The pseudodifferential operator square root of the Klein–Gordon equation

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A nonlocal square root of the Klein–Gordon equation is proposed. This nonlocal equation is a special relativistic equation for a scalar field of first order in the time derivative. Its space derivative part is described by a pseudodifferential operator. The usual quantum mechanical formalism can be set up. The nonrelativistic limit and the classical limit in the form of plane wave solutions and the Ehrenfest theorem are correctly included. The nonlocality of the wave equation does not disturb the light cone structure, and the relativity principle of special relativity is fulfilled. Uniqueness and existence of solutions of the Cauchy problem for this equation can be proved. The second quantized version of this theory turns out to be macrocausal.

I. INTRODUCTION

Point particles are a good idealization for doing physics in the classical domain. However, there already occurs problems with the pointlikeness of particles because the electrostatic energy of a charged point particle is infinite. Also in the problem of the radiation reaction of an accelerated point particle in some electromagnetic field, the pointlikeness seems to cause the problem. And one of the main problems in physics is the occurrence of divergences in quantum field theory, for which the locality of the underlying field equation was made responsible. For most of the field equations this problem was solved by renormalization.

Therefore the study of nonlocal field equations is of interest and may contribute to the solution of some of these problems. There are many proposals of nonlocal theories, theories of extended particles, appearing in the literature (see, e.g., the monograph by Namsrai[1] and the literature cited therein). Also the string theory is a theory of extended particles and in this sense a nonlocal one.

Here we propose another way to introduce a nonlocality in field theory: by means of pseudodifferential operator equations. Pseudodifferential operators are first introduced to get solutions of partial differential equations with variable coefficients. These operators give a parametric for a differential operator, that is, an inverse of the differential operator up to C^∞ functions. For applications in physics and treating the subject by an intrinsic calculus see Fulling and Kennedy.[2] In addition, pseudodifferential operators can also be used to formulate generalizations of differential equations, namely, pseudodifferential equations. This latter application is what we want to describe in the following.

We will construct a special nonlocal (pseudodifferential operator) field equation, which is based on the usual Klein–Gordon equation, and discuss the properties of this new equation. Although the nonlocality of pseudodifferential equations do not cure divergences in quantum field theory, there are advantages which make such types of equations worth studying. Some of the advantages of this pseudodifferential equation are the following: (i) It is a scalar relativistic equation of first order in the time derivative. (ii) It possesses a quantum mechanical interpretation, that is, it leads to a conserved quantity and leads automatically to the correct Ehrenfest

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theorem. (iii) It preserves the light cone structure. (iv) All observable quantities are in accordance with the relativity principle. (v) The classical limit as well as the nonrelativistic limit is correctly included. (vi) It possesses plane wave solutions of positive energy only.

Therefore this theory has all the good properties of a Schrödinger theory and avoids many problems connected with the interpretation of usual relativistic field equations.

In this article we want to consider how to take the square root of the Klein–Gordon-equation (KGE). At first we will state this problem in Minkowski space without interaction. The coupling to electromagnetism and gravitation will be considered in further publications.

The KGE

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi+m^2\varphi=0 \tag{1.1}$$

($\mu=0,...,3$) with the Minkowski metric $\eta=\mathrm{diag}(+---)$ and φ as a scalar field, is a relativistic field equation. However, because of the following reasons it cannot be considered as a quantum mechanical equation, that is, as a quantum mechanically interpreted classical field equation:

(i) The KGE (1.1) does not obey the usual quantum principles because it is not of the first order in the time derivative.

(ii) The probability current $j^\mu=(\hbar/2mi)(\varphi^+\partial^\mu\varphi-(\partial^\mu\varphi^+)\varphi)$ derived from Eq. (1.1) is not positive definite, that is, negative probabilities in finding a particle within a given space region may occur.

(iii) Solutions of the KGE can possess negative energies. This leads to the occurence of the *Zitterbewegung*[3] and generally to difficulties for the interpretation of this equation. [Both problems (ii) and (iii) can be overcome by demanding that the KG particles be charged. Also second quantization removes these difficulties.]

(iv) There is no good canonical position operator.

These are reasons which force one to look for another relativistic field equation. Writing Eq. (1.1) as $-\partial_t^2\varphi=(-\Delta+m^2)\varphi$, then, in order to get only one time derivative on the left hand side, one might somehow try to take the square root of the operator on the right hand side: $i\partial_t\varphi = \sqrt{-\Delta+m^2}\varphi$. This means, one is looking for an operator A that fulfills

$$AA\varphi=(-\Delta+m^2)\varphi=-\partial_t^2\varphi. \tag{1.2}$$

The usual way to solve this problem formally is to change to a matrix-valued equation. This procedure leads to the Dirac equation.

In this article we intend to go another way: The equation $AA\varphi=\partial_t^2\varphi$ can also be solved in interpreting A as a pseudodifferential operator. By doing so, all the difficulties with the KGE mentioned above can be avoided. Furthermore, some problems which occur with the Dirac equation do not occur in our case. However, we want to emphasize that our theory pertains to a scalar field and is no replacement for the Dirac equation.

The outline of this article is as follows: At first we will shortly discuss the Dirac square root of the KGE. Then we will give a short introduction to pseudodifferential operators to the extent we need. In Sec. IV we use this mathematical tool to get the pseudodifferential operator square root of the free KGE. In the rest of this article this pseudodifferential operator field equation and its solutions will be discussed.

II. THE DIRAC EQUATION

A. Derivation of the Dirac equation

The equation $AA\varphi=(-\Delta+m^2)\varphi$ can be solved for A by looking for a matrix-valued equation in writing $A=\alpha^a i\partial_a+\beta m$ ($a=1,2,3$) with matrices α^a and β and requiring $\alpha^{(a}\alpha^{b)}=\delta^{ab}$, $\beta^2=1$, and $\alpha^a\beta+\beta\alpha^a=0$. This procedure especially means that one alters the nature of the considered scalar field φ in replacing it by a complex vector valued field $\varphi \in \mathbb{C}^s$ with $s>1$. Since the relations for the matrices α and β possess nontrivial solutions, this procedure leads to a square root of the KGE. Therefore we get $\partial_t \varphi = \alpha^a \partial_a \varphi + \beta m \varphi$, or equivalently,

$$0 = i\gamma^\mu \partial_\mu \varphi - m\varphi,$$

the Dirac equation.

