

Is there a stable hydrogen atom in higher dimensions?

Frank Burgbacher^{a)}

Fakultät für Physik, Universität Konstanz, Fach M674, D-78457 Konstanz, Germany

Claus Lämmerzahl^{b)}

*Fakultät für Physik, Universität Konstanz, Fach M674, D-78457 Konstanz, Germany
and Laboratoire de Physique des Lasers, Institut Galilée, Université Paris 13,
F-93430 Villetaneuse, France*

Alfredo Macias^{c)}

*Departamento de Física, Universidad Autónoma Metropolitana—Iztapalapa,
Apartado Postal 55-534, C.P. 09340, México, D.F., México*

(Received 23 July 1998; accepted for publication 19 October 1998)

The Schrödinger equation in higher dimensions is considered. It consists of the kinetic energy part given by the corresponding Laplace operator, and a term describing the interaction with the electrostatic field of a point charge. From Rutherford-type scattering experiments one can conclude that the potential of a point charge is $\sim 1/r$ irrespective of the dimension of the space where the experiment is carried through. Also the structure of the kinetic energy is unchanged in higher dimensions so that one is lead to the result that there exist stable atoms in higher spatial dimensions $d \geq 4$. The solutions and energy eigenvalues to this Schrödinger equation in higher dimensions are presented. As a consequence, the dimensionality of space can be read off from the spectral scheme of atoms: The three-dimensionality of space is a consequence of the existence of the Lyman series. Another consequence is that the Maxwell equations in higher dimensions must be modified in order to have the $1/r$ -potential as solution for a point charge.

© 1999 American Institute of Physics. [S0022-2488(99)00502-2]

I. INTRODUCTION

The idea of extra space–time dimensions continues to pervade current attempts to unify the fundamental forces, but in ways somewhat different from that originally envisaged. A modern perspective on the role of internal dimensions in physics comes mainly from the superstring theory, which is the most promising candidate for a unified field theory. The appearance of extra space–time dimensions at high energy scales is a generic feature of string theory. Typically these extra dimensions remain compactified at the Planck scale, but it is possible for new dimensions to have an effect below the Planck scale. In particular, large-radius compactification schemes have recently been discussed in a number of theoretical and phenomenological contexts.^{1,2} Similarly, the effects of extra dimensions below the Planck scale have played a role in understanding the strong-coupling behavior of string theory.³ Even the old pioneer Kaluza–Klein theory is embedded in a super-string theory; their states persist as a subset of the full string spectrum. However, string theory comes to rescue and ensures correct high-energy behavior.⁴ Then, we can regard this theory as an effective “medium” energy model coming from finite string field theories. Therefore, the study of different higher-dimensional models is of importance for the understanding of more general theories.

One of the most interesting questions addressed to the higher-dimensional approaches concerns the stability of atoms in higher spatial dimensions, i.e., $d > 3$. These investigations started

^{a)}Electronic mail: frank.burgbacher@uni-konstanz.de

^{b)}Electronic mail: claus.laemmerzahl@uni-konstanz.de

^{c)}Electronic mail: amac@xanum.uam.mx

with the well-known paper of Ehrenfest⁵ and has inspired many additional interesting investigations. For reviews see Refs. 6 and 7, and for a recent paper on this problem see Ref. 8 where the dimensionality of space–time has been related to physical phenomena which are accessible to experiment.

According to the analysis of Ehrenfest, see also Ref. 9, there are statements in all papers that in higher dimensions it is not possible to have stable atoms. It is one of our purposes in this paper to show that it is indeed possible to have *stable atoms in higher dimensions*. The main point is that first the kinetic energy in the Schrödinger has the usual form described by the d -dimensional Laplacian and that the electrostatic interaction in the Schrödinger equation has the same form irrespective of the spatial dimension. This of course leads to modified Maxwell equations in higher dimensions. While the main characteristics of these new Maxwell equations in higher dimensions remains the same as compared with the Maxwell equations in three dimensions (the solutions have the same structure and the force between charges is the same as in three dimensions), these modified Maxwell equations do not lead to a Gaussian law for charges. This may sound strange but the results of scattering experiments, the stability of atoms in higher dimensions, and the structure of the force between charges is certainly of more basic physical content.

A second point in our paper is that the spectra of atoms are influenced by the spatial dimension. That means, as we shall show, that we *can decide from a spectroscopic experiment the dimension of our configuration space*. To be more concrete, the ratio of the frequencies of two distinguished spectral lines leads to a number from which we uniquely infer the three-dimensionality of our space. If this ratio gives a different number we would be led to four or another number of spatial dimensions.

