The Lense–Thirring Effect: From the Basic Notions to the Observed Effects

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Abstract. A pedagogical derivation is given of the Lense–Thirring effect using basic notions from the motion of point particles and light rays. First, the notion of rotation is introduced using the properties of light rays only. Second, two realizations for a non–rotating propagation of space–like directions are presented: the gyroscope and the spin of elementary particles. Then the gravitational field around a rotating body is specified which is taken for determining the various effects connected with a point particle or a gyroscope: the deSitter precession (geodesic precession) and the Lense–Thirring effect ('frame dragging'). The results are applied to the precession of gyroscopes and to the motion of satellites around the earth.

1 Introduction

In the Lense–Thirring effect three rotations are involved: the rotation of the gravitating body, the rotation of the test body around its own axis, and the rotation of the axis of rotation of the test body. The properties of the gravitating body are prescribed, the rotating test body can be shown to move approximately on a geodesics around the gravitating body, and its axis of rotation can be shown to be Fermi propagated along the path of the test body resulting in a precession of the axis of rotation with respect to distant stars. In this note we want to derive all these notions and equations of motion from scratch in order to indicate clearly that everything is provided by General Relativity: Everything follows from the geodesic equation for point particles and the validity of Einstein's equations; we don't have to use additional assumptions. At the end we will discuss several experimental approaches to test the various effects related to rotating bodies.

In the following we (i) introduce the notion of rotation, (ii) derive the equation of motion of the spinning axis of a rotating test body, (iii) derive the gravitational field of a rotating gravitating body, and (iv) use these results for discussing and analyzing the equation of motion of the test body and of its spinning axis.

All of what we assume is that gravity is described by means of a Riemannian geometry endowed with a space–time metric g and that light rays and freely falling point particles move along geodesics of that metric,

$$D_v v = \alpha v \,, \tag{1}$$

where D is the unique metric compatible, $Dg = 0$, torsion–free, $D_u v - D_v u - [u, v] = 0$, covariant derivative. In components, $(D_w v)^\nu = w^\mu(\partial_\mu v^\nu + \{ {}^{\nu}_{\mu\sigma} \} v^\sigma)$,

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with the Christoffel symbol $\{ {}^{\nu}_{\mu\sigma} \} := \frac{1}{2} g^{\nu\rho} \left(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\rho g_{\mu\sigma} \right)$. We do not assume any normalization condition of the 4–velocity. Light rays with tangents l obey the same equation of motion (1), but with the additional condition $g(l, l) = 0$. – In addition, we will assume Einstein's field equations.

2 Rotation

In order to define the notion of rotation, we use the so–called *zig–zag construction*, or the *bouncing photon*, as introduced by Pirani [1]. This construction uses a central point particle which moves along an arbitrary path, see Fig.1. In the neighborhood of that central particle there are two other point particles equipped with a mirror. At first, the central point particle emits a flash of light which hits the auxilary point particles. These auxilary point particles reflect this flash of light in such a way that it again meets the central point particle and, in addition, the other auxilary particle positioned appropriately behind the central particle. Then the auxilarty particles again reflect the flash of light so that it again meets the central particle and the other auxilary particle, and so on, see Fig. 1. For this construction we assume that the two satellites are "near" to the central particle which means that no curvature effects should be involved. This is a condition which can always be fulfilled.

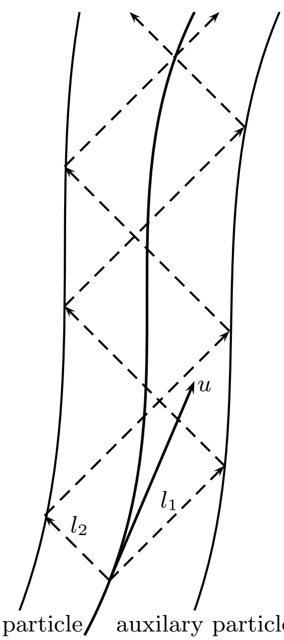

Fig. 1. Geometry of the *bouncing photon.* Two auxilary particles communicate via light rays in such a way that all light rays (dashed lines) have to meet the central particle. No particle is assumed to be in geodesic motion.

It is not assumed that the central particle moves along a geodesic. And even if the path of this particle is geodesic, then the two auxilary particles in general are not geodesic because they always have to be re–positioned in order to meet the above construction.

In this way the particles define a time–like 2–surface, or, after projection into the rest space of the central particle, a direction in the rest space propagating along the path of the central particle. This construction defines the propagation of the direction of the light rays in the rest space of the central particle. It turns out that this propagation can be used to define the notion of a "non–rotating" propagation of a vector. This notion coincides with the notion of Fermi–displacement which usually is used for the description of non–rotating propagation. However, in our case we get this notion from an *operational* procedure.

We now turn to the mathematical description of this procedure. We make use of the equation of motion (1). The condition that the light rays with tangent l_1 and l_2 lie in the same plane with the central particle u is

$$u = \sigma_1 l_1 + \sigma_2 l_2 \,, \qquad \text{for some } \sigma_1, \sigma_2 \in \mathbb{R} \,. \tag{2}$$

The condition that, after reflection, the light rays will cross the central worldline again, is secured by

$$\mathcal{L}_{l_1} l_2 = \epsilon_1 l_2 + \epsilon_2 l_1 \,, \qquad \text{for some } \epsilon_1, \epsilon_2 \in \mathbb{R} \,. \tag{3}$$

Now we derive the equation which governs the transport of the directions

$$V_1 := P_u l_1 \quad \text{or} \quad V_2 := P_u l_2 \,, \tag{4}$$

along u, where

$$P_u A := A - \frac{g(A,u)}{g(u,u)} u \tag{5}$$

is the projection operator onto the rest space of the world line of the central particle.

Using $\sigma_1 V_1 = -\sigma_2 V_2$ we get

$$P_{V_1} P_u D_u V_1 = \frac{1}{\sigma_1} P_{V_1} P_u D_u (\sigma_1 V_1) = \frac{1}{2\sigma_1} P_{V_1} P_u D_u (\sigma_1 V_1 - \sigma_2 V_2) \,. \tag{6}$$

Inserting (2, 3, 4) and the equation of motion (1) for the light rays l_1 and l_2, we finally get

$$P_{V_1} P_u D_u V_1 = 0 \,. \tag{7}$$

The expression $F_u V := P_V P_u D_u V$ is the so–called *Fermi–derivative* of the vector V along u. Eq.(7) is invariant against reparametrization of the paths, so that it indeed describes the propagation of the direction V. This is precisely the characterization of the bouncing photon which we take as definition for a non–rotating propagation of a vector along a given path [1]. If for another vector W defined along P the above expression does not vanish, then the operator Ω, which is a $\binom{1}{1}$-tensor defined by $P_W P_u D_u W = \Omega(P_u W)$, is called the *rotation* of W. This is the characterization of the notion "rotation" we announced.

Now we turn to the question whether, beside this bouncing photon, there are other physical realizations of a non–rotating propagation.

3 Equation of Motion for Angular Momentum

There are indeed two further realizations for a non–rotating propagation which are experimentally easier to handle with than the bouncing photon: One realization is given by a rotating gyroscope possessing orbital angular momentum, the other is the elementary particle with spin.

3.1 The Motion of Gyroscopes

The metrical energy–momentum tensor $T^{\mu\nu}$ which, within Einstein's theory, is the source of the gravitational field, is symmetric and divergence–free

$$D_\mu T^{\mu\nu} = 0\,. \tag{8}$$

This is the equation of motion for matter in General Relativity.