According to Bjorken and Drell[4] one is not forced, either for mathematical or for physical reasons, to solve the above problem by transforming it to a matrix-valued equation. The KGE as well as the Dirac equation are local field equations. This fact is responsible for the occurence of divergences in calculating processes within the framework of second quantization.

B. Discussion of the Dirac equation

Although one has solved the first two of the above problems, with respect to the Dirac equation, there remains problems (iii) and (iv), that is, especially the problems of interpretation. When calculating the velocity operator along the usual lines this operator turns out to possess only the light velocity as an eigenvalue. This difficulty might be removed by the process of a Foldy–Wouthuysen transformation or by constructing a position operator which is different from the canonical position operator.[5] For more interpretation see, e.g., Refs. 3 and 6.

III. PSEUDODIFFERENTIAL OPERATORS

In the following we want to show that the Dirac equation is not the only possible square root of the KGE. The other square root can be performed by pseudodifferential operators, which are the natural (nonlocal) generalization of the usual partial differential operators.

A. Motivation and definition for pseudodifferential operators

1. Motivation for pseudodifferential operators

Let $P(x,D)$ be a linear partial differential operator ($D := -i\partial$) of order m with C^∞ coefficients. Then $P(x,D)$ is a mapping $\mathscr{E}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$.

The Fourier transform (FT) of $u \in \mathscr{E}(\mathbb{R}^n)$ with compact support is denoted by $\mathscr{F}u$. For such a u we can reformulate $P(x,D)u$ by means of the FT (in this section we work in an n-dimensional space x, $\xi \in \mathbb{R}^n$ and $\xi \cdot x = \xi_a x^a$, $a=1,\ldots,n$)

$$P(x,D)u(x) = P(x,D)\left(\frac{1}{(2\pi)^{n/2}} \int (\mathscr{F}u)(\xi) e^{i\xi\cdot x}\, d\xi\right) = \frac{1}{(2\pi)^{n/2}} \int P(x,\xi)(\mathscr{F}u)(\xi) e^{i\xi\cdot x}\, d\xi. \tag{3.1}$$

The right hand side does not contain any differential operator. Consequently, differentiation is replaced by integration. Integration is superior to differentiation.

This is now the point to generalize Eq. (3.1): At first we can replace the polynomial $P(x,\xi)$ in ξ by an arbitrary function $a(x,\xi)$ in such a way, that the integral still makes sense. This procedure defines a new operator $\mathscr{A}$:

$$(\mathscr{A}u)(x) := \frac{1}{(2\pi)^{n/2}} \int a(x,\xi)(\mathscr{F}u)(\xi) e^{i\xi\cdot x}\, d\xi. \tag{3.2}$$

Operators of this kind are called pseudodifferential operators (PDO); the functions $a(x,\xi)$ are called symbols. Each symbol $a(x,\xi)$ corresponds to a pseudodifferential operator $\mathscr{A}$. Writing explicitly the FT in Eq. (3.2) we get

$$(\mathscr{A}u)(x):=\frac{1}{(2\pi)^n}\iint a(x,\xi)u(y)e^{i\xi\cdot(x-y)}\,dy\,d\xi. \tag{3.3}$$

(This expression can be easily further generalized by allowing the symbol to depend on y too.) An extended exposition of the theory of PDOs can be found in Hörmander,[7] Dieudonné,[8] Taylor,[9] and Treves.[10]

PDOs are a basic tool for the mathematical description of the Weylian quantization procedure, see, e.g., Folland[11] and Unterberger.[12]

2. Symbol classes and definition of PDOs

To be able to define the integration in Eq. (3.2) resp. Eq. (3.3) the symbols must fulfill some requirements. We do not define the most general symbol classes. For all $x\in K\subset\mathbb{R}^n$, K compact, and $\xi\in\mathbb{R}^n$ the functions $a(x,\xi)$ are not allowed to grow faster with respect to ξ than some polynomial, that is, there are constants m, $c^{\nu_1\cdots\nu_j}_{\mu_1\cdots\mu_i}(K)\in\mathbb{R}$ so that

$$\left|\partial_{\mu_1}\cdots\partial_{\mu_i}\frac{\partial^j}{\partial\xi_{\nu_1}\cdots\partial\xi_{\nu_j}}a(x,\xi)\right|\leqslant c^{\nu_1\cdots\nu_j}_{\mu_1\cdots\mu_i}(K)(1+|\xi|)^{m-j},\quad\forall i,j. \tag{3.4}$$

These are the symbols of the order m. Their class is denoted by S^m. We also define $S^{-\infty}:=\bigcap_{m\in\mathbb{R}}S^m$.

Each of these symbols define via Eq. (3.3) a PDO.

Especially the polynomials $P(x,\xi)=\Sigma_{i=0}^m a^{\mu_1\cdots\mu_i}(x)\xi_{\mu_1}\cdots\xi_{\mu_i}$ appearing in Eq. (3.1) are symbols of the order m.

Equality of two symbols up to an $S^{-\infty}$ part, $a-a'\in S^{-\infty}$, defines an equivalence relation $a\sim a'$.

The symbol classes of smaller m are contained in the symbol classes of greater m: $S^m\subset S^{m'}$ for $m<m'$. We can also show $a\in S^m\Rightarrow\partial a/\partial\xi_\mu\in S^{m-1}$ and $a\in S^m$, $b\in S^{m'}\Rightarrow ab\in S^{m+m'}$.

One of the most important properties of symbols is the fact that they can be expanded in asymptotic series, that is, in a sum of symbols of decreasing order $a\sim\Sigma_{i=m}^{-\infty}a_i$ with $a_i\in S^i$.

