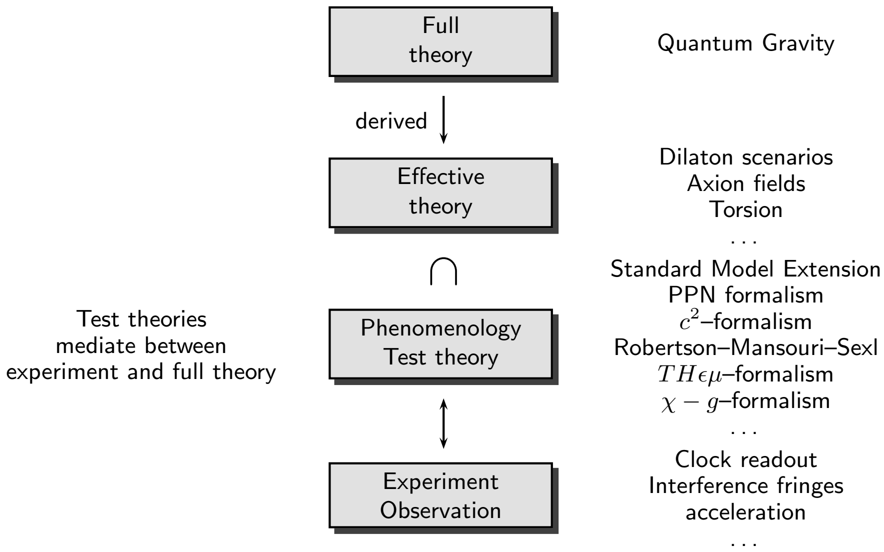


Fig. 1. The hierarchy of descriptions of the physical world. The effective theories are a subset of the phenomenological theories

and Stilwell are identified which are needed in order to verify SR. For dynamical test theories this classification turns out to be not sufficient, in particular since only the behavior of light is considered.

2.3 Dynamical Test Theories

Dynamical test theories start with generalized equations of motion which are used in order to describe experiments. This means that generalized equations of motion for the present standard model are needed, in particular generalized Maxwell and Dirac equations. There are of course infinitely many ways to generalize equations. However, for each kind of phenomenon related to the violation of one of the principles underlying SR, one can begin with very simple modifications. Starting from the standard Maxwell and Dirac equations these modifications in the following (5) and (5) may consist of introducing

- terms $\chi^{\mu\rho\nu\sigma}$, M, and X^{ab} violating Lorentz invariance (see the contribution of R. Bluhm in this volume),
- terms $\chi^{\mu\rho\sigma}$ violating charge non–conservation [12] which also violate Lorentz invariance,
- higher derivatives which in general also violate Lorentz invariance,
- non–linearities.

These modifications then yield the generalized Maxwell and Dirac equations

$$\begin{aligned} 4\pi j^\mu &= \eta^{\mu\rho}\eta^{\nu\sigma}\partial_\nu F_{\rho\sigma} + \chi^{\mu\rho\nu\sigma}\partial_\nu F_{\rho\sigma} + \chi^{\mu\rho\sigma} F_{\rho\sigma} \\ &\qquad + \chi^{\mu\rho\nu\sigma\tau}\partial_\nu\partial_\tau F_{\rho\sigma} + \ldots + \zeta^{\mu\rho\sigma\tau\nu} F_{\rho\sigma} F_{\tau\nu} + \ldots \qquad (5) \\ 0 &= i\gamma^a D_a\psi + m\psi + M\psi + \gamma^{ab} D_a D_b \psi + \ldots + N(\psi)\psi \qquad (6) \end{aligned}$$

where $D_a = \partial_a - ieA_a$ and

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} + X^{ab}\,. \qquad (7)$$

The possible effects which can be derived from the above generalized equations are

- Birefringence
- Anisotropic speed of light
- Anisotropy in quantum fields
- Charge non-conservation
- Anomalous dispersion
- Decoherence, space-time fluctuations
- Modified interference
- Non-localities

In general, as in the Standard Model Extension, for example, the parameters are assumed to be constant.

In this contribution we in extenso treat the kinematical test theories, make some remarks on dynamical test theories and, finally, present a comparison between these test theories.

3 Model-Independent Descriptions of LI Tests

Before we enter the description of the tests of LI in terms of kinematical tests theories, we describe them in a model independent way. Here "model independent" means that we do not assume anything related to the space-time geometry. We of course employ models related to wave propagation, resonators, etc. which – and we like to emphasize this once more – do not anticipate any results on the Lorentzian structure of space–time.

3.1 Isotropy of the Speed of Light

There are two main experimental schemes for testing the isotropy of the speed of light: rotating Michelson interferometers and rotating resonators. We describe both.

Interference Experiments

The Setup

The setup of the experiment by Michelson and Morley [13] uses a Michelson interferometer mounted on a turn table, see Fig. 2. Light from a source is split

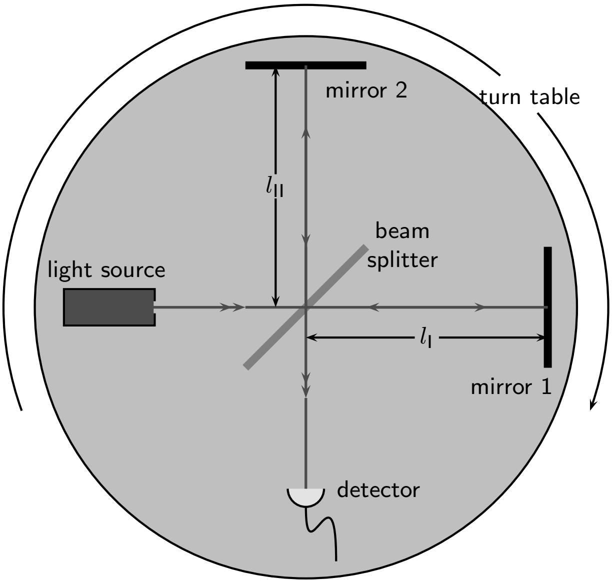


Fig. 2. Setup of the Michelson–Morley experiment. A Michelson interferometer is mounted on a turn table. One looks for a variation of the intensity for varying orientation

coherently and propagates along two different directions. After reflection by mirrors the light rays recombine and interfere. The intensity of the interfering light rays is observed in the detector.

Model Independent Description

We assume the interferometer is in the $x - y$–plane. The incoming light ray is described by a plane wave[1] with frequency ω $e^{i(kx-\omega t)}$. The two split light waves are given by $e^{i(k_{1,2\pm}x-\omega t)}$, where k_{1+} is the wave vector of the wave propagating from the beam splitter to the mirrors, and $k_{1,2-}$ is the wave vector of the reflected waves. Stationarity requires a unique frequency.

The intensity of the interfering waves is

$$I = \frac{1}{2}\left|e^{i(k_{1+}l_1+k_{1-}l_1+\omega t)} + e^{i(k_{2+}l_2+k_{2-}l_2+\omega t)}\right|^2 , \qquad (8)$$

where l_1 and l_2 are the lengths of the interferometer arms. We use the dispersion relations[2] $\omega = k_{1\pm}\, c_{1\pm}$ and $\omega = k_{2\pm}\, c_{2\pm}$, where c_{1+} and c_{2+} are the velocities

[1] Here we assume that light can be described by a plane wave. This assumption is independent from any results concerning Lorentz invariance.

[2] Dispersion relations are a consequence of dynamical equations like the wave equation if one discusses plane wave solutions.

of light (phase velocity) propagating from the beam splitter to the mirrors and c_{1-} and c_{2-} the corresponding velocities in opposite direction, and obtain

$$I = \frac{1}{2}\left|e^{i\omega\left(\frac{l_1}{c_{1+}}+\frac{l_1}{c_{1-}}-t\right)} + e^{i\omega\left(\frac{l_2}{c_{2+}}+\frac{l_2}{c_{2-}}-t\right)}\right|^2 = \frac{1}{2}\left[1+\cos\left(\frac{2\omega l_2}{c_2} - \frac{2\omega l_1}{c_1}\right)\right] . \tag{9}$$

Here c_1 and c_2 are the synchronization independent two-way velocities

$$\frac{2}{c_{1,2}} = \frac{1}{c_{1,2+}} + \frac{1}{c_{1,2-}} \tag{10}$$

along the two interferometer arms. The observable phase shift is

$$\Delta\phi = \omega\left(\frac{2l_2}{c_2} - \frac{2l_1}{c_1}\right) , \tag{11}$$

We assume a small variation of the speed of light, $c_{1,2} = c + \delta c_{1,2}$ with $\delta c_{1,2} \ll c$ and obtain

$$\Delta\phi = 2\omega\left(\frac{l_2 - l_1}{c} + \frac{l_1}{c}\frac{\delta c_1}{c} - \frac{l_2}{c}\frac{\delta c_2}{c}\right) . \tag{12}$$

The variation $\delta c_{1,2}$ may depend on the orientation θ of the interferometer. Since the interferometer arms are orthogonal, $\delta c_2 = \delta c(\theta)$ and $\delta c_1 = \delta c(\theta + \frac{\pi}{2})$. Then

$$\Delta\phi(\theta) = 2\omega\left(\frac{l_2 - l_1}{c} + \frac{l_1}{c}\frac{\delta c(\theta + \frac{\pi}{2})}{c} - \frac{l_2}{c}\frac{\delta c(\theta)}{c}\right) . \tag{13}$$

Upon rotating the interferometer an orientation dependent speed of light yields the phase shift

$$\delta\Delta\phi(\theta) = 2\omega\left(\frac{l_1}{c}\frac{\delta c(\theta + \frac{\pi}{2})}{c} - \frac{l_2}{c}\frac{\delta c(\theta)}{c}\right) \overset{l_1 = l_2}{=} 2\frac{\omega l}{c}\left(\frac{\delta c(\theta + \frac{\pi}{2})}{c} - \frac{\delta c(\theta)}{c}\right) , \tag{14}$$

where in the last step we also assumed an equal arm interferometer.

In the derivation we assumed that the speed of light might depend on the orientation. It can already be seen from (11) that an orientation dependent arm length give the same effect. Operationally one cannot distinguish between a variation of the speed of light and the a variation of the arm length. What can be observed is the difference in the changes. In fact, in dynamical approaches both has to be taken into account [14]. This also means that in experiments the length of the interferometer has to be controlled very carefully. Any thermal change of the length may simulate a varying speed of light. It is a general agreement to formally assigning any result to the speed of light, that is, we *define* the length of the interferometer arm as constant provided any external influence has been ruled out (what sometimes is subject to some debates as, e.g., in the case of the Miller experiment [15]).

In the case $\delta c(\theta) = \delta c \cos\theta$ we obtain

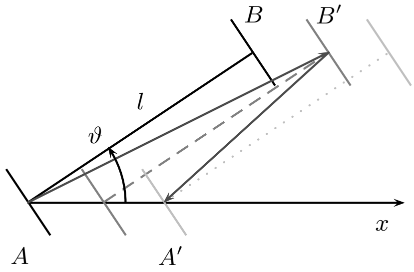


Fig. 3. The paths of light in an interferometer arm moving to the right in an ether frame. The angle between the velocity of the interferometer with respect to the ether and the orientation of the interferometer arm is ϑ

$$\delta\phi(\theta) = -2\omega\frac{l}{c}\left(\sin\theta + \cos\theta\right)\frac{\delta c}{c} = \frac{2l\omega}{c}\sqrt{2}\sin(\theta + \tfrac{\pi}{4})\frac{\delta c}{c}\,. \tag{15}$$

This is the expected phase shift for $\delta c \neq 0$. If no phase shift is observed, then $\delta c = 0$ within the accuracy of the interferometer. (For an interferometer with orthogonal arms a variation $\delta c(\theta) = \delta c \cos(4n\theta)$, $n \in \mathbb{N}$, cannot be detected.)

