PHYS131: Vectors and Vector Algebra

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- Office Hour: Wed 1-2pm

Literature:

- FLAP, Sec. 2.4 – 2.7
- Jordan and Smith: "Mathematical Techniques", Chapt. 9 – 11
- Young and Freedman: "University Physics"

I. Basic Vector Algebra

In physics one distinguishes scalars (number with a unit) and vectors (number with a unit and direction).

Examples:	Scalars	Vectors
	mass of a particle	velocity of a particle
	charge of a particle	acceleration of a particle
	speed of a particle	force upon a particle
	temperature at a point	electric field vector at a point

Remark: In this course we will deal with *real* scalars and *real* vectors only. You will learn about *complex* scalars and *complex* vectors later.

It is recommendable to distinguish vectors from scalars by notation. Three different ways of how to do this can be found in the literature:

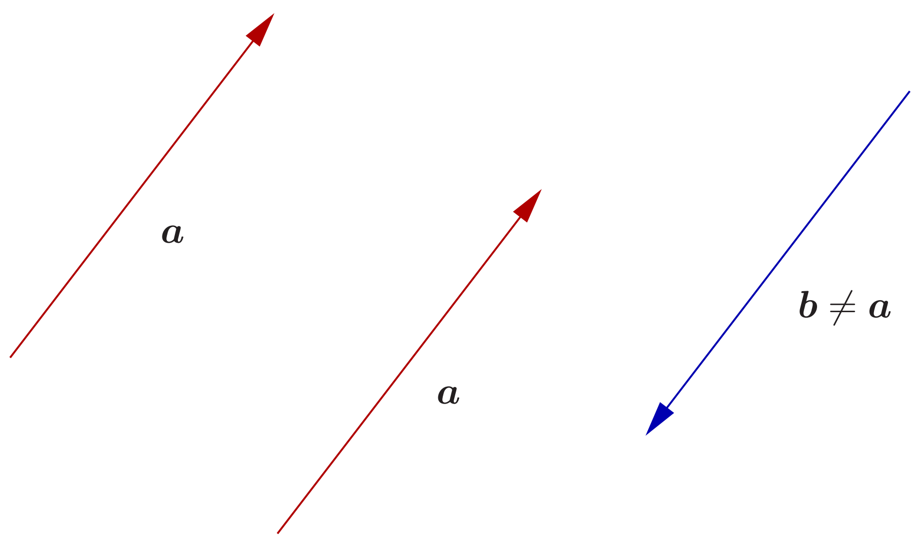
1) Use boldface letters for vectors in print: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \ldots$
and underlined letters for vectors in hand-writing: $\underline{a}, \underline{b}, \underline{c}, \underline{d} \ldots$
In LATEX math mode, $\boldsymbol{a}$ is encoded as `\boldsymbol{a}`.

2) Use letters with arrows for vectors: $\vec{a}, \vec{b}, \vec{c}, \vec{d} \ldots$
In LATEX math mode, $\vec{a}$ is encoded as `\vec{a}`.

3) Use german (gothic) letters for vectors: $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \ldots$
This is no longer usual but can be found in older books. In LATEX math mode, $\mathfrak{a}$ is encoded as `\mathfrak{a}`.

In this course we will use the first notation, i.e. boldface letters for vectors in print and underlined letters in hand-written text.

It is usual to depict vectors as arrows. Parallel arrows of equal length represent the same vector (but anti-parallel arrows do not). So, a vector corresponds to a *class* of arrows.



The length (= modulus = magnitude = norm) of a vector $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$.

$|\boldsymbol{a}|$ is a non-negative scalar.

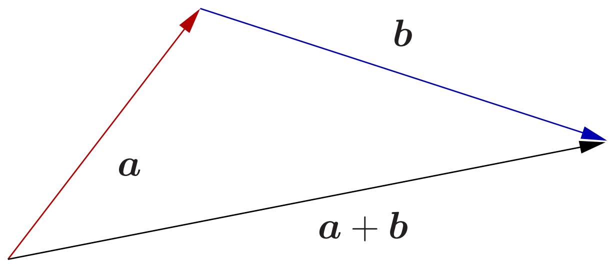
A vector $\boldsymbol{a}$ with $|\boldsymbol{a}| = 1$ is called a *unit vector*.

The vector $\mathbf{0}$ which is represented by an arrow whose head coincides with its tail is called the *zero vector*. It has zero length, $|\mathbf{0}| = 0$.

In the following the four basic algebraic vector operations are geometrically introduced. Later we will rewrite these operations in terms of vector components.

1) Adding two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ together:

   Represent $\boldsymbol{a}$ and $\boldsymbol{b}$ by arrows such that the head of $\boldsymbol{a}$ coincides with the tail of $\boldsymbol{b}$. Then $\boldsymbol{a}+\boldsymbol{b}$ is represented by the arrow from the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$.



   Rules: • $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$

   • $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$, so brackets can be dropped.

2) Multiplying a vector $\boldsymbol{a}$ by a scalar s:
   $s\boldsymbol{a}$ is the vector resulting from stretching $\boldsymbol{a}$ with factor s.

   Rules: • $s\boldsymbol{a}=\boldsymbol{a}s$ (by convention)

   • $s(\boldsymbol{a}+\boldsymbol{b})=s\boldsymbol{a}+s\boldsymbol{b}$

   • $(s+t)\boldsymbol{a}=s\boldsymbol{a}+t\boldsymbol{a}$

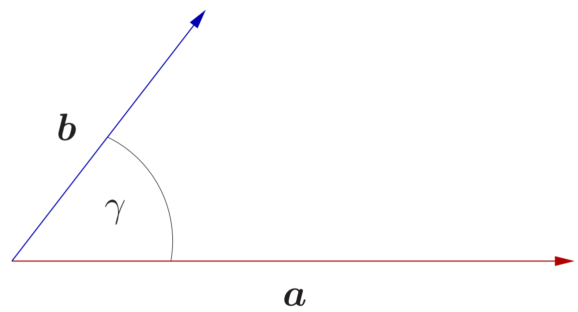
   The subtraction of a vector from another is now defined as

$$\boldsymbol{a}-\boldsymbol{b} \;=\; \boldsymbol{a} \;+\; (-1)\,\boldsymbol{b}\,.$$

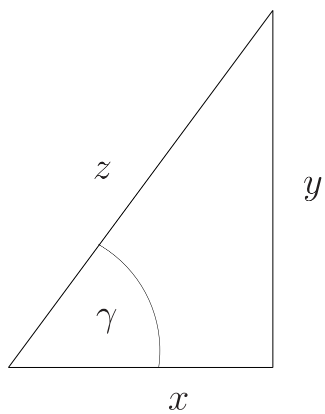
3) Scalar product (or dot product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

The scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by the equation

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\gamma$$



Recall the definition of sin, cos, tan and cot:



$$\sin\gamma = \tfrac{y}{z}$$
$$\cos\gamma = \tfrac{x}{z}$$
$$\tan\gamma = \tfrac{y}{x}$$
$$\cot\gamma = \tfrac{x}{y}$$

Rules:
- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot (s\,\boldsymbol{b}) = s\,(\boldsymbol{a} \cdot \boldsymbol{b})$

Notes:
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$, so $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$
- $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal to each other. (This includes the case that one of the two vectors is the zero vector. The zero vector is orthogonal to any vector.)

4) The vector product (or cross product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

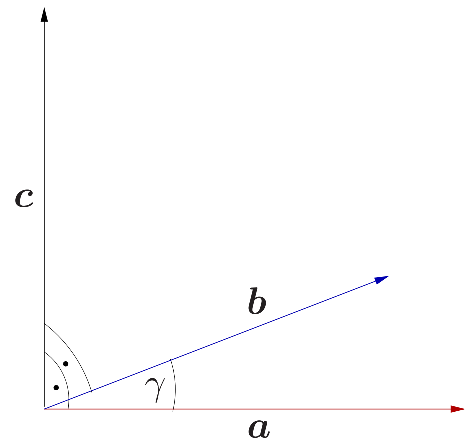
$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ is defined by

(i) $|\boldsymbol{c}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\gamma$

(ii) $\boldsymbol{c}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$

(iii) $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are right-handed, i.e.

$$\left.\begin{array}{rc} \boldsymbol{a}: & \text{thumb} \\ \boldsymbol{b}: & \text{index finger} \\ \boldsymbol{c}: & \text{middle finger} \end{array}\right\} \text{ of right hand}$$



Rules:
- $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
- $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$
- $\boldsymbol{a} \times (s\boldsymbol{b}) = s\,(\boldsymbol{a} \times \boldsymbol{b})$

We now want to rewrite these four basic algebraic vector operations in terms of components (or coordinates). To that end we need the notion of linear independence:

Definition: Three vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are called *linearly independent* if

$$u\boldsymbol{a} + v\boldsymbol{b} + w\boldsymbol{c} = \boldsymbol{0} \qquad \text{implies} \qquad u = v = w = 0\,.$$

Geometrically, this means that three vectors are linearly indendent if there is no plane such that all three vectors are tangent to this plane.

Three vectors which are linearly independent are called a *basis* for the three-dimensional space. If we have chosen a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, every vector $\boldsymbol{a}$ in three-dimensional space can be written as a linear combination of the basis vectors,

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3\ .$$

The numbers a_1, a_2, a_3 are called the *components* (or *coordinates*) of $\boldsymbol{a}$ with respect to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

For most applications it is recommendable to choose a socalled Cartesian basis:

Definition: The three vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called a *Cartesian basis* if

(i) all three vectors are unit vectors,

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$$

(ii) they are orthogonal to each other,

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = 0$$

(iii) they are right-handed,

$$\left.\begin{array}{rc} \boldsymbol{i}: & \text{thumb} \\ \boldsymbol{j}: & \text{index finger} \\ \boldsymbol{k}: & \text{middle finger} \end{array}\right\} \text{ of right hand}$$

A Cartesian basis is also called a "right-handed orthonormal basis". We will not work with non-Cartesian bases in this course (although they are sometimes used, e.g. in crystal physics).

