Covariant Equations of Motion of Extended Bodies with Mass and Spin Multipoles

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Content of lecture:

- Motivations
- Statement of the problem
- Notable issues
- Mathematical techniques
- Field equations
- Gravitational field and multipoles
- Equations of motion in the local chart
- Covariant equations of motion

Motivation for the present work

- ✓ LIGO-Virgo direct detection of gravitational waves from coalescing BH-BH, BH-NS and HS-NS binaries
- ✓ Needs GWs templates with accounting for higher-order post-Newtonian approximations
- ✓ The last stage of coalescence is sensitive to higher-order multipole moments. Open access to study the internal structure of NS and BH.
- ✓ Gravitational field of planets has been measured up to a significant number of multipole moments (e.g. 2190 harmonics in case of the earth - GRACE). Incorporation of the multipole moments to EIH equations of motion seems timely.

Statement of the problem

We consider N-body problem in an external gravitational environment

Here, we assume the background spacetime to be asymptotically flat.

The problem can be treated also in case of an expanding Friedmann universe taken as a background spacetime (Kopeikin et al., PRD, 2012-17).

Each extended body is to be represented as a point-like particle carrying out a set of its own internal mass and spin multipole moments.

The particle moves along time-like world line on a background spacetime manifold ("external universe").

The world line of the particle is split in geodesic motion and acceleration that is due to the interaction of the internal multipoles of the particle with the external universe.

Our objective is to find out a covariant description of the world line of the particle and equation of propagation of the particle's spin.

Notable issues:

Relativistic definition of multipole moments of the "particle"

Building the background spacetime manifold – separation of spacetime into the "particle world" and "external universe"

Back-reaction problem

Bootstrap effect (self-force) problem

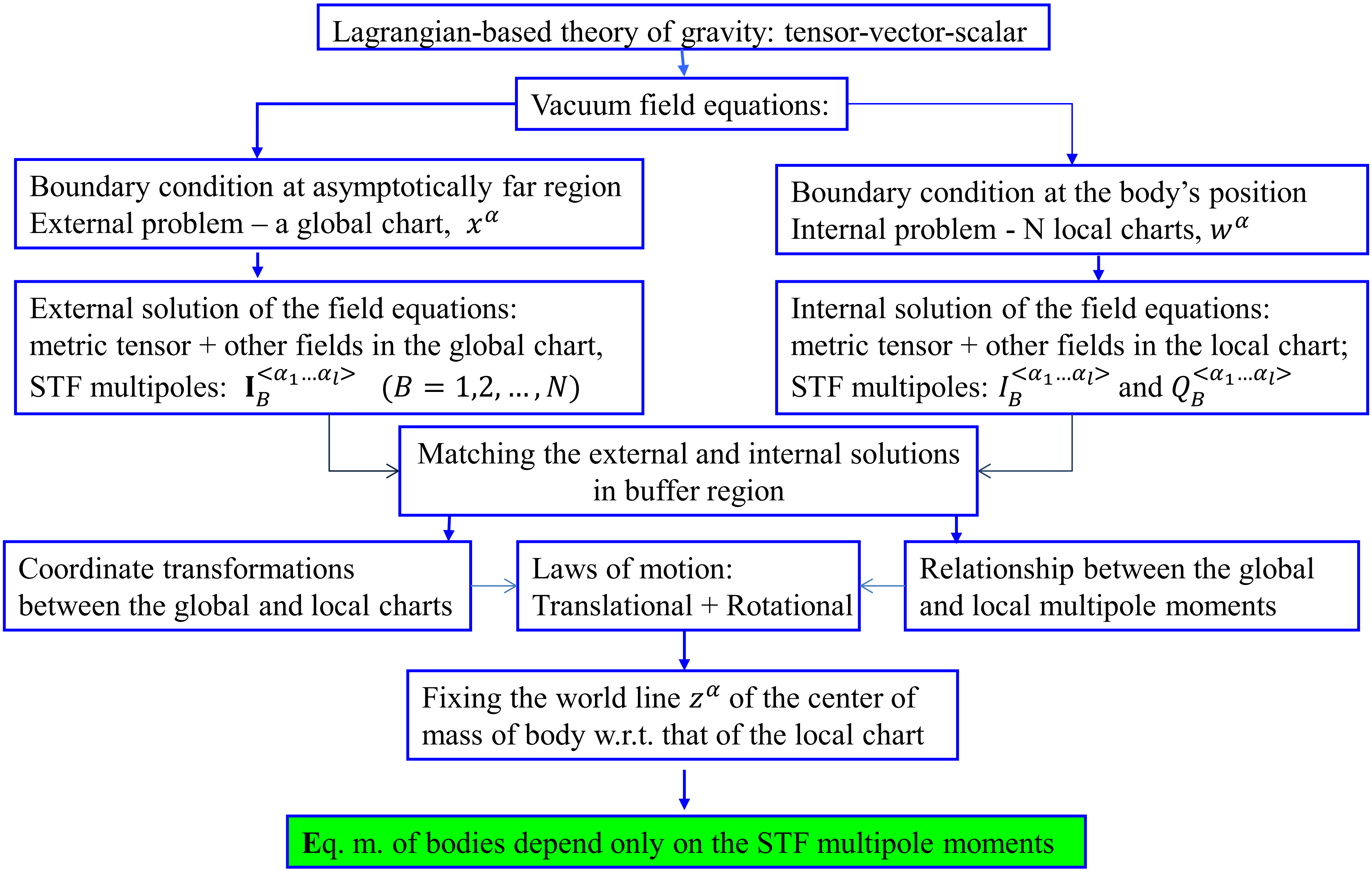
Effacing principle problem (a minimal number of multipoles entering equations of motion)

The best mathematical definition of the center of mass of the body (the world line of the particle crucially depends on it).