Interpretation within the Ether Theory

In the ether frame the calculation of the time $t(\vartheta, v)$ light needs to propagate from the beam splitter to one mirror and back to the beam splitter immediately yields from Fig. 3

$$t(l,\vartheta) = \frac{2lc}{c^2 - v^2}\sqrt{1 - \frac{v^2}{c^2}\left(1 - \cos^2\vartheta\right)}\,. \tag{16}$$

where ϑ is the angle between the interferometer arm and $\boldsymbol{v}$. The difference of the time for light moving along two orthogonal interferometer arms is $\Delta t = t(l,\vartheta) - t(l,\vartheta + \pi/2)$. This gives the phase shift

$$\Delta\phi = \frac{2l\omega}{c}\frac{1}{1 - \frac{v^2}{c^2}}\left(\sqrt{1 - \frac{v^2}{c^2}\left(1 - \cos^2\vartheta\right)} - \sqrt{1 - \frac{v^2}{c^2}\left(1 - \sin^2\vartheta\right)}\right) \tag{17}$$

$$= \frac{2l\omega}{c}\frac{v^2}{c^2}\left(\cos^2\vartheta - \sin^2\vartheta\right) = \frac{2l\omega}{c}\frac{v^2}{c^2}\cos(2\vartheta)\,. \tag{18}$$

The same result comes out when we perform the calculation in the frame of the interferometer and make use of the speed of light given by (4).

For an interferometer with an arm length of 11 m as used by Michelson and Morley, and a wavelength of 550 nm one obtains a phase shift of $\Delta\phi = 0.8\pi$ if one uses the velocity of approx. 30 km/s of the Earth around the Sun. Today one would have taken the velocity of the Earth of approx. 360 km/s with respect to the cosmological background which is one order of magnitude larger and, thus, yields a phase shift of approx 10π. The sensitivity of the original Michelson–Morley interferometer was $\Delta\phi \sim 0.01\pi$ so that this effect should be measurable. However, nothing has been seen which means that $v \leq 8$ km/s.

This null result has been explained by a drag of the ether. Another hypothesis was the length contraction suggested by Lorentz and FitzGerald. Since this contraction should be universal, experiments have been carried through with different materials for the interferometer arms [16, 17].

A comparison of the phase shifts gives a relation of velocity of the motion of the reference frame with respect a hypothetical ether to the orientation dependent variation of the speed of light

$$\frac{v^2}{c^2} = \sqrt{2}\frac{\delta_\theta c}{c} \, . \tag{19}$$

Experiments with Resonators

The Setup

In 1955 Essen for the first time used (microwave–) resonators instead of interferometers in order to search for an anisotropic speed of light [18], see Fig. 4. The frequency of a standing electromagnetic wave inside the resonator is determined by the length of the resonator and the speed of light. This frequency can be measured. A varying frequency during turning around the resonator signals an

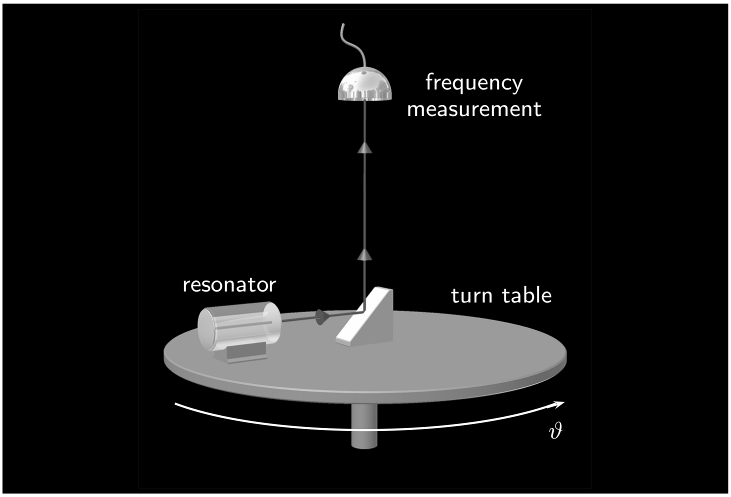


Fig. 4. The principal setup for a test of the isotropy of light using resonators. The frequency of electromagnetic radiation inside the resonators is given by the ratio of the speed of light and the length of the resonator. A change of the frequency during a change of the resonators implies that either the speed of light or the length of the resonator changes with orientation

orientation dependent speed of light (or an orientation dependent length of the resonator). In a modified setup one can use two orthogonally oriented resonators what resembles the Michelson–Morley setup.

Why resonators? Experiments with resonators are much more precise than experiments using interferometers. There are two main reasons: (i) The high finesse (quality factor) of the resonator which today is of the order 10^5 to 10^6. This means that a photon can travel 10^5 to 10^6 times back and forth between the mirrors before leaving the resonator. Therefore, the effective optical path length is much longer than in interferometers. Therefore, a photon can accumulate much more information on an anisotropic speed ob light than in interferometers. For a resonator of 10 cm this amounts to 10 to 100 km compared to 10 m arm length of a typical interferometer. (ii) Resonators are much smaller than interferometers so that much better temperature, vibration, etc. control can be applied. These two reasons lead to the present accuracy of these devices. As an illustration: the distance between the two mirrors can be controlled to up to 1/100 of a proton radius.

Model Independent Description

We have to determine the frequency of the standing electromagnetic wave inside the resonator. This wave consists of two parts traveling back and forth

$$\varphi = A e^{-i(\omega_+ t - k_+ x)} + B e^{-i(\omega_- t + k_- x)} \,. \tag{20}$$

For a stationary problem we have $\omega = \omega_+ = \omega_-$. Again we use the dispersion relation $\omega = k_\pm c_\pm$ (see footnote on page 355). The velocities of light $c_\pm$ may depend on the orientation related to the orientation of the resonator. Then

$$\varphi = A e^{-i\omega\left(t - \frac{x}{c_+}\right)} + B e^{-i\omega\left(t + \frac{x}{c_-}\right)} \,. \tag{21}$$

The amplitudes A and B have to be determined using the ordinary boundary conditions $\varphi(0) = 0$ and $\varphi(L) = 0$. The first condition yields $B = -A$ so that

$$\varphi = A e^{-i\omega t} \left(e^{i\omega \frac{x}{c_+}} - e^{-i\omega \frac{x}{c_-}} \right) \,. \tag{22}$$

The boundary condition at $x = L$

$$0 = e^{i\omega \frac{L}{c_+}} - e^{-i\omega \frac{L}{c_-}} \tag{23}$$

is fulfilled if

$$\sin\left(\omega \left(\frac{1}{c_+} + \frac{1}{c_-} \right) \frac{L}{2} \right) = 0 \,, \tag{24}$$

or, equivalently,

$$\frac{\omega}{c} = \frac{n\pi}{L} \,, \qquad n \in \mathbb{N} \tag{25}$$

with the two-way velocity c. This corresponds to a frequency

$$\nu(\theta) = \frac{n}{2L} c(\theta) \,, \tag{26}$$

where we assumed an orientation dependent two-way speed of light. While turning the resonator on a turn table the frequency of the outcoupled electromagnetic wave is compared with a stationary mounted frequency standard.

In the case of two orthogonally oriented resonators one can observe the beat frequency

$$\nu(\theta + \tfrac{\pi}{2}) - \nu(\theta) = \frac{n}{2L}\left(c(\theta + \tfrac{\pi}{2}) - c(\theta)\right) . \tag{27}$$

For $c(\theta) = c + \delta c \cos\theta$ this yields

$$\nu(\theta + \tfrac{\pi}{2}) - \nu(\theta) = -\frac{n}{2L}(\sin\theta + \cos\theta)\delta c , \tag{28}$$

which reproduces (12).

Interpretation within the Ether Theory

We can use the above calculations and just replace the speed of light by its value (4) given within ether theory, that is, we use

$$c_+ = c'(\theta, v) \qquad \text{and} \qquad c_- = c'(\theta + \pi, v) \tag{29}$$

and obtain for the observed frequency

$$\nu(\theta', v) = \frac{nc}{2L} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}(1 - \cos^2\vartheta')}} \approx \frac{nc}{L}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\left(1 + \cos^2\vartheta'\right)\right) , \tag{30}$$

what corresponds to the time light needs to propagate back and forth an interferometer arm. Comparison with the model independent calculation again gives a relation between the velocity with respect to the ether and the orientation dependence of the velocity of light

$$\frac{1}{2}\frac{v^2}{c^2} = \frac{\delta c}{c} . \tag{31}$$

For two orthogonally oriented resonators we obtain from (30) for the beat frequency

$$\begin{aligned} \nu(\theta' + \tfrac{\pi}{2}, v) - \nu(\theta', v) &= \frac{nc}{2L} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}(1 - \sin^2\theta')}} - \frac{nc}{2L} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}(1 - \cos^2\vartheta')}} \\ &= \frac{nc}{4L}\frac{v^2}{c^2}\cos(2\vartheta') + \mathcal{O}(v^4/c^4) , \end{aligned} \tag{32}$$

which is sensitive to the same quantity as interference experiments. In this case the comparison with the model independent calculation gives

$$\frac{1}{2}\frac{v^2}{c^2} = \sqrt{2}\frac{\delta c}{c} . \tag{33}$$

3.2 Constancy of Speed of Light

This class of experiments explore whether the outcome of experiments depends, via a velocity-dependent speed of light, on the velocity of the laboratory. As for the isotropy, this has been tested with interferometers as well as with resonators.

Interference Experiments

The Setup

The setup is essentially the same as for the Michelson–Morley experiment. The only difference is that we need unequal interferometer arm lengths, see Fig. 5. In the course of the experiment one varies the state of motion (velocity) of the apparatus and looks for associated variations in the intensity of the interfering light rays. For simplicity, we assume the interferometer arms do be orthogonal.

Model Independent Description

The intensity for an unequal arm Michelson interferometer has been given in (12). Now we assume that the speed of light may possibly depend on the velocity of the apparatus, too. The velocity is measured with respect to some inertial system. The result will not depend on the choice of this system.