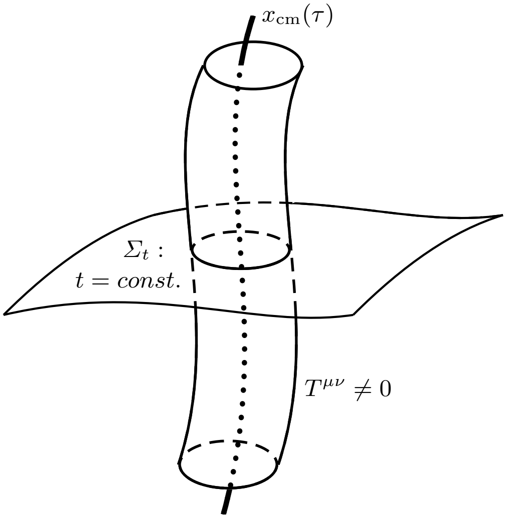

Fig. 2. The world–tube of space–time points for which the energy–momentum tensor in non–vanishing. The center–of–mass wordline is dotted.

In the case that matter is fairly localized (that is, $T^{\mu\nu} \neq 0$ for a compact space–like region only, within which the gravitational field in terms of the curvature does not vary too much), it is possible to extract from this tensor a center–of–mass and correspondingly the equation of motion of this center–of–mass. We assume that this center–of–mass $x_{\rm cm}(\tau)$, where τ is the proper time, lies within the body, that is, $\{x_{\rm cm}(\tau) | \tau \in \mathbb{R}\} \subset \operatorname{supp} T^{\mu\nu}$, see Fig.2. With this center–of–mass worldline there is connected the 4–velocity $v = \frac{d}{d\tau} x_{\rm cm}$.

In addition, one can define an angular momentum with respect to this distinguished point is space–time and derive its temporal evolution. In doing so we follow the procedure originated by Papapetrou [2,3] and developed further by Dixon and [4] and Ehlers and Rudolph [5].

We first introduce a 3+1–slicing of the space–time by introducing hypersurfaces Σ_t with normal n_μ, see Fig. 2. From the energy momentum tensor we can define various moments ($\mathfrak{T}^{\mu\nu} = \sqrt{-g}\, T^{\mu\nu}$)

$$P^{\mu_1\mu_2\ldots\mu_n\nu} := \int_{\Sigma_t} \delta x^{\mu_1} \delta x^{\mu_2} \cdots \delta x^{\mu_n} \mathfrak{T}^{\nu 0} d^3x\,, \tag{9}$$

$$M^{\rho_1\rho_2\ldots\rho_n\mu\nu} := \int_{\Sigma_t} \delta x^{\rho_1} \delta x^{\rho_2} \cdots \delta x^{\rho_n} \mathfrak{T}^{\mu\nu} d^3x\,, \tag{10}$$

where $\delta x^\mu = x^\mu - x^\mu_{\rm cm}$ with $x, x_{\rm cm} \in \Sigma_t$. We use coordinates so that $t = const.$ over Σ_t. Consequently, $\delta x^0 = 0$ in (9) and $n_\mu = \delta^0_\mu$.

We can distinguish between various types of particles: If all momenta (9) but the symmetrical

$$M^{\mu\nu} = \int_{\Sigma_t} \mathfrak{T}^{\mu\nu} d^3x \tag{11}$$

vanish, then we have a particle which possesses no internal structure and thus is called a *point particle*. It will become clear in the following that in this case

$$P^\mu = \int_{\Sigma_t} \mathfrak{T}^{\mu 0} d^3x \tag{12}$$

can be identified with the momentum of that point particle. — If all momenta but $M^{\mu\nu}$ and

$$M^{\mu\nu\rho} = \int_{\Sigma_t} \delta x^\mu \mathfrak{T}^{\nu\rho} d^3x \tag{13}$$

vanish, then we have a particle with mass and, in addition, an orbital angular momentum which in this case is given by

$$L^{\mu\nu} := 2P^{[\mu\nu]} = 2\int_{\Sigma_t} \delta x^{[\mu} \mathfrak{T}^{\nu]0} d^3x \, . \tag{14}$$

That this quantity indeed describes the angular momentum will become clear later, too. This quantity characterizes a special type of internal motion, so that we call this type of matter a *spinning particle* (please note, that this notion "spin" does not mean the elementary particle spin), or spinning top. It can be shown that the order of the highest moment is an invariant [2]. That means that a spinning particle cannot become a point particle by a coordinate transformation.

We assume in the following that all other moments but (11) and (13) vanish. From these definitions and the basic equation of motion (8) we first derive the equations of motion for a point particle, and second for a spinning particle.

The point particle. From (8) we have $0 = \partial_0 \mathfrak{T}^{\mu 0} + \partial_i \mathfrak{T}^{\mu i} + \{ {}^{\,\mu}_{\nu\sigma} \} \mathfrak{T}^{\nu\sigma}$ from which we get by integration

$$\frac{d}{dt} P^\mu = \int_{\Sigma_t} \partial_0 \mathfrak{T}^{\mu 0} d^3x = -\{ {}^{\,\mu}_{\nu\sigma} \} M^{\nu\sigma} \, . \tag{15}$$

In an analogous way we analyze the quantity $\int x^\rho \mathfrak{T}^{\mu\nu} d^3x$. With (8) we get

$$\partial_0 (x^\rho \mathfrak{T}^{\mu 0}) + \partial_i (x^\rho \mathfrak{T}^{\mu i}) = \mathfrak{T}^{\mu\rho} - x^\rho \{ {}^{\,\mu}_{\nu\sigma} \} \mathfrak{T}^{\nu\sigma} \tag{16}$$

and thus, by integration,

$$M^{\mu\rho} = \frac{d}{dt} \int_{\Sigma_t} x^\rho \mathfrak{T}^{\mu 0} d^3x + \int_{\Sigma_t} x^\rho \{ {}^{\,\mu}_{\nu\sigma} \} \mathfrak{T}^{\nu\sigma} d^3x \, . \tag{17}$$

We expand x and $\{ {}^{\,\mu}_{\nu\sigma} \}$ around the coordinate of the worldline $x_{\rm cm}$

$$x = x_{\rm cm} + \delta x \, , \qquad \{ {}^{\,\mu}_{\nu\sigma} \}(x) = \{ {}^{\,\mu}_{\nu\sigma} \}(x_{\rm cm}) + \delta x^\kappa \partial_\kappa \{ {}^{\,\mu}_{\nu\sigma} \}(x_{\rm cm}) \tag{18}$$

and get, taking into account the condition for a point particle, $\int \delta x \mathfrak{T} d^3x = 0$,

$$M^{\mu\rho} = v^\rho P^\mu \, . \tag{19}$$

Since $M^{\mu\nu}$ is symmetric, we must have $P^\mu = mv^\mu$ with $m := P^0/v^0$. Therefore $M^{\mu\nu} = m\ v^\mu v^\nu$, and, consequently, we get from (15) $D_v(mv) = 0$. From this and the fact that v is a normalized 4–velocity, $g(v,v) = -1$, we get $D_v m = 0$ which means that m is the mass of the particle which is constant. Then we also have the geodesic equation for the center–of–mass trajectory:

$$D_v v = 0\,. \tag{20}$$

The spinning particle. This kind of particle is defined by $\int \mathfrak{T}^{\mu\rho} d^3x \neq 0$ and $\int \delta x^\mu \mathfrak{T}^{\nu\rho} d^3x \neq 0$; all other moments vanish. We consider the divergences $\partial_\rho \mathfrak{T}^{\mu\rho}$, $\partial_\rho(x^\mu \mathfrak{T}^{\nu\rho})$, and $\partial_\rho(x^\mu x^\nu \mathfrak{T}^{\sigma\rho})$ and get in a way analogous to above

$$\frac{d}{dt} M^{\mu 0} = -\{{}^{\ \mu}_{\rho\sigma}\} M^{\rho\sigma} - \partial_\kappa \{{}^{\ \mu}_{\rho\sigma}\} M^{\kappa\rho\sigma}\,, \tag{21}$$

$$\frac{d}{dt} M^{\mu\nu 0} = M^{\mu\nu} - v^\mu M^{\nu 0} - \{{}^{\ \nu}_{\rho\sigma}\} M^{\mu\rho\sigma}\,, \tag{22}$$

$$v^\mu M^{\nu\sigma 0} + v^\nu M^{\mu\sigma 0} = M^{\mu\nu\sigma} + M^{\nu\mu\sigma}\,, \tag{23}$$

where all Christoffel symbols are evaluated at the center–of–mass position $x_{\rm cm}$ and where, in addition to (11), we used (13). With definition (13) we also have (note $\delta x^0 = 0$)

$$L^{\mu\nu} = M^{\mu\nu 0} - M^{\nu\mu 0}\,, \tag{24}$$

$$L^{\mu 0} = M^{\mu 00}\,. \tag{25}$$

Now we first express, using (23), $M^{\mu\nu\rho}$ in terms of v and L, and second, using (22), the propagation of L, and, at last, with (21) the equation for the center–of–mass motion.