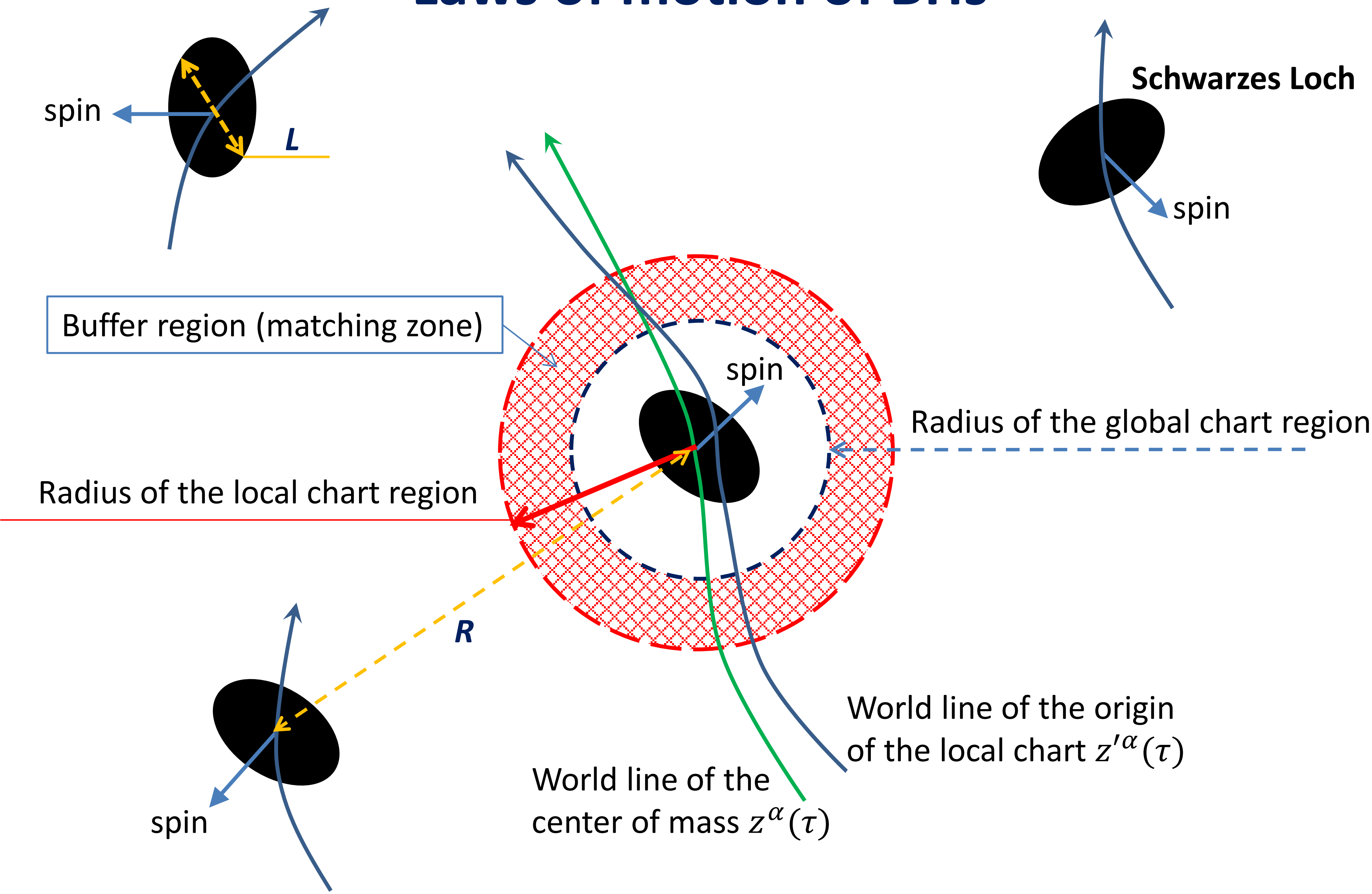
Mathematical Techniques for Deriving Equations of Motion

- Einstein-Infeld-Hoffmann
- Fock-Papapetrou
- D'Eath asymptotic matching
- Mathisson-Dixon-Synge (covariant)
- Thorne-Hurtle (covariant)