The most important starting point of our investigation is the structure of the Schrödinger equation in higher dimensions. One way which fixes the kinetic part of the Schrödinger equation is the quantization scheme arising from the Hamilton–Jacobi equation of a point mass which also in higher dimensions has the usual form $E = p^2/2m + V$, where V is some potential energy. In addition, also from a constructive axiomatic scheme (see, for example, Refs. 8 or 10) one gets a Dirac equation in higher dimensions which nonrelativistic limit¹¹ necessarily possesses a kinetic term which is proportional to the Laplace operator. Therefore, any modification of this term would need a modification of the quantization scheme as well as a violation of fundamental properties (like unique evolution, superposition principle, finite propagation speed, etc., see, for example, Ref. 10) of single particle quantum systems. Since these modifications obviously changes physics drastically we do not change the structure of the usual kinetic term.

As far as the potential energy term is concerned we use results from scattering experiments to fix its form. Indeed, since the results of Rutherford-type scattering experiments are independent of the spatial dimension, we can unambiguously conclude from the experimental data, that in any dimension d the potential must be of the form $\sim 1/r$. This is of course consistent with the analysis of Ref. 5 that atoms with the usual kinetic energy coupled to a modified potential of the form $\sim 1/r^{d-2}$ are not stable (the exponent $d-2$ is due to the requirement that Gauss' law should be still valid in higher dimensions). Since our result for the electrostatic potential is not compatible with a Gaussian law for electrostatics, we conclude that we have to modify the structure of Maxwell's equations in higher dimensions.

Consequently, we take as general ansatz for the Hamilton operator for the hydrogen atom in higher dimensions,

$$H = \frac{p^2}{2m} + V(r), \quad (1)$$

where $V(r)$ is the spherically symmetric potential given by

$$V(r) = \frac{\alpha}{r}. \quad (2)$$

In comparison to other work on the problem of physics in higher dimensions, we do not consider the usual physical laws like the Maxwell equations (see, for example, Refs. 12 and 13) or the Schrödinger equation or Newton's field equations (see Ref. 5), or the Einstein equations (see Refs. 14–16) to be valid in higher dimensions and discuss physical implications of the solutions. Instead, we start with general *physical* properties of the class of phenomena under consideration and then try to get information of the structure of the physical laws. In general, these equations in higher dimensions are very different from the equations in three dimensions describing the same effects. An interesting approach,¹⁷ which is in the line of our reasoning, is based on the causal structure of space–time events. It deduces the four-dimensionality of space–time from a set of axioms which do not use the notion of a differentiable manifold or of the dimensionality. Another approach having some similarities to our reasoning is given in Ref. 18 where it is shown that for a gravitational theory based on a quadratic Lagrangian the usual Newtonian limit and Huygens' principle is valid only if this theory is formulated in six space–time dimensions. In Ref. 8, a very general approach to a generalized Dirac equation in arbitrary dimensions has been used and the dimensions of space–time has been inferred from the propagation of helicity states and from the validity of Huygen's principle. In this work we do not consider the fractal dimension; see, for example, Ref. 19.

In earlier work^{5–7} it has been shown that there are no stable hydrogen atoms in higher dimensions. Essential for that was the assumption that also in higher dimensions Maxwell's equations were assumed to be valid leading to a potential of a point charge of the form $\sim 1/r^{d-2}$ where d is the spatial dimension. In our approach we do not assume the usual Maxwell equations to be valid. We only use the results of scattering experiments to get information about the potential of a point charge. We use this potential in Sec. III in order to solve the hydrogen atom and then show that even in higher dimensions there are stable atoms. However, from the comparison of the calculated spectrum with the observational data we are able to determine in Sec. IV the dimensionality of our space. In Sec. V we present the full set of modified Maxwell equations in order to show that even our potential violating Gauss' law is part of a consistent set of equations governing electro-dynamical phenomena in a higher dimension. Though being nonlocal in general, they are still Lorentz-covariant.

II. THE POTENTIAL OF A POINT CHARGE IN HIGHER DIMENSIONS

The electrostatic potential of the atomic nucleus which we assume to be pointlike, can be determined by means of scattering processes. Indeed, using the scattering of α -particles at gold atoms, Rutherford was able to deduce that the electrostatic potential within an atom is the Coulomb potential. We will show that this procedure and this result is true independent of the underlying spatial dimensions. This can be seen already from the fact that the classical trajectory of a point charge in a $1/r$ potential does not depend on the spatial dimension so that the relation between the deflection angle and the potential also remains the same.