B. Properties of PDOs

Some of the most important properties of PDOs are the following: (i) The product of two PDOs is again a PDO. (ii) PDOs are pseudolocal, that is,

$$\text{sing supp}(\mathscr{A}\varphi)\subset\text{sing supp}\,\varphi. \tag{3.5}$$

PDOs are not local, they do not fulfill supp $(\mathscr{A}\varphi)\subset$ supp φ. This last property, the locality condition, characterizes partial differential operators.[13,14] (iii) As for symbols, there is an asymptotic expansion for PDOs: $\mathscr{A}=\Sigma\mathscr{A}_i$ where each $\mathscr{A}_i$ is a PDO defined by the symbol $a_i\in S^i$ with decreasing i. (iv) With the help of the asymptotic expansion we can determine the commutator of two PDOs: If $a(x,\xi)$ is the symbol of $\mathscr{A}$ and $b(x,\xi)$ the symbol of $\mathscr{B}$, then the asymptotic expansion of the symbol $\sigma_{\mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}}$ of the PDO $\mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}$ reads[7]

$$\begin{aligned}\sigma_{\mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}}(x,\xi)&\sim\sum_{j=0}^{\infty}\frac{i^j}{j!}\left(\frac{\partial^j a(x,\xi)}{\partial\xi_{a_0}\cdots\partial\xi_{a_j}}\frac{\partial^j b(x,\xi)}{\partial x^{a_0}\cdots\partial x^{a_j}}-\frac{\partial^j b(x,\xi)}{\partial\xi_{a_0}\cdots\partial\xi_{a_j}}\frac{\partial^j a(x,\xi)}{\partial x^{a_0}\cdots\partial x^{a_j}}\right)\\&=i\{a,b\}(x,\xi)-\frac{1}{2}\left(\frac{\partial^2 a(x,\xi)}{\partial\xi_a\partial\xi_b}\frac{\partial^2 b(x,\xi)}{\partial x^a\partial x^b}+\frac{\partial^2 b(x,\xi)}{\partial\xi_a\partial\xi_b}\frac{\partial^2 a(x,\xi)}{\partial x^a\partial x^b}\right)+\cdots,\end{aligned} \tag{3.6}$$

where $\{,\}$ is the Poisson bracket with respect to (x,ξ). The leading term therefore consists in the Poisson bracket. The higher order terms correspond to "quantum corrections." This will be shown after coupling the field equation (4.1) to the Maxwell field. If one of the symbols is the canonical variable x or ξ, then in Eq. (3.6) all higher terms vanish so that the general formula reduces to[7]

$$\sigma_{\mathscr{A}\hat{x}^a-\hat{x}^a\mathscr{A}}(x,\xi)=i\{a(x,\xi),x^a\}=i\frac{\partial a(x,\xi)}{\partial\xi_a}, \tag{3.7a}$$

$$\sigma_{\mathscr{A}\hat{\xi}_a-\hat{\xi}_a\mathscr{A}}(x,\xi)=i\{a(x,\xi),\xi_a\}=-i\frac{\partial a(x,\xi)}{\partial x^a}. \tag{3.7b}$$

(Entities with a hat are operators.) The commutator relation $[\hat{x}^a,\hat{\xi}_b]=-i\delta^a_b$ easily results from the above general relation.

IV. THE SQUARE ROOT OF THE FREE KLEIN–GORDON EQUATION USING PSEUDODIFFERENTIAL OPERATORS

A. The definition of the square root of the free KGE

At first we note that the operator $-\Delta+m^2$ is strongly elliptic. For operators of this kind, the square root is well-defined and necessarily results in a PDO.[9] We can construct the respective PDO by taking as the symbol the square root of the original symbol.

In our case we then have to take as the symbol of the PDO the function $\sqrt{\xi^2+m^2}$ with $\xi^2:=\delta^{ab}\xi_a\xi_b$, $(a,\ b=1,\ 2,\ 3)$. Therefore our square root of the KGE reads

$$i(\partial_t\varphi)(t,x)=(\mathscr{H}\varphi)(t,x)=\frac{1}{(2\pi)^3}\int\int\sqrt{\xi^2+m^2}e^{i\xi\cdot(x-y)}\varphi(t,y)d^3y\,d^3\xi. \tag{4.1}$$

This equation shows new effects in comparison to the KGE (in the same sense as the Dirac equation shows new effects in comparison to the KGE). As for the Dirac equation, the KGE just serves as motivation for introducing a new field equation meeting the quantum mechanical requirements. Equation (4.1) is a *relativistic equation for a scalar field φ which is of the first order in the time derivative.* In the following we want to discuss this equation with respect to its physical content.

First we make some comments on Eq. (4.1):

(i) The function $\sqrt{\xi^2+m^2}$ is a symbol from S^1.

(ii) Since the right hand side of Eq. (4.1) consists in a PDO, Eq. (4.1) is a nonlocal equation.

(iii) The square root is defined with respect to a chosen spacelike hypersurface. But any hypersurface can be chosen. Therefore Eq. (4.1) is not in contradiction to the relativity principle (that is, no Lorentz system can be distinguished by any physical experiment). Below we will show that measured quantities explicitly obey the relativity principle. [Equation (4.1) is indeed not manifestly covariant. The covariance appears in the arbitrariness of the chosen hypersurface and the covariance of measured quantities.] In addition, it has been shown[15] that there is a representation of the Lorentz group acting on functions so that the respective transformation carrys solutions of Eq. (4.1) into another solution. These transformations also leave the scalar product $\langle\psi|\varphi\rangle:=\int\psi^*\varphi\,d^3x$ (see Sec. IV H 3) invariant.

(iv) While the usual relativistic field equations like the KGE and the Dirac equation in the absence of interactions can be written as real equations (for the Dirac equation this amounts to choosing a special representation of the γ matrices, namely, the Majorana representation) solutions of Eq. (4.1) are necessarily complex valued. Equation (4.1) has this property in common with the Schrödinger equation.

(v) By calculating the square root twice, we arrive again at the KGE (1.2). In the following we discuss some physical implications of the field equation (4.1).

B. Plane wave solutions

At first we will show that Eq. (4.1) possesses plane wave solutions. For doing so we make the ansatz $\varphi(t,x) = ae^{-ip_\mu x^\mu} = ae^{-i(\omega t - p\cdot x)}$ with a, p=const. Insertion into Eq. (4.1) gives

$$\omega a e^{-i(\omega t-p\cdot x)}=\frac{1}{(2\pi)^3}\int\int \sqrt{\xi^2+m^2}e^{i\xi\cdot(x-y)}ae^{-i(\omega t-p\cdot y)}d^3y\, d^3\xi.$$

The integration can be performed and results in

$$\omega=\sqrt{p^2+m^2}. \tag{4.2}$$

Therefore plane waves are solutions of Eq. (4.1). These plane waves always have *positive energy* only. The spectrum of the Hamiltonian (4.1) consists in the set $[m,\infty)$, see Weder.[16]

Since for a given time the plane waves form a complete set, any solution, that is, any physical state, may be decomposed according to

$$\varphi(t,x)=\frac{1}{(2\pi)^{3/2}}\int a(t,p)e^{-ip\cdot x}\, d^3p. \tag{4.3}$$

Inserting Eq. (4.3) into Eq. (4.1) gives the time evolution of the coefficients $i\partial_t a(t,p) = \sqrt{p^2+m^2}a(t,p)$ having the solution

$$a(t,p)=e^{-i\sqrt{p^2+m^2}t}a(p). \tag{4.4}$$

Because of the positivity of the energy the group velocity of any wave packet will be timelike and future directed. Furthermore, there is no *Zitterbewegung*, that is, there is no interference between positive and negative energies in bilinear expressions.