Cyclic permutation of the three indices in (23) and adding two and subtracting the third relation gives

$$2M^{\mu\nu\rho} = v^\mu(M^{\nu\rho 0} + M^{\rho\nu 0}) + v^\nu L^{\mu\rho} + v^\rho L^{\mu\nu}\,, \tag{26}$$

where we used (24). By specifying $\sigma = 0$ in (23), we can express the first part also in terms of the angular momentum, $M^{\mu\nu 0} + M^{\nu\mu 0} = v^\mu L^{\nu 0} + v^\nu L^{\mu 0}$, so that we finally find

$$2M^{\mu\nu\rho} = v^\mu(v^\nu L^{\rho 0} + v^\rho L^{\nu 0}) + v^\nu L^{\mu\rho} + v^\rho L^{\mu\nu}\,. \tag{27}$$

Choosing $\nu = 0$ in (22) and reinserting this into (22) gives

$$0 = \frac{d}{dt} M^{\mu\nu 0} + \{{}^{\ \nu}_{\rho\sigma}\} M^{\mu\rho\sigma} - M^{\mu\nu} + v^\mu \left(v^\nu M^{00} + \frac{d}{dt} M^{\nu 00} + \{{}^{\ 0}_{\rho\sigma}\} M^{\nu\rho\sigma} \right). \tag{28}$$

Antisymmetrization leads to an equation of motion for L:

$$0 = \frac{d}{dt} L^{\mu\nu} - \{{}^{\ \mu}_{\rho\sigma}\} M^{\nu\rho\sigma} + \{{}^{\ \nu}_{\rho\sigma}\} M^{\mu\rho\sigma} + \left[v^\mu \left(\frac{d}{dt} L^{\nu 0} + \{{}^{\ 0}_{\rho\sigma}\} M^{\nu\rho\sigma} \right) - (\mu \leftrightarrow \nu) \right]. \tag{29}$$

Inserting $M^{\mu\rho\sigma}$ in terms of the center–of–mass velocity and the angular momentum, Eq.(27), we obtain a covariant equation of motion for the angular momentum L:

$$0 = D_v L^{\mu\nu} + v^\mu D_v L^{\nu 0} - v^\nu D_v L^{\mu 0} \,. \tag{30}$$

Multiplication with v_ν gives $D_v L^{\mu 0} = -v_\nu D_v L^{\mu\nu} - v_\nu v^\mu D_v L^{\nu 0}$ which can be inserted into (30)

$$0 = D_v L^{\mu\nu} + v^\mu v_\rho D_v L^{\rho\nu} + v^\nu v_\rho D_v L^{\mu\rho} = P_v D_v L^{\mu\nu} \,. \tag{31}$$

This is the equation of motion for the angular momentum.

In a similar fashion [2] we derive from (21) the equation of motion for the path. We get

$$D_v \left(m v^\mu - v_\rho D_v L^{\mu\rho} \right) = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} v^\nu L^{\rho\sigma} \,. \tag{32}$$

By counting the degrees of freedom, it is clear that (31) and (32) are 6 equations for 3 components of v^μ and 6 components of $L^{\mu\nu}$. Therefore we have to reduce the numer of unknown components in the angular momentum. What is still unspecified in our approach is the center–of–mass coordinate. The center–of–mass coordinate can be determined by the so–called *Frenkel condition*

$$0 = L^{\mu\nu} v_\nu = v_\nu \int \delta x^{[\mu} \mathfrak{T}^{\nu]0} d^3 x \,, \tag{33}$$

which leads to an expression of the form $0 = \int \rho \delta x^\mu d^3 x +$ relativistic corrections, where $\rho = \mathfrak{T}^{00}$ is the energy density, see [5], e.g., for a detailed treatment of the center–of–mass problem.

If the Frenkel condition is valid, then it makes sense to introduce a vector for the angular momentum $L^\mu := \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} v_\nu L_{\rho\sigma}$. In terms of this vector, Eq.(30) means that the angular momentum vector is Fermi propagated,

$$F_v L = P_v D_v L = 0 \tag{34}$$

and thus is non–rotating.

3.2 Motion of an Elementary Particle with Spin $\frac{1}{2}$

A spin–$\frac{1}{2}$–particle ψ is assumed to obey an equation of motion which can be derived from the minimally coupled Lagrangian for the Dirac field in a Riemannian geometry:

$$\mathcal{L} = \sqrt{-g} \left[\frac{i\hbar}{2} \left(\bar{\psi} \gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \gamma^\mu \psi \right) - m \bar{\psi} \psi \right] \,. \tag{35}$$

The parameter m is the mass of the Dirac particle. The matrices γ^μ are given by $\gamma^\mu = h^\mu_a \gamma^a$ where the γ^a are the special relativistic Dirac matrices obeying the Clifford algebra $\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}$ ($\eta^{ab} = \mathrm{diag}(-1, +1, +1, +1)$). The tetrads h^μ_a are defined by $g_{\mu\nu} h^\mu_a h^\nu_b = \eta_{ab}$. Therefore

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \,. \tag{36}$$

D_μ is the covariant spinorial derivative $D_\mu = \partial_\mu + \Gamma_\mu$ with the spinorial connection $\Gamma_\mu := -\frac{1}{2}(D_\mu h^\nu_a) h_{b\nu} G^{ab}$ (the covariant derivative D_μ acts on the vectorial index μ in $D_\mu h^\nu_a$ only). The $G^{ab} := \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ are the generators of the Lorentz–group. The *adjoint* spinor is defined by $\bar\psi := \psi^+ \gamma^{(0)}$ (in this case $\bar\psi\psi$ transforms as a scalar). Here $\gamma^{(0)}$ is the zeroth special relativistic Dirac–matrix.

Variation of the above Lagrangian with respect to $\bar\psi$ gives the field equation

$$0 = i\hbar \gamma^\mu D_\mu \psi - m\psi \,. \tag{37}$$

This is the Dirac equation in curved space–time. Here we use $c = 1$.