Starting with (1,2), conventional quantum mechanics gives the asymptotics of scattered waves according to

$$u^+(\vec{r}) = \frac{1}{(2\pi)^{d/2}} \left(e^{i\vec{k}\vec{r}} + f_{\vec{k}}(\vec{e}) \frac{e^{ikr}}{r^{(d-1)/2}} \right), \tag{3}$$

with $\vec{e} = \vec{r}/r$. This can be shown by calculating the Green's function in the energy representation

$$G(\vec{r}, \vec{r}') = \frac{1}{(2\pi\hbar)^d} \lim_{\epsilon \rightarrow 0^+} \int \frac{e^{(i/\hbar)(\vec{r}-\vec{r}')\cdot\vec{p}}}{E_0 + i\epsilon - p^2/2m} d^d p$$

which results in a position dependence of the form $\sim |\vec{r} - \vec{r}'|^{-(d-2)}$ with factors depending on the dimension d and an integration over a spherical Bessel function. The scattering amplitude is then given by

$$f_{\vec{k}}(\vec{e}) \sim \int e^{-ik\vec{e}\vec{r}'} V(\vec{r}') u^+(\vec{r}') d^d x', \quad (4)$$

where $\vec{e} = \vec{r}/r$. In the Born approximation we have

$$f_{\vec{k}}(\vec{e}) \sim \int e^{i(\vec{k}-\vec{q})\vec{r}} V(\vec{r}) d^d x. \quad (5)$$

In a scattering experiment the measured quantity is the differential cross section $\sigma(\vec{e}, \vec{k}_0)$ which is related to the scattering amplitude by

$$\sigma(\vec{e}, \vec{k}_0) = |f_{\vec{k}_0}(\vec{e})|^2. \quad (6)$$

This is a relation which is independent of the dimension of the underlying space. In the Born approximation there is a one-to-one correspondence between the differential cross section and the potential $V(\vec{r})$. Therefore, by analyzing the standard Rutherford-type experiments we can uniquely conclude that a point charge, or the nucleus of a hydrogen atom, possess a potential of the form $\sim r^{-1}$, independent of the spatial dimension.

III. THE HYDROGEN ATOM IN HIGHER DIMENSIONS

We start with the Hamilton operator (1) in the external spherically symmetric potential (2) which gives, in position representation,

$$[\Delta - \phi + \epsilon] \psi = \psi, \quad (7)$$

where we introduced the abbreviations

$$\phi(r) = \frac{2m}{\hbar^2} V(r), \quad (8)$$

$$\epsilon = \frac{2m}{\hbar^2} E. \quad (9)$$

The following calculations are analogous to that in three dimensions. Also, in a higher dimension we can separate the Laplace operator into a radial and an angular part:

$$\Delta = \hat{R} - \frac{1}{r^2} \hat{L}, \quad (10)$$

where we introduced

$$\hat{R} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \quad (11)$$

$$\hat{L} = \hat{L}(\Theta_2, \dots, \Theta_n).$$

With the corresponding ansatz,

$$\psi = R(r) Y(\Theta_2, \dots, \Theta_d), \quad (12)$$

we get from the Schrödinger equation in d dimensions,

$$\frac{r^2}{R} \hat{R}R + r^2(\epsilon - \phi) = \frac{1}{Y} \hat{L}Y = l(l + d - 2). \tag{13}$$

$Y(\Theta_2, \dots, \Theta_d)$ represents the spherical harmonics in d dimensions. They are eigenstates of the angular momentum operator \hat{L} with the eigenvalues $l(l + d - 2)$. Thus we get for the radial part of the wave function,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \epsilon - \phi - \frac{l(l+d-2)}{r^2} \right] R = 0. \tag{14}$$

We introduce

$$r' = \frac{r}{r_0}, \quad -\epsilon = \frac{1}{r_0^2}, \quad \alpha' = \frac{2m\alpha r_0}{\hbar^2} \frac{r_0}{2}, \tag{15}$$

and assume for the potential the form

$$\phi(r') r_0^2 = -\frac{\alpha'}{r'}. \tag{16}$$

We also introduce a new variable $f(r')$ through

$$R = e^{-(1/2)r'} r'^{\gamma} f(r'), \tag{17}$$

and get an equation for the function f :

$$0 = r \frac{d^2 f(r')}{dr'^2} + [2\gamma + d - 1 - r'] \frac{df(r')}{dr'} + \left[\frac{\gamma(\gamma + d - 2) - l(l + d - 2)}{r'} + \frac{\alpha'}{r'} - \frac{2\gamma + d - 1}{2} \right] f(r'). \tag{18}$$