From now on we will assume that a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ has been chosen. We can then write every vector $\boldsymbol{a}$ in three-dimensional space as a linear combination of the basis vectors,

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \ .$$

We will identify the vector $\boldsymbol{a}$ with its components a_1, a_2, a_3, written in the form

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Some authors write this as a row, rather than as a column, i.e. $\boldsymbol{a} = (a_1, a_2, a_3)$. However, for calculations the column notation is usually more convenient.

We are now ready to rewrite the four basic algebraic vector operations in coordinate notation.

1) Adding two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ together.

$$\boldsymbol{a}+\boldsymbol{b} = a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k} + b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k} = (a_1+b_1)\boldsymbol{i}+(a_2+b_2)\boldsymbol{j}+(a_3+b_3)\boldsymbol{k}$$

i.e., the addition is done by adding the components,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} .$$

2) Multiplying a vector $\boldsymbol{a}$ by a scalar s:

$$s\,\boldsymbol{a} = s\,(a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = s\,a_1\boldsymbol{i} + s\,a_2\boldsymbol{j} + s\,a_3\boldsymbol{k}$$

i.e., multiplication by a scalar is done my multiplying each component,

$$s\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} s\,a_1 \\ s\,a_2 \\ s\,a_3 \end{pmatrix} .$$

Example: Check whether the vectors $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{j} + \boldsymbol{k}$ are linearly independent!
Let us assume that

$$u\boldsymbol{a} + v\boldsymbol{b} + w\boldsymbol{c} = \boldsymbol{0}\,.$$

We have to check whether this implies that $u = v = w = 0$.

$$u\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

This vector equation gives three scalar equations:

$$u + v = 0$$
$$v + w = 0$$
$$w = 0$$

As, by the last equation, $w = 0$, the second equation yields $v = 0$, hence the first equation yields $u = 0$. So the three vectors are, indeed, linearly independent.

3) Scalar product (or dot product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} \;=\; (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \;=$$

$$a_1\,b_1\,\underbrace{\boldsymbol{i}\cdot\boldsymbol{i}}_{=1} \;+\; a_1\,b_2\,\underbrace{\boldsymbol{i}\cdot\boldsymbol{j}}_{=0} \;+\; a_1\,b_3\,\underbrace{\boldsymbol{i}\cdot\boldsymbol{k}}_{=0} +$$

$$a_2\,b_1\,\underbrace{\boldsymbol{j}\cdot\boldsymbol{i}}_{=0} \;+\; a_2\,b_2\,\underbrace{\boldsymbol{j}\cdot\boldsymbol{j}}_{=1} \;+\; a_2\,b_3\,\underbrace{\boldsymbol{j}\cdot\boldsymbol{k}}_{=0} +$$

$$a_3\,b_1\,\underbrace{\boldsymbol{k}\cdot\boldsymbol{i}}_{=0} \;+\; a_3\,b_2\,\underbrace{\boldsymbol{k}\cdot\boldsymbol{j}}_{=0} \;+\; a_3\,b_3\,\underbrace{\boldsymbol{k}\cdot\boldsymbol{k}}_{=0} \;=$$

$$a_1\,b_1 \;+\; a_2\,b_2 \;+\; a_3\,b_3\,.$$

So in component notation the scalar product is calculated as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \;=\; a_1\,b_1 \;+\; a_2\,b_2 \;+\; a_3\,b_3\,.$$

Example: Calculate the angle between the vectors $\boldsymbol{a} = \boldsymbol{i}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$!

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\gamma$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \;=\; \sqrt{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}\;\sqrt{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}\;\cos\gamma$$

$$1 + 0 + 0 \;=\; \sqrt{1+0+0}\,\sqrt{1+1+0}\,\cos\gamma$$

$$1 = \sqrt{2}\cos\gamma \quad\Rightarrow\quad \cos\gamma \;=\; 1/\sqrt{2} \quad\Rightarrow\quad \gamma \;=\; 45^o\,.$$

4) The vector product (or cross product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

First observe that $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ because (i) $\underbrace{|\boldsymbol{k}|}_{=1} = \underbrace{|\boldsymbol{i}|}_{=1}\,\underbrace{|\boldsymbol{j}|}_{=1}\,\underbrace{\sin\sphericalangle(\boldsymbol{i},\boldsymbol{j})}_{=1}$,

(ii) $\boldsymbol{k}$ is orthogonal to $\boldsymbol{i}$ and $\boldsymbol{j}$,

(iii) $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ form a right-handed basis.

Analogously one verifies that $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.
(Mind the order of the factors in the vector product!)

Now we calculate:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1\,\boldsymbol{i} + a_2\,\boldsymbol{j} + a_3\,\boldsymbol{k}) \times (b_1\,\boldsymbol{i} + b_2\,\boldsymbol{j} + b_3\,\boldsymbol{k}) =$$

$$= a_1b_1\,\underbrace{\boldsymbol{i} \times \boldsymbol{i}}_{=\boldsymbol{0}} + a_1b_2\,\underbrace{\boldsymbol{i} \times \boldsymbol{j}}_{=\boldsymbol{k}} + a_1b_3\,\underbrace{\boldsymbol{i} \times \boldsymbol{k}}_{=-\boldsymbol{j}} +$$

$$a_2b_1\,\underbrace{\boldsymbol{j} \times \boldsymbol{i}}_{=-\boldsymbol{k}} + a_2b_2\,\underbrace{\boldsymbol{j} \times \boldsymbol{j}}_{=\boldsymbol{0}} + a_2b_3\,\underbrace{\boldsymbol{j} \times \boldsymbol{k}}_{=\boldsymbol{i}} +$$

$$a_3b_1\,\underbrace{\boldsymbol{k} \times \boldsymbol{i}}_{=\boldsymbol{j}} + a_3b_2\,\underbrace{\boldsymbol{k} \times \boldsymbol{j}}_{=-\boldsymbol{i}} + a_3b_3\,\underbrace{\boldsymbol{k} \times \boldsymbol{k}}_{=\boldsymbol{0}} =$$

$$= (a_2b_3 - a_3b_2)\,\boldsymbol{i} + (a_3b_1 - a_1b_3)\,\boldsymbol{j} + (a_1b_2 - a_2b_1)\,\boldsymbol{i}\,,$$

Hence in column notation:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

which can be memorised by noting the cyclic order of indices in each row: 123, 231, 312.

Another way of memorising this formula is by writing it as a determinant.

Determinants are defined in the following way.

2-by-2 determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\,.$$

3-by-3 determinant:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} .$$

Mind the alternating sign!

n-by-n determinants, for $n > 3$, can be defined iteratively in a similar fashion but we will not need them in this course.

With determinants at hand, we can rewrite the cross procuct in the following form:

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} .$$

Indeed, calculating

$$\det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \boldsymbol{i} \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} - \boldsymbol{j} \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} + \boldsymbol{k} \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} =$$

$$= \boldsymbol{i}\,(a_2b_3 - a_3b_2) - \boldsymbol{j}\,(a_1b_3 - a_3b_1) + \boldsymbol{k}\,(a_1b_2 - a_2b_1)$$

reproduces our above expression for $\boldsymbol{a} \times \boldsymbol{b}$.

Remarks:

- Our coordinate formulas for scalar product and cross product hold in a *Cartesian* basis only. For this reason it is not recommendable to use the column notation for non-Cartesian coordinates.

- The notion of *dividing* by a vector cannot be defined. The reason is that neither the scalar product nor the vector product is an invertible operation:

  If $\boldsymbol{a}$ and s are given, the equation $\boldsymbol{a} \cdot \boldsymbol{x} = s$ does not determine a *unique* vector $\boldsymbol{x}$; if this equation is satisfied by some vector $\boldsymbol{x}$, it is also satisfied by all vectors $\boldsymbol{x} + \boldsymbol{y}$ where $\boldsymbol{y}$ is any vector orthogonal to $\boldsymbol{a}$.

  If $\boldsymbol{a}$ and $\boldsymbol{c}$ are given, the equation $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{c}$ does not determine a *unique* vector $\boldsymbol{x}$; if this equation is satisfied by some vector $\boldsymbol{x}$, it is also satisfied by all vectors $\boldsymbol{x} + t\boldsymbol{a}$ where t is any scalar.

  For this reason, any equation with a vector in the denominator is nonsensical. (Of course, one can divide by the *length* of a vector.)

Triple products: In which way can we combine scalar and/or vector products to multiply three vectors together? Up to changing order of factors, there are four possibilities of combining three vectors with two multiplication signs:

- $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is a scalar
- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is a vector
- $\boldsymbol{a} \times (\boldsymbol{b} \cdot \boldsymbol{c})$ is a nonsense
- $\boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c})$ is a nonsense

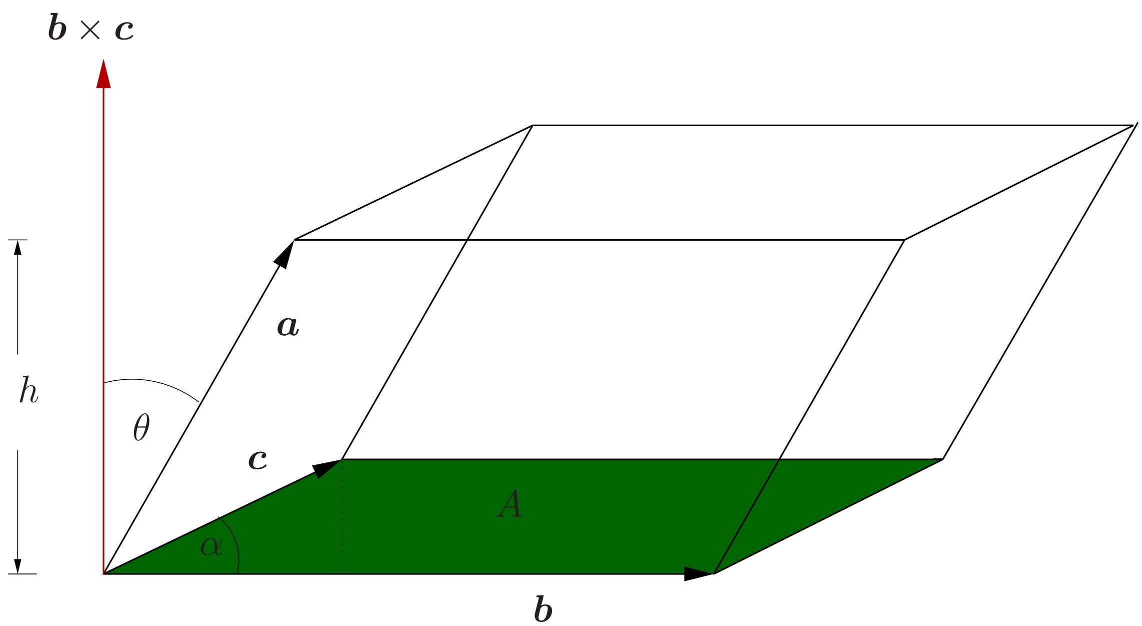
Of course, $\boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c})$ does make sense as a product of the vector $\boldsymbol{a}$ with the scalar $\boldsymbol{b} \cdot \boldsymbol{c}$ (and gives a vector), but we do not write this as either $\boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c})$ or $\boldsymbol{a} \times (\boldsymbol{b} \cdot \boldsymbol{c})$; the sign $\cdot$ is reserved for the scalar product of two vectors and the sign $\times$ is reserved for the vector product of two vectors.