We assume that δc may depend on the state of motion, too

$$\delta c_1 = \delta c(\theta, v), \quad \delta c_2 = \delta c(\theta + \tfrac{\pi}{2}, v)\,. \tag{34}$$

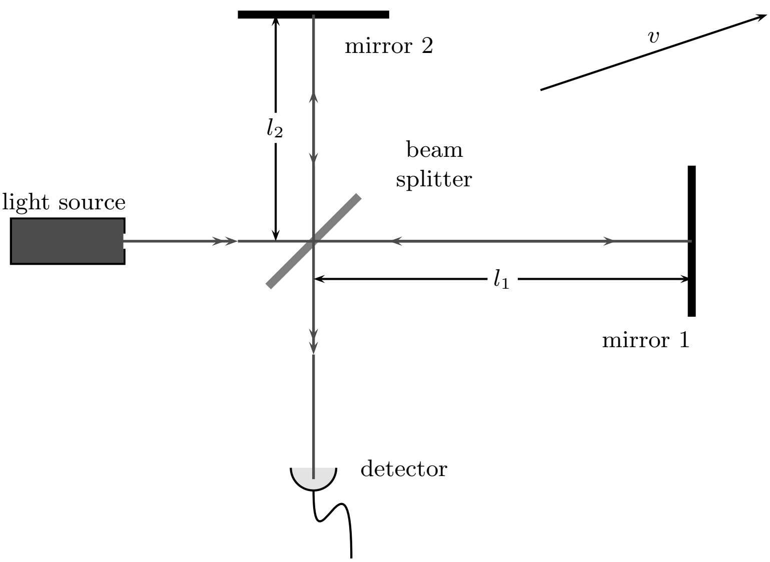


Fig. 5. The experiment of Kenndy and Thorndike uses a Michelson interferometer with different arm lengths $l_1 \neq l_2$

Then (12) yields

$$\Delta\phi(\theta, v) = \omega \left(\frac{l_1 - l_2}{c} + \frac{l_2}{c} \frac{\delta c(\theta + \frac{\pi}{2}, v)}{c} - \frac{l_1}{c} \frac{\delta c(\theta, v)}{c} \right) . \tag{35}$$

In the case that the speed of light does not depend on the direction this simplifies

$$\Delta\phi(v) = \omega \frac{\Delta l}{c} \left(1 + \frac{\delta c(v)}{c} \right) , \tag{36}$$

where $\Delta l = l_1 - l_2$. Therefore, a velocity dependence of the speed of light can be detected only if the interferometer arms are of unequal length.

What has been searched for in this type of experiments is the variation of the intensity while varying the velocity,

$$\delta\Delta\phi = \Delta\phi(v + \delta v) - \Delta\phi(v) = \omega \frac{\Delta l}{c} \frac{\delta c(v + \delta v) - \delta c(v)}{c} = \omega \frac{l_y - l_x}{c} \frac{\delta_v c}{c} . \tag{37}$$

Interpretation within the Ether Theory

The same calculation as for the Michelson–Morley experiment gives for an interferometer with unequal arm lengths $l_1 \neq l_2$ the phase shift

$$\Delta\phi = \frac{2\omega}{c} \frac{1}{1 - \frac{v^2}{c^2}} \left(l_1 \sqrt{1 - \frac{v^2}{c^2} \left(1 - \cos^2 \vartheta\right)} - l_2 \sqrt{1 - \frac{v^2}{c^2} \left(1 - \sin^2 \vartheta\right)} \right) \tag{38}$$

$$= \frac{2\omega}{c} \left(l_1 - l_2 + \frac{v^2}{c^2} \frac{1}{2} \left(l_1 - l_2 + l_1 \cos^2 \vartheta - l_2 \sin^2 \vartheta\right) \right) + \mathcal{O}(v^4/c^4) . \tag{39}$$

A change in the velocity with respect to the ether should result in a phase shift. (This is also the case for Michelson–Morley experiments, but there the velocity term is connected with the orientation which obscures a unique interpretation. Here the effect is related to a different arm length.) For a change of the velocity $\boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v}$ we obtain from (39) to first order in the variation $\delta\boldsymbol{v}$

$$\delta\phi = \frac{2\omega}{c} \frac{\boldsymbol{v} \cdot \delta\boldsymbol{v}}{c^2} \left(l_1 - l_2 + l_1 \cos^2 \vartheta - l_2 \sin^2 \vartheta\right) . \tag{40}$$

For a given $\delta\boldsymbol{v}$ and a measured phase shift $\delta\phi$ one can conclude the value of $\boldsymbol{v}$. The larger the variation of the velocity, the better estimates will be.

A comparison with the model independent calculation gives

$$\frac{\delta_v c}{c} = 2 \frac{v}{c} \frac{\delta v}{c} . \tag{41}$$

Experiments with Resonators

The Setup

The setup is the same as described above. The only difference is that the setup will not be rotated but will change its state of motion. While changing the velocity of he setup, one looks for a change of the frequency of the outcoupled electromagnetic wave. The variation of the state of motion of the laboratory is provided by the rotation of the Earth or its motion around the sun. For using the latter one has to use long term stable resonators.

Here we have to add an important remark of caution. The measurement of the frequency consists of a comparison of two frequencies, one frequency is given by the outcoupled wave, the other by some frequency standard. The frequency standard is defined by some atomic or molecular transition, for example. In principle, the frequency standard may also depend on its state of motion. This means that the present experiment explores whether two frequency standards, one given by the resonator, the other given by some atom or molecular transition, depend in the same or in a different way on the state of motion. In any case, as above any change of the measured frequency is, by convention, assigned to a change of the velocity of the light. Any definite statement regarding the 'true' cause of a (hypothetical) dependence of the signal from the state of motion can be made only by using a dynamical theory.

Model Independent Description

The frequency of the electromagnetic wave in the resonator is again given by (26) with the only modification that now the speed of light may depend on the state of motion of the apparatus, too, $c = c(\theta, v)$. Then

$$\nu(v, \theta) = \frac{n}{2L} c(\theta, v) \,. \tag{42}$$

A variation of the state of motion shows up in a variation of the measured frequency,

$$\delta\nu = \nu(v+\delta v, \theta) - \nu(v, \theta) = \frac{n}{2L} \left(c(v+\delta v, \theta) - c(v, \theta)\right) = \nu(v, \theta) \frac{\delta_v c(v, \theta)}{c} \,. \tag{43}$$

If no effect can be seen then $\delta_v c = 0$ within limits given by the accuracy of the apparatus. Also in this case one cannot distinguish a variation of the speed of light from a velocity dependent variation of the length of the resonator.

Interpretation within the Ether Theory

A change of the velocity results with (30) in the frequency shift

$$\begin{aligned} \delta\nu &= \nu(\theta', v+\delta v) - \nu(\theta', v) \\ &= \frac{nc}{2L} \frac{1 - \frac{(v+\delta v)^2}{c^2}}{\sqrt{1 - \frac{(v+\delta v)^2}{c^2}(1-\cos^2\vartheta')}} - \frac{nc}{2L} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}(1-\cos^2\vartheta')}} \\ &\approx \frac{nc}{L} \frac{\boldsymbol{v}\cdot\delta\boldsymbol{v}}{c^2} \left(1 + \cos^2\vartheta'\right) \,, \end{aligned} \tag{44}$$

and a comparison with the model independent calculation gives

$$\frac{\delta_v \nu}{\nu} = \frac{\delta_v c}{c} = \frac{\boldsymbol{v}\cdot\delta\boldsymbol{v}}{c^2} \,. \tag{45}$$

4 The General Frame for Kinematical Test Theories

4.1 The Setting

In our kinematical test theory the consequences of transformations

$$t' = t'(t, \boldsymbol{x}) \,, \qquad \boldsymbol{x}' = \boldsymbol{x}'(t, \boldsymbol{x}) \tag{46}$$

between the time and spatial coordinates of two observers are analyzed. On physical grounds we restrict to transformations which obey the following three requirements: The transformation

1. maps a force-free motion into a force-free motion, that is,

$$\frac{d^2\boldsymbol{x}}{dt^2} = 0 \quad \Leftrightarrow \quad \frac{d^2\boldsymbol{x}'}{dt'^2} = 0 \,, \tag{47}$$

2. is a one-to-one mapping, and
3. the mapping depends on the relative velocity between the two observers only.

The first requirement implies a projective transformation [19] which with the second requirement gives the linearity of the transformation. From the third requirement we conclude that the linear transformation must have the particular structure

$$t' = a(v)t + e(v)\boldsymbol{v} \cdot \boldsymbol{x} \tag{48}$$

$$\boldsymbol{x}' = d(v)\boldsymbol{x} + b(v)\frac{\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{x})}{v^2} + f(v)\boldsymbol{v}t \,, \tag{49}$$

with undetermined function $a(v)$, $b(v)$, $d(v)$, $e(v)$, and $f(v)$. One function can be fixed by specifying the relative velocity between the observers and one function is related to the synchronization. Only three functions are of true physical nature and are related to the outcome of experiments.

The essential assumption now is that there exist a preferred frame Σ with coordinates $\boldsymbol{X}$ and T. In this frame light it assumed to propagate isotropically

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 = 0 \,, \tag{50}$$

Usually, this preferred frame is identified with the cosmological frame in which the microwave background radiation is isotropic.

4.2 The General Transformation

The transformation between the preferred frame and another frame S with coordinates $(t', \boldsymbol{x})$ is described through (48,49)

$$t' = a(v)T + e(v)\boldsymbol{v} \cdot \boldsymbol{X} \tag{51}$$

$$\boldsymbol{x} = d(v)\boldsymbol{X} + b(v)\frac{\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{X})}{v^2} + f(v)\boldsymbol{v}T \,. \tag{52}$$

The velocity $\boldsymbol{v}$ between S and Σ is defined by the trajectory of the origin of S with respect to Σ, that is, $\boldsymbol{x}=0$ is given by $\boldsymbol{X}=\boldsymbol{v}T$. That means

$$f(v) = -b(v) - d(v)\,. \tag{53}$$

Then we obtain the transformation

$$T = \frac{1}{a(v)}\left(t' - e(v)\boldsymbol{v}\cdot\boldsymbol{x}\right) \tag{54}$$

$$\boldsymbol{X} = \frac{1}{d(v)}\boldsymbol{x} - \left(\frac{1}{d(v)} - \frac{1}{b(v)}\right)\frac{\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x})}{v^2} - \frac{\boldsymbol{v}}{a(v)}\left(t' - \boldsymbol{v}e(v)(\boldsymbol{v}\cdot\boldsymbol{x})\right)\,. \tag{55}$$

We have the freedom to introduce in S' another synchronization through $t = t' + \boldsymbol{\epsilon}'\cdot\boldsymbol{x}$. The coordinates in the corresponding system S are denoted by $(t,\boldsymbol{x})$. As a result, we obtain the transformations between Σ and S with arbitrary synchronization

$$T = \frac{1}{a(v)}\left(t - \boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) \tag{56}$$

$$\boldsymbol{X} = \frac{1}{d(v)}\boldsymbol{x} - \left(\frac{1}{d(v)} - \frac{1}{b(v)}\right)\frac{\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x})}{v^2} - \frac{1}{a(v)}\boldsymbol{v}(\boldsymbol{\epsilon}\cdot\boldsymbol{x}) + \frac{1}{a(v)}\boldsymbol{v}t\,, \tag{57}$$

where

$$\boldsymbol{\epsilon} := e(v)\frac{\boldsymbol{v}}{v} + \boldsymbol{\epsilon}'\,. \tag{58}$$

The line element in S comes out as

$$\begin{aligned} T^2 - \boldsymbol{X}^2 = {} & \frac{1-v^2}{a^2}t^2 - 2\left(\frac{1-v^2}{a^2}\boldsymbol{\epsilon} + \frac{1}{ab}\boldsymbol{v}\right)\cdot\boldsymbol{x}\,t \\ & -\frac{x^2}{d^2} + \frac{1-v^2}{a^2}(\boldsymbol{\epsilon}\cdot\boldsymbol{x})^2 + \frac{2}{ab}(\boldsymbol{v}\cdot\boldsymbol{x})(\boldsymbol{\epsilon}\cdot\boldsymbol{x}) + \left(\frac{1}{d^2} - \frac{1}{b^2}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{x})^2}{v^2}\,. \end{aligned} \tag{59}$$

The light cone in S is defined by the vanishing of (59). We denote by θ the angle between the direction of light propagation and $\boldsymbol{v}$ and by θ' the angle between the speed of light and $\boldsymbol{\epsilon}$. Then in S the modulus of the speed of light

$$\begin{aligned} c(\theta, v, \epsilon) = \frac{|\boldsymbol{x}|}{t} = & \\ \frac{bd(1-v^2)}{adv\cos\theta + bd\epsilon(1-v^2)\cos\theta' - a\sqrt{b^2(1-v^2) + (d^2 - b^2(1-v^2))\cos\theta}} & \end{aligned} \tag{60}$$

depends on the direction, on the velocity of the observer system and on the synchronization. This velocity will be used to describe the Michelson–Morley, Kennedy–Thorndike, and Ives-Stilwell experiments.