This equation is valid for arbitrary d . In order to solve this equation we specify the value of γ by the requirement that the term $\sim 1/r'$ should vanish:

$$\gamma(\gamma + d - 2) - l(l + d - 2) = 0. \tag{19}$$

This gives the two possibilities

$$\gamma_+ = l, \tag{20}$$

$$\gamma_- = -(l + d - 2), \tag{21}$$

and from (18),

$$zf''' - [\vartheta - z]f' - \beta f = 0, \tag{22}$$

with

$$\vartheta := \pm(2l + d - 2) + 1, \quad \beta := \frac{\pm(2l + d - 2) + 1}{2} - \alpha'. \tag{23}$$

Equation (22) is the confluent hypergeometric differential equation with the solution²⁰

$$f(\beta, \vartheta, z) = \sum_{\nu=0}^{\infty} \frac{(\beta + \nu)! \vartheta! z^{\nu}}{\beta! (\vartheta + \nu)! \nu!}, \quad (24)$$

which is appropriate for our problem.

It is clear that, in order to get no infinite terms, ϑ is not allowed to be a negative integer: $\vartheta \neq -1, -2, \dots$. Therefore we cannot use the solution (21). In addition, if the sum does not terminate, then the solution diverges for large r faster than $\exp(\frac{1}{2}r')$ which leads to non-normalizable solutions. The condition for a termination of the sum is $\beta \in \mathbb{Z}^-$, or

$$\beta = l + \frac{d-1}{2} - \alpha' = -k, \quad k \in \mathbb{N}. \quad (25)$$

Here α' is connected with the energy eigenvalues (1,8,15). Therefore we get for the energy eigenvalues E ,

$$E = \frac{2m\alpha^2}{\hbar^2} \frac{1}{\alpha'^2} = -\frac{2m\alpha^2}{\hbar^2} \frac{1}{(l + [(d-1)/2] + k)^2} = -Ry \frac{1}{n^2} =: E_n, \quad (26)$$

where the principal quantum number n is given by the series

$$n = \frac{d-1}{2}, \frac{d-1}{2} + 1, \frac{d-1}{2} + 2, \frac{d-1}{2} + 3, \dots \quad (27)$$

We also introduced the Rydberg constant Ry which, in general, may depend through α on the dimension d . In the case $d=3$ we recover the usual expressions. Note that, in general, the principal quantum number n must not be an integer.

Consequently, we have shown that for a potential of the form $\sim 1/r$ even in higher dimensions there is a lowest energy level, that is, there are stable atoms.

IV. THE INFLUENCE OF THE DIMENSION ON THE SPECTRUM

We discuss now the spectrum of stable hydrogen atoms in higher dimensions. It is clear that the spectrum depends on the dimension d . An interesting question is whether this dependence is accessible to observations. In an experiment only the difference of two energy eigenvalues,

$$\Delta E_{n',n} = E_{n'} - E_n, \quad n' > n, \quad (28)$$

can be measured. For a fixed n one gets an atomic series which now depends on the dimension d . In three dimensions $d=3$ one gets for $n=1$ the Lyman series, for $n=2$ the Balmer series, for $n=3$ the Paschen series, etc. In 4 dimension, for example, according to (27) it is not possible to have $n=1$, so that in this case there is no Lyman series. In $d=6$ dimensions there is also no Balmer series.

However, since the Rydberg constant Ry may depend on the dimension d in an unknown way, we are not able to draw any conclusions about the dimensionality of space from testing the atomic spectral series. Therefore we are forced to restrict ourselves to the ratio of two energy differences which is also independent of any unit conventions. In our case it is enough to take the ratio of the difference between the three lowest energy levels of one series characterized by n :

$$D(n) = \frac{\Delta E_{n+2,n}}{\Delta E_{n+1,n}} = \frac{4(1+n)^3}{(2+n)^2(1+2n)}. \quad (29)$$

Because this function $D(n)$ is one-to-one, the value of $D(n)$ uniquely characterizes the corresponding series. For the first few values we get $D(0) = 1$, $D(\frac{1}{2}) = \frac{27}{25} = 1.08$, $D(1) = \frac{32}{27} = 1.18519$, $D(\frac{3}{2}) = \frac{125}{98} = 1.275$, $D(2) = \frac{27}{20} = 1.35$, $D(\frac{5}{2}) = \frac{343}{243} = 1.41152$, etc.