So there are only two meaningful triple products, a scalar-valued one and a vector-valued one.

Discussion of the scalar-valued triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$:

Assume first that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form a right-handed basis.

Consider the parallel-epiped (slant sided box) spanned by these three vectors:



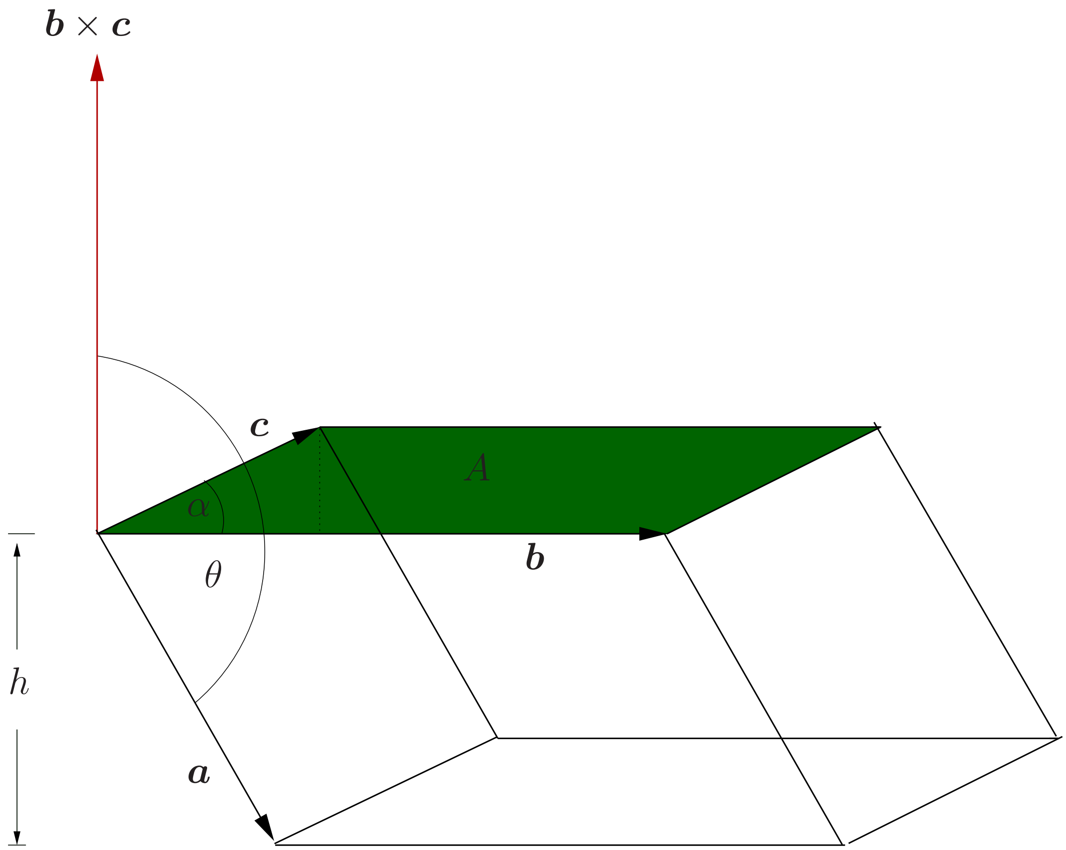
Then

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \;=\; |\boldsymbol{a}|\;\underbrace{|\boldsymbol{b}\times\boldsymbol{c}|}_{|\boldsymbol{b}|\,|\boldsymbol{c}|\sin\alpha}\cos\theta \;=$$

$$\underbrace{|\boldsymbol{b}|\;\;|\boldsymbol{c}|\sin\alpha}_{\text{area}\,A}\,\underbrace{|\boldsymbol{a}|\cos\theta}_{\text{height}\,h} \;=\; \underbrace{V}_{\text{volume of parallel-epiped}}\;,$$

so for a right-handed basis $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ the scalar-valued triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ gives the volume of the parallel-epiped spanned by these three vectors.

If $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is a left-handed basis, the only thing that changes is that now the angle θ is bigger than 90^o, so $\cos\theta$ is negative.



Hence

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \;=\; \underbrace{|\boldsymbol{b}|\;\;|\boldsymbol{c}|\sin\alpha}_{\text{area }A}\,\underbrace{|\boldsymbol{a}|\cos\theta}_{-\text{height }h} \;=\; -\underbrace{V}_{\text{volume of parallel-epiped}}\;.$$

Finally, if $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are linearly dependent the parallel-epiped degenerates into a two-dimensional figure with zero volume, so $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = 0\,$.

In summary

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\;\begin{cases} >0 \;\text{ if } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \text{ is a right-handed basis}\,,\\ <0 \;\text{ if } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \text{ is a left-handed basis}\,,\\ =0 \;\text{ if } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \text{ are linearly dependent}\,.\end{cases}$$

Hence the scalar-valued triple product can be used for checking if three vectors are linearly independent and if they are right-handed or left-handed.

In Cartesian coordinates, the scalar-valued triple product can be calculated as a determinant:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \boldsymbol{a} \cdot \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \;=$$

$$= \; (a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k})\cdot\Big(\boldsymbol{i}\,(b_2c_3-c_2b_3)-\boldsymbol{j}\,(b_1c_3-c_1b_3)+\boldsymbol{k}\,(b_1c_2-c_1b_2)\Big) \;=$$

$$= \; a_1(b_2c_3-c_2b_3)+0+0+0-a_2(b_1c_3-c_1b_3)+0+0+0+a_3(b_1c_2-c_1b_2) \;=$$

$$= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \; .$$

From this result one easily verifies that

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) \;=\; \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \;=$$

$$-\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b}) \;=\; -\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) \;=\; -\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})$$

which is geometrically evident from the interpretation of the scalar-valued triple product as a (signed) volume.

Example: Check with the help of the triple product whether $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$, $\boldsymbol{c} = -\,2\,\boldsymbol{k}$ form a right-handed basis.

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \;=\; \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \;=$$

$$= \det \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} + 0 \;=\; 0-0-(-2-0) \;=\; 2 > 0$$

so $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form, indeed, a right-handed basis.

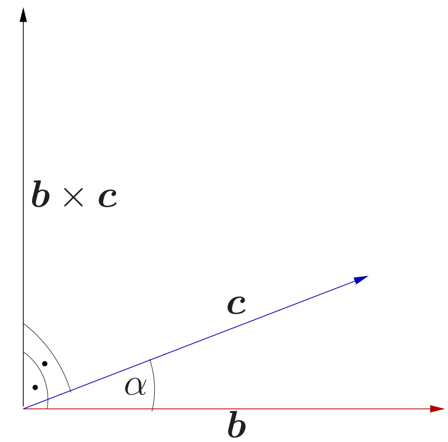
Discussion of the vector-valued triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$:

As $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is orthogonal to $\boldsymbol{b} \times \boldsymbol{c}$, it must lie in the plane spanned by $\boldsymbol{b}$ and $\boldsymbol{c}$, i.e.

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \;=\; u\,\boldsymbol{b} + v\,\boldsymbol{c}$$

with some scalars u and v.

How depend u and v on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$?



Claim: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})$ (“bac-cab rule”)

Proof:

Left-hand side:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \boldsymbol{a} \times \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \;=$$

$$(a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k})\times\Big(\boldsymbol{i}\,(b_2c_3-c_2b_3)-\boldsymbol{j}\,(b_1c_3-c_1b_3)+\boldsymbol{k}\,(b_1c_2-c_1b_2)\Big) \;=$$

$$= \; -a_1\boldsymbol{k}\,(b_1c_3 - c_1b_3) - a_1\boldsymbol{j}\,(b_1c_2 - c_1b_2) - a_2\boldsymbol{k}\,(b_2c_3 - c_2b_3)$$

$$+\, a_2\boldsymbol{i}\,(b_1c_2 - c_1b_2) + a_3\boldsymbol{j}\,(b_2c_3 - c_2b_3) + a_3\boldsymbol{i}\,(b_1c_3 - c_1b_3) \;=$$

$$= \; \boldsymbol{i}(a_2b_1c_2 - a_2c_1b_2 + a_3b_1c_3 - a_3c_1b_3)$$

$$+\,\boldsymbol{j}(-a_1b_1c_2 + a_1c_1b_2 + a_3b_2c_3 - a_3c_2b_3)$$

$$+\,\boldsymbol{k}(-a_1b_1c_3 + a_1c_1b_3 - a_2b_2c_3 + a_2c_2b_3)$$

Right-hand side

$$\boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) \;=$$

$$= \; (b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k})(a_1c_1+a_2c_2+a_3c_3)-(c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k})(a_1b_1+a_2b_2+a_3b_3) \;=$$

$$= \; \boldsymbol{i}(\cancel{b_1a_1c_1} + b_1a_2c_2 + b_1a_3c_3 - \cancel{c_1a_1b_1} - c_1a_2b_2 - c_1a_3b_3)$$

$$+\boldsymbol{j}(b_2a_1c_1 + \cancel{b_2a_2c_2} + b_2a_3c_3 - c_2a_1b_1 - \cancel{c_2a_2b_2} - c_2a_3b_3)$$

$$+\boldsymbol{k}(b_3a_1c_1 + b_3a_2c_2 + \cancel{b_3a_3c_3} - c_3a_1b_1 - c_3a_2b_2 - \cancel{c_3a_3b_3})$$

So the left-hand side and the right-hand side are equal.