C. The light cone

One characteristic feature of the propagation phenomena described by partial differential equations (in the hyperbolic case) is the occurence of light cones (compare Refs. 7 and 17). The structure of the light cones is exhibited in the singular support of the fundamental solution of the differential equation.

By taking the pseudodifferential operator square root of the KGE this characteristic feature of the KGE is not disturbed. This can be seen as follows (we restrict to $t>0$, see Sec. IV I): If $E_{\rm KG}$ is a fundamental solution of the KGE, i.e., if $(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)E_{\rm KG}=\delta$, then, because of $(i\partial_t+\mathscr{H})(i\partial_t-\mathscr{H})E_{\rm KG}=(-\partial_t^2+\Delta-m^2)E_{\rm KG}=(i\partial_t-\mathscr{H})(i\partial_t+\mathscr{H})E_{\rm KG}=\delta$, a fundamental solution of Eq. (4.1) is given by $E_{\sqrt{\rm KG}} = (i\partial_t + \mathscr{H})E_{\rm KG}$. From this result it is clear that (the singular support, sing supp, of a function f is the set of points $x\in\mathbb{R}^n$ which have no open neighborhood to which the restriction of f is C^∞) sing supp $E_{\sqrt{\rm KG}}$=sing supp $((i\partial_t+\mathscr{H})E_{\rm KG})$=sing supp $(\partial_t E_{\rm KG})\cup$sing supp $(\mathscr{H}E_{\rm KG})\subset$sing supp $E_{\rm KG}$ because of the pseudolocality of $\mathscr{H}$ (3.5) and sing supp $(\partial_t E_{\rm KG})\subset$sing supp $E_{\rm KG}$. This means that the light cone structure is not disturbed by taking the pseudodifferential operator square root of the KGE.

The result that even sing supp $E_{\sqrt{\rm KG}}$=sing supp $E_{\rm KG}$ for $t>0$ will be explicitly established in Sec. IV I.

D. The nonrelativistic limit

As we have seen, plane waves with momentum p are solutions of Eq. (4.1). We can now introduce a set of functions $S_{\rm nr}$ by means of the requirement that the support of the FT of these functions is restricted to a neighborhood $U_1 \subset \mathbb{R}^3$ of zero. To be more precise, $S_{\rm nr} := \{\varphi \in \mathscr{S}' \mid \text{supp}(\mathscr{F}\varphi) \subset U_1\}$ with $U_1 := \{\xi \in \mathbb{R}^3 \mid \xi^2 < m^2\}$. This means that in the decomposition of these functions with respect to plane waves, only plane waves with momentum $|p| < m$ contribute. p is the momentum as measured by the observer introduced by the $(3+1)$ slicing. This means in addition, that the group velocity of wave packets built up of plane waves with momenta $\xi \in U_1$ is smaller than the velocity of light. In this case the square root appearing in Eq. (4.1) can be expanded with respect to ξ^2/m^2. This gives

$$i(\partial_t \varphi)(x) = \frac{1}{(2\pi)^3} \int\int m\left(1 + \frac{\xi^2}{2m^2} + \frac{\xi^4}{8m^4} + \cdots\right) e^{i\xi\cdot(x-y)} \varphi(y) d^3y\, d^3\xi$$

$$= \left(m + \frac{1}{2m}\Delta + \frac{1}{8m^3}\Delta^2 + \cdots\right)\varphi. \tag{4.5}$$

Therefore we get to the first order (except the term $m\varphi$ which however can be absorbed into the function φ by substitution) the usual Schrödinger equation. Consequently, the nonrelativistic limit is correctly contained in Eq. (4.1).

Of course, the expansion (4.5) is valid only when applied to $\varphi \in S_{\rm nr}$. Otherwise the sum of symbols of increasing order will not converge.

E. The ultrarelativistic limit

If in the opposite case the FT of the particle contains large momenta $p^2 > m^2$, then the square root in Eq. (4.1) can be expanded with respect to m^2/ξ^2 resulting in

$$i(\partial_t \varphi)(x) = \frac{1}{(2\pi)^3} \int\int |\xi| \left(1 + \frac{1}{2}\frac{m^2}{\xi^2} + \frac{1}{8}\frac{m^4}{\xi^4} + \cdots\right) e^{i\xi\cdot(x-y)} \varphi(y) d^3y\, d^3\xi.$$

Inserting a plane wave $e^{i(p\cdot x - \omega t)}$ gives the correct dispersion relation for the ultrarelativistic limit $\omega = |p| + (m^2/2|p|) + \cdots$. This expansion describes a light cone expansion which means the approximation of geometrical optics. The first term gives the light cone structure.

From the mathematical point of view the above expansion is always defined because each term $m^{2j}/|\xi|^{2j-1}$ is a symbol of order $1-2j$. It is an asymptotic expansion of the symbol $\sqrt{\xi^2+m^2}$.

F. The nonlocality

At first we have to distinguish between different notions of nonlocality appearing in physics. There are (i) Aharonov–Bohm-like interactions; (ii) Einstein–Podolsky–Rosen Paradox (EPR) correlations; (iii) the impossibility to localize a particle, that is, δ functions are not eigenfunctions of a position operator; (iv) the field equation contains an infinite order of derivatives; (v) coupling of a field to derivatives of potentials, e.g., Darwin-like terms; (vi) locality conditions in quantized theories, that is, the commutator of fields vanishes for spacelike distances.

Here we are not concerned with (i) and (ii). We have no problems with (iii) either, because it has already been shown that Eq. (4.1) provides us with a complete set of plane waves so that it is possible to localize a particle. The nonlocality of Eq. (4.1) is of the form (iv). PDOs are so to say differential operators with an infinite order of derivatives. After coupling the field equation (4.1) to the Maxwell field, nonlocalities of the kind (v) also occur. Therefore we have to discuss the nonlocality of type (iv). In Sec. IV J it is shown that a locality in the sense of (vi) does not hold either; nevertheless, the fields will prove to be macrocausal.