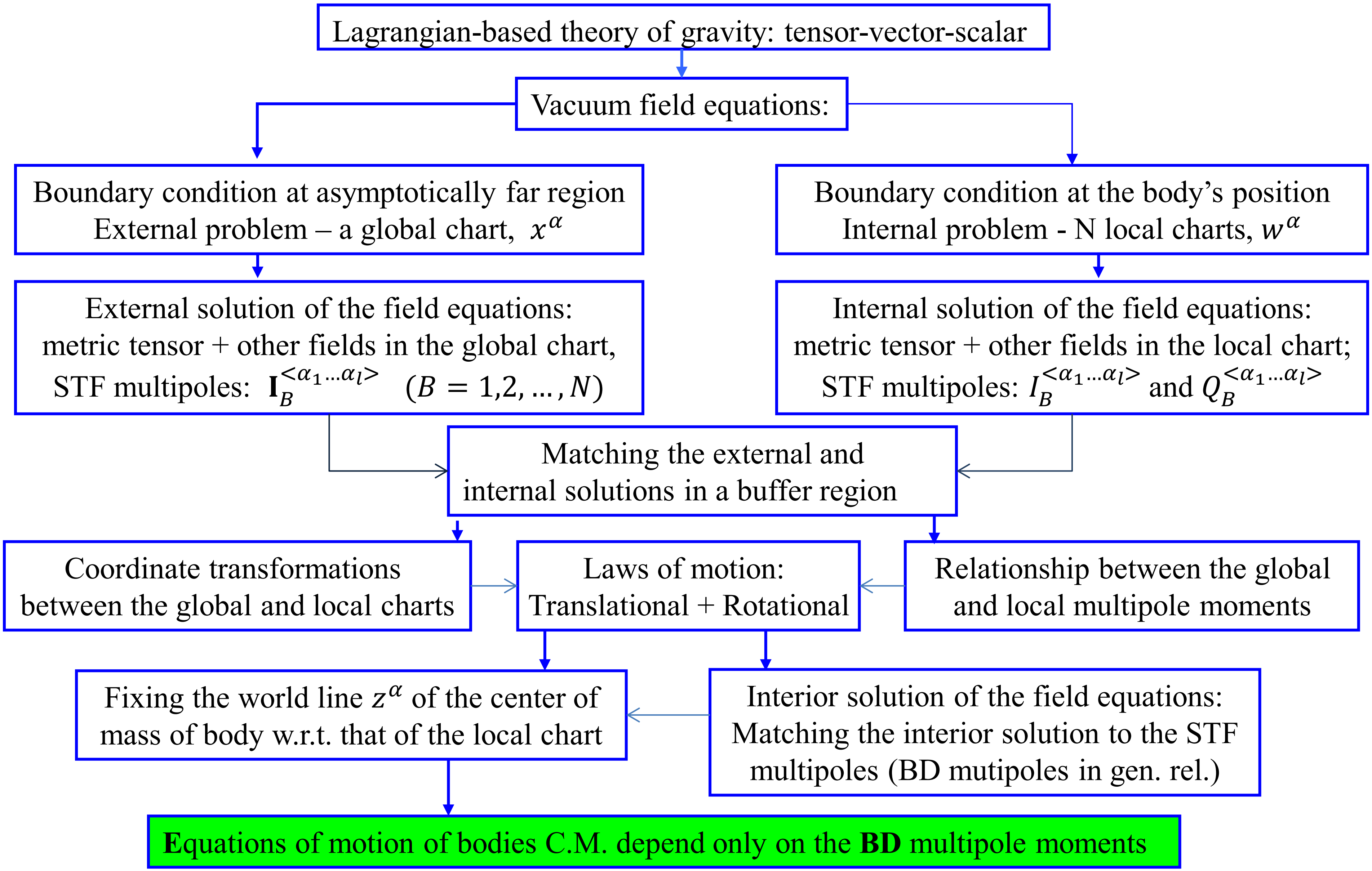
Deriving equations of motion for compact bodies (black holes).