When applying the bac-cab rule, the brackets must be around the second and third factor. If they are around the first and second factor, one gets something else:

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} \;=\; -(\boldsymbol{b}\times\boldsymbol{a})\times\boldsymbol{c} \;=$$

$$= \underbrace{\boldsymbol{c}}_{\boldsymbol{a}}\times(\underbrace{\boldsymbol{b}}_{\boldsymbol{b}}\times\underbrace{\boldsymbol{a}}_{\boldsymbol{c}}) \;=\; \underbrace{\boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{a})}_{\boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c})} - \underbrace{\boldsymbol{a}\,(\boldsymbol{c}\cdot\boldsymbol{b})}_{\boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})}$$

So $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ is a linear combination of $\boldsymbol{b}$ and $\boldsymbol{a}$, whereas $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is a linear combination of $\boldsymbol{b}$ and $\boldsymbol{c}$.

As $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}\neq\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$, the expression $\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c}$ is nonsensical.

Example:

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\boldsymbol{a}}\times\Bigg(\underbrace{\begin{pmatrix}2\\2\\2\end{pmatrix}}_{\boldsymbol{b}}\times\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\boldsymbol{c}}\Bigg) = \underbrace{\begin{pmatrix}2\\2\\2\end{pmatrix}}_{\boldsymbol{b}}\Bigg(\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\boldsymbol{a}}\cdot\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\boldsymbol{c}}\Bigg) - \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\boldsymbol{c}}\Bigg(\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\boldsymbol{a}}\cdot\underbrace{\begin{pmatrix}2\\2\\2\end{pmatrix}}_{\boldsymbol{b}}\Bigg) =$$

$$= \begin{pmatrix}2\\2\\2\end{pmatrix}(0+0+0) - \begin{pmatrix}0\\0\\1\end{pmatrix}(2+0+0) = \begin{pmatrix}0\\0\\-2\end{pmatrix}.$$

Two important inequalities:

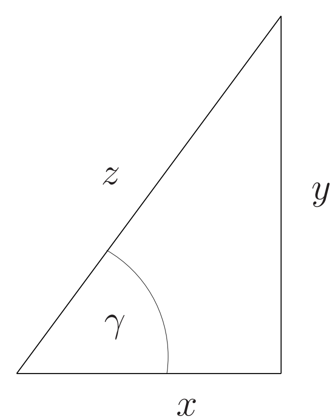
1) Schwarz inequality:

$(\boldsymbol{a}\cdot\boldsymbol{b})^2 \;\leq\; |\boldsymbol{a}|^2|\boldsymbol{b}|^2$

Proof:

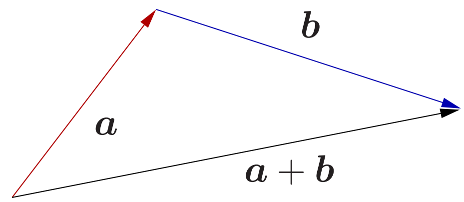
$(\boldsymbol{a}\cdot\boldsymbol{b})^2 \;=\; (|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\gamma)^2 \;=\; |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\gamma\;.$

As $\cos^2\gamma \;=\; \dfrac{x^2}{z^2} \;\leq\; 1$, the Schwarz inequality follows.



2) Triangle inequality:

$$|\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}|+|\boldsymbol{b}|$$



Proof: Although the triangle inequality is geometrically evident, we give a formal proof. As $|\boldsymbol{a}||\boldsymbol{b}|$ is non-negative, the Schwarz inequality implies

$$\begin{aligned} 2\boldsymbol{a}\cdot\boldsymbol{b} &\leq 2|\boldsymbol{a}|\,|\boldsymbol{b}| \\ \Longrightarrow \quad |\boldsymbol{a}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{b}|^2 &\leq |\boldsymbol{a}|^2+2|\boldsymbol{a}|\,|\boldsymbol{b}|+|\boldsymbol{b}|^2 \\ \Longrightarrow \quad \big(\boldsymbol{a}+\boldsymbol{b}\big)\cdot\big(\boldsymbol{a}+\boldsymbol{b}\big) &\leq \big(|\boldsymbol{a}|+|\boldsymbol{b}|\big)^2 \\ \Longrightarrow \quad |\boldsymbol{a}+\boldsymbol{b}|^2 &\leq \big(|\boldsymbol{a}|+|\boldsymbol{b}|\big)^2 \end{aligned}$$

As both $|\boldsymbol{a}+\boldsymbol{b}|$ and $|\boldsymbol{a}|+|\boldsymbol{b}|$ are non-negative numbers, we can take the square-root on both sides and we get the triangle inequality.

Collection of important formulas:

Definition of scalar product:

$$\boxed{\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\gamma}$$

Norm of vector product:

$$\boxed{|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\gamma}$$

Antisymmetry property of vector product:

$$\boxed{\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}}$$

Properties of Cartesian basis vectors:

$$\boldsymbol{i}\cdot\boldsymbol{i} \,=\, \boldsymbol{j}\cdot\boldsymbol{j} \,=\, \boldsymbol{k}\cdot\boldsymbol{k} \,=\, 1\,, \qquad \boldsymbol{i}\cdot\boldsymbol{j} \,=\, \boldsymbol{j}\cdot\boldsymbol{k} \,=\, \boldsymbol{k}\cdot\boldsymbol{i} \,=\, 0$$

$$\boldsymbol{i}\times\boldsymbol{j} \,=\, \boldsymbol{k}\,, \quad \boldsymbol{j}\times\boldsymbol{k} \,=\, \boldsymbol{i}\,, \quad \boldsymbol{k}\times\boldsymbol{i} \,=\, \boldsymbol{j}\,.$$

Scalar product in Cartesian coordinates:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \,=\, a_1 b_1 + a_2 b_2 + a_3 b_3$$

Vector product in Cartesian coordinates:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \,=\, \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\boldsymbol{a}\times\boldsymbol{b} \,=\, \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Cyclic property of scalar-valued triple product:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \,=\, \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$$

Scalar-valued triple product in Cartesian coordinates:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \,=\, \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

“bac-cab rule”

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \;=\; \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})$$

Schwarz inequality

$$(\boldsymbol{a} \cdot \boldsymbol{b})^2 \;\leq\; |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$$

Triangle inequality

$$|\boldsymbol{a} + \boldsymbol{b}| \;\leq\; |\boldsymbol{a}| + |\boldsymbol{b}|$$

II. Applications of Vector Algebra to Physics

II.1 Kinematics of a point particle

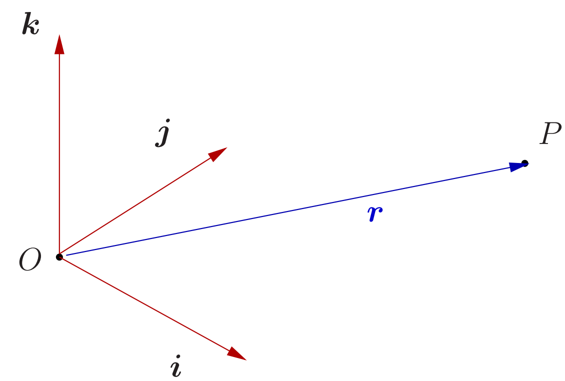
Kinematics: Description of motion without asking for the forces that cause the motion

To assign a *position vector* to any point in three-dimensional space, one chooses an origin O and a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

The position vector $\boldsymbol{r}$ of a point P is the vector represented by the arrow from O to P. Its Cartesian coordinates x, y, z characterize the point P unambiguously,

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,.$$

Of course, it depends on the choice of O and of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ which coordinates are assigned to which point.



Example: Consider a cube whose edges have length a. Choose O as one of the corners of the cube and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along the adjacent axes. What are the Cartesian coordinates of the point P at the corner opposite to O?

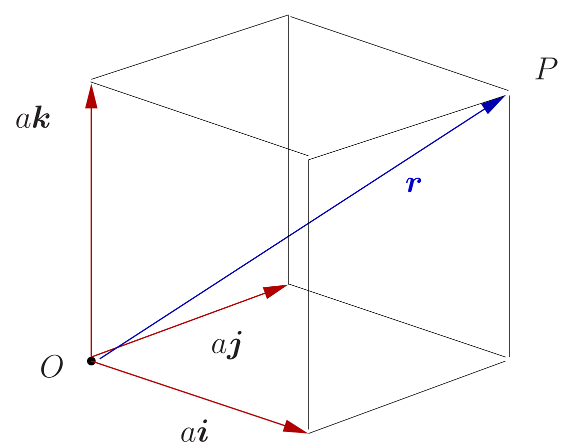
From the picture we read

$$\boldsymbol{r} = a\,\boldsymbol{i} + a\,\boldsymbol{j} + a\,\boldsymbol{k}\,,$$

or, in column notation,

$$\boldsymbol{r} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}\,.$$

So $x = y = z = a$.