For later use we note

$$\lim_{v\to0} a(v) = 1\,, \quad \lim_{v\to0} b(v) = 1\,, \quad \lim_{v\to0} d(v) = 1\,, \quad \lim_{v\to0} \boldsymbol{\epsilon}(v) = 0\,, \tag{61}$$

what can be inferred from the property $t \to T$ and $\boldsymbol{x} \to \boldsymbol{X}$ for $\boldsymbol{v} \to 0$ in (56) and (56).

Special Relativity with arbitrary synchronization is characterized by

$$a(v) = \sqrt{1 - v^2}\,, \quad d(v) = 1\,, \quad b(v) = \frac{1}{\sqrt{1 - v^2}}\,, \tag{62}$$

and for standard Einstein synchronization we have in addition

$$\boldsymbol{\epsilon} = \boldsymbol{v}\,. \tag{63}$$

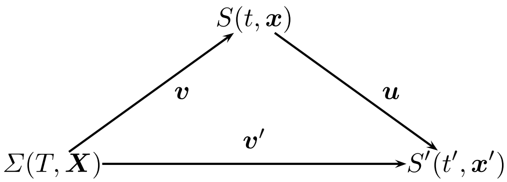
Then (56,57) give the Lorentz–transformations

$$T = \frac{1}{\sqrt{1 - v^2}}\,(t + \boldsymbol{v} \cdot \boldsymbol{x}) \tag{64}$$

$$\boldsymbol{X} = \boldsymbol{x}_\perp + \frac{1}{\sqrt{1 - v^2}}\,\left(\boldsymbol{x}_\parallel + \boldsymbol{v}t\right)\,. \tag{65}$$

4.3 Addition of Velocities

For the description of clock transport and the time dilation effects we need the addition of velocities in our general frame. We have three systems Σ, S and S' with corresponding relative velocities



(66)

The task is to represent $\boldsymbol{v}'$ as function of $\boldsymbol{v}$ and $\boldsymbol{u}$.

For that we insert $\boldsymbol{x}' = 0$ into the transformation $\Sigma \to S'$, and $\boldsymbol{x} = \boldsymbol{u}t$ into the transformation $\Sigma \to S$. Elimination of T and $\boldsymbol{X}$ gives

$$\frac{1}{a(v')} t' = \frac{1}{a(v)}\,(t - \boldsymbol{\epsilon} \cdot \boldsymbol{u}t) \tag{67}$$

$$\frac{1}{a(v')} \boldsymbol{v}' t' = \frac{1}{d(v)} \boldsymbol{u}t - \left(\frac{1}{d(v)} - \frac{1}{b(v)}\right) \frac{\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{u}t)}{v^2} - \frac{1}{a(v)} \boldsymbol{v}(\boldsymbol{\epsilon} \cdot \boldsymbol{u}t) + \frac{1}{a(v)} \boldsymbol{v}t \tag{68}$$

which yields

$$\boldsymbol{v}' = \boldsymbol{v} + \frac{\dfrac{a(v)}{d(v)} \boldsymbol{u}_\perp + \dfrac{a(v)}{b(v)} \boldsymbol{u}_\parallel}{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}}\,. \tag{69}$$

For the choice (62) of the parameters we obtain the special relativistic expression. For small velocities $\boldsymbol{u}$,

$$\boldsymbol{v}' \approx \boldsymbol{v} + \frac{a(v)}{d(v)} \boldsymbol{u} - \left(\frac{a(v)}{d(v)} - \frac{a(v)}{b(v)}\right) \frac{\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{u})}{v^2}\,. \tag{70}$$

5 The Test Theory of Robertson

The test theory of Robertson [8] now specializes the above formalism to the special case of Einstein synchronization. The resulting theory is physically equivalent to the original one.

5.1 The Einstein–synchronization

In order to determine the coefficient $\boldsymbol{\epsilon}$ for the Einstein–synchronization we consider two clocks A and B which are at rest in a system S. This system S moves with a velocity $\boldsymbol{v}$ with respect to Σ. At $t = 0$, a signal is sent from A an arrives in B at $t = t_1$. This signal will be sent back immediately and reaches A at t_2, see Fig. 6. Einstein synchronization now requires (compare, e.g., [9])

$$t_2 = 2\,t_1\,. \tag{71}$$

According to the diagram (66) and the relations (56,57) we represent the events E_1 and E_2 in the relations between S and Σ as well as in the relations between S' and Σ. Since the clock A is at rest in the moving system S, we have $\boldsymbol{x}_2 = 0$ and $\boldsymbol{X}_2 = \boldsymbol{v}T_2$. Therefore,

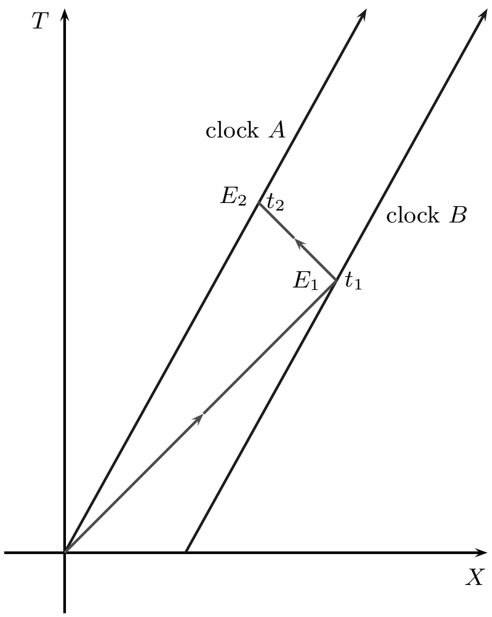


Fig. 6. The Einstein–synchronization: A and B are worldlines of two clocks at rest in a system which moves with respect to Σ. At $t = 0$, the observer A sends a light signal to B, where it arrives at time t_1. The signal sent back immediately arrives in A at time t_2. The Einstein–synchronization now requires $t_1 = \frac{1}{2}t_2$

$$T_2 = \frac{1}{a(v)} t_2 \qquad \boldsymbol{X}_2 = \boldsymbol{v} T_2 \,. \tag{72}$$

From the equations for light propagation

$$|\boldsymbol{X}_1|^2 = T_1^2 \,, \qquad |\boldsymbol{X}_2 - \boldsymbol{X}_1|^2 = (T_2 - T_1)^2 \,, \tag{73}$$

we then obtain after some calculations

$$\boldsymbol{\epsilon} = -\boldsymbol{v} \frac{a(v)}{b(v)(1-v^2)} \,. \tag{74}$$

Using this condition we obtain for our transformations (56,57)

$$T = \frac{1}{a(v)} \left(t + \frac{a(v)}{b(v)(1-v^2)} \boldsymbol{v} \cdot \boldsymbol{x} \right) \tag{75}$$

$$\boldsymbol{X} = \frac{1}{d(v)} \boldsymbol{x}_\perp + \frac{1}{b(v)(1-v^2)} \boldsymbol{x}_\parallel^2 + \frac{1}{a(v)} \boldsymbol{v} t \,, \tag{76}$$

where $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$ with $\boldsymbol{x}_\parallel = \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{v^2} \boldsymbol{v}$. The line element S will be

$$T^2 - \boldsymbol{X}^2 = \frac{1-v^2}{a^2(v)} t^2 - \frac{1}{d^2(v)} \boldsymbol{x}_\perp^2 - \frac{1}{b^2(v)\,(1-v^2)} \boldsymbol{x}_\parallel^2 \tag{77}$$

which has the structure

$$ds^2 = g_0^2(v) t^2 - \left(g_1^2(v) d\boldsymbol{x}_\parallel^2 + g_2^2(v) d\boldsymbol{x}_\perp^2 \right) \tag{78}$$

with

$$g_0^2(v) = \frac{1-v^2}{a^2(v)} , \quad g_1^2(v) = \frac{1}{b^2(v)(1-v^2)} , \quad g_2^2(v) = \frac{1}{d^2(v)} \,. \tag{79}$$

This is (59) in the case of the Einstein–synchronization. The measurements of length depend on the velocity of the frame which violates the relativity principle. For SR $g_0(v) = g_1(v) = g_2(v) = 1$ for all $\boldsymbol{v}$.

The speed of light is

$$c(\theta, v) = \frac{1}{B(v)} \frac{1}{\sqrt{1 + A^2(v) \cos^2 \theta}} \,, \tag{80}$$

where we defined the modulus $1/B(v)$ and the anisotropy $A(v)$

$$\frac{1}{B(v)} = \frac{d(v)\sqrt{1-v^2}}{a(v)} = \frac{g_0(v)}{g_2(v)} \tag{81}$$

$$A(v) = \frac{d(v)}{b(v)\sqrt{1-v^2}} - 1 = \frac{\sqrt{g_1^2(v) - g_2^2(v)}}{g_2(v)} \,. \tag{82}$$

In the subspace orthogonal to $\boldsymbol{v}$ the speed of light is isotropic. The relative velocity $\boldsymbol{v}$ defines a preferred direction. The function $a(v)$ is not related to a possible anisotropy of c. If we define the speed of light in direction of and orthogonal to $\boldsymbol{v}$

$$c_\parallel(v) = c(0,v) = \frac{b(v)(1-v^2)}{a(v)} = \frac{\sqrt{1-v^2}}{A(v)} \tag{83}$$

$$c_\perp(v) = c(\tfrac{\pi}{2},v) = \frac{1}{a(v)} d(v)\sqrt{1-v^2} = \frac{1}{B(v)} \tag{84}$$

then

$$c(\theta,v) = \frac{c_\perp(v)}{\sqrt{1+\frac{c_\perp^2(v)-c_\parallel^2(v)}{c_\parallel^2(v)}\cos^2\theta}} . \tag{85}$$

Therefore the anisotropy is given by the relative difference of $c_\parallel$ and $c_\perp$.

Using $c_\parallel$ and $c_\perp$ we also can express the transformations (56,57)

$$T = \frac{1}{a(v)}\left(t - \boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) \tag{86}$$

$$\begin{aligned} \boldsymbol{X} &= \frac{\sqrt{1-v^2}}{a(v)}\left(\frac{\boldsymbol{x}_\perp}{c_\perp} + \frac{\sqrt{1-v^2}}{c_\parallel}\boldsymbol{x}_\parallel - \frac{\boldsymbol{v}\,(t-\boldsymbol{\epsilon}\cdot\boldsymbol{x})}{\sqrt{1-v^2}}\right) \\ &= \frac{\sqrt{1-v^2}}{a(v)}\left(B(v)\boldsymbol{x}_\perp + A(v)\boldsymbol{x}_\parallel - \frac{\boldsymbol{v}\,(t-\boldsymbol{\epsilon}\cdot\boldsymbol{x})}{\sqrt{1-v^2}}\right) , \end{aligned} \tag{87}$$

and the line element (59)

$$\begin{aligned} T^2 - \boldsymbol{X}^2 = \frac{1-v^2}{a^2(v)}\Bigg(& t^2 - 2\left(\boldsymbol{\epsilon} + \frac{\boldsymbol{v}}{c_\parallel}\right)\cdot\boldsymbol{x}\,t - \frac{1}{c_\perp^2}x^2 \\ & + \left(\left(\boldsymbol{\epsilon}+\frac{\boldsymbol{v}}{c_\parallel}\right)\cdot\boldsymbol{x}\right)^2 + \left(\frac{1}{c_\perp^2} - \frac{1}{c_\parallel^2}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{x})^2}{v^2}\Bigg) . \end{aligned} \tag{88}$$

Here we like to add some remarks on other synchronizations. We show (i) that the synchronization by slow-clock transport yields a different $\boldsymbol{\epsilon}$ and (ii) that the requirement of coincidence of Einstein with slow-clock transport synchronization is only possible for $a(v) = \sqrt{1-v^2}$ [9].