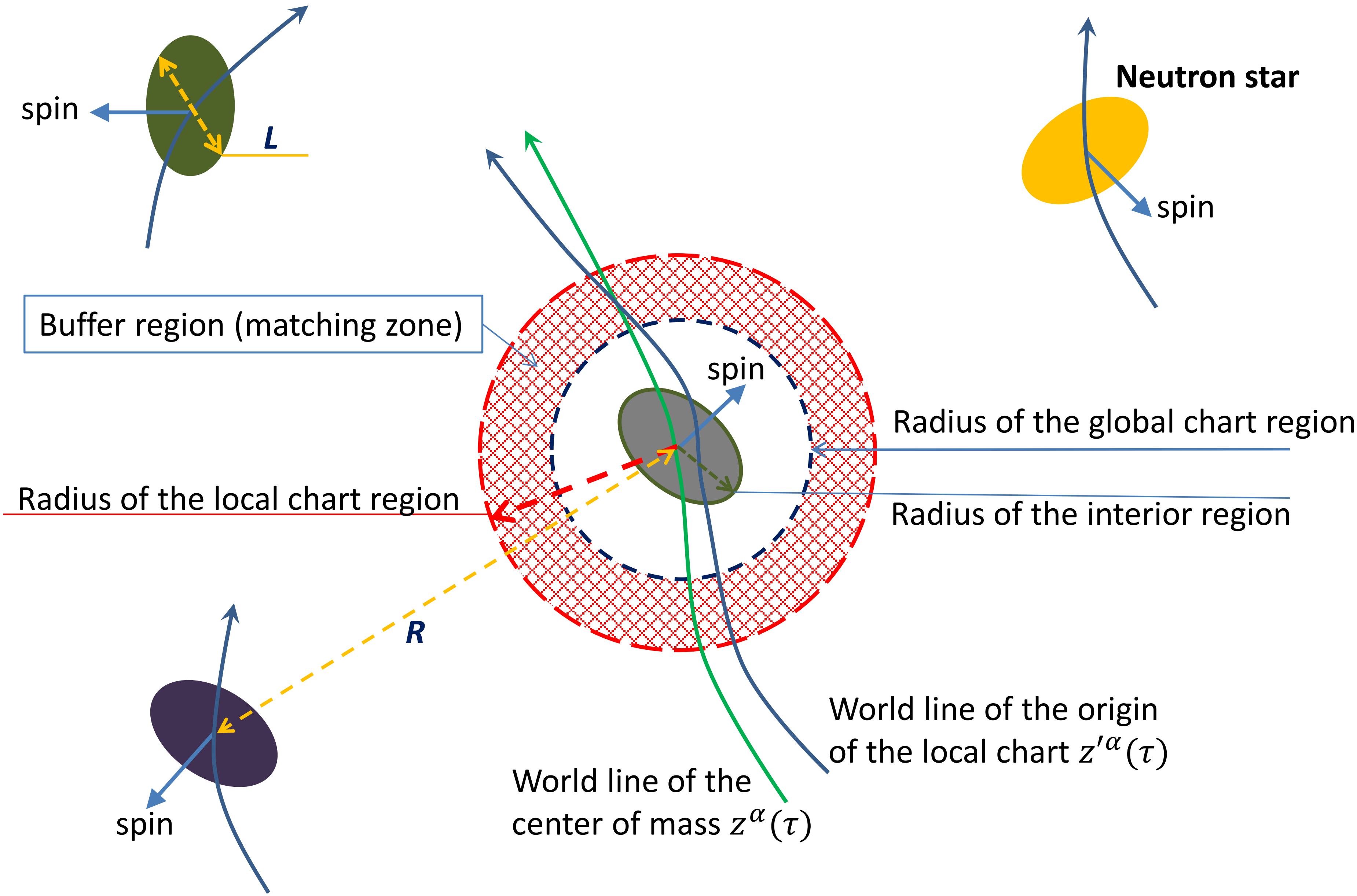
Laws of motion of BHs



Deriving equations of motion for extended bodies (neutron stars).



Laws of motion of extended bodies



Gravitational Field Equations (scalar-tensor theory)

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} - \frac{\theta(\phi)}{\phi^2} \phi_{,\mu} \phi_{,\nu} - \frac{1}{\phi} \left[\phi_{;\mu\nu} - \frac{1}{2} \frac{d\ln\big(2 + 3\theta(\phi)\big)}{d\phi} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right]$$

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{\alpha\beta} \phi_{,\alpha} \right)_{,\beta} = \frac{8\pi T}{2 + 3\theta(\phi)} - \frac{d\ln\big(2 + 3\theta(\phi)\big)}{d\phi} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$$

Harmonic (conformal) gauge

$$\left(\phi \sqrt{-g} g^{\alpha\beta} \right)_{,\beta} = 0$$

Post-Newtonian Approximations

slow motion **weak field**

Small parameters: $\epsilon_{\rm i} \sim v_{\rm i}/c$, $\epsilon_{\rm e} \sim v_{\rm e}/c$, and $\eta_{\rm i} \sim U_{\rm i}/c^2$, $\eta_{\rm e} \sim U_{\rm e}/c^2$

$U_{\rm i} \simeq GM/L$ and $U_{\rm e} \simeq GM/R$,

$\delta \simeq L/R$ **point-likeness parameter (multipole expansion)**

$$\sqrt{-g}g^{\alpha\beta} = \eta^{\alpha\beta} + \mathfrak{h}^{\alpha\beta}(x)$$

metric tensor decomposition

body **external universe** **non-linear interaction**

$$\mathfrak{h}^{\alpha\beta} = \mathfrak{h}_B^{\alpha\beta} + \mathfrak{h}_E^{\alpha\beta} + \mathfrak{h}_I^{\alpha\beta}$$

where $h_I^{\mu\nu} = O\left\{h_B^{\beta(\mu} h_{E\beta}^{\nu)}\right\}$

$$\Delta\mathfrak{h}^{\alpha\beta} = 16\pi W^{\alpha\beta} + \partial_t^2 \mathfrak{h}^{\alpha\beta}$$

$$W^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right) + \mathfrak{h}^{\mu\nu}\mathfrak{h}^{\alpha\beta}{}_{,\mu\nu} - \mathfrak{h}^{\alpha\mu}{}_{,\nu}\mathfrak{h}^{\beta\nu}{}_{,\mu}$$ + scalar field quadratic terms

$\Delta\phi = \partial_t^2\phi + 8\pi(\gamma - 1)T +$ scalar field and the metric perturbation quadratic terms

Internal Region: vacuum gravitational field
(solution of a homogeneous Laplace equation)

mass — dipole — STF internal mass multipoles

$$\mathfrak{h}^{00}_B = -\frac{4}{c^2}\left[\frac{m}{r} + \frac{\mathcal{I}_i x^i}{r^3} + \sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}\mathcal{I}_{<L>}\frac{x^{<L>}}{r^{2l+1}}\right] ,$$

momentum

STF internal spin multipoles

$$\mathfrak{h}^{0i}_B = -\frac{4}{c^3}\left[\frac{\dot{\mathcal{I}}_i}{r} + \sum_{l=2}^{\infty}\frac{(2l-3)!!}{l!}\dot{\mathcal{I}}_{<iL-1>}\frac{x^{<L-1>}}{r^{2l-1}}\right] + \frac{4}{c^3}\sum_{l=1}^{\infty}\frac{l(2l-1)!!}{(l+1)!}\epsilon_{iab}\mathcal{S}_{<bL-1>}\frac{x^{<aL-1>}}{r^{2l+1}} ,$$

$$\mathfrak{h}^{ij}_B = -\frac{4}{c^4}\sum_{l=2}^{\infty}\frac{(2l-5)!!}{l!}\ddot{\mathcal{I}}_{<ijL-2>}\frac{x^{<L-2>}}{r^{2l-3}} + \frac{8}{c^4}\sum_{l=2}^{\infty}\frac{l(2l-3)!!}{(l+1)!}\epsilon_{ab(i}\dot{\mathcal{S}}_{<j)bL-2>}\frac{x^{<aL-2>}}{r^{2l-1}} .$$

External Region: gravitational field of "external universe" (solution of a homogeneous Laplace equation)

acceleration

$$\mathfrak{h}_E^{00} = \frac{4}{c^2}\left[Q_i x^i + \sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}Q_{<L>}x^{<L>}\right],$$

$$\mathfrak{h}_E^{0i} = -\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(2l-1)!!}{(l+1)!}\frac{2l+1}{2l+3}\left[\dot{Q}_{<L>}x^{<L>}x^i - \frac{l}{2l+1}\dot{Q}_{<iL-1>}x^{<L-1>}r^2\right]$$