For a moving point particle, the position vector is a function of time,

$$\boldsymbol{r}(t) \;=\; x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} + z(t)\,\boldsymbol{k}\,.$$

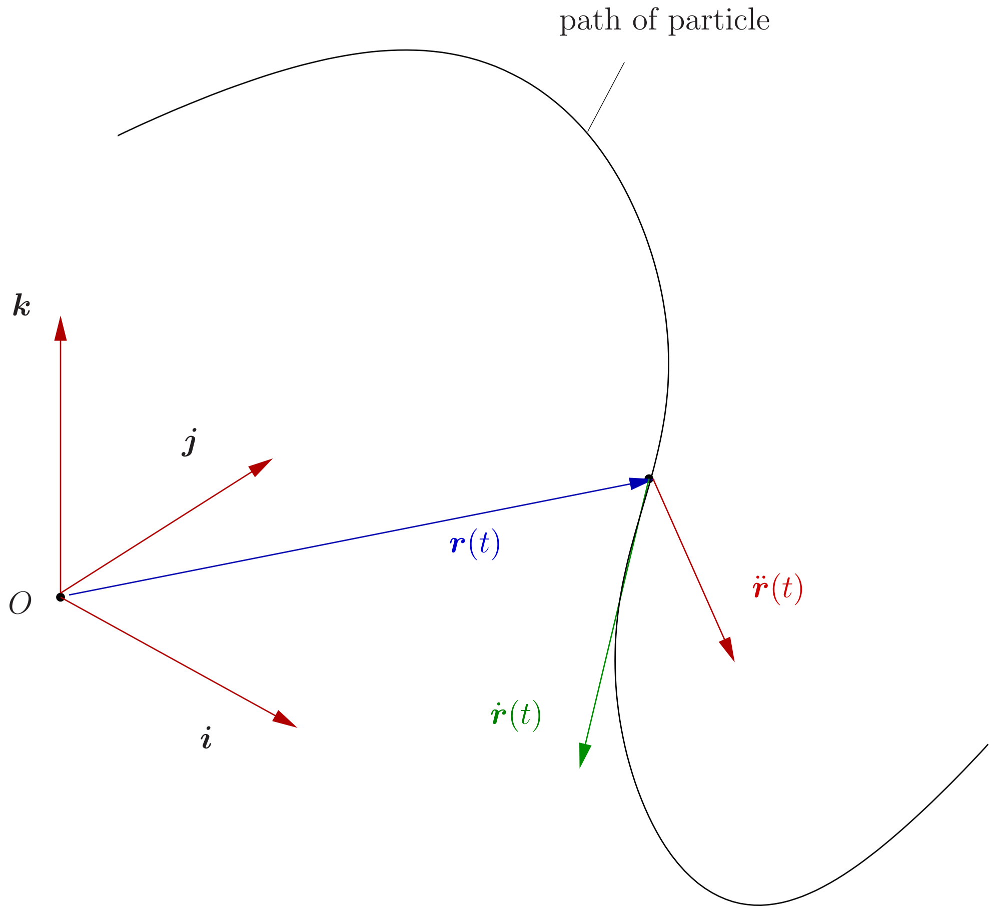
For the time being, we assume that the origin O and the basis vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ do *not* depend on time. Then we define the

$$\text{velocity vector} \quad \dot{\boldsymbol{r}}(t) \;=\; \dot{x}(t)\,\boldsymbol{i} + \dot{y}(t)\,\boldsymbol{j} + \dot{z}(t)\,\boldsymbol{k}$$

and the

$$\text{acceleration vector} \quad \ddot{\boldsymbol{r}}(t) \;=\; \ddot{x}(t)\,\boldsymbol{i} + \ddot{y}(t)\,\boldsymbol{j} + \ddot{z}(t)\,\boldsymbol{k}\,.$$

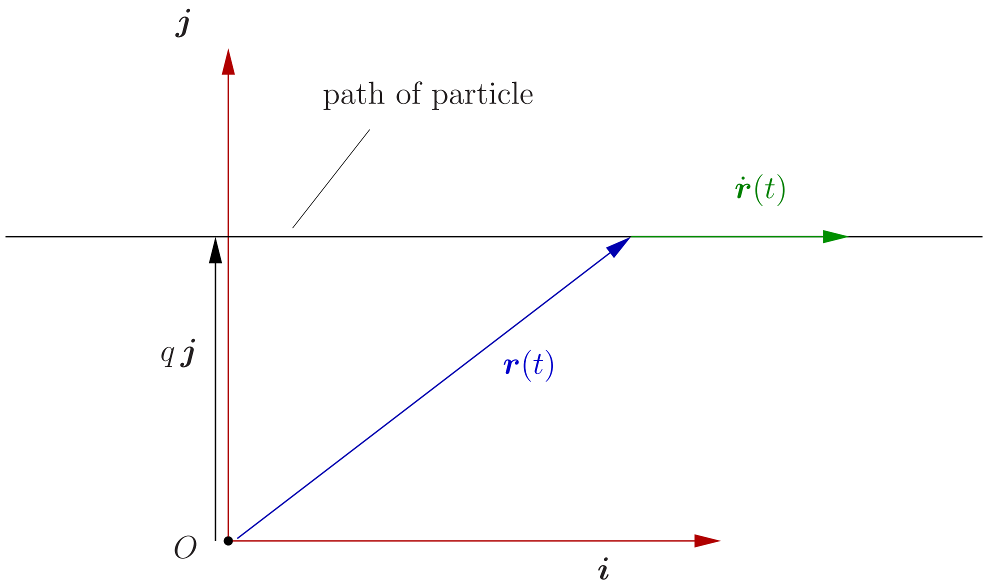
Here and in the following, the overdot means differentiation with respect to time, $\dot{x}(t) = \dfrac{dx(t)}{dt}$ etc.



Examples:

1) $\boldsymbol{r}(t) = v\,t\,\boldsymbol{i} + q\,\boldsymbol{j}$ where v and q are positive constants.

This is a straight line:



In column notation:

$$\boldsymbol{r}(t) = \begin{pmatrix} v\,t \\ q \\ 0 \end{pmatrix} , \quad \dot{\boldsymbol{r}}(t) = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} , \quad \ddot{\boldsymbol{r}}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

2) $\boldsymbol{r}(t) = R\cos(\omega t)\,\boldsymbol{i} + R\sin(\omega t)\,\boldsymbol{j}$ where R and ω are positive constants.

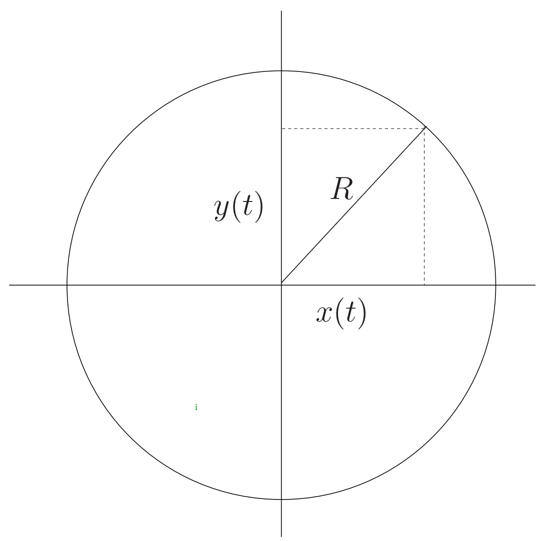
Particle moves on a circle of radius R:

$x(t) = R\cos(\omega t)$, $\qquad y(t) = R\sin(\omega t)$

$x(t)^2 + y(t)^2 = R^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right) = R^2$

where we used the identity

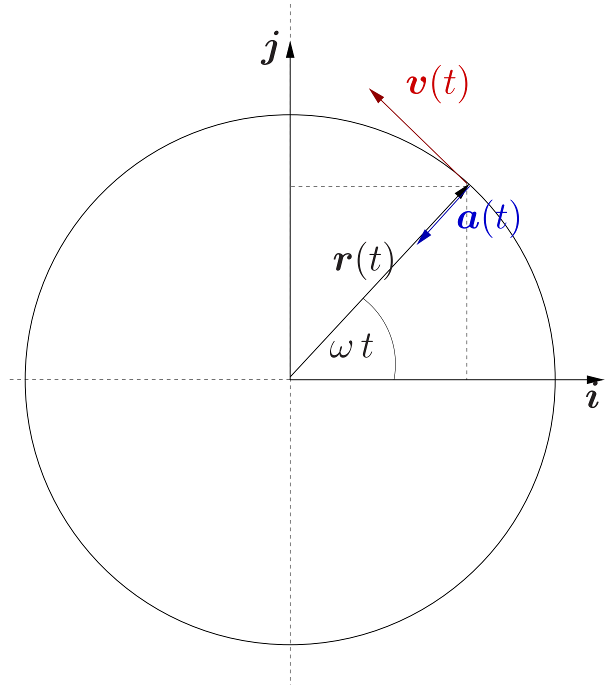
$\cos^2\varphi + \sin^2\varphi = 1$.



Uniform circular motion:

$R =$ radius of circle [unit=m]

$\omega =$ magnitude of angular velocity [unit $= \frac{1}{s}$]



$$\boldsymbol{r}(t) \;=\; \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ 0 \end{pmatrix}$$

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) \;=\; \begin{pmatrix} -R\,\omega\sin(\omega t) \\ R\,\omega\cos(\omega t) \\ 0 \end{pmatrix}$$

$$\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) \;=\; \begin{pmatrix} -R\,\omega^2\cos(\omega t) \\ -R\,\omega^2\sin(\omega t) \\ 0 \end{pmatrix}$$

Note that

$$\boldsymbol{a} = -\omega^2 \boldsymbol{r}\,;$$

$$\boldsymbol{v} \cdot \boldsymbol{r} = 0\,;$$

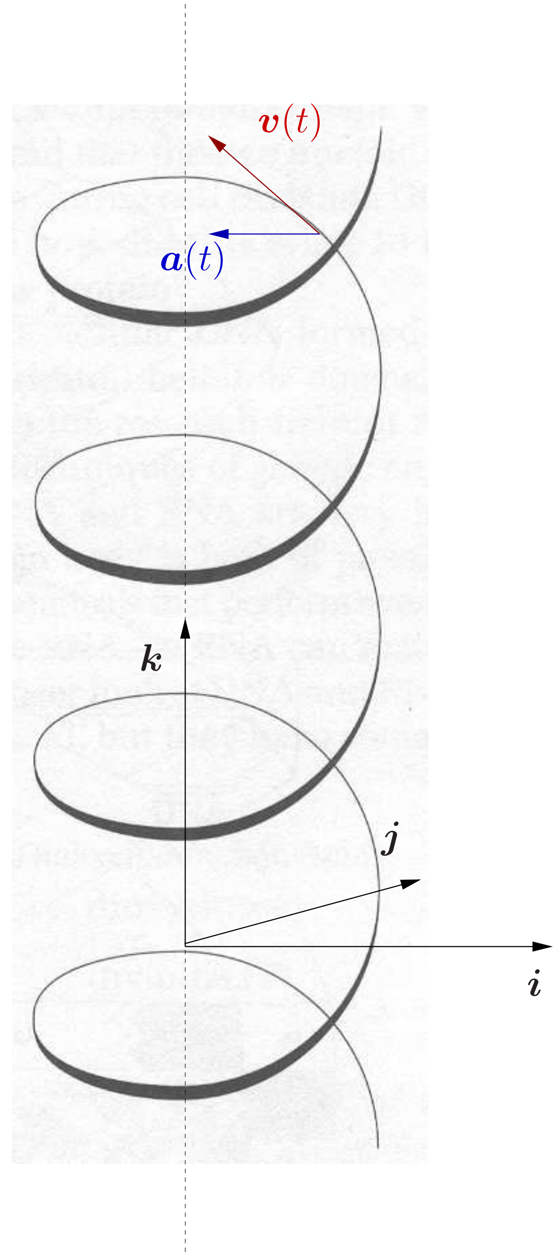
$$|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = R^2\omega^2,$$

so $|\boldsymbol{v}| = R\omega$ is constant whereas $\boldsymbol{v}$ is not.