We first observe that the nonlocality induced by the pseudodifferential operator appears only with respect to the space coordinates x. There is no nonlocality with respect to the time coordinate.

One approach to the nonlocality of particles described by Eq. (4.1) is to observe that in an expansion of the square root [compare Eq. (4.5)] there is so to say an infinite order of differentiations. Equation (4.5) can be compared with an operator of the form $e^{-\lambda^2\Delta}$ which is the prototype of a nonlocal operator with a characteristic nonlocality given by the length λ. This comparison up to the order four gives that λ is equal to the Compton-wavelength $\lambda_C = \hbar/mc$.

This is true for all choices of hypersurfaces which one choses for the formulation of the square root. Therefore, although it amounts to the measurement of a length in the chosen spacelike hypersurface, this nonlocality is a covariant phenomenon; no hypersurface, that is, no Lorentz observer, can be distinguished by this measurement. Also in this sense (4.1) is a relativistic equation.

The above nonlocality of the field equation does not imply that fields cannot be localized. Indeed, by means of Eq. (4.3) it is possible to construct with plane wave solutions of Eq. (4.1) a δ function in x space. δ functions are eigenfunctions of the position operator x. This is in contrast to the usual relativistic field equations, that is, the KGE and Dirac equation.[4] In these theories the impossibility to localize a particle is connected with the existence of positive and negative energy solutions.

In addition, the nonlocality of the field equation (4.1) means the following: If an initial state φ_0 with compact support is given, then the time derivative of the solution evolving from this initial state is not bound to supp φ_0, as it is for partial differential equations, see Audretsch and Lämmerzahl.[14] If, for example, the initial state is given by $\varphi_0=\delta_0$, then we get

$$i(\partial_t\varphi)(0,x)=\frac{1}{(2\pi)^3}\int \sqrt{\xi^2+m^2}e^{i\xi\cdot x}\,d^3\xi$$

and observe that the support of this function is larger than the point $\{0\}=\text{supp }\varphi_0$. See Sec. IV H.

The nonlocality of Eq. (4.1) also means that there is no finite maximum speed for the propagation of solutions. Requiring finite propagation speed implies that the underlying evolution equation is a partial differential equation[7,18,19] which then necessarily turns out to be weakly hyperbolic.[7] Nonfinite propagation speed must not be in contradiction to the principles of special relativity because of the following reasons: (i) The light cone structure is not disturbed by Eq. (4.1) as was shown above. (ii) The special theory of relativity is valid in the classical domain as it is shown by the plane wave solutions and by the Ehrenfest theorem. Equation (4.1) is macrocausal (see Sec. IV J). (iii) All observables agree with the relativity principle.

G. Ehrenfests's theorem

1. The velocity operator

The Hamilton function in six-dimensional phase space corresponding to Eq. (4.1), that is, the symbol of the Hamilton operator, is $H(x,p) = \sqrt{p^2+m^2}$. The velocity operator is obtained by means of the canonical relation

$$\hat{v}^a=\frac{d}{dt}\hat{x}^a=\frac{1}{i}\,[\mathcal{H}(x,p),\hat{x}^a].$$

This commutator can be evaluated according to Eq. (3.7a). We get for the symbol of the velocity operator

$$v^a(x,p)=\frac{\partial H(x,p)}{\partial p_a}=\frac{\delta^{ab}p_b}{\sqrt{p^2+m^2}}. \tag{4.6}$$

The velocity operator $\hat{v}^a$ then reads

$$(\hat{v}^a\varphi)(t,x)=\frac{1}{(2\pi)^3}\int\int v(\xi)e^{i\xi\cdot(x-y)}\varphi(t,y)d^3y\, d^3\xi.$$

Eigenfunctions of $\hat{v}^a$ are plane waves $a(t)e^{-i\eta\cdot x}$ with eigenvalue $\eta^a/\sqrt{\eta^2+m^2}$.

Expanding the square root one gets the expansion for slow velocities

$$v^a(x,p)=\delta^{ab}\frac{p_b}{m}\left(1-\frac{p^2}{2m^2}+\mathcal{O}(p^3)\right).$$

Therefore in our theory there appears no difficulties with the velocity operator in contrast to the Dirac theory. We do not have to construct a new position operator, the canonical is already the right one.

2. Acceleration and force operator

For the acceleration operator $\hat{a}^a=(d/dt)\hat{v}^a=(1/i)[\mathcal{H},\hat{v}^a]$ we of course get $\hat{a}^a=0$ because the Hamiltonian does not depend on x. The force operator $\hat{f}_a=(d/dt)\hat{p}_a=(1/i)[\mathcal{H},\hat{p}_a]$ vanishes also.

H. Lagrange formalism

1. A Lagrangian

The field equation (4.1) can be derived from a Lagrangian. Since the field equation is of the first order in the time derivative, the kinetic term of the Lagrangian must contain $\frac{1}{2}\varphi^*i\overleftrightarrow{\partial}_t\varphi$ where φ^* is the complex conjugate of φ. It is then easy to see that the Lagrangian density must be

$$\mathcal{L}(t,x)=\frac{1}{2}\varphi^*(t,x)i\overleftrightarrow{\partial}_t\varphi(t,x)-\frac{1}{(2\pi)^3}\int\int\varphi^*(t,x)\sqrt{\xi^2+m^2}e^{i\xi\cdot(x-y)}\varphi(t,y)d^3y\, d^3\xi.$$

The Lagrange function will then be

$$L(t)=\int\mathcal{L}\, d^3x$$

$$=\int\frac{1}{2}\varphi^*(t,x)i\overleftrightarrow{\partial}_t\varphi(t,x)d^3x-\frac{1}{(2\pi)^3}\int\int\int\varphi^*(t,x)\sqrt{\xi^2+m^2}e^{i\xi\cdot(x-y)}\varphi(t,y)d^3x\, d^3y\, d^3\xi.$$

Introducing the Schwartz kernel

$$K(x,y):=\frac{1}{(2\pi)^3}\int\sqrt{\xi^2+m^2}e^{i\xi\cdot(x-y)}\, d^3\xi=K(x-y)=K^*(y,x)$$

this can be written as

$$L=\int \frac{1}{2}\varphi^*(t,x)i\overleftrightarrow{\partial_t}\varphi(t,x)d^3x-\int\int \varphi^*(t,x)K(x,y)\varphi(t,y)d^3x\,d^3y.$$