STF spin external multipoles

$$+\frac{4}{c^3}\sum_{l=2}^{\infty}\frac{l(2l-1)!!}{(l+1)!}\epsilon_{iab}C_{<bL-1>}x^{<aL-1>},$$

$$\mathfrak{h}_E^{ij} = \frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(2l-1)!!}{(l+2)!}\frac{2l+1}{2l+5}\left[\ddot{Q}_{<L>}x^{<L>}x^i x^j - \frac{1}{2l+3}\ddot{Q}_{<L>}x^{<L>}r^2\delta_{ij}\right.$$

$$\left.-\frac{2l}{2l+3}x_{(i}\ddot{Q}_{<j)L-1>}x^{<L-1>}r^2 + \frac{l(l-1)}{(2l+1)(2l+3)}\ddot{Q}_{<ijL-2>}x^{<L-2>}r^4\right]$$

$$-\frac{4}{c^3}\sum_{l=2}^{\infty}\frac{l(2l-1)!!}{(l+2)!}\frac{2l+1}{2l+3}\left[x_{(i}\epsilon_{j)ab}\dot{C}_{<bL-1>}x^a x^{<L-1>} - \frac{l-1}{2l+1}\epsilon_{ab(i}\dot{C}_{<j)bL-2>}x^a x^{<L-2>}r^2\right].$$

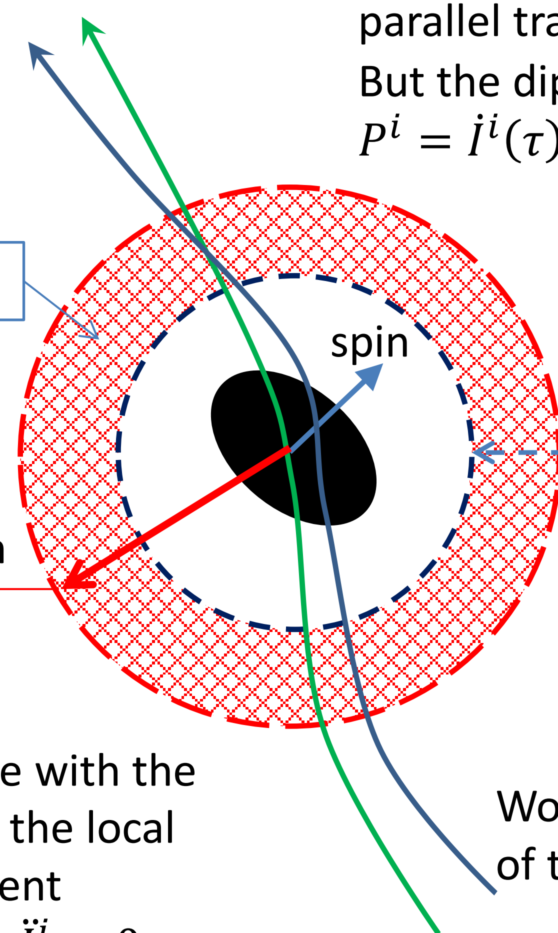
Gravitational field of non-linear interaction
(solution of an inhomogeneous Poisson equation)

$$
\begin{aligned}
\Delta \mathfrak{h}^{ij}_{I} &= \frac{m}{r^3} \sum_{l=1}^{\infty} \frac{(2l-1)!!}{(l-1)!} \left(\mathcal{Q}_{<iL-1>} x^j x^{<L-1>} + \mathcal{Q}_{<jL-1>} x^i x^{<L-1>} - \delta_{ij} \mathcal{Q}_{<L>} x^{<L>} \right) \\
&+ \sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \frac{(2l+1)!!}{l!} \frac{(2p-1)!!}{(p-1)!} \frac{\mathcal{I}_{<L>} x^{<L>}}{r^{2l+3}} \left(\mathcal{Q}_{<iP-1>} x^j x^{<P-1>} + \mathcal{Q}_{<jP-1>} x^i x^{<P-1>} - \delta_{ij} \mathcal{Q}_{<P>} x^{<P>} \right) \\
&- \sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \frac{(2l-1)!!}{(l-1)!} \frac{(2p-1)!!}{(p-1)!} \frac{x^{<L-1>} x^{<P-1>}}{r^{2l+1}} \left(\mathcal{I}_{<iL-1>} \mathcal{Q}_{<jP-1>} + \mathcal{I}_{<jL-1>} \mathcal{Q}_{<iP-1>} - \delta_{ij} \mathcal{I}_{<kL-1>} \mathcal{Q}_{<kP-1>} \right) ,
\end{aligned}
$$

Laws and equations of motion

$Q_i(\tau) = 0$ – a local inertial frame with a parallel transport of axes of the local chart. But the dipole moment $I^i(\tau) \neq 0$; $P^i = \dot{I}^i(\tau) \neq 0$; $\dot{P}^i = \ddot{I}^i \neq 0$.

Buffer region (matching zone)



Radius of the global chart region

Radius of the local chart region

$Q_i(\tau) \neq 0$ – a local inertial frame with the Fermi-Walker transport of axes of the local chart. Allows for the dipole moment $I^i(\tau) = 0$; $P^i = \dot{I}^i(\tau) = 0$; $\dot{P}^i = \ddot{I}^i = 0$.