3) $\boldsymbol{r}(t) \;=\; R\cos(\omega t)\,\boldsymbol{i} \,+\, R\sin(\omega t)\,\boldsymbol{j} \,+\, b\,t\,\boldsymbol{k}$

where R, ω and b are positive constants.

This is a helix:



$$\boldsymbol{r}(t) \;=\; \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ b\,t \end{pmatrix} \;,$$

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) \;=\; \begin{pmatrix} -R\,\omega\sin(\omega t) \\ R\,\omega\cos(\omega t) \\ b \end{pmatrix} \;,$$

$$\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) \;=\; \begin{pmatrix} -R\,\omega^2\cos(\omega t) \\ -R\,\omega^2\sin(\omega t) \\ 0 \end{pmatrix} \;,$$

so the acceleration is horizontal and points towards the axis

II.2 Dynamics of a point particle

Dynamics: How to calculate motion as a result of forces

In the following we consider a point particle with constant mass m.

(Particles with time-dependent mass, e.g. rockets, will be discussed later).

Newton's second law: $\boldsymbol{F}(t) = m\,\boldsymbol{a}(t)\,.$

$m =$ mass of the particle [scalar, unit = kg]

$\boldsymbol{a} =$ acceleration of the particle [vector, unit $= \frac{m}{s^2}$]

$\boldsymbol{F} =$ total force acting on the particle [vector, unit $= \frac{kg\,m}{s^2} = N$]

The total force is the vectorial sum of all forces acting on the particle,

$$\boldsymbol{F} = \boldsymbol{F}_1 + \ldots + \boldsymbol{F}_N\,.$$

Examples:

1) $\boldsymbol{F} = \boldsymbol{0}$

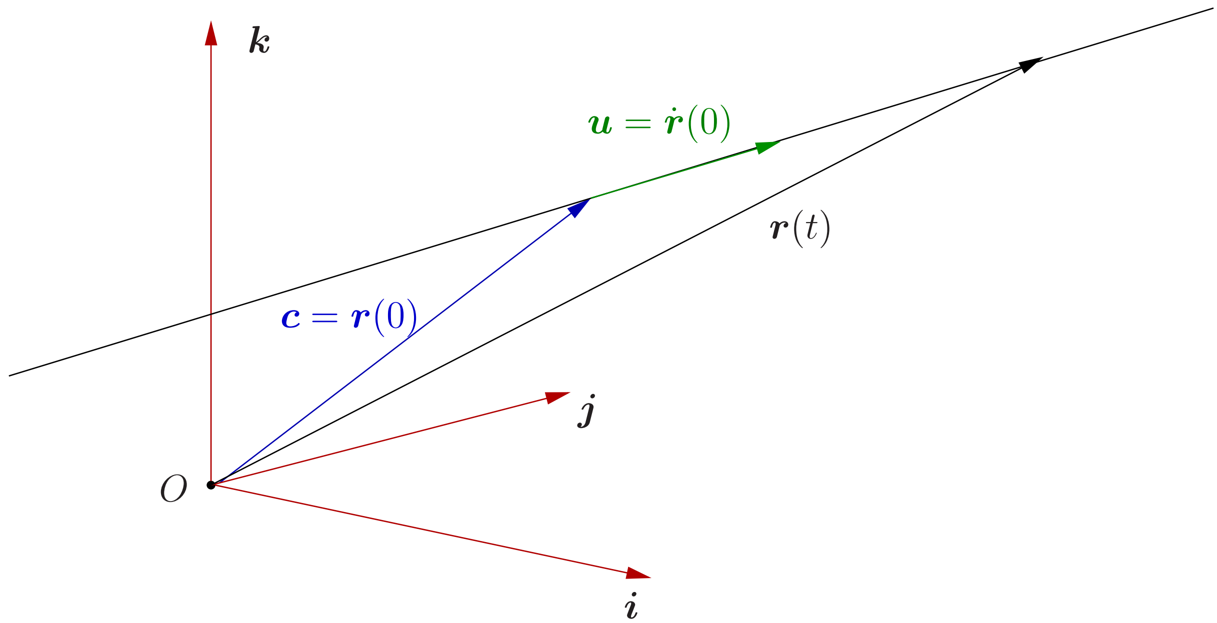
As the mass is non-zero, Newton's second law gives

$$\ddot{\boldsymbol{r}}(t) = \boldsymbol{0} \quad \Longrightarrow \quad \dot{\boldsymbol{r}}(t) = \boldsymbol{u} \quad \Longrightarrow \quad \boldsymbol{r}(t) = \boldsymbol{u}\,t + \boldsymbol{c}$$

with constant vectors $\boldsymbol{u}$ and $\boldsymbol{c}$. The same calculation reads in column notation:

$$\begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} u_1\,t + c_1 \\ u_2\,t + c_2 \\ u_3\,t + c_3 \end{pmatrix}$$

This is a uniform linear motion. The vectors $\boldsymbol{c}$ and $\boldsymbol{u}$ can be chosen arbitrarily and reflect the freedom of choosing arbitrary initial conditions for the position and the velocity, $\boldsymbol{c} = \boldsymbol{r}(0)$ and $\boldsymbol{u} = \dot{\boldsymbol{r}}(0)\,.$



So we have found that, if the total force on a particle is zero, the particle moves uniformly along a straight line. This is Newton's first law. It is, formally, a consequence of Newton's second law.

2) Homogeneous gravitational field

$$\boldsymbol{F} = -m\,g\,\boldsymbol{k}\,, \quad g \approx 9.81\,\frac{m}{s^2}\,.$$

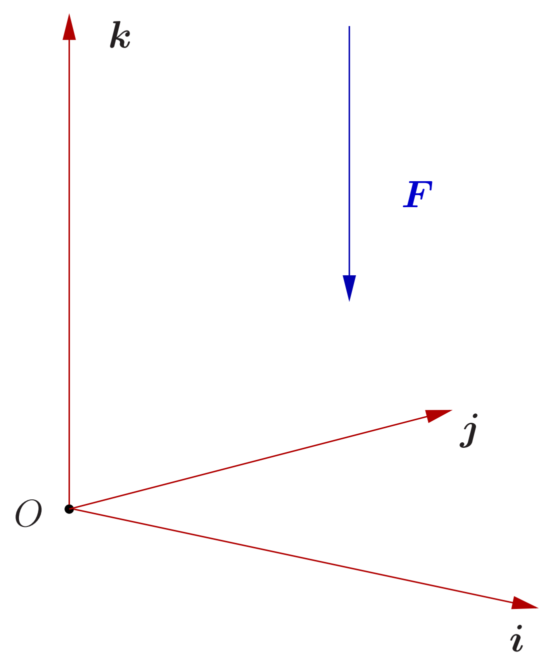
For this case Newton's second law gives

$$-\underbrace{m}_{\substack{\text{gravitational}\\ \text{mass}}}\,g\,\boldsymbol{k} = \underbrace{m}_{\substack{\text{inertial}\\ \text{mass}}}\boldsymbol{a}\,.$$

As the gravitational mass is equal to the inertial mass:

$$-g\,\boldsymbol{k} = \boldsymbol{a}\,,$$

so m has dropped out.



$$\begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ -gt + u_3 \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} u_1 t + c_1 \\ u_2 t + c_2 \\ -\frac{1}{2}gt^2 + u_3 t + c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}g \end{pmatrix} t^2 + \boldsymbol{u}\, t + \boldsymbol{c}\, .$$

Again, $\boldsymbol{u}$ and $\boldsymbol{c}$ are arbitrary vectors and reflect the freedom of choosing arbitrary initial conditions, $\boldsymbol{r}(0) = \boldsymbol{c}$ and $\dot{\boldsymbol{r}}(0) = \boldsymbol{u}\, .$

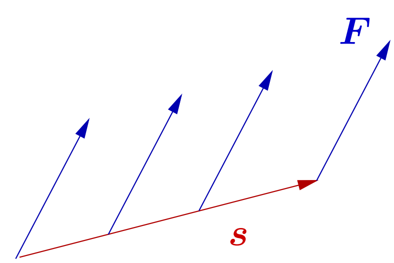
We can choose the initial conditions such that $u_2 = 0$ and $c_2 = 0$, then the motion is in the $\boldsymbol{i} - \boldsymbol{k}-$plane,

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} u_1 t + c_1 \\ 0 \\ -\frac{1}{2}gt^2 + u_3 t + c_3\ . \end{pmatrix}$$

In the following we need some more vocabulary:

"work", "power", "(linear) momentum" and "angular momentum".