Starting from these expressions we can carry through the usual canonical formalism. So we get the canonical momentum

$$\pi(t,x)=\frac{\delta L(t)}{\delta\partial_t\varphi(t,x)}=\frac{i}{2}\varphi^*(t,x). \tag{4.7}$$

2. The field equations

By variation with respect to φ^* we arrive again at the field equation (4.1). A general variation of the action gives

$$\delta S=\frac{i}{2}\left(\int_{\Sigma_2}-\int_{\Sigma_1}\right)(\varphi^*\delta\varphi-\delta\varphi^*\varphi)d^3x+i\int(\delta\varphi^*\partial_t\varphi-\partial_t\varphi^*\delta\varphi)d^3x\,dt$$

$$-\int\int(\delta\varphi^*(t,x)K(x,y)\varphi(t,y)+\varphi^*(t,x)K(x,y)\delta\varphi(t,y))d^3x\,d^3y\,dt,$$

where Σ_2 and Σ_1 are two hypersurfaces of constant $t_2>t_1$. Now we can insert various variations:

(i) Choosing especially $\delta\varphi^*=\delta\varphi=0$ at t_1 and t_2 we recover the field equation (4.1) and its complex conjugate.

3. Symmetries

In addition we can get conserved quantities by choosing other variations:

(ii) Choosing $\delta\varphi=i\alpha\varphi$ for $\alpha\in\mathbb{R}$, and constant, and using the field equation, we get probability conservation

$$\frac{d}{dt}\int\varphi^*\varphi\,d^3x=0$$

in the sense that there is a globally conserved probability which is positive definite. Therefore one can define a scalar product $\langle\psi|\varphi\rangle:=\int\psi^*\varphi\,d^3x$ which is conserved for solutions of Eq. (4.1). A Hilbert space formalism can be established.

For a field obeying Eq. (4.1) no current conservation in the sense of $\partial_\mu j^\mu=0$ can be found because this current conservation would be a local law.

(iii) Choosing $\delta\varphi=\alpha^a\partial_a\varphi$ for $\alpha\in\mathbb{R}^3$, and constant, and using the field equation, we get momentum conservation

$$P_a=\frac{i}{2}\int(\partial_a\varphi^*\varphi-\varphi^*\partial_a\varphi)d^3x=-\int\varphi^*i\partial_a\varphi\,d^3x=\text{const.}$$

for fields falling off sufficiently fast outside a bound region. Therefore $\hat{P}:=-i\nabla$ is the momentum operator.

(iv) Choosing in the same way $\delta\varphi=\alpha\partial_t\varphi$, $\alpha\in\mathbb{R}$, we get energy conservation

$$E:=\int\varphi^*i\partial_t\varphi\,d^3x=\text{const.}$$

and $\mathscr{H}=i\partial_t$ is the Hamilton operator giving the energy of the quantum system.

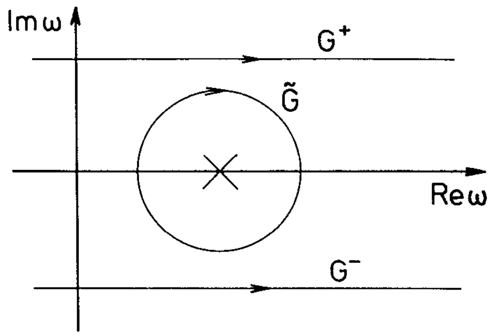

FIG. 1. Contours of the various Green's functions.

(v) And for $\delta\varphi = \alpha^b_a x^a \partial_b \varphi$ with $\alpha_{ab} = -\alpha_{ba}$ we get angular momentum conservation

$$L_{ab} = \frac{1}{2} \int \varphi^* i(x_a \partial_b - x_b \partial_a)\varphi \, d^3x = \text{const.}$$

All integrals are to be evaluated with respect to an arbitrary spacelike hypersurface. That means that the observables of our theory are in accordance with the relativity principle.

I. The fundamental solution, Green's functions

1. The Green's functions

By means of standard Fourier transformation techniques it is possible to construct the fundamental solutions, i.e., Green's functions, $G(t,x)$ for Eq. (4.1). That is, we are constructing the solution the source of which is a delta function

$$i(\partial_t G)(t,x) - \frac{1}{(2\pi)^3} \int\int \sqrt{\xi^2 + m^2} e^{i\xi \cdot (x-y)} G(t,y) d^3y \, d^3\xi = -\delta^3(x)\delta(t).$$

The Fourier transform of this equation gives

$$\frac{1}{(2\pi)^2} \int (\omega G(\omega,p) - \sqrt{p^2+m^2} G(\omega,p)) e^{-i(\omega t - p \cdot x)} \, d^3p \, d\omega = -\frac{1}{(2\pi)^4} \int e^{-i(\omega t - p \cdot x)} \, d^3p \, d\omega.$$

This leads to an equation for $G(\omega,p)$ in momentum space

$$G(\omega,p) = -\frac{1}{(2\pi)^2} \frac{1}{\omega - \sqrt{p^2+m^2}}.$$

Transforming back to x space we get

$$G(t,x) = -\frac{1}{(2\pi)^4} \int\int \frac{1}{\omega - \sqrt{p^2+m^2}} e^{-i(\omega t - p \cdot x)} \, d\omega \, d^3p.$$

The integration over ω can be performed by choosing a path in the complex ω plane (see Fig. 1). This procedure amounts in replacing the denominator $\omega - \sqrt{p^2+m^2}$ by $\omega - (1 \mp i\epsilon)\sqrt{p^2+m^2}$ with $\epsilon > 0$. With

$$\theta(t)=\begin{cases}1, & \text{if } t\geqslant 0\\ 0, & \text{if } t<0\end{cases}$$

integration along a path in the upper half plane gives $G^+(t,x)$, the other path gives $G^-(t,x)$

$$G^+(t,x)=\frac{i}{(2\pi)^3}\lim_{\epsilon\to 0}\theta(t)\int e^{i(p\cdot x-(1-i\epsilon)\sqrt{p^2+m^2}t)}\,d^3p,$$

$$G^-(t,x)=-\frac{i}{(2\pi)^3}\lim_{\epsilon\to 0}\theta(-t)\int e^{i(p\cdot x-(1+i\epsilon)\sqrt{p^2+m^2}t)}\,d^3p.$$

These are the fundamental solutions. We additionally can define

$$\widetilde{G}(t,x)=G^+(t,x)-G^-(t,x)=\frac{i}{(2\pi)^3}\int e^{i(px-\sqrt{p^2+m^2}t)}\,d^3p,$$

which later proves to be the commutator function.