World line of the origin of the local chart $z'^{\alpha}(\tau)$

World line of the center of mass $z^{\alpha}(\tau)$

Active mass multipoles

$$\mathcal{I}^{<L>} = \int\limits_{V_B} \sigma(u,w) w^{<L>} d^3w + \frac{\epsilon^2}{2(2l+3)} \left[\frac{d^2}{du^2} \int\limits_{V_B} \sigma(u,w) w^{<L>} w^2 d^3w \right.$$

$$\left. - 4(1+\gamma) \frac{2l+1}{l+1} \frac{d}{du} \int\limits_{V_B} \sigma^i(u,w) w^{<iL>} d^3w \right]$$

$$- \epsilon^2 \int\limits_{V_B} d^3w \, \sigma(u,w) \left\{ A + (2\beta - \gamma - 1)\mathcal{P} + \sum_{k=1}^{\infty} \frac{1}{k!} \left[Q_K + 2(\beta - 1)\mathcal{P}_K \right] w^{<K>} \right\} w^{<L>}$$

Conformal mass multipoles

$$I^{<L>} = \int\limits_{V_B} \varrho(u,w) \left\{ 1 - \epsilon^2 \left[A + (1-\gamma)\mathcal{P} + \sum_{k=1}^{\infty} \frac{1}{k!} Q_K w^{<K>} \right] \right\} w^{<L>} \, d^3w$$

$$+ \frac{\epsilon^2}{2(2l+3)} \left[\frac{d^2}{du^2} \int\limits_{V_B} \varrho(u,w) w^{<L>} w^2 \, d^3w \right.$$

$$\left. - \frac{8(2l+1)}{l+1} \frac{d}{du} \int\limits_{V_B} \sigma^i(u,w) w^{<iL>} \, d^3w \right],$$

Dipole moment and Linear momentum

$$I^i = \int_{V_B} \rho^* w^i \left[1 + \epsilon^2\left(\frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}_B\right)\right] d^3w$$
$$-\epsilon^2\left\{[A + (1-\gamma)\mathcal{P}] \int_{V_B} \rho^* w^i d^3w + \sum_{l=1}^{\infty} \frac{l+1}{l!} Q_L \mathcal{J}^{<iL>}\right.$$
$$\left. + \frac{1}{2}\sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} Q_{iL}\mathcal{N}^L\right\} + \mathcal{O}(\epsilon^4) \, ,$$

$$\mathcal{N}^L = \int_{V_B} \rho^* w^2 w^{<L>} d^3w \, ;$$

$$\hat{W}^i_B = G \int_{V_B} \frac{\rho^*(u, w')'^k (w^k - w'^k)(w^i - w'^i)}{|w - w'|^3} d^3w'$$

$$P^i = \int_{V_B} \rho^* \nu^i \left[1 + \epsilon^2\left(\frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}_B\right)\right] d^3w$$
$$+\epsilon^2 \int_{V_B} \left[\pi_{ik}\nu^k - \frac{1}{2}\rho^* \hat{W}^i_B\right] d^3w$$
$$-\epsilon^2 \frac{d}{du}\left\{[A + (1-\gamma)\mathcal{P}] I^i + \sum_{l=1}^{\infty} \frac{l+1}{l!} Q_L \mathcal{J}^{<iL>} + \frac{1}{2}\sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} Q_{iL}\mathcal{N}^L\right\}$$
$$+\epsilon^2 \sum_{l=1}^{\infty} \frac{1}{l!}\left[Q_L \dot{\mathcal{J}}^{<iL>} + \frac{l}{2l+1} Q_{iL-1}\dot{\mathcal{N}}^{L-1} - Q_L \int_{V_B} \rho^* v^i w^{<L>} d^3w\right] + \mathcal{O}(\epsilon^4) \, ,$$

The law of translational motion

$$\dot{P}^i = \mathcal{M}Q^i + \sum_{l=1}^{\infty}\frac{1}{l!}Q_{iL}\mathcal{I}^{<L>}$$

$$-\epsilon^2\Big\{\sum_{l=2}^{\infty}\frac{1}{(l+1)!}[(l^2+l+4)Q_L + 2(\gamma-1)\mathcal{P}_L]\ddot{\mathcal{I}}^{<iL>}$$

$$+\sum_{l=2}^{\infty}\frac{2l+1}{(l+1)(l+1)!}[(l^2+2l+5)\dot{Q}_L + 2(\gamma-1)\dot{\mathcal{P}}_L]\dot{\mathcal{I}}^{<iL>}$$

$$+\sum_{l=2}^{\infty}\frac{2l+1}{(2l+3)(l+1)!}[(l^2+3l+6)\ddot{Q}_L + 2(\gamma-1)\ddot{\mathcal{P}}_L]\mathcal{I}^{<iL>}$$

$$+[3Q_k + (\gamma-1)\mathcal{P}_k]\ddot{\mathcal{I}}^{<ik>} + \frac{3}{2}[4\dot{Q}_k + (\gamma-1)\dot{\mathcal{P}}_k]\dot{\mathcal{I}}^{<ik>}$$

$$+\frac{3}{5}[5\ddot{Q}_k + (\gamma-1)\ddot{\mathcal{P}}_k]\mathcal{I}^{<ik>} + \sum_{l=2}^{\infty}\frac{1}{l!}\dot{Z}_{iL}\mathcal{I}^{<L>}$$

$$+\sum_{l=1}^{\infty}\frac{1}{(l+1)!}\varepsilon_{ipq}\left[\dot{C}_{pL}\mathcal{I}^{<qL>} + \frac{l+2}{l+1}C_{pL}\dot{\mathcal{I}}^{<qL>}\right]$$

$$-2\sum_{l=1}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\left[(2Q_{pL} + (\gamma-1)\mathcal{P}_{pL})\dot{\mathcal{S}}^{<qL>}\right.$$