Now we describe a particle in a quasiclassical approximation. That means, we look for a solution of the Dirac equation (37) which locally has the form of a plane wave:

$$\psi = e^{\frac{i}{\hbar} S(x)} a(x) \,. \tag{38}$$

Inserting this ansatz into the field equations (37) gives

$$0 = -(\gamma^\mu \partial_\mu S + m) a + i\hbar \gamma^\mu D_\mu a \,. \tag{39}$$

The main step of the quasiclassical approximation consists in the assumption that the external fields are weak enough, so that, in first approximation, the derivatives of the amplitudes can be neglected. That means $(-\gamma^\mu \partial_\mu S + m)a + i\hbar\gamma^\mu D_\mu a \approx (-\gamma^\mu \partial_\mu S + m)a$, or $|\hbar\gamma^\mu D_\mu a| \ll |ma|$, where $|\cdot|$ denotes some norm on a complex vector space. If we use this condition, then we get from (39) with $p_\mu := -\partial_\mu S$

$$0 = (\gamma^\mu p_\mu - m) a \,. \tag{40}$$

Here p_μ is the momentum of the plane wave. Eq (40) is an algebraic condition which possesses a solution for p_μ if and only if the determinant of the coefficient matrix vanishes, $0 = \det(\gamma^\mu p_\mu - m)$. This leads to the condition

$$0 = \left(g^{\mu\nu} p_\mu p_\nu + m^2\right)^2 \,. \tag{41}$$

(The exponent 2 characterizes the fact that we have for both spin states the same mass shell.) Eq (41) is a Hamilton–Jacobi partial differential equation for the phase $S(x,t)$ which always possess a solution.

From the plane wave ansatz (38) we can define a wave packet by superposition of plane waves from a continuous spectrum of momenta peaked around $\widehat{p}_\mu$. Then one can show that the tangent vector of the path of the peak of this wave packet is given by the group velocity $v^\mu := \frac{1}{m} g^{\mu\nu} p_\nu |_{p=\widehat{p}}$ which fulfills the normalization condition, $g(v,v) = -1$. Differentiating (41) once more yields immediately the geodesic equation for this group velocity

$$0 = D_v v \,. \tag{42}$$

The integral curves of this geodesic equation are the paths of the peaks of wave packets.

If we have a solution of the first order equation, then the first part in (39) vanishes and we get as equation for the next order of approximation

$$0 = \gamma^\mu D_\mu a\,. \tag{43}$$

What we are looking for is a propagation equation for the amplitude, that is, an equation of the form $D_v a = f(x, v)a$ which describes the evolution of a along the path given by v. For this we multiply (43) with $(\gamma^\nu p_\nu + m)/m$ and get

$$\begin{aligned} 0 &= \frac{1}{m}(p_\nu\gamma^\nu + m)\gamma^\mu D_\mu a \\ &= \frac{1}{m}\left(\gamma^\mu \left(D_\mu \left((p_\nu\gamma^\nu + m)a\right) - D_\mu(\gamma^\nu p_\nu + m)a\right) + p_\nu[\gamma^\nu, \gamma^\mu]D_\mu a\right) \\ &= -D_\mu v^\mu a - 2v^\mu D_\mu a\,. \end{aligned} \tag{44}$$

Here we used $\partial_{[\mu}p_{\nu]} = 0$. For obtaining this result, the existence of a Clifford algebra is important. With the definition for the expansion $\theta := D_\mu v^\mu$ we finally find

$$D_v a = -\frac{1}{2}\theta a\,. \tag{45}$$

The same holds true for the adjoint spinor: $D_v\bar{a} = -\frac{1}{2}\theta\bar{a}$. Within the frame of the theory of congruences [6] (see also the Appendix), θ is interpreted as the divergence of the trajectories given by the phase $S(x)$. If we define a normalized spinorial amplitude $b := a/\sqrt{\bar{a}a}$, then we get [7,8]

$$D_v b = 0\,, \qquad D_v\bar{b} = 0\,. \tag{46}$$

That means that the normalized spinors b and $\bar{b}$ are parallely propagated along the path of the center of the wave packet.

With these propagation equations for the spinors b and $\bar{b}$, we can calculate propagation equations for the bilinears [9] $S := \bar{b}b$, $P := \bar{b}i\gamma_5 b$, $j^\mu := \bar{b}\gamma^\mu b$, $S^\mu := \bar{\gamma}_5\gamma^\mu b$, and $S^{\mu\nu} := \bar{b}2iG^{\mu\nu}b$. Using (40) one can derive

$$Sp^\mu = mj^\mu\,, \tag{47}$$

$$S^\mu p_\mu = 0\,, \tag{48}$$

$$P = 0\,, \tag{49}$$

$$\epsilon^{\mu\nu\rho}{}_\sigma S^\sigma p_\rho = -mS^{\mu\nu}\,. \tag{50}$$

The last relation can be inverted:

$$v^a S^b - v^b S^a = \frac{1}{2}\epsilon^{ab}{}_{cd}S^{cd}\,. \tag{51}$$

From these identities we get an interpretation of these bilinear quantities. Eq.(47) yields $j^a = Sv^a$ so that S is the intensity of the Dirac field. Since S^a is an axial vector and, according to (48) a rest–frame quantity, it is identified with the spin of the Dirac particle.

Therefore, the only independent normalized quantities are the normalized current $\hat{j}^\mu = \bar{b}\gamma^\mu b = v^\mu$ and the normalized spin–vector $S^\mu = \bar{b}\gamma_5\gamma^\mu b$. The propagation equations (46) for b and $\bar{b}$ then give propagation equations for j^μ and S^μ:

$$D_v v = 0\,, \tag{52}$$
$$D_v S = 0\,. \tag{53}$$

Therefore the direction of the spin is parallely propagated along the path of the Dirac particle. The spin behaves in the same way as a spinning top. (For a gravitational theory with torsion it can be shown that the spin couples to torsion while the orbital angular momentum does not [10].)

We note without proof that in the next approximation it is possible to get an influence of the spin on the path of the wave packet [8]:

$$D_v v = \frac{1}{2}\lambda_{\mathrm{C}} R^*(\cdot, v, S, v)\,, \tag{54}$$

where R^* is the right–dual of the curvature tensor and $_{\mathrm{C}}$ the Compton wavelength of the Dirac particle.

4 Gravitational Field of a Rotating Body

In this section we want to derive the general features of a gravitational field which is created by a rotating body. The gravitational field, that is, the space–time metric $g_{\mu\nu}$, is given by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^2}\,T_{\mu\nu}\,, \tag{55}$$

where $G = 6.673(10) \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is Newton's gravitational constant [11]. We now analyze two aspects of a rotation in the gravitational field: (i) we discuss the gravitational field of an arbitrary stationary situation and (ii) discuss the general structure of the gravitational field created from a rotating mass given in form of the energy momentum tensor.

4.1 Stationary Gravitational Field

A stationary gravitational field is characterized by a time–like Killing vector ξ, $g(\xi,\xi) < 0$, with $\pounds_\xi g = 0$, whereas a gravitational field with an axial symmetry is characterized by a space–like Killing vector η, $g(\eta,\eta) > 0$, with $\pounds_\eta g = 0$ and the integral curves of η are space–like closed curves. An axisymmetric space–time possess both Killing vectors ξ and η which, in addition, should commute $[\xi,\eta] = 0$. For any Killing vector field then there exists a coordinate system, which coordinate lines are the integral curves of the Killing vector field, so that the metric does not depend on the coordinates corresponding to the Killing field. For a stationary gravitational field this means $g(x) \stackrel{*}{=} g(x^1, x^2, x^3)$ and, if the gravitational field possesses an additional axial symmetry, then $g(x) \stackrel{*}{=} g(x^1, x^3)$, where x^0 plays the role of the time coordinate and x^2 the role of the angle φ. Therefore $ds^2 = g_{\mu\nu}(x^1, x^3)dx^\mu dx^\nu$. If we choose $x^1 = \rho$ and $x^3 = \vartheta$, then

$$ds^2 \stackrel{*}{=} g_{00}(\rho, \vartheta)\, dt^2 + 2g_{0i}(\rho, \vartheta)\, dt\, dx^i + g_{ij}(\rho, \vartheta)\, dx^i\, dx^j\, . \tag{56}$$

We call a gravitational field static if the rotation of the time–like Killing congruence vanishes (cf. Appendix): $\omega = 0$ or $\epsilon^{\mu\nu\rho\sigma}\xi_\nu\partial_\rho\xi_\sigma = 0$. The gravitational field is stationary, if the Killing congruence rotates: $\omega \neq 0$ or $\epsilon^{\mu\nu\rho\sigma}\xi_\nu\partial_\rho\xi_\sigma \neq 0$.