The integration with respect to d^3p can also be performed. We write $p\cdot x=\mathrm{px}\cos\theta$ $(p:=|p|,x:=|x|)$ and integrate with respect to ϕ and θ

$$\begin{aligned}G^\pm(x)&=\pm\frac{i}{(2\pi)^3}\lim_{\epsilon\to 0}\theta(\pm t)\int e^{i(\mathrm{px}\cos\theta-(1\mp i\epsilon)\sqrt{p^2+m^2}t)}\mathrm{p}^2\,d\mathrm{p}\sin\theta\,d\theta\,d\phi\\ &=\pm\frac{2i}{(2\pi)^2}\theta(\pm t)\lim_{\epsilon\to 0}\frac{1}{x}\int \mathrm{p}\sin(\mathrm{px})e^{-it(1\mp i\epsilon)\sqrt{p^2+m^2}}\,d\mathrm{p}.\end{aligned}$$

We formally write $\mathrm{p}\sin(\mathrm{px})=-(d/d\mathrm{x})\cos(\mathrm{px})$, commute integration and differentiation, and integrate[20]

$$G^\pm(t,x)=\pm\frac{2}{(2\pi)^2}\theta(\pm t)\lim_{\epsilon\to 0}\frac{1}{\mathrm{x}}\frac{d}{d\mathrm{x}}\left(\frac{mt(1\mp i\epsilon)}{\sqrt{\mathrm{x}^2-t^2(1\mp i\epsilon)^2}}K_1\big(m\sqrt{\mathrm{x}^2-t^2(1\mp i\epsilon)^2}\big)\right). \tag{4.8}$$

We can replace $t^2(1\mp i\epsilon)^2-\mathrm{x}^2$ by $t^2-\mathrm{x}^2\mp i\epsilon$ and note that $K_1(z)=H_{-1}^{(1)}(iz)=-(\pi/2)H_1^{(1)}(iz)$ giving

$$\begin{aligned}G^\pm(t,x)&=\pm\frac{i}{4\pi}\theta(\pm t)\lim_{\epsilon\to 0}\frac{-m^2t}{\mathrm{x}}\frac{d}{d\mathrm{x}}\left(\frac{H_1^{(1)}(-m\sqrt{t^2-\mathrm{x}^2\mp i\epsilon})}{-m\sqrt{t^2-\mathrm{x}^2\mp i\epsilon}}\right)\\ &=\pm\frac{i}{4\pi}t\theta(\pm t)m^4\lim_{\epsilon\to 0}\frac{1}{z}\frac{d}{dz}\left(\frac{H_1^{(1)}(z)}{z}\right)\Bigg|_{z=-m\sqrt{t^2-\mathrm{x}^2\mp i\epsilon}}.\end{aligned} \tag{4.9}$$

These are the two Green's functions, the advanced and the retarded one. Up to the $t\theta(\pm t)$ function these Green's functions are Lorentz invariants. For a given $(3+1)$ slicing the Green's functions are given by a Lorentz invariant function which will be projected by $t\theta$ into the corresponding Lorentz system.

It would be of course possible to calculate the Green's function according to $G=(i\partial_t+\mathscr{H})E_{\mathrm{KG}}$ which however seems not to be as easy as the calculation above.

We also get sing supp $G^{\pm}=\Sigma_0\cup\{(t,x)\in\mathbb{R}^4\,|\,t^2-x^2=0\}$, that is, the union of the $t=0$ hypersurface and the light cone. This is a feature our equation has in common with other relativistic field equations as the Maxwell equations[17] or the equation of linear perturbation of a background solution of Einstein's equation.[21]

2. Light cone expansion

For small z, that is, near the light cone, we can expand the Hankel function

$$H_1^{(1)}(z)=J_1(z)+iN_1(z),$$

$$J_1(z)=\frac{z}{2}-\frac{z^3}{2^2\cdot 4}+\frac{z^5}{2^2\cdot 4^2\cdot 6}+\cdots,$$

$$N_1(z)=\frac{2}{\pi}\left[-\frac{J_0(z)}{z}+\left(\gamma+\ln\frac{z}{2}\right)J_1(z)-\frac{z}{2}+\frac{3}{2}\frac{z^3}{2^2\cdot 4}-\frac{11}{6}\frac{z^5}{2^2\cdot 4^2\cdot 6}+\cdots\right],$$

$$J_0(z)=1-\frac{z^2}{2^2}+\frac{z^4}{2^2\cdot 4^2}+\cdots,$$

where γ is Euler's number. Inserting these expansions into Eq. (4.9) and using

$$\frac{1}{z^2+i\epsilon}=\frac{1}{z^2}-i\pi\delta(z^2),\quad \frac{1}{(z^2+i\epsilon)^2}=\frac{d}{dz^2}\left(-\frac{1}{z^2}+i\pi\delta(z^2)\right),\quad \ln(z^2+i\epsilon)=\ln z^2+i\pi\theta(z^2)$$

we finally get $(z^2=m^2(t^2-x^2))$

$$G^{\pm}(z)=\pm\frac{i}{4\pi}\theta(\pm t)tm^4\left\{2\frac{d}{dz^2}\delta(z^2)+\delta(z^2)+\left[1-\theta\left(\frac{z^2}{4}\right)\right]\left(\frac{1}{8}-\frac{z^2}{96}\right)\right.$$
$$\left.+\frac{2i}{\pi}\left[\left(\gamma+\frac{1}{2}\ln\frac{z^2}{4}\right)\left(\frac{1}{8}-\frac{z^2}{96}\right)+\frac{d}{dz^2}\frac{1}{z^2}-\frac{1}{z^2}-\frac{1}{32}+\frac{z^2}{72}\right]+\mathscr{O}(z^3)\right\}. \quad (4.10)$$

Therefore the Green's functions do not vanish outside the light cone. The appearance of a derivative is in accordance with $G=(i\partial_t+\mathscr{H})E_{\mathrm{KG}}$.