Therefore we have the following experimental method at hand in order to determine the dimensionality of our space: We consider that series which belongs to the lowest energy state. From this series we take the two highest frequency spectral lines and calculate the ratio. This gives the value $D(n_{\min})$. From this value we can calculate the corresponding n_{\min} and, using (27), the dimension $d = 2n_{\min} + 1$ of our space. Here we used that in each dimension the lowest series contains only transitions with $l = k = 0$.

We know from spectroscopy of the hydrogen atom that the two spectral lines coming from transitions to the lowest energy level have (see, for example, Ref. 21) 1215.67 Å and 1025.73 Å, so that $D(n_{\min}) = 1216/1026 = 1.18518$. A comparison with the values of $D(n)$ shows that this implies $n_{\min} = 1$, and from Eq. (27) that $d = 3$. Therefore we have *proven by a spectroscopic experiment that our space is three-dimensional*. In other words, because we know the spectrum of the hydrogen atom we are able to determine the dimensionality of space.

We want to stress once more that it is not the stability of the atom which one may use as argument in favor of three spatial dimensions. In our approach the stability of atoms is secured in any dimension. It is only the structure of the spectral series which leads us to the conclusion that space is three-dimensional.

V. MAXWELL EQUATIONS IN HIGHER DIMENSIONS

We have seen that the electric potential of a point charge in the Schrödinger equation in higher dimensions must be of the form $U \sim 1/r$ independent of the dimension d . Since the usual Laplacian has the same form in any dimension, the above potential cannot be a solution of the Poisson equation in $d > 3$ dimensions. However, we show that it is indeed possible to present a consistent set of equations governing the electromagnetic phenomena in higher dimensions which violates no fundamental principle of electrodynamics and, in addition, possesses the above electrostatic solutions for a point charge. Of course, the structure of the Maxwell equations will be not the same as in three dimensions.

In order to determine the structure of the stationary Maxwell equation for the electric field, we use results of Riesz distributions, see, for example, Refs. 22, 23. In doing so we first define the distribution

$$G_\lambda := \frac{e^{-i\pi\lambda} \Gamma[(d/2) - \lambda] r^{2\lambda - d}}{4^\lambda \pi^{d/2} \Gamma(\lambda)}, \tag{30}$$

where r as usual is the distance $r^2 = \sum_{i=1}^d x_i^2$. The properties of these distributions G_λ are given by the composition law

$$G_\mu * G_\lambda = G_{\lambda + \mu}, \tag{31}$$

and an explicit representation in the case of negative integers,

$$G_k = \Delta^{-k} \delta, \quad k = 0, -1, -2, \dots, \tag{32}$$

where Δ is again the Laplace operator in an arbitrary dimension, δ the usual Dirac delta distribution, and the star $*$ the convolution operation.

We formally introduce operators $\bar{\Delta}^\lambda$ by

$$\bar{\Delta}^\lambda := G_{-\lambda}, \tag{33}$$

so that the following composition law holds:

$$\bar{\Delta}^{\mu*}G_{\lambda} = G_{\lambda-\mu} \tag{34}$$

An important special case is given by $\mu = \lambda$:

$$\bar{\Delta}^{\mu*}G_{\mu} = \delta. \tag{35}$$

This means that G_{μ} is a Green's function corresponding to the operator $\bar{\Delta}^{\mu*}$.

Now we come back to our problem of finding the field equations which are required to possess the solution $\sim r^{-1}$ in any dimension d . That means that we require in any dimension $G_{\mu} \sim 1/r$ which implies $\mu = (d-1)/2$. Consequently,

$$\bar{\Delta}^{(d-1)/2*}G_{(d-1)/2} = \delta, \tag{36}$$

or

$$\bar{\Delta}^{(d-1)/2*}\frac{1}{r} = (4\pi)^{(d-1)/2}e^{[i\pi(d-1)/2]}\Gamma\left(\frac{d-1}{2}\right)\delta. \tag{37}$$

This means that the equation for the electric potential, or the generalized Poisson equation, reads as

$$(\bar{\Delta}^{(d-1)/2*}\phi)(x) = (4\pi)^{(d-1)/2}e^{[i\pi(d-1)/2]}\Gamma\left(\frac{d-1}{2}\right)\rho(x), \tag{38}$$

where $\rho(x)$ is the charge density in d dimensions. The operator $\bar{\Delta}^{(d-1)/2}$ replaces the Laplacian in three dimensions. In general, this operator is no differential operator.