The standard example for a static space–time is given by the Schwarzschild solution, and an example for a stationary space–time is given by the Kerr solution, see [12], e.g., or the space–time determined from a thin rotating disk [13].

4.2 Gravitational Field of a Rotating Source

It is intuitively clear what a rotating source is: The source consists of a set of point particles (a gas, or a rigid body, for example), which form a rotating congruence. The particles of the source may interact with one another. Therefore, the source is a conguence of point particles moving on trajectories with 4–velocity u. This congruence may possess rotation, acceleration, expansion, and shear. If we have, as a very simple example, a perfect fluid, then we have as source of the gravitational field the energy–momentum tensor $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$, where ρ is the energy density and p the pressure. For $p = 0$ (dust) one can show that the geodesic equation (1) follows from (8). If the vector field u belongs to a rotating congruence, then this energy–momentum tensor describes a rotating source.

We now calculate the gravitational field which is created by such a rotating source. For this purpose, we split the metric into two parts, $g = g_0 + g_1$. The curvature associated with g can be split into a term corresponding to g_0 and terms depending on g_1 [12]: $R(g) = R(g_0) + \delta R(g_0, g_1)$. We assume that the curvature associated with g_0 vanishes, $R(g_0) = 0$. From the Einstein equations (55) we finally get a differential equation for the part $\widetilde{g}_1 := g_1 - \frac{1}{2}\bar{g}_1 g_0$ with $\bar{g} := g_0^{\mu\nu} g_{1\mu\nu}$:

$$\Box\, \widetilde{g}_1 = \kappa T\, , \tag{57}$$

where $\Box$ is the d'Alambertian with respect to the metric g_0 and where we have chosen a coordinate system such that $\partial_\nu g_1^{\mu\nu} = 0$, with $g_1^{\mu\nu} := g_0^{\mu\rho} g_0^{\nu\sigma} g_{1\rho\sigma}$. Because the curvature associated with g_0 vanishes, it is possible to introcduce a global coordinate system such that $g_{0\mu\nu} = \eta_{\mu\nu}$. If the source is stationary, then g_1 does not depend on the time and (57) reduces to the Poisson equation $\Delta\, \widetilde{g}_1 = \kappa T$ which can be integrated,

$$\widetilde{g}_1 = \kappa \int \frac{T(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3x'\, , \tag{58}$$

provided T falls off appropriately at spatial infinity.

The component $T_{00} =: \rho$ of the energy–momentum tensor is interpreted as the energy density, the components $T_{i0} = T_{i0} =: \rho v_i$ as the energy flux, and the T_{ij} as the stress tensor $(i, j, \ldots = 1, 2, 3)$. Therefore,

$$(\widetilde{g}_1)_{00} = G \int \frac{\rho(t, \boldsymbol{r}')}{|\boldsymbol{x} - \boldsymbol{r}'|} d^3x' \,, \tag{59}$$

$$(\widetilde{g}_1)_{i0} = G \int \frac{\rho(t, \boldsymbol{r}') v_i(t, \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3x' \,, \tag{60}$$

$$(\widetilde{g}_1)_{ij} = G \int \frac{T_{ij}(t, \boldsymbol{r}')}{|\boldsymbol{x} - \boldsymbol{r}'|} d^3x' \,. \tag{61}$$

It is clear that $(\widetilde{g}_1)_{i0}$ is smaller than $(\widetilde{g}_1)_{00}$ by a factor v/c and $(\widetilde{g}_1)_{ij}$ by a factor $(v/c)^2$, cf. [12]. In the case of an isolated body and large distances, we have

$$U := (\widetilde{g}_1)_{00} = G \, \frac{M}{r} \,, \qquad h_i := (\widetilde{g}_1)_{i0} = -\frac{G}{2} \, \frac{(\boldsymbol{r} \times \boldsymbol{L})_i}{r^3} \,, \tag{62}$$

where M is the total mass and $\boldsymbol{L}$ the angular momentum of the gravitating body.

In a coordinate system where the components of the metric are isotropic, we have as line element

$$ds^2 = -(1 - 2U + 2U^2)\, dt^2 + (1 + 2U)(dx^2 + dy^2 + dz^2) - 4h_i \, dx^i \, dt \,, \tag{63}$$

or

$$g_{\mu\nu} = \begin{pmatrix} -1 + 2U - U^2 & -2h_i \\ -2h_i & (1 + 2U)\delta_{ij} \end{pmatrix} . \tag{64}$$

This metric is time–independent. Thus a time–like Killing vector $\xi = \partial_t$ exists, in components $\xi^\mu = \delta^\mu_0$. The different components of this vector ξ are given by

$$\xi^\mu = \delta^\mu_0 \,, \qquad \xi_0 = g_{0\nu}\xi^\nu = g_{00} \,, \qquad \xi_i = g_{i\mu}\xi^\mu = g_{0i} = -2h_i \,. \tag{65}$$

Hence the curl of the Killing vector field (see Appendix) is connected with $\partial_{[i}h_{j]}$.

The tetrads ϑ^a (one–forms, $a = 0, \ldots 3$) connected with the Killing vector field ξ are given by

$$\vartheta^{(0)}_0 = -1 + U - \frac{1}{2}U^2 \,, \qquad \vartheta^{(0)}_i = -h_i \tag{66}$$

$$\vartheta^{\hat{a}}_0 = -h_{\hat{a}} \,, \qquad \vartheta^{\hat{a}}_i = (1 + U)\delta^{\hat{a}}_i \,. \tag{67}$$

Later we need to boost this tetrad to a comoving (with the gyroscope) tetrad $\bar{\vartheta}^a = L^a{}_b(\dot{x})\vartheta^b$ where $\dot{x}$ is the relative velocity of the gyroscope with respect to the tetrad ϑ^a. Since for a pure boost with small velocity v we have $L = \begin{pmatrix} 1 - \frac{1}{2}\dot{x}^2 & \dot{x} \\ \dot{x} & \delta^i_j + \frac{1}{2}\dot{x}^i\dot{x}_j \end{pmatrix}$, we find for the new tetrad

$$\bar{\vartheta}^{(0)}_0 = -1 + U - \frac{1}{2}U^2 - \frac{1}{2}\dot{x}^2 \,, \qquad \bar{\vartheta}^{(0)}_i = -h_i + \dot{x}_i \tag{68}$$

$$\bar{\vartheta}^{\hat{a}}_0 = -h_{\hat{a}} + \dot{x}_{\hat{a}} \,, \qquad \bar{\vartheta}^{\hat{a}}_i = (1 + U)\, \delta^{\hat{a}}_i + \dot{x}^{\hat{a}}\dot{x}_i \,. \tag{69}$$

5 Lense–Thirring Effect

Let us combine the results and notions derived above in order to describe the dynamics of a rotating test body in the neighborhood of a gravitating rotating body. We consider a stationary situation: The gravitating body rotates with a constant angular velocity. In the field of such a body we will consider the motion of a point particle as well as the motion of a gyroscope.