For slow–clock synchronization we consider a clock moving with a small velocity with respect to the system S. By passing the clocks at rest in S, these clocks will be given the time of the slowly moving clock, see Fig. 7. Since the moving clocks are at rest in S' we have $T = t'/a(v')$. The same clocks is described in S by $T = (t - \boldsymbol{\epsilon}\cdot\boldsymbol{x})t$. The prescription of synchronization by slow clock transport now is $t' = t$ from which we immediately obtain $\boldsymbol{\epsilon}\cdot\boldsymbol{x} = (a(v) - a(v'))T$.

Furthermore, from $\boldsymbol{X} = \boldsymbol{v}'T$ in (57) and the addition of velocities (70) for small $\boldsymbol{u}$ we obtain $a(v)(\boldsymbol{v}\cdot\boldsymbol{u})T = \boldsymbol{v}\cdot\boldsymbol{x}$. Then

$$\begin{aligned} \boldsymbol{\epsilon}\cdot\boldsymbol{x} &= -\frac{1}{a(v)}\big(a(v') - a(v)\big)\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\boldsymbol{u}\cdot\boldsymbol{v}} \\ &\approx -\frac{1}{a(v)}\big((\boldsymbol{v}'-\boldsymbol{v})\cdot\boldsymbol{\nabla}_v a(v)\big)\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\boldsymbol{u}\cdot\boldsymbol{v}} \\ &= -\frac{1}{b(v)}\frac{1}{v}\frac{da(v)}{dv}\boldsymbol{v}\cdot\boldsymbol{x} , \end{aligned} \tag{89}$$

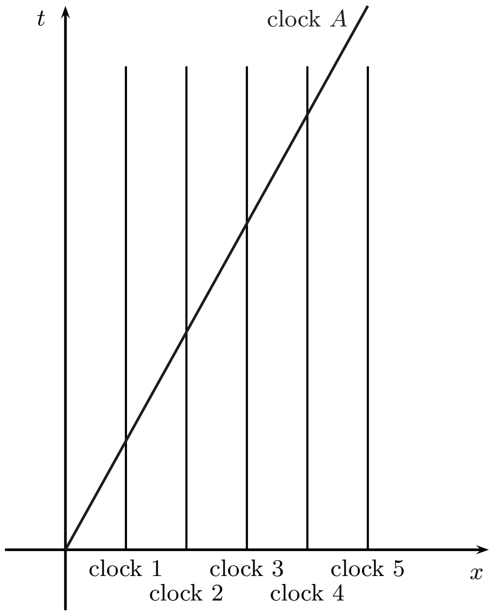


Fig. 7. Synchronization by using slow clocks: a clock A moves slowly in S and sets all clocks in S at its own time

where we again used (70) for small $\boldsymbol{u}$. Since this should hold for all $\boldsymbol{x}$ we obtain for the synchronization parameter

$$\boldsymbol{\epsilon} = \frac{1}{b(v)} \frac{da(v)}{dv} \frac{\boldsymbol{v}}{v} \,. \tag{90}$$

This is different from the result (74) for the Einstein synchronization.

If we require that both methods of synchronization should lead to the same synchronization parameter

$$\frac{da(v)}{dv} = -\frac{v}{(1-v^2)} a(v) \,, \tag{91}$$

we obtain, after integration,

$$a(v) = \sqrt{1-v^2} \,. \tag{92}$$

This is the ordinary time dilation factor. Only in this case both synchronization schemes coincide [9].

5.2 Discussion of the Experiments

Based on (78) we discuss the three classes of experiments, namely the experiments testing the isotropy of the speed of light (Michelson–Morley-experiments), the experiments testing the independence of the speed of light from the velocity of the apparatus (Kennedy–Thorndike-experiments) and the experiments measuring the time dilatation in terms of the Doppler effect (Ives–Stilwell–experiments). These three experiments together imply the Lorentz–transformations and, thus, Lorentz invariance.

Isotropy of the Speed of Light

For a Michelson–Morley experiment with interferometer arms of equal length l the general phase shift (11) with (80) yields

$$\delta\phi = 2\omega l \left(\frac{1}{c_2} - \frac{1}{c_1}\right) = 2\frac{\omega l}{B(v)} \left(\sqrt{1 + A(v)\sin^2\theta} - \sqrt{1 + A(v)\cos^2\theta}\right) . \quad (93)$$

This is independent from the orientation only if the anisotropy $A(v)$ vanishes,

$$A(v) = 0 \quad \Leftrightarrow \quad g_1(v) = g_2(v) \quad \Leftrightarrow \quad d(v) = b(v)\sqrt{1 - v^2} \quad (94)$$

The effect does not depend on the time dilation factor g_0. As a consequence the speed of light (80) reduces to

$$c(\theta, v) = c(v) = 1/B(v) , \quad (95)$$

what still may depend on v. In principle, these experiments have to carried through for all v.

For the description of experiments with resonators we use (26) and (80) and obtain

$$\nu(\theta, v) = \frac{n}{2L} c(\theta, v) = \frac{n}{2LB(v)} \frac{1}{\sqrt{1 + A(v)\cos^2\theta}} . \quad (96)$$

This again does not depend on the orientation if (94) holds.

In the case of a setup with two orthogonally oriented resonators the relative change of the two frequencies is

$$\frac{\nu(\vartheta + \frac{\pi}{2}, v) - \nu(\vartheta, v)}{\nu(\frac{\pi}{2}, v)} = \sqrt{\frac{1 + A(v)\cos^2\vartheta}{1 + A(v)\sin^2\vartheta}} - 1 . \quad (97)$$

The lack of any signal again yields (94).

Constancy of the Speed of Light

Here we take the isotropy of the speed of light as granted, that is, we assume (94).

For an unequal arm interferometer we obtain from (11) and (95)

$$\delta\phi = 2\omega \left(\frac{l_2}{c_2} - \frac{l_1}{c_1}\right) = 2\frac{\omega(l_2 - l_1)}{B(v)} . \quad (98)$$

This phase shift is independent from the velocity v of the apparatus if $B(v) = K$, that is, $g_0(v) = K g_1(v)$, where K is some constant. The condition $\lim_{v\to 0} g_1(v) = \lim_{v\to 0} g_0(v) = 1$ implies $K = 1$. As a consequence

$$B(v) = 1 \quad \Leftrightarrow \quad g_0(v) = g_1(v) . \quad (99)$$

Fig. 8. Example for isotropic (*left*) and anisotropic (*right*) propagation

Also the frequency of the radiation outcoupled from a resonator

$$\nu(v) = \frac{n}{2L} c(v) = \frac{n}{2LB(v)} \tag{100}$$

does not depend on the velocity of the resonator if (99) holds.

The independence from the orientation and the velocity of the apparatus yields with (79)

$$b^2(v) = \frac{a^2(v)}{(1-v^2)^2} \qquad \text{and} \qquad d(v) = \frac{a(v)}{\sqrt{1-v^2}} \,. \tag{101}$$

or, equivalently,

$$g_0(v) = g_1(v) = g_2(v) \tag{102}$$

In this case the line element is $ds^2 = t^2 - \boldsymbol{x}^2 = 0$. The function $a(v)$ is the only unknown function remaining in the transformations (75,76)

$$T = \frac{1}{a(v)} (t + \boldsymbol{v} \cdot \boldsymbol{x}) \tag{103}$$

$$\boldsymbol{X} = \frac{\sqrt{1-v^2}}{a(v)} \left(\boldsymbol{x}_\perp + \frac{1}{\sqrt{1-v^2}} (\boldsymbol{x}_\parallel + \boldsymbol{v}t) \right) \,. \tag{104}$$

Time Dilation

We still need a further experiment which can determine the remaining function $g_0(v)$ or $a(v)$ which gives the time dilatation. Such an experiment is the Doppler shift, for example. In these experiment the frequency of radiation emitted from moving sources will be measured. In this setup, both the laboratory as well as the source will move with respect to the preferred frame. Therefore we need the transformations between frames S and S' moving with v and v' with respect to Σ. This transformation can be easily derived and reads

$$t' = \frac{a(v')(1 + \boldsymbol{u} \cdot \boldsymbol{v})}{a(v)(1-u^2)} (t - \boldsymbol{x} \cdot \boldsymbol{u}) \tag{105}$$

$$\boldsymbol{x}' = \frac{a(v')(1 + \boldsymbol{u} \cdot \boldsymbol{v})}{a(v)\sqrt{1-u^2}} \left(\boldsymbol{x} - \left(1 - \frac{1}{\sqrt{1-u^2}} \right) \frac{\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x})}{u^2} + \frac{\boldsymbol{u}}{\sqrt{1-u^2}} t \right) , \tag{106}$$

where $\boldsymbol{u}$ is the velocity of S' with respect to S.

Since experiments on time dilation measure the relation between t and t' one can determine the last unknown function $g_0(v)$ or $a(v)$. For doing so we observe the frequency of a radiating atom moving with velocity $\boldsymbol{u}$ in the laboratory S or, equivalently, with velocity $\boldsymbol{v}'$ with respect to Σ. We define a system S' in which the atom is at rest. For the determination of $a(v)$ it is enough to have $\boldsymbol{v}$, $\boldsymbol{v}'$, and thus $\boldsymbol{u}$, in x-direction. S and S' are related to Σ via (103,104) $\boldsymbol{v}$ and $\boldsymbol{v}'$ as relative velocities.

We emphasize that we do know neither the magnitude not the direction of the velocity $\boldsymbol{v}$ with respect to the preferred system Σ. Therefore we carry through the following calculations in full generality.