3. Zero-mass Green's functions

Since $m\to 0$ implies $z\to 0$ we can use Eq. (4.10). Defining $z^2=:m^2s^2$ then the only m-independent term appearing in Eq. (4.10) is the term $(d/dz^2)\delta(z^2)$ giving as Green's functions for the $m=0$ case $(s^2=t^2-x^2)$

$$G^{\pm}_{m=0}(t,x)=\pm\frac{i}{2\pi}\theta(\pm t)t\frac{d}{ds^2}\delta(s^2).$$

4. On the Cauchy problem

Since the Green's functions are known, the homogeneous and inhomogeneous Cauchy problems can be solved. For the case $i\partial_t\varphi-\mathscr{H}\varphi=f$ with supp $f\subset\mathbb{R}^4_+$ with $\varphi_0(x)=\varphi(0,x)=0$ we have as unique solution $\varphi(t,x)=\int G^+(t-t',x-x')f(t',x')d^3x'\,dt'$. For the case $i\partial_t\varphi-\mathscr{H}\varphi=0$ and $\varphi_0(x)=\varphi(0,x)$ we get as solution $\varphi(t,x)=\int G^+(t,x-x')\varphi_0(x')d^3x'$.

Equation (4.1) is an evolution equation for which the question of uniqueness and existence of the Cauchy problem should be answered. It is the PDO generalization of the same question for partial differential equations with x-independent coefficients. In the latter case uniqueness and existence is equivalent to the demand of hyperbolicity of the differential operator, see, e.g., Ref. 7. However there does not seem to exist a complete theory of pseudodifferential equations with constant symbols seems not to exist. What can be shown is that a Cauchy–Kowalewski theorem for analytical PDOs[22] holds and that there are existence and uniqueness theorems for symmetric hyperbolic equations.[9]

It should be noted once more that nonfinite propagation speed does not contradict the relativity principle: If two Lorentz systems prepare the same initial state, then each of them will also see the same dynamical evolution.

J. On second quantization

We will shortly remark that our theory can be second quantized by means of the canonical formalism. To that purpose we introduce the operators $\hat{\varphi}(t,x)$, $\hat{\pi}(t,x)=(i/2)\hat{\varphi}^+(t,x)$ and the canonical commutation relations

$$[\hat{\varphi}(t,x),\hat{\varphi}(t,x')]=[\hat{\varphi}^+(t,x),\hat{\varphi}^+(t,x')]=0,\quad [\hat{\varphi}(t,x),\hat{\varphi}^+(t,x')]=2\delta(x-x'). \tag{4.11}$$

$\hat{\varphi}(t,x)$ fulfills Eq. (4.1). Since plane waves with arbitrary spacial momentum are solutions of Eq. (4.1), a complete set of solutions is available, namely, the energy-eigenfunctions $e^{-i(\omega t-p\cdot x)}$. Therefore an arbitrary solution can be expanded according to Eq. (4.3) with the coefficients $a(t,p)$ replaced by operators $\hat{a}(t,p)$. From Eq. (4.11) we get

$$[\hat{a}(t,p),\hat{a}(t,p')]=[\hat{a}^+(t,p),\hat{a}^+(t,p')]=0,\quad [\hat{a}(t,p),\hat{a}^+(t,p')]=\delta(p,p').$$

The time dependence of the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$ is given by $d\hat{a}(t,p)/dt=(1/i)[\hat{H},\hat{a}(t,p)]$ leading to $\hat{a}(t,p)=e^{-i\sqrt{p^2+m^2}t}\hat{a}(p)$ in accordance with Eq. (4.4). Now we are able to build up the whole quantum field theoretical formalism.

We can also show $[\hat{\phi}(t,x),\hat{\phi}^+(t',x')]=[\hat{\varphi}(t,x),\hat{\varphi}^+(t',x')]-[\hat{\varphi}(t,x),\hat{\varphi}^+(t',x')]^+ =2i\tilde{G}(t'-t,x'-x)$ with $\hat{\phi}:=\hat{\varphi}+\hat{\varphi}^+$. This commutator function is also a characteristic feature of the nonlocality of the fields. Because $\tilde{G}=G^+-G^-$ we infer from Eq. (4.8) that $\tilde{G}$ does not vanish for spacelike separated points for $t>0$. However, by means of the expansion of

$$K_1(m\sqrt{x^2-t^2})=\sqrt{\frac{\pi}{\bar{z}}}\,e^{-\bar{z}}\left(1+\frac{3}{8}\frac{1}{\bar{z}}+\cdots\right),$$

with $\bar{z}:=m\sqrt{x^2-t^2}$ for $t>0$ we can see that $\tilde{G}$ approaches zero for large spacelike distances. Since the leading term is $e^{-m\sqrt{x^2-t^2}}$, the characteristic length is the Compton wavelength. Therefore the commutator function fulfills a *macrocausality condition*.

In quantum mechanics the vanishing of the commutator of two Hermitian operators has the consequence that the related observed quantities can be measured simultaneously with arbitrary accuracy and without mutual influence. Accordingly, in local relativistic quantum theories the commutator of field operators at two spacelike separated points are required to vanish. In our case the commutator does not vanish for spacelike separations. This implies that the measurement of the field at two spacelike points can influence one another. However, since the commutator function falls off with the characteristic length λ_C this influence is only effective within a distance of λ_C. Since a violation of causality at such small distances is not measurable by any macroscopic device and since this violation is of the order $\hbar$, a macroscopic causality is still valid.

The fact that the canonical formalism works without essential modification is due to the fact that the field equation (4.1) is differential with respect to the time.

Therefore, after coupling our field equation to the Maxwell field, we may perform quantum electrodynamics as usual. The coupling to gravitational fields may be also described in this way. And finally even the dynamics of weak gravitational fields which is described by an equation $\eta^{\mu\nu}\partial_\mu\partial_\nu h_{\rho\sigma}+\cdots=0$ may be treated using this formalism. These problems will be postponed to a later publication.

V. CONCLUSION

By the method of taking the pseudodifferential operator square root of the KGE we have derived a *relativistic field equation for a spinless particle of first order in the time derivative* (4.1). This equation proved to be nonlocal and meets all the requirements of quantum mechanics and special relativity. This field equation can be properly interpreted.

The coupling of the field equation (4.1) to the Maxwell field can be performed by the usual minimal coupling procedure as will be shown later. For describing an equation of the form (4.1) in a gravitational field a more powerful method is needed, namely, Fourier integral operators. Of course, one expects that the main features of the present theory, the nonlocality or the violation of microcausality, will persist or become even more complicated.

Note added in proof: Another related work to ours but using a different formalism is E. Trübenbacher, Z. Naturforsch. **44a**, 801 (1989).

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