Testing General Relativity with clocks in space

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Models of Gravity

Universität Bremen*
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CENTER OF
APPLIED SPACE TECHNOLOGY
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Introduction

Notion of time

- In Newtonian theory time is absolute; all clocks tick at the same rate
- In Special Relativity we have to distinguish between coordinate time and proper time; different standard clocks tick with different rates if they are in relative motion
- In General Relativity proper time does in addition depend on the gravitational field

Clock effects in General Relativity (not complete)

- Gravitational redshift
- Shapiro delay
- Gravitomagnetic clock effect

Shapiro delay

Consider a photon moving radially in a Schwarzschild spacetime

$$0 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$
$$\Rightarrow \quad \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)$$

- This can be easily integrated to

$$t = r - r_0 + 2M \ln \frac{r - 2M}{r_0 - 2M}$$

- Newtonian travel time: $t = r - r_0$
- In addition there appears a logarithmic term
- This is the Shapiro delay

Observation of clock effects

In the weak field regime

- The gravitational redshift has been measured on Earth by GPA
  $\rightarrow$ talk by S. Herrmann on a new test of the redshift
- The gravitational redshift can be used to define and determine the Earth's geoid
  $\rightarrow$ talk by D. Philipp
- The Shapiro delay has been measured in the Solar System
- There is an additional temporal effect (due to frame dragging) which has never been tested in the weak field: the gravitomagnetic clock effect

Observation of clock effects

In the strong field regime

- The Shapiro delay and the redshift (encoded in the "Einstein delay") have been tested to first post–Newtonian order by observing pulsars in binary systems
- There is an ongoing search for pulsars orbiting a black hole

But:

- Is the 1st order PN approximation still valid in the strong gravitational field of a black hole?
- In the case of a pulsar orbiting Sgr A* (extreme mass ratio) we can find an exact expression
- Dhani, Master thesis 2017: At least 2nd order PN should be used

The gravitomagnetic clock effect has never been tested in the strong field regime

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The gravitomagnetic clock effect

The setup

- Two clocks on circular orbits in the equatorial plane of a rotating astronomical object
- One clock on prograde orbit, one on retrograde orbit
- Compare the measured time after a full revolution of 2π

Also called observer-dependent two-clock clock effect

Cohen and Mashhoon 1993 (Phys. Lett. A, 181:353)

- $\tau_+ - \tau_- \approx \frac{4\pi J}{mc^2}$
- For the Earth: time difference of about $10^{-7}\mathrm{sec}$ per revolution

$\rightarrow$ Large effect!?

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Problems and Goals

Problems

- Identical initial conditions required
- Identical orbits required
- Idealized circular orbits required

$\rightarrow$ Generalisations to eccentric and inclined orbits exist

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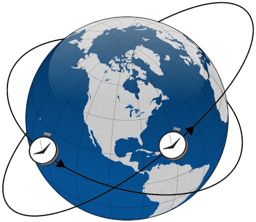

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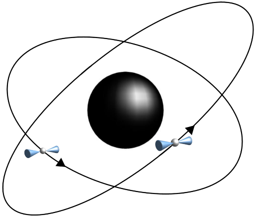

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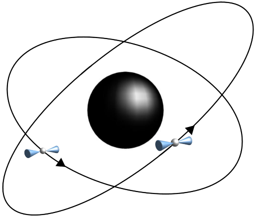

Generalisation: Fully general relativistic definition

$\rightarrow$ Consider bound geodesic orbits in Kerr spacetime

$\rightarrow$ Derive an expression for $\tau(\pm 2\pi)$, τ proper time

$\rightarrow$ Use fundamental frequencies

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Kerr spacetime

in Boyer-Lindquist (BL) coordinates

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}(adt - (r^2+a^2)d\varphi)^2 + \rho^2 d\theta^2$$

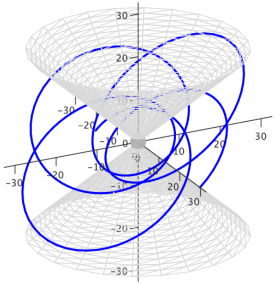
where $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$,
$M = \frac{Gm}{c^2}$ the mass, $a = J/(mc)$ the spin.