The Doppler Formula

In the preferred frame Σ light rays obey the usual relation

$$(T_\mathrm{r} - T_\mathrm{s})^2 = |\boldsymbol{X}_\mathrm{r} - \boldsymbol{X}_\mathrm{s}|^2 \tag{107}$$

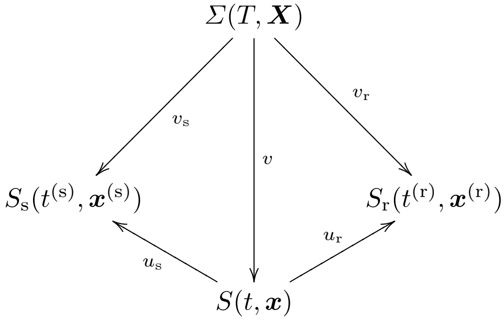
With (103,104) we can transform this to relations for the coordinates in S and obtain

$$t_\mathrm{rs} = x_\mathrm{rs}\,, \tag{108}$$

where we defined

$$t_\mathrm{rs} = t_\mathrm{r} - t_\mathrm{s}\,, \quad x_\mathrm{rs} = |\boldsymbol{x}_\mathrm{rs}|\,, \quad \boldsymbol{x}_\mathrm{rs} = \boldsymbol{x}_\mathrm{r} - \boldsymbol{x}_\mathrm{s} \tag{109}$$

Now we consider the situation shown in the diagram



(110)

and two light rays emitted at events $(t_\mathrm{s}^{(1)}, \boldsymbol{x}_\mathrm{s}^{(1)})$ and $(t_\mathrm{s}^{(2)}, \boldsymbol{x}_\mathrm{s}^{(2)})$ and received at $(t_\mathrm{r}^{(1)}, \boldsymbol{x}_\mathrm{r}^{(1)})$ and $(t_\mathrm{r}^{(2)}, \boldsymbol{x}_\mathrm{r}^{(2)})$, see Fig. 9. $\boldsymbol{u}_\mathrm{s}$ and $\boldsymbol{u}_\mathrm{r}$ are the velocities of the sender and receiver with respect to S. Then in S

$$\boldsymbol{x}_\mathrm{rs}^{(2)} - \boldsymbol{x}_\mathrm{rs}^{(1)} = \boldsymbol{x}_\mathrm{r}^{(2)} - \boldsymbol{x}_\mathrm{r}^{(1)} - \boldsymbol{x}_\mathrm{s}^{(2)} + \boldsymbol{x}_\mathrm{s}^{(1)} = \boldsymbol{u}_\mathrm{r}\Delta t_\mathrm{r} - \boldsymbol{u}_\mathrm{s}\Delta t_\mathrm{s}\,, \tag{111}$$

where $\Delta t_\mathrm{s} = t_\mathrm{s}^{(2)} - t_\mathrm{s}^{(1)}$ and $\Delta t_\mathrm{r} = t_\mathrm{r}^{(2)} - t_\mathrm{r}^{(1)}$. From (108) and (111) we obtain

$$\begin{aligned}\Delta t_\mathrm{r} &= t_\mathrm{s}^{(2)} + x_\mathrm{rs}^{(2)} - (t_\mathrm{s}^{(1)} + x_\mathrm{rs}^{(1)}) \\ &= \Delta t_\mathrm{s} + \boldsymbol{n}\cdot(\boldsymbol{x}_\mathrm{rs}^{(2)} - \boldsymbol{x}_\mathrm{rs}^{(1)}) \\ &= \Delta t_\mathrm{s} + \boldsymbol{n}\cdot(\boldsymbol{u}_\mathrm{r}\Delta t_\mathrm{r} - \boldsymbol{u}_\mathrm{s}\Delta t_\mathrm{s})\,,\end{aligned} \tag{112}$$

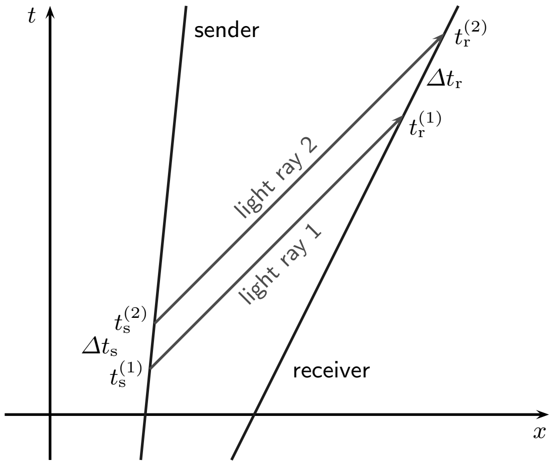


Fig. 9. The observed frequency $1/\Delta t_{\rm r}$ is function of the emitted frequency $1/\Delta t_{\rm s}$

where we used $|\boldsymbol{x}_{\rm rs}^{(2)}| = |\boldsymbol{x}_{\rm rs}^{(1)}| + (\boldsymbol{x}_{\rm rs}^{(2)} - \boldsymbol{x}_{\rm rs}^{(1)})\boldsymbol{\nabla}|\boldsymbol{x}_{\rm rs}^{(1)}|$ and $\boldsymbol{n} = \boldsymbol{x}_{\rm rs}/|\boldsymbol{x}_{\rm rs}|$. This gives

$$\Delta t_{\rm r} = \frac{1 - \boldsymbol{n} \cdot \boldsymbol{u}_{\rm s}}{1 - \boldsymbol{n} \cdot \boldsymbol{u}_{\rm r}} \Delta t_{\rm s} \,. \tag{113}$$

We still have to replace the coordinate times $\Delta t_{\rm r}$ and $\Delta t_{\rm s}$ by the corresponding eigentimes $\Delta t_{\rm r}^{(\rm r)}$ and $\Delta t_{\rm s}^{(\rm s)}$. For doing so we use (103)

$$T_{\rm s,r} = \frac{1}{a(v_{\rm s,r})} \left(t^{(\rm s,r)} + \boldsymbol{v}_{\rm s,r} \cdot \boldsymbol{x}^{(\rm s,r)} \right) \,, \tag{114}$$

where $\boldsymbol{v}_{\rm s}$ is the relative velocity of the sender with respect to the preferred frame Σ. Since the clock of the sender/receiver is at rest in $S_{\rm s,r}$ we have $\boldsymbol{x}_{\rm s,r} = 0$ and

$$\Delta T_{\rm s,r} = \frac{1}{a(v_{\rm s,r})} \Delta t_{\rm s,r}^{(\rm s,r)} \,. \tag{115}$$

We furthermore get from (103)

$$\Delta T_{\rm s,r} = \frac{1}{a(v)} \left(\Delta t_{\rm s,r} + \boldsymbol{v} \cdot \Delta \boldsymbol{x}_{\rm s,r} \right) = \frac{1}{a(v)} \left(1 + \boldsymbol{v} \cdot \boldsymbol{u}_{\rm s,r} \right) \Delta t_{\rm s,r} \,. \tag{116}$$

With that we can eliminate $\Delta T_{\rm s,r}$ and, thus, can express the eigentime $\Delta t_{\rm s,r}^{(\rm s,r)}$ by $\Delta t_{\rm s,r}$

$$\Delta t_{\rm s,r}^{(\rm s,r)} = \frac{a(v'_{\rm s,r})}{a(v)} \left(1 + \boldsymbol{v} \cdot \boldsymbol{u}_{\rm s,r} \right) \Delta t_{\rm s,r} \,. \tag{117}$$

In terms of the frequencies defined by $\nu_{\rm s,r} = 1/\Delta t_{\rm s,r}^{(\rm s,r)}$ we thus have

$$\frac{\nu_{\rm r}}{\nu_{\rm s}} = \frac{\Delta t_{\rm s}^{(\rm s)}}{\Delta t_{\rm r}^{(\rm r)}} = \frac{a(v_{\rm s}) \left(1 + \boldsymbol{v} \cdot \boldsymbol{u}_{\rm s} \right) \left(1 - \boldsymbol{n} \cdot \boldsymbol{u}_{\rm r} \right)}{a(v_{\rm r}) \left(1 + \boldsymbol{v} \cdot \boldsymbol{u}_{\rm r} \right) \left(1 - \boldsymbol{n} \cdot \boldsymbol{u}_{\rm s} \right)} \,. \tag{118}$$

We specialize to a receiver at rest in S, $\boldsymbol{u}_\mathrm{r} = 0$, and finally obtain the Doppler formula we need

$$\frac{\nu_\mathrm{r}}{\nu_\mathrm{s}} = \frac{a(v'_\mathrm{s})\,(1 + \boldsymbol{v} \cdot \boldsymbol{u}_\mathrm{s})}{a(v)(1 - \boldsymbol{n} \cdot \boldsymbol{u}_\mathrm{s})} \,. \tag{119}$$

The Experiment

If one measures the frequency emitted by a moving atom parallel and anti–parallel to the velocity of the atom, then the experiments gives (and this is also what SR predicts) that the product is just the square of the frequency of the atom at rest

$$\nu_\mathrm{r}^+ \nu_\mathrm{r}^- = \nu_\mathrm{s}^2 \,. \tag{120}$$

With (119) this means (for ν_r^+ we chose the direction $\boldsymbol{n} = \boldsymbol{n}_0$, and for ν_E^- the direction $\boldsymbol{n} = -\boldsymbol{n}_0$)

$$\frac{\nu_\mathrm{r}^+}{\nu_\mathrm{s}} \frac{\nu_\mathrm{r}^-}{\nu_\mathrm{s}} = \frac{a(v'_\mathrm{s})\,(1 + \boldsymbol{v} \cdot \boldsymbol{u}_\mathrm{s})}{a(v)(1 - \boldsymbol{n}_0 \cdot \boldsymbol{u}_\mathrm{s})} \frac{a(v'_\mathrm{s})\,(1 + \boldsymbol{v} \cdot \boldsymbol{u}_\mathrm{s})}{a(v)(1 + \boldsymbol{n}_0 \cdot \boldsymbol{u}_\mathrm{s})} = \frac{a^2(v'_\mathrm{s})\,(1 + \boldsymbol{v} \cdot \boldsymbol{u}_\mathrm{s})^2}{a^2(v)(1 - u_\mathrm{s}^2)} = 1 \tag{121}$$

with $u_\mathrm{s} = \boldsymbol{n}_0 \cdot \boldsymbol{u}_S$, so that

$$\frac{a(v'_\mathrm{s})(1 + \boldsymbol{v} \cdot \boldsymbol{u}_\mathrm{s})}{a(v)} = \sqrt{1 - u_\mathrm{s}^2} \,. \tag{122}$$

We abbreviate $\boldsymbol{v}_\mathrm{s} = \boldsymbol{v}$ and $\boldsymbol{u}_\mathrm{s} = \boldsymbol{u}$ and use this result in the transformations (105,106)

$$t' = \frac{1}{\sqrt{1 - u^2}}\,(t - \boldsymbol{x} \cdot \boldsymbol{u}) \tag{123}$$

$$\boldsymbol{x}' = \boldsymbol{x} - \left(1 - \frac{1}{\sqrt{1 - u^2}}\right) \frac{\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x})}{u^2} + \boldsymbol{u}t \,. \tag{124}$$

This are the ordinary Lorentz–transformations. Any information about the state of motion with respect to Σ disappeared. The preferred frame Σ plays no role anymore.

We are also in the position to determine the function $a(v)$: All considerations above hold for all systems S and S', so that we can assume $\boldsymbol{v} = 0$ for a particular system S. Then, from (122) we infer $a(u) = \sqrt{1 - u^2} a(0)$. Since we should obtain the identity for $\boldsymbol{u} = 0$, compare (61), we finally obtain $a(0) = 1$. Therefore, with three experiments we were able to determine all three parameter functions $a(v)$, $b(v)$ and $d(v)$.

5.3 Linearization of the Robertson Test Theory

Since in all laboratory experiments the velocities are relatively small compared to the speed of light and since also the velocity of the Earth with respect to the Sun and the cosmological preferred frame is small, we can expand the functions $g_r(v)$ $(r = 1, 2, 3)$ with respect to the velocities

$$g_r(v) = 1 + \frac{1}{2} g_r^0 v^2 + \dots , \tag{125}$$

where we used (61). As a consequence, the determination of the functions $g_r(v)$ reduces to the determination of three parameters g_r^0.