5.1 Motion of a Point Particle

The equation of motion for a point particle is the geodesic equation $D_v v = 0$. The four components of this equation can be evaluated by using the solution for a static spherically symmetric mass distribution, that is the Schwarzschild solution, or the solution for rotating bodies, like the Kerr metric or the metric for a rotating disk of dust [13]. However, in our approach, we restrict ourselves to the case of a weak stationary gravitational field. In this case, for a point particle without spin and in 3–notation, the geodesic equation reads

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{\nabla} U + \boldsymbol{F} - 2\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{h}) \,. \tag{70}$$

The first term is the Newtonian part whereas $\boldsymbol{F}$ symbolizes nonlinear contributions of the gravitostatic field U which are responsible for the perihelion shift, for example. The last term is the gravitomagnetic part due to the rotation of the gravitating body. Since the motion of a satellite around the earth represents a gyroscope, too, this interaction results in a precession of the angular momentum of the satellite around the earth. Therefore, the plane of the path of the satellite is no longer fixed, as it is in the Schwarzschild case, but starts to precess instead. Thus the pericenter (perihelion) or the nodes (intersections of the paths of the satellite with the equatorial plane of the earth) move. This should be observable in an experiment proposed by Ciufolini [14] according to which two excentric satellites orbit around the earth.

5.2 Motion of a Gyroscope

For the description of the motion of a direction attached to a gyroscope we use Eqs.(34) and (32) or (53) and (54). For simplicity we assume that the path is geodesic which is very well fulfilled because, according to Eqs.(32) and (54), all non–geodesic terms can be neglected for weak gravitational fields.

The non–rotating frame defined by the gyroscope will be compared with a direction given by a fixed star far away from the gravitating body. The light from that star comes from a fixed direction. Thus the tangent vector l of light rays of this star is stationary: $\pounds_\xi l = 0$. Accordingly, we can introduce a stationary space–like unit vector: $e_{(1)} = P_u l/(g(P_u l, P_u l))^{1/2}$, which again is stationary $\pounds_\xi e_{(1)} = 0$. We can complete this unit vector to give a 3–bein by adding two more spatial unit vectors which are orthogonal to $e_{(1)}$ and to each other and which are stationary, too: $g(e_{\hat a}, e_{\hat b}) = \delta_{\hat a\hat b}$, $\pounds_\xi e_{\hat a} = 0$. If we take $e_{(0)} = u$, then $e_a = (u, e_{\hat a})$ defines a tetrad with $\pounds_\xi e_a = 0$. Accordingly, also for the dual basis ϑ^a we have $\pounds_\xi \vartheta = 0$.

In order to determine the behaviour of the gyroscope with respect to the direction given by the fixed star, we project the spin of the gyroscope on the comoving basis connected with the fixed star: $\bar S^a := \bar\vartheta^a(S)$. This projection is the quantity observed. We calculate the dynamics of this projection $\dot{\bar S}^a = \partial_v \bar S^a$, where S is parallely displaced along the path of the gyroscope, while the basis ϑ^a is Lie–displaced along ξ:

$$\begin{array}{ccc} \bar\vartheta & \text{Lorentz–transf. } L^a{}_b & \vartheta \\ \text{comoving with gyro} & \longrightarrow & \text{attached to distant stars} \\ \updownarrow & & \updownarrow \\ D_v S = 0 & & \pounds_\xi \vartheta = 0 \end{array}$$

The four–velocity of the gyroscope is related to the fourth leg u via $v = \gamma u + \gamma \dot x^{\hat a} e_{\hat a}$ where γ is the Lorentz factor $(1 - \dot x^2)^{-1/2}$ and $\dot x^{\hat a}$ is the relative velocity measured between u and v. Moreover, because of $\bar\vartheta^{(0)}(S) = 0$, we have $0 = L^{(0)}{}_b \vartheta^b(S) = L^{(0)}{}_{(0)} \vartheta^{(0)}(S) + L^{(0)}{}_{\hat a} \vartheta^{\hat a}(S)$ so that

$$\vartheta^{(0)}(S) = -\dot x_{\hat a} \vartheta^{\hat a}(S)\,. \tag{71}$$

We calculate, using (34) and (53), respectively, ($\hat a = 1, 2, 3$),

$$\begin{aligned} \dot{\bar S}^{\hat a} &= D_v(\bar\vartheta^{\hat a}(S)) \\ &= (D_v \bar\vartheta^{\hat a})(S) + \bar\vartheta^{\hat a}(D_v S) \\ &= (D_v \bar\vartheta^{\hat a})(S) \\ &= (D_v(L^{\hat a}{}_b \vartheta^b))(S) \\ &= D_v L^{\hat a}{}_b \vartheta^b(S) + L^{\hat a}{}_b (D_v \vartheta^b)(S)\,. \end{aligned} \tag{72}$$

The first term can be evaluated by using $L^a{}_b = \begin{pmatrix} 1 + \frac12 \dot x^2 & \dot x_{\hat b} \\ \dot x^{\hat a} & \delta^{\hat a}_{\hat b} + \frac12 \dot x^{\hat a} \dot x_{\hat b} \end{pmatrix}$ and $D_v L^a{}_b = \begin{pmatrix} \dot x_{\hat c} \ddot x^{\hat c} & \ddot x_{\hat b} \\ \ddot x^{\hat a} & \frac12 (\ddot x^{\hat a} \dot x_{\hat b} + \dot x^{\hat a} \ddot x_{\hat b}) \end{pmatrix}$. With (71) this yields

$$D_v L^{\hat a}{}_b \vartheta^b(S) = D_v L^{\hat a}{}_{(0)} \vartheta^{(0)}(S) + D_v L^{\hat a}{}_{\hat b} \vartheta^{\hat b}(S) \approx -\frac12 (\dot v^{\hat a} v_{\hat b} - v^{\hat a} \dot v_{\hat b}) \bar\vartheta^{\hat b}(S)\,. \tag{73}$$

For the seond term

$$L^{\hat a}{}_b (D_v \vartheta^b)(S) = L^{\hat a}{}_b (D_{\gamma u + \gamma \dot x} \vartheta^b)(S) \approx L^{\hat a}{}_b (D_u \vartheta^b)(S) = L^{\hat a}{}_b e^{-U} (D_\xi \vartheta^b)(S) \tag{74}$$

we use the fact that the frame ϑ is stationary: $\pounds_\xi \vartheta^a = 0$. In components: $0 = \xi^\nu D_\nu \vartheta^a_\mu + \vartheta^a_\nu D_\mu \xi^\nu$, so that $(D_\xi \vartheta^a)_\mu = -\vartheta^a_\nu D_\mu \xi^\nu$. Thus

$$(D_\xi \vartheta^a)_\mu = -\vartheta^a_\nu D_\mu \xi^\nu = -\vartheta^a_\nu (e^U \omega_\mu{}^\nu + \overset{\xi}{a}_\mu \xi^\nu - \overset{\xi}{a}{}^\nu \xi_\mu)\,. \tag{75}$$

Therefore we find for the total precession of the spin

$$\begin{aligned}\dot{\bar{S}}^{\hat{a}} &= -\frac{1}{2}(\ddot{x}^{\hat{a}}\dot{x}_{\hat{b}} - \dot{x}^{\hat{a}}\ddot{x}_{\hat{b}})\,\bar{\vartheta}^{\hat{b}}(S) - L^{\hat{a}}{}_{b}e^{-U}\vartheta^{b}_{\nu}\,(e^{U}\omega^{\nu}{}_{\mu} + \overset{\xi}{a}_{\mu}\xi^{\nu} - \overset{\xi}{a}{}^{\nu}\xi_{\mu})\,S^{\mu} \\ &= -\omega^{a}{}_{b}\bar{S}^{b} - \left[\left(\overset{\xi}{a}{}^{\hat{a}} + \frac{1}{2}\ddot{x}^{\hat{a}}\right)\dot{x}_{\hat{b}} - \dot{x}^{\hat{a}}\left(\overset{\xi}{a}_{\hat{b}} + \frac{1}{2}\ddot{x}_{\hat{b}}\right)\right]\bar{S}^{\hat{b}}\,, \end{aligned} \tag{76}$$

with $u^{\hat{a}} = 0$ and where we neglected terms with "velocity $\times$ gravitomagnetic field" and terms of the order $\dot{x}^2$. Using $\ddot{x}^{\hat{a}} = a^{\hat{a}} + \overset{\xi}{a}{}^{\hat{a}}$, where a is any non–gravitational acceleration, we finally arrive in 3–notation at

$$\frac{d}{d\tau}\boldsymbol{S} = \boldsymbol{\Omega} \times \boldsymbol{S}\,, \tag{77}$$

with

$$\boldsymbol{\Omega} = \boldsymbol{v} \times \left(-\frac{1}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{\nabla}U\right) + \boldsymbol{\nabla} \times \boldsymbol{h}\,. \tag{78}$$