We briefly discuss this new form of the Poisson equation in electrostatics.

- (1) It is possible to reformulate the field equation for the potential ϕ in terms of the electric field strength $\mathbf{E} = -\nabla\phi$. For doing so we use (31) and (32):

$$\bar{\Delta}^{(d-1)/2*}\phi = (\bar{\Delta}*\bar{\Delta}^{(d-3)/2})*\phi = \Delta\delta*(\bar{\Delta}^{(d-3)/2*}\phi) = \delta*(\bar{\Delta}^{(d-3)/2*}\delta\phi) = (\bar{\Delta}^{(d-3)/2*}\nabla\cdot\mathbf{E}), \tag{39}$$

so that we get as field equation for the electric field strength,

$$((\bar{\Delta}^{(d-3)/2*}\nabla)\cdot\mathbf{E})(x) = (4\pi)^{(d-1)/2}e^{[i\pi(d-1)/2]}\Gamma\left(\frac{d-1}{2}\right)\rho(x). \tag{40}$$

- (2) The force between two charges still has the same form as in 3 dimensions, namely $f \sim q_1q_2/r^2$.
- (3) For all charge densities $\rho(x)$ the solution for the potential looks as usual, i.e., $1/r*\rho$.
- (4) In odd dimensions $d=1,3,5,\dots$, the above equation reduces via (31) and (32) to a differential equation:

$$\bar{\Delta}^{(d-1)/2*}\frac{1}{r} = \delta*\Delta^{(d-1)/2}\frac{1}{r} = \Delta^{(d-1)/2}\frac{1}{r} = (4\pi)^{(d-1)/2}e^{[i\pi(d-1)/2]}\Gamma\left(\frac{d-1}{2}\right)\delta. \tag{41}$$

For a three-dimensional space, $d=3$, we get the usual Laplace equation $\Delta\phi(x) = -4\pi\rho(x)$ and in a five-dimensional space we get $\Delta^2\phi(x) = (4\pi)^2\rho(x)$.

- (5) In even dimensions, the operator $\bar{\Delta}^{(d-1)/2}$ is no differential operator but instead a pseudo-differential operator. Therefore the corresponding field equations are pseudo-differential operator equations. These operators are nonlocal. (Indeed, differential operators are the only local operators acting linearly and surjective on C^∞ ; see Ref. 24. For a physical discussion, see, for example, Ref. 25.)
- (6) An essential difference to the usual properties of the electric field in 3 dimensions is that now the Gauss' law is no longer valid. This is easy to see by integrating the fundamental solution r^{-1} in an arbitrary dimension over the surface of a sphere with radius R :

$$\int_R \mathbf{E}\cdot d\mathbf{A} = \int_R \nabla\frac{1}{r}\cdot\hat{\mathbf{r}}R^{d-1}d\Omega = (4\pi)^{(d-1)/2}e^{[i\pi(d-1)/2]}\Gamma\left(\frac{d-1}{2}\right)R^{d-3}, \tag{42}$$

where $\hat{\mathbf{r}}$ is the unit vector in radial direction and $d\Omega$ is the surface element in d dimensions. The result depends on the radius of the sphere so that indeed Gauss' law does not hold. It is only in three spatial dimensions that the quantity \mathbf{E} which plays the role of a force on a charged particle, is also that quantity which integral over the area enclosing a volume gives the total charge which acts as source of \mathbf{E} (the field strength \mathbf{E} is defined by means of the force acting on a charged particle; whether this quantity obeys a law like Gauss' law is a deduced property which holds in three dimensions).

However, Gauss' law is valid for a quantity deduced from \mathbf{E} , namely for $\mathfrak{E} = \bar{\Delta}^{(d-3)/2} \mathbf{E}$:

$$\nabla \cdot \mathfrak{E} = (4\pi)^{(d-1)/2} e^{i\pi[(d-1)/2]} \Gamma\left(\frac{d-1}{2}\right) \rho(x) \Leftrightarrow \oint \mathfrak{E} \cdot d\mathbf{A} = (4\pi)^{(d-1)/2} e^{i\pi[(d-1)/2]} Q, \tag{43}$$

where $Q = \int_V \rho(x) d^d x$ is the charge contained in the volume V .