In this approximation the important combinations are given by

$$\frac{g_2(v) - g_1(v)}{g_1(v)} = \frac{1}{2} \left(g_2^0 - g_1^0 \right) v^2 , \quad \left| \frac{g_0(v)}{g_1(v)} - 1 \right| = \frac{1}{2} \left| g_0^0 - g_1^0 \right| v^2 \tag{126}$$

and $g_0(v) = 1 + \frac{1}{2} g_0^0 v^2$. In this approximation the Michelson–Morley experiment implies $g_1^0 = g_2^0$ and the Kennedy–Thorndike-experiment $g_0^0 = g_1^0$. The time dilation experiment yields $g_0^0 = 0$. The parameter combinations $g_2^0 - g_1^0$ and $g_0^0 - g_1^0$ will show up again in the Mansouri–Sexl test theory.

6 The General Formalism

Based on the transformations (56,57), the line element (59) and the corresponding speed of light (60) we describe now all experiments without the assumption of Einstein synchronization. We will show that again the three previous experiments are enough to characterize Lorentz invariance. However, the basic physical quantities will be slightly different, namely the two-way speed of light and the two-way Doppler shift.

6.1 The Frame

Though the one-way velocity of light depends on the synchronization parameter, the two-way velocity defined by

$$\frac{2}{c_{(2)}(v, \epsilon, \vartheta, \vartheta')} = \frac{1}{c(v, \epsilon, \vartheta, \vartheta')} + \frac{1}{c(v, \epsilon, \vartheta + \pi, \vartheta')} \tag{127}$$

has the same form as the one-way velocity of light under the assumption of Einstein–synchronization

$$c_{(2)}(v, \epsilon, \vartheta, \vartheta') = c_{(2)}(\vartheta, v) = \frac{1}{B(v)} \frac{1}{\sqrt{1 + A(v) \cos^2 \vartheta}} , \tag{128}$$

which no longer depends on $\boldsymbol{\epsilon}$.

Since the two-way velocity of light is exactly the same as the one-way velocity for Einstein synchronization, the results (101) for the experiments testing the isotropy and constancy of the speed of light are also the same and, thus, need not to be repeated. The only nontrivial experiment is the time dilation experiment.

We use the results (101) in order to eliminate $b(v)$ and $d(v)$ in (56,57)

$$T = \frac{1}{a(v)}\left(t - \boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) \tag{129}$$

$$\boldsymbol{X} = \frac{\sqrt{1-v^2}}{a(v)}\left(\boldsymbol{x} - \left(1-\sqrt{1-v^2}\right)\frac{\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x})}{v^2}\right) - \frac{1}{a(v)}\left(\boldsymbol{v}(\boldsymbol{\epsilon}\cdot\boldsymbol{x}) - \boldsymbol{v}t\right) , \tag{130}$$

and in the expression for the light cone in S

$$0 = T^2 - \boldsymbol{X}^2 = \frac{1-v^2}{a^2(v)}\left(t^2 - 2\left(\boldsymbol{\epsilon}+\boldsymbol{v}\right)\cdot\boldsymbol{x}t - \boldsymbol{x}^2 + \left((\boldsymbol{\epsilon}+\boldsymbol{v})\cdot\boldsymbol{x}\right)^2\right) , \tag{131}$$

which is the SR lightcone for arbitrary synchronization. Again, we have to determine the remaining function $a(v)$.

In order to discuss time dilation effects we again have to consider the transformation between two systems S and S' moving with $\boldsymbol{v}$ and $\boldsymbol{v}'$ with respect to the preferred frame Σ. Using (129,130), a lengthy calculation yields

$$t' = \frac{a(v')}{a(v)}\left(t - \boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) + \boldsymbol{\epsilon}'\cdot\boldsymbol{x}' \tag{132}$$

$$\begin{aligned}\boldsymbol{x}' = \frac{a(v')\gamma'}{a(v)\gamma}&\left(\boldsymbol{x} + \boldsymbol{v}\left(\gamma\left(t-\boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) - \left(1-\frac{1}{\gamma}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{x})}{v^2}\right)\right.\\ &+\boldsymbol{v}'\left(\frac{\gamma'-1}{v'^2}\left(\boldsymbol{v}'\cdot\boldsymbol{x} - \left(1-\frac{1}{\gamma}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{x})(\boldsymbol{v}'\cdot\boldsymbol{v})}{w^2} + (\boldsymbol{v}'\cdot\boldsymbol{v})\gamma\left(t-\boldsymbol{\epsilon}\cdot\boldsymbol{x}\right)\right)\right.\\ &\left.\left.-\gamma'\gamma\left(t-\boldsymbol{\epsilon}\cdot\boldsymbol{x}\right)\right)\right)\end{aligned} \tag{133}$$

where $\gamma' = \gamma(v')$.

6.2 Discussion of the Experiments: Time Dilatation

The Doppler Formula

Again we calculate and use the Doppler effect for moving atoms in order to determine the time dilatation. As before, we have to calculate the time differences between the emission and reception times of two light rays, see Fig. 9 and diagram (110). Again we can start from (107), use (129,130), and obtain

$$\Delta t_{\mathrm{r}} = \frac{1-(\boldsymbol{n}+\boldsymbol{w}+\boldsymbol{\epsilon})\cdot\boldsymbol{v}_{\mathrm{s}}}{1-(\boldsymbol{n}+\boldsymbol{w}+\boldsymbol{\epsilon})\cdot\boldsymbol{v}_{\mathrm{r}}}\Delta t_{\mathrm{s}} . \tag{134}$$

what generalizes (113). We replace the time differences Δt_{r} and Δt_{s} by the times shown by moving clocks, that is, by the time of the moving system. For that we need (129,130). We get

$$T = \frac{1}{a(v_{\mathrm{r,s}})}\left(t^{(\mathrm{r,s})} - \boldsymbol{\epsilon}^{(\mathrm{r,s})}\cdot\boldsymbol{x}^{(\mathrm{r,s})}\right) \quad \text{and} \quad T = \frac{1}{a(v)}\left(t - \boldsymbol{\epsilon}\cdot\boldsymbol{x}\right) . \tag{135}$$

The clocks are at rest in $S_{\mathrm{r,s}}$ and move with $\boldsymbol{u}_{\mathrm{r,s}}$ with respect to $S_{\mathrm{r,s}}$. This implies $\boldsymbol{x}^{(\mathrm{r,s})}_{\mathrm{r,s}} = 0$ and $\boldsymbol{x}_{\mathrm{r,s}} = \boldsymbol{u}_{\mathrm{r,s}}t_{\mathrm{r,s}}$. Therefore we have two relations

$$T_{\mathrm{r,s}} = \frac{1}{a(v_{\mathrm{r,s}})} t^{(\mathrm{r,s})}_{\mathrm{r,s}} \qquad \text{and} \qquad T_{\mathrm{r,s}} = \frac{1}{a(v)} \left(1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{r,s}}\right) t_{\mathrm{r,s}} \tag{136}$$

from which we obtain the measured time differences $\Delta t^{(\mathrm{r,s})}_{\mathrm{r,s}}$ in terms of the coordinate time differences $\Delta t_{\mathrm{r,s}}$

$$\Delta t^{(\mathrm{r,s})}_{\mathrm{r,s}} = \frac{a(v_{\mathrm{r,s}})}{a(v)} \left(1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{r,s}}\right) \Delta t_{\mathrm{r,s}} \,. \tag{137}$$

With the frequencies $\nu_{\mathrm{r,s}} = 1/\Delta t^{(\mathrm{r,s})}_{\mathrm{r,s}}$ we thus obtain from (113)

$$\frac{\nu_{\mathrm{r}}}{\nu_{\mathrm{s}}} = \frac{a(v)}{a(v_{\mathrm{r}})} \frac{a(v_{\mathrm{s}})}{a(v)} \frac{1 - (\boldsymbol{n} + \boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{v}_{\mathrm{r}}}{1 - (\boldsymbol{n} + \boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{v}_{\mathrm{s}}} \frac{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}}{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{r}}} \,. \tag{138}$$

For a receiver at rest in S ($\boldsymbol{u}_{\mathrm{r}} = 0$) we finally obtain the Doppler formula for arbitrary synchronization

$$\frac{\nu_{\mathrm{r}}}{\nu_{\mathrm{s}}} = \frac{a(v_{\mathrm{s}})}{a(v_{\mathrm{r}})} \frac{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}}{1 - (\boldsymbol{n} + \boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_{\mathrm{s}}} \,. \tag{139}$$

The Experiment

Again we calculate the product of the frequencies measured parallel and antiparallel to the velocity of a moving radiating atom

$$\begin{aligned} \frac{\nu^+_{\mathrm{r}} \nu^-_{\mathrm{r}}}{\nu^2_{\mathrm{s}}} &= \frac{a(v_{\mathrm{s}})}{a(v_{\mathrm{r}})} \frac{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}}{1 - (\boldsymbol{n} + \boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_{\mathrm{s}}} \frac{a(v_{\mathrm{s}})}{a(v_{\mathrm{r}})} \frac{1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}}{1 - (-\boldsymbol{n} + \boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_{\mathrm{s}}} \\ &= \frac{a^2(v_{\mathrm{s}})}{a^2(v_{\mathrm{r}})} \frac{\left(1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}\right)^2}{\left(1 - (\boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_{\mathrm{s}}\right)^2 - u^2_{\mathrm{s}}} \,. \end{aligned} \tag{140}$$

The result of the experiment is $\nu^+_{\mathrm{r}} \nu^-_{\mathrm{r}} = \nu^2_{\mathrm{s}}$ so that

$$\frac{a(v_{\mathrm{s}})}{a(v)} \left(1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_{\mathrm{s}}\right) = \sqrt{\left(1 - (\boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_{\mathrm{s}}\right)^2 - u^2_{\mathrm{s}}} \,. \tag{141}$$

This result allows us to determine the factor $a(v')\gamma'/(a(v)\gamma)$ in (132,133)

$$\frac{a(v_{\mathrm{s}})\gamma(v_{\mathrm{s}})}{a(v)\gamma(v)} = \frac{\gamma(v_{\mathrm{s}})\sqrt{\left(1 - (\boldsymbol{v} + \boldsymbol{\epsilon}) \cdot \boldsymbol{u}_S\right)^2 - u^2_S}}{\gamma(v)\left(1 - \boldsymbol{\epsilon} \cdot \boldsymbol{u}_S\right)} = 1 \,, \tag{142}$$

where we used (69) in order to calculate $\gamma(v_{\mathrm{s}})$ as function of $\gamma(v)$. With this result the transformations (132,133) become the Lorentz transformations between two arbitrarily synchronized reference frames. From the validity of (142) for all v and v_{s} we again infer that $a(v) = a(0)\sqrt{1 - v^2}$. Together with $a(0) = 1$ from (61) we finally get $a(v) = \sqrt{1 - v^2}$. Furthermore, from (101) we obtain $b(v) = 1/\sqrt{1 - v^2}$ and $d(v) = 1$.

As a result, we obtain the Lorentz transformations for arbitrary synchronization [20–25].