Equations of motion (using $d\tau = \rho^2 d\lambda$)

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r)\,, \quad \frac{d\varphi}{d\lambda} = \Phi_r(r) + \Phi_\theta(\theta)\,,$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta)\,, \quad \frac{dt}{d\lambda} = T_r(r) + T_\theta(\theta)$$

Periodic motion



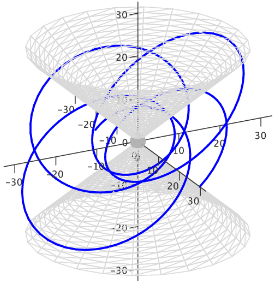
For bound orbits outside the horizons:

- The radial motion is periodic, $r \in [r_\mathrm{p}, r_\mathrm{a}]$
- The θ motion is periodic, $\theta \in [\theta_\mathrm{min}, \theta_\mathrm{max}]$

From $\left(\frac{dr}{d\lambda}\right)^2 = R$, $\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta$:

- Radial period Λ_r: $r(\lambda + \Lambda_r) = r(\lambda)$, $\Lambda_r = 2\int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{dr}{\sqrt{R}}$, $\Upsilon_r = \frac{2\pi}{\Lambda_r}$
- θ period Λ_θ: $\theta(\lambda + \Lambda_\theta) = \theta(\lambda)$, $\Lambda_\theta = 2\int_{\theta_\mathrm{min}}^{\theta_\mathrm{max}} \frac{d\theta}{\sqrt{\Theta}}$, $\Upsilon_\theta = \frac{2\pi}{\Lambda_\theta}$

Fundamental Frequencies



- φ, t, and τ are not periodic
- can be expressed as a linear function in λ + periodic oscillations
- Ansatz: $\varphi(\lambda) = \Upsilon_\varphi \lambda + \Phi^r_{osc} + \Phi^\theta_{osc}$
  Υ_φ infinite $\lambda-$average
- Analogously: $\tau(\lambda) = \Upsilon_\tau \lambda +$ osc.;
  $t(\lambda) = \Upsilon_t \lambda +$ osc.

Proper time as function of φ:

- Use averaged $\tau = \Upsilon_\tau \lambda$ and $\varphi = \Upsilon_\varphi \lambda$

$\rightarrow$ $\tau : \varphi \mapsto \tau(\lambda(\varphi)) = \Upsilon_\tau \Upsilon_\varphi^{-1} \varphi$

- In the Newtonian limit we obtain from this the Keplerian time of revolution

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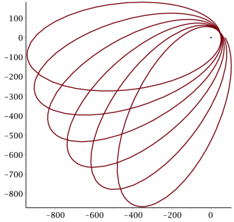
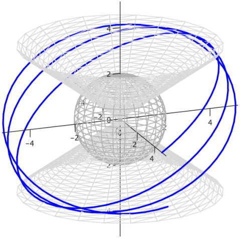
Periapsis precession and Lense-Thirring effect





Periapsis precession

- mismatch of radial and angular frequency wrt coordinate time
- $\dot{\omega} = \Omega_r - \Omega_\varphi = \frac{\Upsilon_r}{\Upsilon_t} - \frac{\Upsilon_\varphi}{\Upsilon_t}$
  $= (2\pi - \Lambda_r \Upsilon_\varphi)/P_r$
- $P_r = \Lambda_r \Upsilon_t$ anomalistic period

Lense-Thirring effect

- mismatch of polar and angular frequency wrt coordinate time
- $\dot{\Omega} = \Omega_\theta - \Omega_\varphi = (2\pi - \Lambda_\theta \Upsilon_\varphi)/P_\theta$
- $P_\theta = \Lambda_\theta \Upsilon_t$ draconitic period

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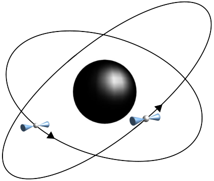
The gravitomagnetic clock effect



Consider two clocks on arbitrary geodesics

- Orbital parameters $r_{\mathrm{p},n}$, $r_{\mathrm{a},n}$, $\theta_{\mathrm{max},n}$, $n = 1, 2$
- Proper time of a full revolution: $\tau_n(\pm 2\pi, J)$

Generalised definition

- *Gravitomagnetic* clock effect:

$$\Delta\tau_{\mathrm{gm}} = \tau_1(\pm 2\pi, J) + \alpha\tau_2(\pm 2\pi, J)$$

- with α such that *gravitoelectric* effects cancel: $\Delta\tau_{\mathrm{gm}} = 0$ for $J = 0$, i.e. $\alpha = -\frac{\tau_1(\pm 2\pi, 0)}{\tau_2(\pm 2\pi, 0)}$

Post-Newtonian expansion

For a one-year orbit around Sgr A*: $a/r \leq M/r \lesssim 5 \times 10^{-4}$

- Expansion for small $\frac{a}{r} = \frac{J}{mcr}$ and small $\frac{M}{r} = \frac{Gm}{c^2 r}$