$$\left.+\frac{l+1}{l+2}(2\dot{Q}_{pL} + (\gamma-1)\dot{\mathcal{P}}_{pL})\mathcal{S}^{<qL>}\right] - \sum_{l=1}^{\infty}\frac{l(l+2)}{(l+1)(l+1)!}C_{iL}\mathcal{S}^{<L>}$$

$$-\frac{1}{2}\varepsilon_{ikq}[(4Q_k + 2(\gamma-1)\mathcal{P}_k)\dot{\mathcal{S}}^q + (2\dot{Q}_k + (\gamma-1)\dot{\mathcal{P}}_k)\mathcal{S}^q]$$

$$+(\mathcal{P}^i - Q^i)\left[\frac{1}{2}\eta\int_{V_B}\rho^*\hat{U}^{(\mathrm{B})}d^3w - \frac{1}{6}(\gamma-1)\ddot{\mathcal{I}}^{(2)} +\right.$$

$$\left.+2(\beta-1)\left(\mathcal{M}\mathcal{P} + \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{P}_L\mathcal{I}^{<L>}\right) + (\gamma-1)\sum_{l=1}^{\infty}\frac{1}{(l-1)!}Q_L\mathcal{I}^{<L>}\right]\Big\}$$

Active mass

Active STF mass multipoles

Scalar field STF multipoles

PPN parameter gamma

Gauge-dependent STF multipoles

Active STF mass multipoles

Nordtvedt parameter eta

Nordtvedt effect terms

PPN parameter beta

Acceleration of the local inertial frame

GR mass   GR mass   Active mass

Active STF mass multipoles

$$
\begin{aligned}
M Q^i = (M - \mathcal{M})\,\mathcal{P}^i - \sum_{l=1}^{\infty} \frac{1}{l!} Q_{iL} \mathcal{I}^{<L>} \\
+\epsilon^2 \Bigg\{ \sum_{l=2}^{\infty} \frac{1}{(l+1)!} [(l^2 + l + 4) Q_L + 2(\gamma - 1)\mathcal{P}_L] \ddot{\mathcal{I}}^{<iL>} \\
+ \sum_{l=2}^{\infty} \frac{2l+1}{(l+1)(l+1)!} [(l^2 + 2l + 5)\dot{Q}_L + 2(\gamma - 1)\dot{\mathcal{P}}_L] \dot{\mathcal{I}}^{<iL>} \\
+ \sum_{l=2}^{\infty} \frac{2l+1}{(2l+3)(l+1)!} [(l^2 + 3l + 6)\ddot{Q}_L + 2(\gamma - 1)\ddot{\mathcal{P}}_L] \mathcal{I}^{<iL>} \\
+[3Q_k + (\gamma - 1)\mathcal{P}_k] \ddot{\mathcal{I}}^{<ik>} + \frac{3}{2}[4\dot{Q}_k + (\gamma - 1)\dot{\mathcal{P}}_k] \dot{\mathcal{I}}^{<ik>} \\
+\frac{3}{5}[5\ddot{Q}_k + (\gamma - 1)\ddot{\mathcal{P}}_k] \mathcal{I}^{<ik>} + \sum_{l=2}^{\infty} \frac{1}{l!} \dot{Z}_{iL} \mathcal{I}^{<L>} \\
+ \sum_{l=1}^{\infty} \frac{1}{(l+1)!} \varepsilon_{ipq} \left[\dot{C}_{pL} \mathcal{I}^{<qL>} + \frac{l+2}{l+1} C_{pL} \dot{\mathcal{I}}^{<qL>} \right] \\
-2 \sum_{l=1}^{\infty} \frac{l+1}{(l+2)!} \varepsilon_{ipq} \Bigg[(2Q_{pL} + (\gamma - 1)\mathcal{P}_{pL}) \dot{\mathcal{S}}^{<qL>} \\
+\frac{l+1}{l+2} (2\dot{Q}_{pL} + (\gamma - 1)\dot{\mathcal{P}}_{pL}) \mathcal{S}^{<qL>} \Bigg] - \sum_{l=1}^{\infty} \frac{l(l+2)}{(l+1)(l+1)!} C_{iL} \mathcal{S}^{<L>} \\
-\frac{1}{2} \varepsilon_{ikq} [(4Q_k + 2(\gamma - 1)\mathcal{P}_k) \dot{\mathcal{S}}^q + (2\dot{Q}_k + (\gamma - 1)\dot{\mathcal{P}}_k) \mathcal{S}^q],
\end{aligned}
$$

Covariant equations of motion

Replace each extended body with a point-like particle carrying out an (infinite) set of the body's mass and spin multipole moments that have been uniquely defined in the course of matching of asymptotic expansions of gravitational field

Write down the world line of the particle on background spacetime manifold $\bar{g}_{\mu\nu}$ formed by gravitational action of N-1 particles ("external universe") in the local coordinates

Establish relationship between the Riemann tensor and its covariant derivatives with the multipole moments of the "external universe"

Split the world line of the particle in time-like geodesic and acceleration-dependent perturbation that is induced due to the interaction of the intrinsic multipoles of the particle with the multipole tidal moments of the "external universe".