Introduce work and power:



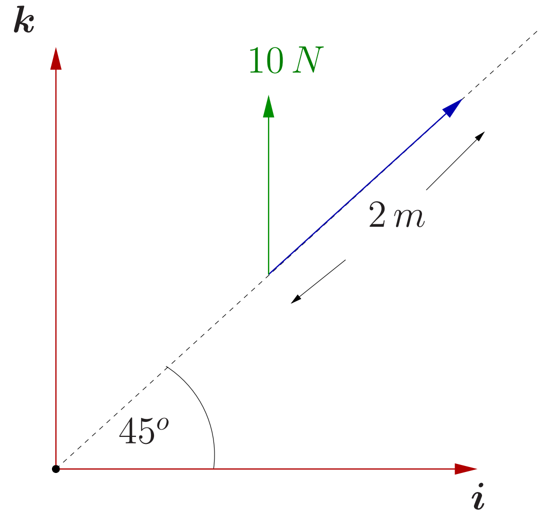
Assume that we carry a particle in a straight line along the vector $\boldsymbol{s}$ under the influence of a *constant* force $\boldsymbol{F}$. Then we define the work

$$W = \boldsymbol{F} \cdot \boldsymbol{s} \ \ [\text{scalar, unit} = N\, m \ = \frac{kg\, m^2}{s^2}]$$

If the process takes the time t, we define the power

$$P = \frac{\boldsymbol{F} \cdot \boldsymbol{s}}{t} \ \ [\text{scalar, unit} = \frac{N\, m}{s} \ = \frac{kg\, m^2}{s^3}\ ]$$

Example:



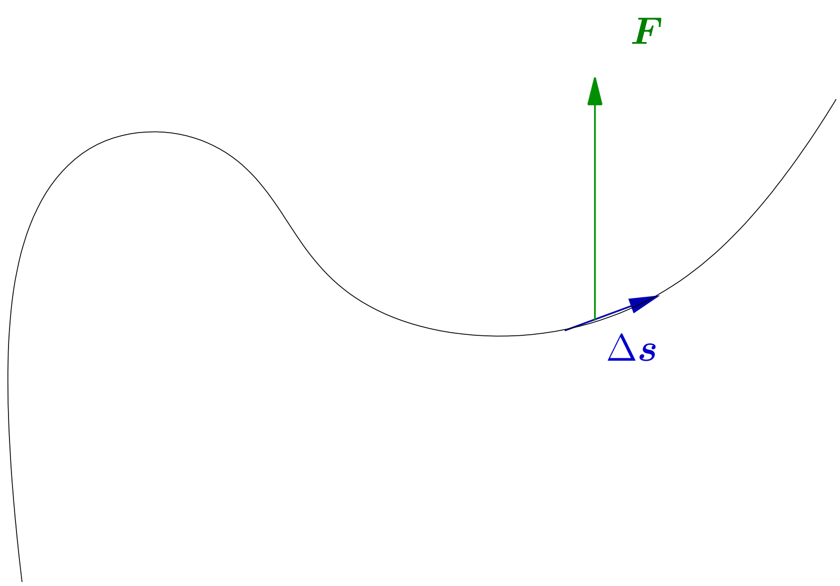
For carrying a particle along the blue arrow:

$$W \;=\; \boldsymbol{F}\cdot\boldsymbol{s} \;=$$

$$=\; 10\,N\,\boldsymbol{k}\cdot 2\,m\,\underbrace{\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{k})}_{\text{unit vector}} \;=$$

$$=\; \sqrt{2}\,10\,N\,m \;\approx\; 14\,N\,m\,.$$

If the force is not constant, or if the path along which the particle is transported is not a straight line, the work can be defined by the above method for sufficiently short sections of the path, $\Delta W = \boldsymbol{F}\cdot\Delta\boldsymbol{s}$



By making the sections shorter and shorter, and adding up the contributions from all sections, the total work results as a *curve integral*, $W \;=\; \int \boldsymbol{F}\cdot d\boldsymbol{s}$. You will learn about this type of integral in PHYS115. We will not deal with them here.

Introduce (linear) momentum and angular momentum:

For a particle of mass m, one defines the (linear) momentum by

$\boldsymbol{p}(t) \;=\; m\,\boldsymbol{v}(t)$ [vector, unit$=\dfrac{kg\,m}{s}$]

and the angular momentum by

$\boldsymbol{L}(t) \;=\; \boldsymbol{r}(t)\times\boldsymbol{p}(t)$ [vector, unit$=\dfrac{kg\,m^2}{s}$]

where $\boldsymbol{r}$ is the position vector and $\boldsymbol{v}=\dot{\boldsymbol{r}}$ is the velocity vector.

Example:

Calculate the angular momentum $\boldsymbol{L}$ for a particle in uniform circular motion.

Recall (p.22) that for uniform circular motion

$\boldsymbol{r}(t) \;=\; R\cos(\omega t)\,\boldsymbol{i}+R\sin(\omega t)\,\boldsymbol{j}$ and $\boldsymbol{v}(t) \;=\; -\,\omega\,R\sin(\omega t)\,\boldsymbol{i}+\omega\,R\cos(\omega t)\,\boldsymbol{j}$

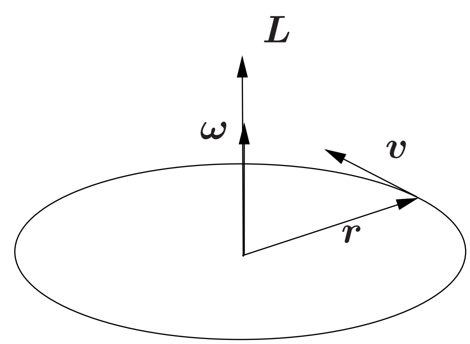
If we introduce the angular velocity vector $\boldsymbol{\omega}=\omega\,\boldsymbol{k}$, we find

$$\boldsymbol{\omega}\times\boldsymbol{r} \;=\; \det\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ R\cos(\omega t) & R\sin(\omega t) & 0 \end{pmatrix} \;=\; \boldsymbol{v}$$

Hence $\boldsymbol{L} \;=\; \boldsymbol{r}\times\boldsymbol{p} \;=\; \boldsymbol{r}\times m\,\boldsymbol{v} \;=\; m\,\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})$.

With the bac-cab rule: $\boldsymbol{L} \;=\; m\big(\boldsymbol{\omega}(\boldsymbol{r}\cdot\boldsymbol{r})-\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{\omega})\big)$.

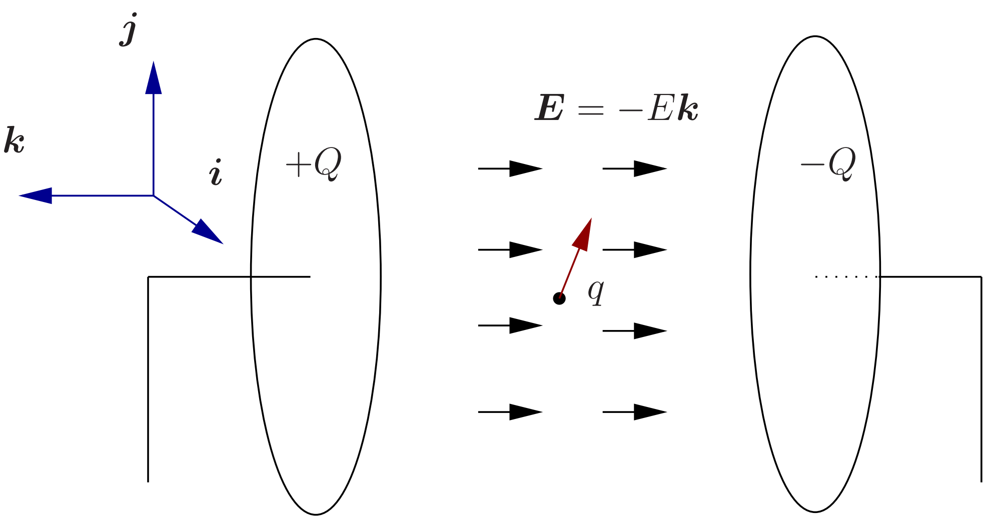
As $\boldsymbol{r}\cdot\boldsymbol{r} \;=\; R^2$ and $\boldsymbol{r}\cdot\boldsymbol{\omega} \;=\; 0$, this results in $\boldsymbol{L} \;=\; m\,R^2\,\boldsymbol{\omega}$.



Two more examples of how to apply Newton's second law:

1) Charged particle in a homogeneous electric field

   A homogeneous electric field can be produced with a capacitor (=condenser):



   Force on particle (found by experiment): $\boldsymbol{F} = q\,\boldsymbol{E}$

   q = charge of particle [scalar, unit = C =Coulomb]

   $\boldsymbol{E}$ = electric field vector [vector, unit = $\dfrac{N}{C} = \dfrac{kg\,m}{s^2\,C}$]

   Newton's second law:

$$m\begin{pmatrix}\ddot{x}(t)\\ \ddot{y}(t)\\ \ddot{z}(t)\end{pmatrix} = q\begin{pmatrix}0\\ 0\\ -E\end{pmatrix}$$

   Compare with particle in homogeneous gravitational field (p.26):

$$\begin{pmatrix}\ddot{x}(t)\\ \ddot{y}(t)\\ \ddot{z}(t)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -g\end{pmatrix}$$

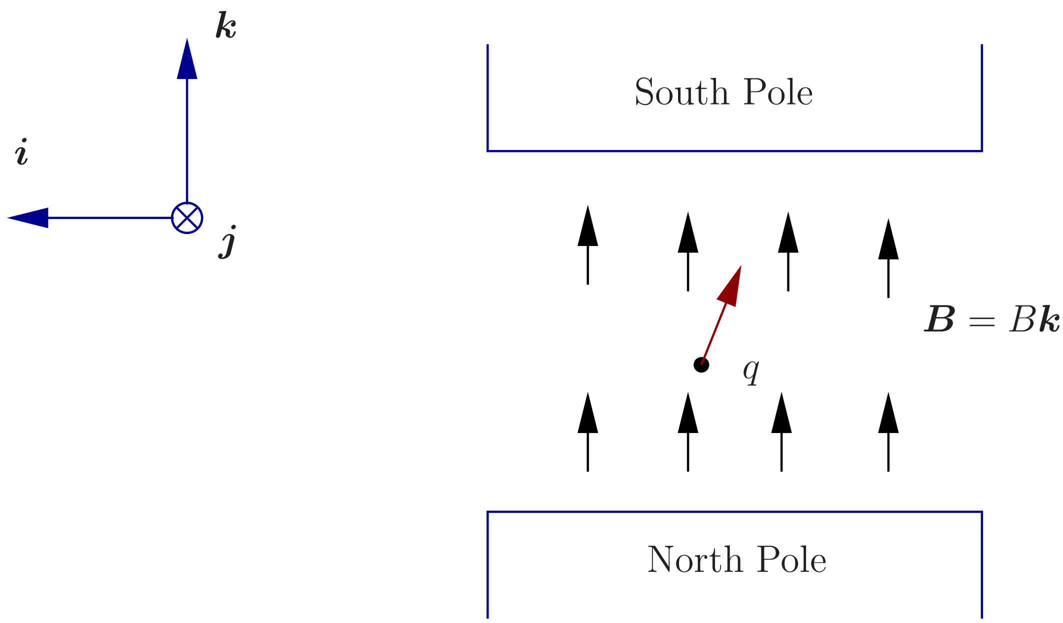
   Both equations have the same form, with g corresponding to $q\,E/m$. Therefore we get the motion of a charged particle in an electric field by taking the motion of a particle in a homogeneous gravitational field and replacing g with $q\,E/m$:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{q\,E}{2\,m} \end{pmatrix} t^2 + \boldsymbol{u}\,t + \boldsymbol{c}\,.$$

This gives a parabolic motion, quite analogous to the trajectory of a particle in a homogeneous gravitational field. Note, however, that for a particle in an electric field the acceleration is not universal; it depends on q/m. So if E is known, measuring the trajectory allows to determine q/m (but not q and m individually).