It is also straightforward to give the full set of Maxwell's equations such that their static limit give the Poisson equation discussed above: Since the covariant generalization of the Laplace operator Δ is given by the d'Alambert operator \square , the covariant generalization of Poisson's equation is $\bar{\square}^{(d-1)/2} * \phi = (4\pi)^{(d-1)/2} e^{i\pi[(d-1)/2]} \Gamma((d-1)/2) \rho$. We complete the quantities to covariant 4-vectors, namely the 4-potential A^a and the 4-current j^a . Then we have, using the same methods as above,

$$\begin{aligned} j^a &= \bar{\square}^{(d-1)/2} * A^a = (\bar{\square} * \bar{\square}^{(d-3)/2}) * A^a = \square \delta * (\bar{\square}^{(d-3)/2} * A^a) = \delta * (\bar{\square}^{(d-3)/2} * \square A^a) \\ &= \bar{\square}^{(d-3)/2} * \partial_b F^{ba} = : \bar{\partial}_b F^{ba}, \end{aligned} \tag{44}$$

where we defined a generalized partial derivative $\bar{\partial}_b := \bar{\square}^{(d-3)/2} * \partial_b$ and, as usual, the Maxwell field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. We also used the Lorentz condition $\partial_a A^a = 0$. By construction, these generalization of Maxwell's equations is covariant. Also current conservation is fulfilled. For even spatial dimensions these equations are nonlocal.

To sum up: despite the fact that the mathematical structure of the equation determining the electric potential from a given charge density changes dramatically when compared with the three-dimensional case, the physical content does not change. The solution has the same form and the force between charges is the same as in three dimensions. Only Gauss' law loses its meaning. However, we think that the specific expression for the force between charged particles and the stability of atoms are of more basic physical importance than the validity of Gauss' law.

VI. SUMMARY AND DISCUSSION

To sum up, we have shown the following.

- (1) From Rutherford-type experiments we can conclude that the potential of the point charge in any spatial dimension must be $\sim 1/r$.
- (2) This potential leads to stable atoms in higher dimensions.
- (3) The dimensionality enters the atomic spectra thus making it possible to infer uniquely from atomic spectra the three dimensionality of space.
- (4) That the Maxwell equations have to be modified in higher dimensions in order to allow solutions of the form $1/r$, leading to nonlocal equations in even spatial dimensions.

In the case that one uses the usual Maxwell equations in higher dimensions the hydrogen atom is proven to be not stable. This has been related to the fact that orbits of classical bodies in a potential derived from the usual Poisson equation in higher dimensions are not stable, as well: small perturbations of the circular orbit leads the body to fall into the central body or to leave the system. Consequently, if one wants to enlarge the above reasoning to the case of Newtonian mechanics, one has to require stable orbits, which gives the $1/r$ potential for gravity also in higher dimensions. This forces one to modify the Poisson equation for the Newtonian potential in the

same way as the Poisson equation for the electrostatic potential in Sec. V. That means, in higher dimension d the field equation for the Newtonian potential $U(x)$ must be of the form $(\Delta^{(d-1)/2} * U)(x) = (4\pi)^{(d-1)/2} e^{i\pi(d-1)/2} \Gamma((d-1)/2) \rho(x)$, where $\rho(x)$ is the mass density. As a consequence, also Einstein's equations should be modified in higher dimensions.

In conclusion, we want to say that our or similar considerations do not rule out the possibility of unifying physics in higher dimensions; we just restrict, from observations, the direct physical applicability of dynamical equations to three spatial dimensions.

ACKNOWLEDGMENTS

We thank J. Audretsch, Ch. J. Bordé, F. W. Hehl, and H. Kastrup for discussions. Moreover, C.L. thanks the German Academic Exchange Service (DAAD) and the CNRS (France) for financial support and Ch. J. Bordé and Ch. Chardonnet for their hospitality at the Laboratoire de Physique des Lasers in Villetaneuse, France. This research was supported by Conacyt, Grant No. 3544-E9311, and by the joint German-Mexican project KFA-Conacyt E130-2924.