7 The Mansouri-Sexl Test Theory

7.1 The Frame

Since most of the experiments are carried through at small velocities, Mansouri and Sexl performed an expansion of the functions $a(v)$,$b(v)$, $d(v)$ and $\boldsymbol{\epsilon}(v)$ with respect to the velocity

$$a(v) = 1 + \left(\alpha - \frac{1}{2}\right) v^2 + \left(\alpha_2 - \frac{1}{8}\right) v^4 + \ldots = 1 + \alpha^{\mathrm{MS}} v^2 + \alpha_2^{\mathrm{MS}} v^4 + \ldots \quad (143)$$

$$b(v) = 1 + \left(\beta + \frac{1}{2}\right) v^2 + \left(\beta_2 + \frac{3}{8}\right) v^4 + \ldots = 1 + \beta^{\mathrm{MS}} v^2 + \beta_2^{\mathrm{MS}} v^4 + \ldots \quad (144)$$

$$d(v) = 1 + \delta v^2 + \delta_2 v^4 + \ldots \quad (145)$$

$$\boldsymbol{\epsilon} = (\epsilon - 1)\boldsymbol{v}\left(1 + \epsilon_2 v^2 + \ldots\right) . \quad (146)$$

The parameter functions are now replaced by a few constant parameters. Here α^{MS} and β^{MS} are parameters originally introduced by Mansouri and Sexl. Our parameters are chosen so that they vanish if SR is valid, compare [26]. In the case of Einstein-synchronization also ϵ vanishes. For simplicity we restrict in the following to first order in v^2. For the next order see [27].

In first order we obtain for the line element

$$\begin{aligned} s^2 = {} & \left[1 - 2\alpha v^2\right] t^2 - 2\left[\epsilon + (\alpha - \beta - 2\alpha\epsilon - \epsilon_2 + \epsilon\epsilon_2) v^2\right] \boldsymbol{v} \cdot \boldsymbol{x} t \\ & - \left[1 - 2\delta v^2\right] x^2 + \left[\epsilon^2 + 2(\beta - \delta)\right] (\boldsymbol{v} \cdot \boldsymbol{x})^2 , \end{aligned} \quad (147)$$

for the one-way velocity of light

$$c(\vartheta, v) = 1 - \epsilon v \cos\vartheta - \left[\delta - \alpha + (\beta - \delta + \epsilon^2) \cos^2\vartheta\right] v^2 , \quad (148)$$

and for the two-way velocity of light defined in (127)

$$c_{(2)}(\vartheta, v) = 1 + \left[\delta - \alpha + (\beta - \delta) \cos^2\vartheta\right] v^2 . \quad (149)$$

The relative change of the speed of light

$$\frac{\delta_\vartheta c}{c} = (\delta - \beta) v^2 \sin^2\vartheta \quad (150)$$

$$\frac{\delta_v c}{c} = 2\left(\delta - \alpha + (\beta - \delta)\cos^2\vartheta\right) \boldsymbol{v} \cdot \delta\boldsymbol{v} \quad (151)$$

relates this formalism to the model independent description.

With (79) and (125) we can relate the linearized Robertson–parameters to the Mansouri–Sexl–parameters

$$g_2^0 - g_1^0 = \delta - \beta , \qquad g_0^0 - g_1^0 = \beta - \alpha , \qquad g_0^0 = -2\alpha . \quad (152)$$

Since synchronization does not play a role in the interpretation of experiments, the Mansouri–Sexl test theory is equivalent to the linearizes Robertson test theory.

Since we now have only three parameters only three experiments are needed in order to fix the theory.

7.2 Discussion of the Experiments

Isotropy of the Speed of Light

With (148) the phase shift for a general interference experiment is

$$\Delta\phi(\vartheta, v) = \frac{\omega}{c} \{2(l_1 - l_2) + [(2\alpha - \beta - \delta)(l_1 - l_2) + (\delta - \beta)(l_1 + l_2)\cos(2\vartheta)]\, v^2\} \,. \quad (153)$$

For a Michelson–Morley experiment we choose $l_1 = l_2 = l$ and obtain

$$\Delta\phi(\vartheta, v) = 2\frac{l\omega}{c} \left[(\delta - \beta)v^2 \cos(2\vartheta)\right] \,. \quad (154)$$

Independence from the orientation implies

$$\delta - \beta = 0\,. \quad (155)$$

For resonators we obtain with (26) and (149) the frequency shift

$$\nu(\vartheta, v) = \frac{n}{L} \left\{1 + \left[\delta - \alpha + (\beta - \delta)\cos^2\vartheta\right] v^2\right\} \,, \quad (156)$$

which in the case of isotropy again implies (155). For the comparison of two orthogonally mounted resonators we obtain the relative beat frequency

$$\frac{\nu(\vartheta, v) - \nu(0, v)}{\nu(0, v)} = (\delta - \beta)v^2 \sin^2\vartheta\,. \quad (157)$$

Constancy of the Speed of Light

If we assume isotropy $\delta = \beta$ then we obtain for a Kennedy–Thorndike-experiment, that is $l_1 \neq l_2$ in (153), the phase shift

$$\Delta\phi(v) = 2(l_1 - l_2)\omega \left(1 + (\alpha - \beta)v^2\right) \,. \quad (158)$$

If this does not depend on the velocity of the apparatus, then

$$\alpha - \beta = 0\,. \quad (159)$$

The frequency in a resonator (156) now is

$$\nu(v) = \frac{cn}{2L} \left[1 + (\beta - \alpha)v^2\right] \,, \quad (160)$$

which does not depend on v if (159) holds.

Time Dilation

Since we discussed these experiments already in the general framework, there is no need to repeat it in this approximation. The result of the easy calculation is is that the exact result $a(v) = \sqrt{1 - v^2}$ in its linearized form now reads $\alpha = 0$. Therefore we obtain from (159) $\beta = 0$ and from (155) $\delta = 0$. Therefore we were able to determine all three parameters α, β and δ from the outcome of three experiments.

8 Discussion

8.1 Summary

We introduces four frames for the discussion of experiments. Three of these frames are special cases of the general framework, see Fig. 10. These special cases are defined by choosing the Einstein synchronization and by a linearization of the theory for small velocities. Physical results should be independent of the chosen synchronization. The description of experiments with arbitrary synchronization forces one to choose appropriate synchronization independent observables.

While this kinematical test theory has the merit for the first time to identify the consequences of certain experiments for the theoretical description, which led to the notion of the "three classical tests" of SR, there are assumptions made which need to be discussed.

8.2 Advantages of Kinematical Test Theories

There are two important and far-reaching advantages:

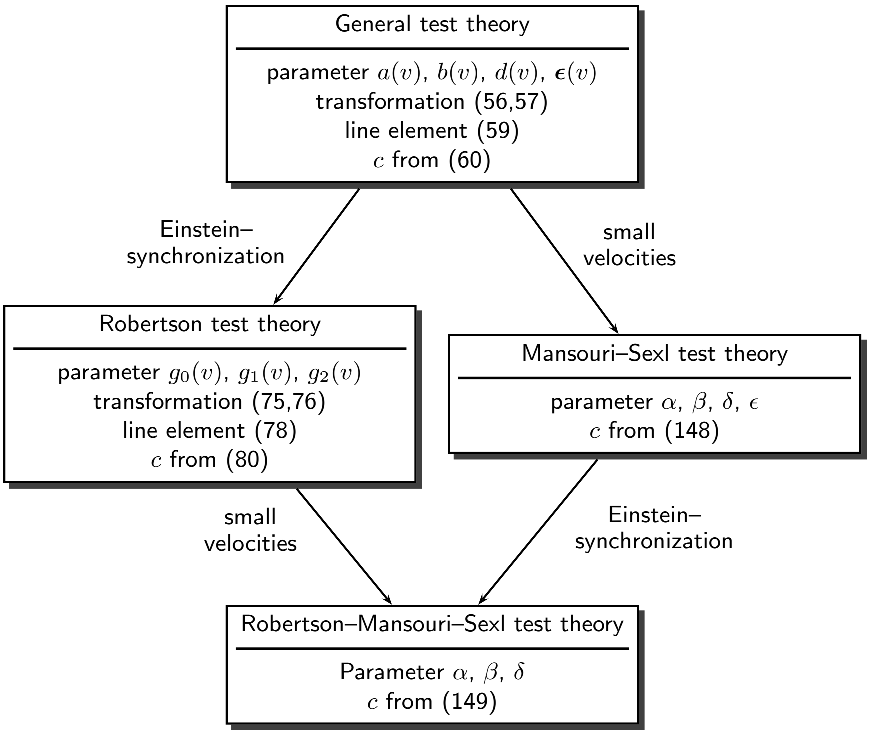


Fig. 10. The relations among the various kinematical test theories. Einstein synchronization connects physically equivalent theories

1. Kinematical test theories are independent from any particle model; they are universal. Since the transformation between frames of reference are under consideration, all physical phenomena are treated in the same way.
2. The test theory is characterized by three parameters only with the consequence that only three experiments are needed in order to fix the theory.

8.3 Disadvantages of Kinematical Test Theories

There are a few severe problems with kinematical test theories based on the assumptions made for setting up this kinematical test theory:

1. The kinematical tests theories need a preferred frame. The choice of a preferred frame may not be unique. Today, one may identify this preferred frame by the cosmological frame defined by the isotropy of the microwave background radiation [28]. Though it is not very probably, it is at least possible in principle, that a stochastic gravitational wave background radiation may define another preferred frame different from the microwave background. Since all estimates describing the degree of validity of SR use the velocity with respect to the preferred frame, the characterization thus depends on our knowledge of cosmology. If we choose another preferred frame, the estimates will change. Therefore, these test theories are intrinsically incomplete. One necessarily needs more input than provided by the kinematical test theory.
2. One assumes a certain geometry of the preferred frame (what has nothing to do with the transformation laws). That means that in Σ one assumes an isotropic speed of light. In principle this also should be subject to experimental proof. One way to handle such a question might be to enlarge the set of parameters by introducing a general propagation through

$$dT^2 = G_1^2 dX^2 - G_2^2 dY^2 - G_3^2 dZ^2 \tag{161}$$

   with undetermined parameters G_1, G_2, and G_3. Even more general propagation structures like that of Finslerian structure $dT = f(d\boldsymbol{x})$, f being homogeneous of degree one in $d\boldsymbol{X}$ and non-degenerate, are possible. It should be no problem to carry through the above calculations for this more general setting. However, then more experiments are needed in order to fix the enlarged set parameters. That means, the discussion of experiments for determining the final structure of space-time will be more intriguing.
3. In kinematical test theories the violation of Lorentz invariance can depend on velocity only. A violation of Lorentz invariance may come in through some cosmologically given vector or tensor fields which may occur, e.g., in string theories with spontaneous broken Lorentz symmetry, where the ground state of the space–time geometry shows a broken symmetry which is not present in the formulation of the theory [29, 30].
4. Kinematical test theories are not only incomplete, they might be even inconsistent if one considers light to be a consequence of the Maxwell equations. This can be seen as follows: If the light depends on the state of motion of the

laboratory, then also the Maxwell equations have to depend on that state of motion. That means that clocks and rods, which both are heavily determined by the Maxwell equations, also depend on the state of motion. Furthermore, they depend in a material-dependent way on th state of motion. Therefore, there is no unique clock and rod. This, however, is part of the scheme of the kinematical test theories.

5. Kinematical test theories cannot describe birefringence in vacuum which also violates Lorentz invariance.
6. Generalizing this idea, kinematical test theories cannot treat a non–unique c, that is, different limiting velocities for different particles. The dynamics of particles is not treated in these kinematical test theories.
7. Furthermore, it is not possible to describe violations of LI by anisotropic masses or anomalous spin couplings of Dirac particles.
8. Since in our scheme of kinematical test theories we assumed in Σ and, thus, also in other frames S a unique light propagation, it is not possible to describe a hypothetical dependence of c from the velocity of source.

All these problems do not occur in dynamical test theories which, by construction, are complete.

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