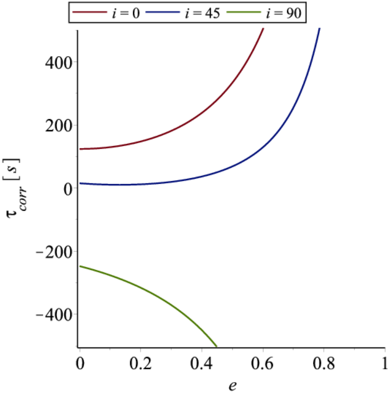
$$\tau(\pm 2\pi) \approx 2\pi \sqrt{\frac{\mathrm{a}^3}{Gm}} \left(1 - \frac{3(1+e^2)}{2(1-e^2)} \frac{M}{\mathrm{a}}\right) \pm \frac{2\pi(\cos i(3e^2+2e+3) - 2e - 2)}{(1-e^2)^{\frac{3}{2}}} \frac{J}{mc^2},$$

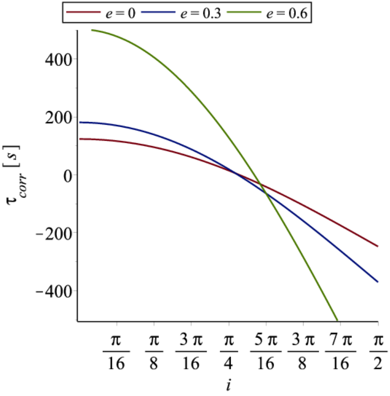
- a semimajor axis, e eccentricity, and i inclination
- $r_{\mathrm{p}} = \mathrm{a}(1-e)$, $r_{\mathrm{a}} = \mathrm{a}(1+e)$, and $\theta_{\max} = \pi/2 + i$

Astronomical object orbiting Sgr A*

Correction to proper orbital period due to frame dragging:

$$\tau_{\text{corr}} \approx \frac{2\pi J}{mc^2} \frac{\cos i(3e^2+2e+3)-2e-2)}{(1-e^2)^{\frac{3}{2}}}$$





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Clock effect for general orbits

For two clocks with arbitrary orbital parameters $\mathrm{a}_{1,2}$, $e_{1,2}$, $i_{1,2}$:

$$\Delta\tau_{\mathrm{gm}} \approx \frac{2\pi J}{mc^2}\Bigg[s_1 \frac{\cos i_1(3e_1^2+2e_1+3)-2e_1-2}{(1-e_1^2)^{\frac{3}{2}}} - s_2\sqrt{\frac{\mathrm{a}_1^3}{\mathrm{a}_2^3}}\frac{\cos i_2(3e_2^2+2e_2+3)-2e_2-2}{(1-e_2^2)^{\frac{3}{2}}}\Bigg]$$

- $s_{1,2}=+1$ for prograde motion, $s_{1,2}=-1$ for retrograde
- In particular: $s_1 = s_2$ possible!
- Identical orbital parameters: $\tau_+ - \tau_- \approx \frac{4\pi J}{mc^2}\frac{\cos i(3e^2+2e+3)-2e-2)}{(1-e^2)^{\frac{3}{2}}}$

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Two examples

First example

- Sgr A* rotates with $J/(mc) = 0.9M$
- First pulsar: 0.5-year orbit, nearly equatorial and circular
- Second pulsar: 1-year orbit, very eccentric and highly inclined
- Result: $\Delta\tau_{\rm gm} \approx 297{\rm s} \approx 2 \times 10^{-5}\, \tau(2\pi; {\rm J} = 0)$

Second example

- Sgr A* rotates with $J/(mc) = 0.5M$
- First pulsar: 1-year orbit, nearly equatorial and circular
- Second pulsar: 2-year orbit, a bit eccentric and quite inclined
- Result: $\Delta\tau_{\rm gm} \approx 59{\rm s} \approx 2 \times 10^{-6}\, \tau(2\pi; {\rm J} = 0)$

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Summary

The gravitomagnetic clock effect for Earth satellites

- satellites orbiting the Earth: effect $\sim 10^{-8} - 10^{-7}\,\mathrm{s}$
- but ultra precise tracking necessary: semi major axis to at least mm accuracy!

The gravitomagnetic clock effect for general astronomical objects

- for arbitrary bound geodesic orbits in Kerr spacetime
- definition via fundamental frequencies
- objects orbiting Sgr A*: effect up to $\sim 10^2\mathrm{s}$
- detectable by pulsars?

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Thank you for your attention!

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