- ¹E. Caceres, V. S. Kaplunovsky, and I. M. Mandelberg, "Large volume string compactification, revisited," Nucl. Phys. B **493**, 73 (1997).
- ²T. Banks and M. Dine, "Couplings and scales in strongly coupled heterotic string theory," Nucl. Phys. B **479**, 173 (1996); "The cosmology of string theoretic axions," *ibid.* **505**, 445 (1997).
- ³P. Hořava and E. Witten, "Heterotic and Type I string dynamics from eleven dimensions," Nucl. Phys. B **460**, 506 (1996).
- ⁴M. J. Duff, "Kaluza–Klein theory in perspective," talk delivered at the Oskar Klein Centenary Nobel Symposium, Stockholm, 19–21 September 1994 (hep-th/9410046).
- ⁵P. Ehrenfest, Proc. Amst. Acad. **20**, 200 (1917); P. Ehrenfest, "Welche Rolle spielt die Dimensionalität des Raumes in den Grundgesetzen der Physik?," Ann. Phys. (Leipzig) **61**, 440 (1920).
- ⁶I. M. Freeman, "Why is space three-dimensional?," based on W. Büchel, "Warum hat der Raum drei Dimensionen?," Am. J. Phys. **37**, 1222 (1969).
- ⁷J. D. Barrow, "Dimensionality," Philos. Trans. R. Soc. London, Ser. A **310**, 337 (1983).
- ⁸C. Lämmerzahl and A. Macias, "On the dimensionality of space–time," J. Math. Phys. **34**, 4540 (1993); Addendum, *ibid.* **36**, 2131 (1995).
- ⁹L. Gurevich and V. Mostepanenko, "On the existence of atoms in n -dimensional space," Phys. Lett. A **35**, 201 (1971).
- ¹⁰J. Audretsch and C. Lämmerzahl, "A new constructive axiomatic scheme for the geometry of space–time," in *Semantical Aspects of Space–Time Geometry*, edited by U. Majer and H.-J. Schmidt (BI Verlag, Mannheim, 1993).
- ¹¹C. Lämmerzahl and U. Bleyer, "A quantum test theory for basic principles of special and general relativity with applications to atom beam interferometry and Hughes–Drever type experiments," preprint, University of Konstanz, 1998.
- ¹²L. M. Sokolowski, F. Occhionero, M. Litterio, and L. Amendola, "Electromagnetism and the dimensionality of space–time," in *Relativistic Astrophysics and Cosmology*, edited by S. Gottlöber, J. P. Mückel, and V. Müller (World Scientific, Singapore, 1992).
- ¹³S. Sonego and V. Faraoni, "Huygen's principle and characteristic propagation property for waves in curved space–times," J. Math. Phys. **33**, 625 (1992).
- ¹⁴F. R. Tangherlini, "Schwarzschild field in n dimensions and the dimensionality of space problem," Nuovo Cimento **27**, 636 (1963).
- ¹⁵B. Müller and A. Schäfer, "Improved bounds on the dimension of space–time," Phys. Rev. Lett. **56**, 1215 (1986).
- ¹⁶R. C. Myers and M. J. Perry, "Black holes in higher dimensional space–times," Ann. Phys. (N.Y.) **172**, 304 (1986).
- ¹⁷H.-J. Seifert, "Zur Dimensionszahl der Raumzeit," in *Festschrift Fachbereich Maschinenbau*, 10 Jahre Hochschule der Bundeswehr Hamburg, 1983.
- ¹⁸C. G. Bollini, J. J. Giambiagi, and O. Obregon, "Are some physical theories related with a specific number of dimensions?," Mod. Phys. Lett. A **7**, 593 (1992).
- ¹⁹A. Zeilinger and K. Svozil, "Measuring the dimension of space–time," Phys. Rev. Lett. **54**, 2553 (1985).
- ²⁰I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1983).
- ²¹G. W. F. Drake, "Atomic Spectroscopy," in *Atomic, Molecular & Optical Physics Handbook* (American Institute of Physics, Woodbury, NY, 1996).
- ²²I. M. Gel'fand, *Generalized Functions* (1966).
- ²³C. G. Bollini and J. J. Giambiagi, "Arbitrary powers of D'Alembertians and the Huygens' principle," J. Math. Phys. **34**, 610 (1993).
- ²⁴J. Peetre, "Rectification à l'article 'Une caractérisation abstraite des opérateurs différentiels,'" Math. Scand. **8**, 116 (1960); J. Dieudonné, *Éléments d'Analyse* (Gauthier-Villars, Paris, 1971).
- ²⁵C. Lämmerzahl, "The pseudo-differential operator square root of the Klein–Gordon equation," J. Math. Phys. **34**, 3918 (1993).