Lift the coordinate-dependent description of equations of motion to covariant (coordinate-independent) form by making use of the rules:

- **geometric objects (multipoles) in the local chart are purely spatial objects being orthogonal to 4-velocity of the world line of the particle;**
- **geometric objects (multipoles) are subject to the Fermi-Walker transport;**
- **partial derivative goes over to the Fermi-Walker derivative.**

External multipoles in terms of the Riemann tensor

$$Q_{<\alpha_1\ldots\alpha_l>} = E_{<\alpha_1\ldots\alpha_l>} + \dot{Z}_{<\alpha_1\ldots\alpha_l>} + 3\sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-2-k)!}E_{<\alpha_1\ldots\alpha_{k+1}}E_{\alpha_{k+2}\ldots\alpha_l>}$$

$$+2\sum_{k=0}^{l-3}\left[\frac{(l-2)!}{k!(l-2-k)!}E_{<\alpha_1\ldots\alpha_{k+2}}E_{\alpha_{k+3}\ldots\alpha_l>} + \sum_{s=0}^{k}\frac{(l-2-k)k!}{s!(k-s)!}E_{<\alpha_1\ldots\alpha_{s+1}}E_{\alpha_{s+2}\ldots\alpha_l>}\right]$$

$$+2(\gamma-1)\left[\sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-2-k)!}E_{<\alpha_1\ldots\alpha_{k+1}}\mathcal{P}_{\alpha_{k+2}\ldots\alpha_l>} + \sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-1)k!}{s!(k-s)!}\mathcal{P}_{<\alpha_1\ldots\alpha_{s+1}}E_{\alpha_{s+2}\ldots\alpha_l>}\right]$$

$$+2(\beta-1)\left[\sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-2-k)!}\mathcal{P}_{<\alpha_1\ldots\alpha_{k+2}}\mathcal{P}_{\alpha_{k+3}\ldots\alpha_l>} + \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-2-k)!}\mathcal{P}_{<\alpha_1\ldots\alpha_{k+1}}\mathcal{P}_{\alpha_{k+2}\ldots\alpha_l>}\right],$$

$$\mathcal{P}_{<\alpha_1\ldots\alpha_l>} = \pi^{\beta_1}_{<\alpha_1}\cdots\pi^{\beta_l}_{\alpha_l>}\bar{\varphi}_{|\beta_1\cdots\beta_l}$$

$$C_{<\alpha_1\ldots\alpha_l>} = \frac{l}{l+1}B_{<\alpha_1\ldots\alpha_l>}$$

$$E_{<\alpha_1\ldots\alpha_l>} \equiv -u^\mu u^\nu \pi^{\beta_1}_{<\alpha_1}\pi^{\beta_2}_{\alpha_2}\ldots\pi^{\beta_l}_{\alpha_l>}\bar{R}_{\mu\beta_1\nu\beta_2|\beta_3\cdots\beta_l}\,,$$

$$B_{<\alpha_1\ldots\alpha_l>} \equiv -u^\nu \varepsilon_{\beta_1}{}^{\rho\sigma}\pi^{\beta_1}_{<\alpha_1}\pi^{\beta_2}_{\alpha_2}\ldots\pi^{\beta_l}_{\alpha_l>}\bar{R}_{\rho\sigma\beta_2\nu|\beta_3\cdots\beta_l}$$

$$\pi^\alpha_\beta \equiv \delta^\alpha_\beta + u^\alpha u_\beta \qquad\qquad \varepsilon_{\alpha\beta\gamma} \equiv \sqrt{-\bar{g}}\,u^\mu \pi^\nu_\alpha \pi^\rho_\beta \pi^\sigma_\gamma E_{\mu\nu\rho\sigma}$$

Covariant translational equations of motion

$$\frac{\mathcal{D}p^\mu}{\mathcal{D}\tau} = F^\mu_D + F^\mu_E + F^\mu_B + F^\mu_\Phi + F^\mu_{EE} + F^\mu_{EB} + F^\mu_{E\Phi} + F^\mu_{\Phi\Phi}$$

$$\mathrm{p}^\mu = M n^\mu$$

Linear momentum

$$n^\mu = u^\mu + \frac{1}{M}\sum_{l=2}^{\infty}\frac{(l^2+l+4)}{(l+1)!}E_{<\alpha_1\ldots\alpha_l>}u^\nu \mathcal{J}^{<\mu\alpha_1\ldots\alpha_l>}{}_{|\nu} + \ldots$$

Dynamic 4-velocity (n^μ)

Kinematic 4-velocity (u^μ)

$$n^\alpha u_\alpha = -1$$

$$F^\mu_D = (\mathcal{M} - M)\,\mathcal{P}^\mu$$

Nordvedt effect

$$F^\mu_E = \sum_{l=1}^{\infty}\frac{1}{l!}\bar{g}^{\mu\nu}E_{<\nu\alpha_1\ldots\alpha_l>}\mathcal{J}^{<\alpha_1\ldots\alpha_l>} + \ldots$$

Dixon force

$$F^\mu_B = \sum_{l=1}^{\infty}\frac{l}{(l+1)!}\bar{g}^{\mu\nu}B_{<\nu\alpha_1\ldots\alpha_l>}\mathcal{S}^{<\alpha_1\ldots\alpha_l>} + \ldots$$

Mathisson force

$$F^\mu_\Phi = 2\sum_{l=1}^{\infty}\frac{1}{(2l+3)(l+1)(l+1)!}u^\nu u^\beta \Phi_{<\alpha_1\ldots\alpha_l>|\nu}\mathcal{J}^{<\mu\alpha_1\ldots\alpha_l>}{}_{|\beta} + \ldots$$

Scalar field force

$$F^\mu_{EE} = 3\sum_{l=1}^{\infty}\sum_{s=0}^{l-1}\frac{1}{s!(l-1-s)!l}\bar{g}^{\mu\nu}E_{<\alpha_1\ldots\alpha_{s+1}}E_{\nu\alpha_{s+2}\ldots\alpha_l>}\mathcal{J}^{<\alpha_1\ldots\alpha_l>} + \ldots$$

Quadratic Riemann force

THANK YOU!