The first term $\boldsymbol{v} \times \boldsymbol{a}$ is a special relativistic term, called the *Thomas precession* which is known from atomic physics. It describes the precession of the spin due to inertial forces. Thus, the second term, $\boldsymbol{v} \times \boldsymbol{\nabla}U$, is a gravity–induced Thomas precession, the so–called *de Sitter precession* or *geodetic precession*. Note that only the Newtonian potential enters this term. The last term is purely post–Newtonian and describes the Lense–Thirring effect. This is the rotation of the locally non–rotating frame with respect to distant fixed stars due to the rotation of a nearby rotating gravitating body ('frame dragging').

6 On the Observation of Gravitomagnetic Effects

The systematic analysis of relativistic effects of planetary motion and motion of the moons of the planets dates back to the first years after the publication of Einstein's theory in 1915. In 1916, W. de Sitter [15] predicted a geodetic precession of the rotating earth-moon system ('earth–moon gyroscope') in the gravitational field of the sun. (The de Sitter term in (78) provides a simple model of the dynamics of that system.) The effect has finally been detected in the late 80's by means of an elaborate combination of lunar ranging and radio interferometry data [16]; refined data can be found in [17]. The accuracy of this verification is of the order of 1%. Three years after de Sitter, J. Lense and H. Thirring published their pioneering work "about the influence of the proper rotation of the central bodies on the motion of the planets and moons according to Einstein's gravitational theory" [18]. Analyzing the equations of motion (70), they excluded measurable effects for the moon orbit as well as for the orbits of the planets, but found considerable secular relativistic perturbations of the orbital parameters of the moons of the outer planets. In particular, Jupiter V evidenced a gravitomagnetic shift of its pericenter of 2.26 arcsec/yr. However, a confirmation of their prediction by observation was not possible at that time.

In the next two sections, we will discuss the theoretical fundamentals of recent satellite experiments. In this context, the earth will be considered to be a sphere (radius R) rotating with a constant angular velocity about its axis, which has a fixed orientation in an inertial system connected with the (distant) stars.

6.1 Lense–Thirring Effect for Point Particles

According to (70) and (62), the equation of motion for a spinless particle (satellite) in the gravitational field of the rotating earth is given by[1]

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{1}{r^3}GM\mathbf{r} + \frac{2G}{c^2r^3}\frac{d\boldsymbol{r}}{dt} \times \left(\boldsymbol{L} - \frac{3\,(\boldsymbol{L}\cdot\boldsymbol{r})}{r^2}\boldsymbol{r}\right) , \qquad (79)$$

where M is the mass of the earth and $\boldsymbol{L}$ is its angular momentum. A detailed discussion of these equations and explicit expressions for the relativistic perturbations of the particle orbit may be found in the original paper of Lense and Thirring [18].

According to the proposal of Ciufolini, it should be possible, with present-day technology, to measure the advance of the pericenter and the nodes of highly eccentric satellites. A first attempt to do this using the LAGEOS I and LAGEOS II satellites, has been carried through in [19–21]. It was possible to verify the gravitomagnetic effect with an precision of about 10%. This poor precision is a result of the low eccentricity of the orbits of the satellites and the difficulties in eliminating the multipoles of the earth which give rise to contributions of comparable order. The idea of a further experiment [14] is to orbit a new LAGEOS satellite with the same orbital parameters as those of an existing LAGEOS, but with supplementary inclinations, and to observe the bisector of the angle between the nodal lines which defines a kind of gyroscope. The expected precision of the verification of the gravitomagnetic effect is of the order of 3% after 3 years of Laser measurements (see also [22]).

6.2 Lense–Thirring Effect for Gyroscopes

Since the early 60's, the Stanford orbiting gyroscope experiment, Gravity Probe B, has been under development [23]. The experimental construction has been completed and the experiment should be performed in 2000/2001. The idea is to put a spacecraft in a polar orbit equipped with four gyroscopes (see Fig.1 on page 53 of Everitts's paper in this volume) and to measure the gravitomagnetic precession of the spins of the gyroscopes. To calculate the expected numerical values for the de Sitter and Lense–Thirring effects, we consider a single gyroscope with spin $\boldsymbol{S}$ at a circular polar orbit. We introduce a co–moving but non–rotating orthogonal coordinate system Σ the z–axis of which is parallel to the earth's angular momentum $\boldsymbol{L}$ and the x–axis of which lies in the orbital plane ($y = 0$), which has a fixed position with respect to the distant stars. The orientation of the co–moving frame can be maintained by two telescopes on board the satellite each of which points at a particular fixed star.

[1] For simplicity, the nonlinear term of (70) has been omitted.

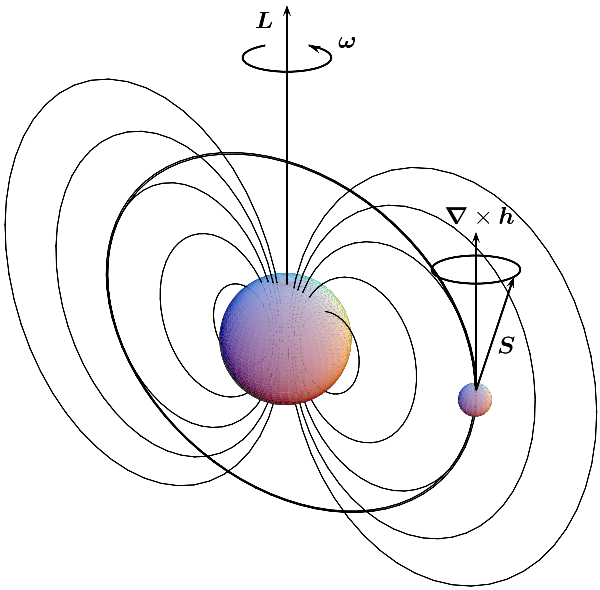

Fig. 3. The Lense–Thirring effect for gyroscopes: The earth rotating with angular velocity $\boldsymbol{\omega}$ and angular momentum $\boldsymbol{L}$ creates a gravitomagnetic field with the shape of a magnetic dipole. A gyroscope with angular momentum or spin $\boldsymbol{S}$ moves around the earth along a geodesic circular polar orbit (thick solid line). The Lense–Thirring effect consists in the precession of $\boldsymbol{S}$ around the direction given by the field lines of $\boldsymbol{\nabla} \times \boldsymbol{h}$.

In order to apply Eqs. (77,78), we have to specify the position vector $\boldsymbol{r}$ of the gyroscope (origin of Σ) and the angular velocity $\boldsymbol{\Omega}$. Obviously,

$$\boldsymbol{r} = (r \cos \omega_0 t, 0, r \sin \omega_0 t) \,. \tag{80}$$

Here r is the constant distance of the gyroscope from the center of the earth and

$$\omega_0 = \frac{1}{r}\sqrt{\frac{GM}{r}} \tag{81}$$

is the orbital angular velocity of the satellite. Then,

$$\boldsymbol{\Omega} = \left(\frac{3GL}{2r^3c^2} \sin 2\omega_0 t, \quad \frac{3}{2}\sqrt{\frac{GM}{r}}\,\frac{GM}{r^2c^2}, \quad \frac{GL}{2r^3c^2}\,(1 - 3\cos 2\omega_0 t) \right) \,. \tag{82}$$

A good approximation for the angular momentum of the earth is

$$L = 0.3306 \cdot MR^2\omega, \tag{83}$$

where $M = (5.974 \pm 0.004)\, 10^{27}$ g is the mass, $R = (6378140 \pm 5)$ m the equatorial radius, and ω the angular velocity of the earth.

Note that the choice of Σ has separated the de Sitter and the Lense–Thirring contributions to $\boldsymbol{\Omega}$: Ω_x and Ω_z are pure Lense–Thirring terms whereas Ω_y is of geodetic origin. After a decomposition of the spin vector $\boldsymbol{S}$ in spherical polar coordinates,

$$S_x = S \cos\varphi \sin\vartheta\,, \quad S_y = S \sin\varphi \sin\vartheta\,, \quad S_z = S\cos\vartheta\,, \quad S = |\boldsymbol{S}|\,, \tag{84}$$

Eqs.(77,78) take the form

$$\dot\varphi = -\Omega_x \cos\varphi \cot\vartheta - \Omega_y \cot\vartheta \sin\varphi + \Omega_z\,, \tag{85}$$

$$\dot\vartheta = -\Omega_x \sin\varphi + \Omega_y \cos\varphi\,. \tag{86}$$

In order to keep the two effects separate, we may start from an equatorial position $(t = 0; \boldsymbol{r} = (r, 0, 0))$ and choose the spin vector $\boldsymbol{S}$ to be perpendicular to the angular momentum $\boldsymbol{L}$ of the earth $(t = 0 : \varphi = 0, \vartheta = \frac{\pi}{2})$. From the linearized Eqs.(85,86)

$$\dot z = i\Omega_x z + (\Omega_y + i\Omega_z)\ , \tag{87}$$

where $z = \vartheta - \frac{\pi}{2} + i\varphi$, we finally obtain the desired secular angular precessions

$$\Delta\vartheta = \Omega_y \Delta t = \frac{3}{2}\sqrt{\frac{GM}{r}}\frac{GM}{r^2c^2}\Delta t \tag{88}$$

$$\Delta\varphi = \overline{\Omega_z}^t \Delta t = \frac{GL}{2r^3c^2}\Delta t\,. \tag{89}$$

Here $\overline{\Omega_z}^t$ is the time–averaged Ω_z as experienced by the gyroscope. For a polar orbit at about 650 km altitude ($r = (6371 + 650)$ km) this leads (note also (83)) to a rate of

$$\frac{\Delta\vartheta}{\Delta t} = 6.6 \text{ arcsec/yr} \tag{90}$$

for the geodetic precession and

$$\frac{\Delta\varphi}{\Delta t} = 0.041 \text{ arcsec/yr} \tag{91}$$

for the Lense–Thirring precession.

The goal for the precision of the Gravity Probe B experiment is about 0.01% for the de Sitter effect and about 1% for the Lense–Thirring effect (in contrast to the Lense–Thirring effect for orbiting point particles the multipole moments of the earth play no role here [24]). The measurement of these effects is based on SQUIDs. The numerical values (90) are illustrated in Fig.1 of Everitt's talk (see p. 53), which, moreover, describes the technicalities o f the sophisticated equipment.

6.3 Lense–Thirring Effect in Quantum Physics

Due to huge improvements in the accuracy of devices based on the quantum properties of matter, it may be useful to estimate the effect of a rotating gravitating body on quantum particles. Two types of effects can be imagined: The effect on a matter wave interferometer and the effect on the spectrum of atoms.

In the first case, the field $\boldsymbol{\Omega}$ acts like a rotation of the interferometer if the interferometer is attached to the fixed stars. The effect of rotation of the interferometer on the phase of the quantum interference is the famous Sagnac effect $\delta\phi_{\text{Sagnac}} = \frac{m}{\hbar}\boldsymbol{\omega} \cdot \boldsymbol{A}$, where $\boldsymbol{\omega}$ is the angular velocity of the interferometer and $\boldsymbol{A}$ its area. Due to great success, e.g., in the cooling of atoms which makes it possible to prepare interfering atoms which stay for a long time inside the interferometer and thus possess a long interaction time, it may be possible to detect the Lense–Thirring effect with atomic interferometry, see [25] for a recent account on the sensitivity of atomic interferometers on rotation.

A realization of this effect is attempted within the HYPER project which is planned to put atomic interferometers in space and to measure, beside the fine structure constant and the quantum gravity induced foam structure of space, the Lense–Thirring effect. For this purpose, two atomic interferometers based on Mg and two based on Cs will be placed in two orthogonal planes. The resolution of rotation rates aimed at is 10^{-14} rad/s for an integration time of 1000 s. Note that, contrary to the GP-B approach where the cumulative effect over apoproximately one year is read out, in this case the *angular velocity* $\boldsymbol{\nabla} \times \boldsymbol{h}$ is measured *locally*. No integration over many days is carried through. The integration takes place for a few minutes only, that is, for a duration, over which the curl $\boldsymbol{\nabla} \times \boldsymbol{h}$ is approximately constant. HYPER is planned to be put into orbit within the next 10 years.

It has been shown that the rotation of the earth has an influence on the spectrum of atoms: While searching for anomalous spin–couplings in atoms [26], one has to compensate for the influence of the earth's rotation on the spin. This in fact establishes [27] an experimental verification of the coupling between spin and rotation, see [28]. However, since the accuracy of this result is not very high, and since the rotation caused by $\boldsymbol{\Omega}$ is about 9 orders of magnitude smaller than the earth's rotation, there is no hope in near future to use this approach for a verification of the Lense–Thirring effect.

Appendix: Theory of Congruences

For a time–like vector field u, $g(u,u) = -1$, which may be interpreted as a field of four velocities being tangents at a set of point–like particles, like dust, we have [6]

$$D_\mu u_\nu = \omega_{\nu\mu} + \sigma_{\nu\mu} + \frac{1}{3}\theta P_{\mu\nu} - u_\mu a_\nu \,, \tag{92}$$

where

$$\omega_{\nu\mu} := (P_u)^\rho_\nu (P_u)^\sigma_\mu D_{[\sigma} u_{\rho]} \,, \tag{93}$$
$$\theta := P_u^{\mu\nu} D_{(\mu} u_{\nu)} \,, \tag{94}$$
$$\sigma_{\nu\mu} := (P_u)^\rho_\nu (P_u)^\sigma_\mu D_{(\sigma} u_{\rho)} - \frac{1}{3}\theta (P_u)_{\mu\nu} \,, \tag{95}$$
$$a := D_u u \,, \tag{96}$$

with the projection operator

$$(P_u)_\mu^\nu := \delta_\mu^\nu + u^\nu u_\mu \,. \tag{97}$$

For a time–like Killing congruence, that is, a congruence the four–velocity of which is proportional to a Killing vector field,

$$\xi = e^U u\,, \qquad \pounds_\xi g = 0\,, \qquad g(u,u) = -1\,, \tag{98}$$

we have $\sigma_{\mu\nu} = 0$ and $\theta = 0$. A Killing congruence possesses only rotation and acceleration. (One can show, that ω is, indeed, a rotation as defined in section 2.) Then the acceleration of u is given by

$$\overset{\xi}{a} := D_u u = dU\,, \qquad D_u U = 0\,, \tag{99}$$

which we get from projecting $0 = D_\mu \xi_\nu + D_\nu \xi_\mu$ onto u^ν and $u^\mu u^\nu$. With this result we get

$$D_\mu \xi_\nu = D_\mu (e^U u_\nu) = e^U \omega_{\nu\mu} + \overset{\xi}{a}_\mu \xi_\nu - \overset{\xi}{a}_\nu \xi_\mu \,. \tag{100}$$

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