2) Charged particle in a homogeneous magnetic field

A homogeneous magnetic field can be produced with two magnets:



Force on particle (found by experiment): $\boldsymbol{F}(t) = q\,\boldsymbol{v}(t) \times \boldsymbol{B}$

q = charge of particle

$\boldsymbol{v}$ = velocity of particle

$\boldsymbol{B}$ = magnetic field vector

[vector, unit $= \dfrac{N\,s}{C\,m} = \dfrac{kg\,\cancel{m}\,\cancel{s}}{s^{\cancel{2}}\,C\,\cancel{m}} = T =$ Tesla]

Newton's second law:

$$m \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} = q \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = q \begin{pmatrix} \dot{y}(t)\, B \\ -\dot{x}(t)\, B \\ 0 \end{pmatrix}$$

This vector equation gives three scalar equations:

$$\ddot{x}(t) = \frac{q\,B}{m}\,\dot{y}(t)\,, \quad \ddot{y}(t) = -\frac{q\,B}{m}\,\dot{x}(t)\,, \quad \ddot{z}(t) = 0$$

The last equation has the solution $z(t) = ut + c$. The constants u and c are determined by initial conditions, $z(0) = c$ and $\dot{z}(0) = u$.

The first two equations are more difficult to evaluate because they are *coupled* equations for $x(t)$ and $y(t)$.

Consider first the special case that $z(0) = c = 0$ and $\dot{z}(0) = u = 0$. Then $z(t) = 0$, i.e., the motion is in the $\boldsymbol{i} - \boldsymbol{j}$−plane.

Two observations:

(i) The acceleration is perpendicular to the velocity:

$$\begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ 0 \end{pmatrix} = \ddot{x}(t)\,\dot{x}(t) + \ddot{y}(t)\,\dot{y}(t) = \frac{qB}{m}\,\dot{y}(t)\,\dot{x}(t) - \frac{qB}{m}\,\dot{x}(t)\,\dot{y}(t) = 0$$

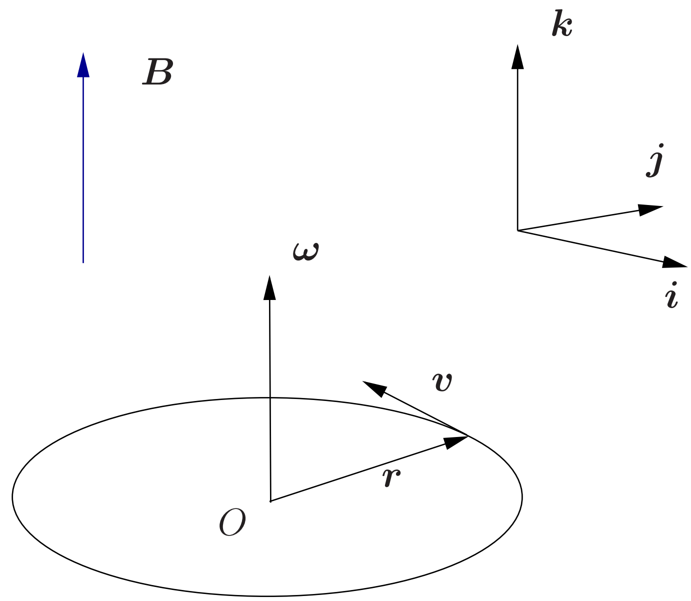
so the speed is constant,

$$\frac{d}{dt}\big(\dot{x}(t)^2 + \dot{y}(t)^2\big) = 2\,\ddot{x}(t)\,\dot{x}(t) + 2\,\ddot{y}(t)\,\dot{y}(t) = 0\,.$$

(ii) The magnitude of the acceleration is constant:

$$\ddot{x}(t)^2 + \ddot{y}(t)^2 = \frac{q^2\,B^2}{m^2}\,\underbrace{\big(\dot{y}(t)^2 + \dot{x}(t)^2\big)}_{=\text{constant}} = \text{constant}$$

The first observation means that the magnetic field changes only the direction but not the magnitude of the velocity; it curves the path of the particle, where the amount of curvature is given by the magnitude of the acceleration. By the second observation, this magnitude is constant, so the curvature must be constant. A curve in a plane with constant curvature is a circle. As the speed is constant, the particle runs through a circle with constant angular velocity $\boldsymbol{\omega}$.



What remains to be done is determining the angular frequency. We can choose the initial conditions for $x(0), y(0), \dot{x}(0)$ and $\dot{y}(0)$ such that the centre of the circle is at the origin O, see picture. Then we have $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ (recall p.29) and, by differentiating,

$$\Rightarrow \quad \dot{\boldsymbol{v}} = \boldsymbol{\omega} \times \dot{\boldsymbol{r}} \quad \Rightarrow \quad \frac{1}{m} \boldsymbol{F} = \boldsymbol{\omega} \times \boldsymbol{v}\,.$$

As $\boldsymbol{\omega}$ is perpendicular to $\boldsymbol{v}$, this implies that

$$F = m\,\omega\,v\,. \qquad (*)$$

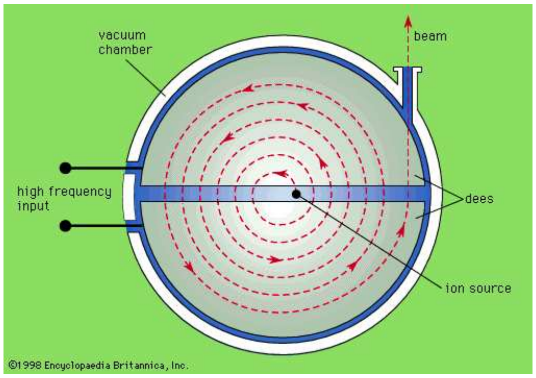
On the other hand $\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$ and, as $\boldsymbol{v}$ and $\boldsymbol{B}$ are perpendicular,

$$F = q\,v\,B\,. \qquad (**)$$

Comparison of $(*)$ and $(**)$ gives

$$\omega = \frac{q\cancel{v}\,B}{m\cancel{v}} = \frac{q\,B}{m}$$

which is known as the "cyclotron frequency". The fact that v drops out is crucial for the cyclotron, invented by E. O. Lawrence (Nobel prize 1939).

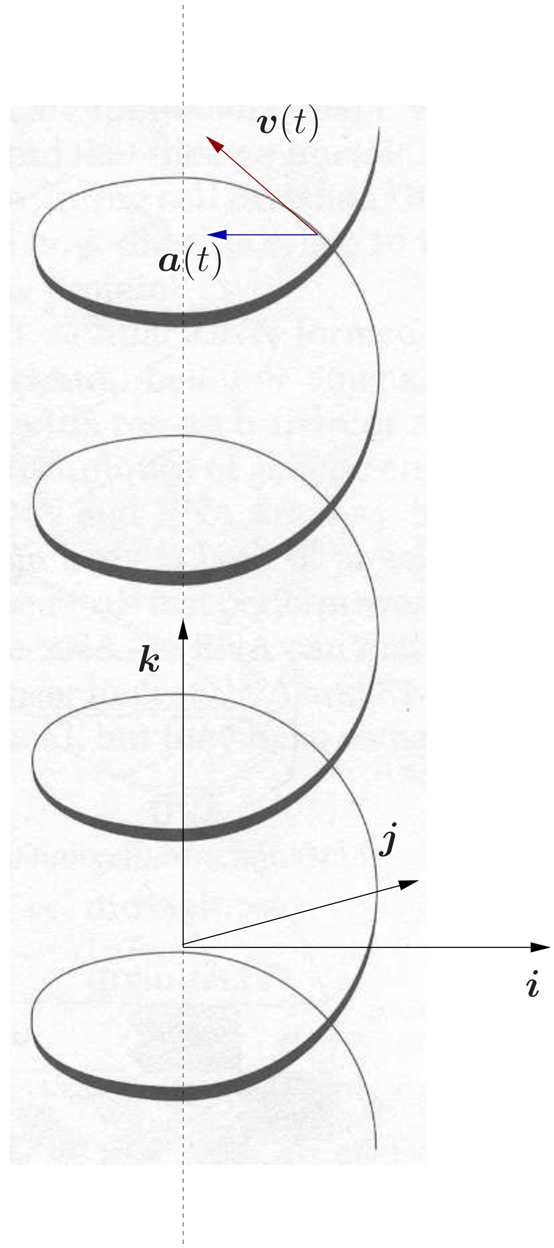


In a cyclotron, an oscillating electric field accelerates charged particles (ions) when they are in the gap between the dees. A magnetic field perpendicular to the sheet forces the particles onto circular paths inside the dees.

We now turn to the general case, $z(0) = c \neq 0$ and $\dot{z}(0) = u \neq 0$. Then we have to superpose linear motion in the $z-$direction, $z(t) = ut + c$, to the circular motion in the plane perpendicular to the magnetic field. The resulting path of the particle is a helix.

The projection into the $\boldsymbol{i} - \boldsymbol{j}-$plane gives a circular motion with the cyclotron frequency $\omega = qB/m$.

The pitch of the helix is determined by $\dot{z}(0) = u$.



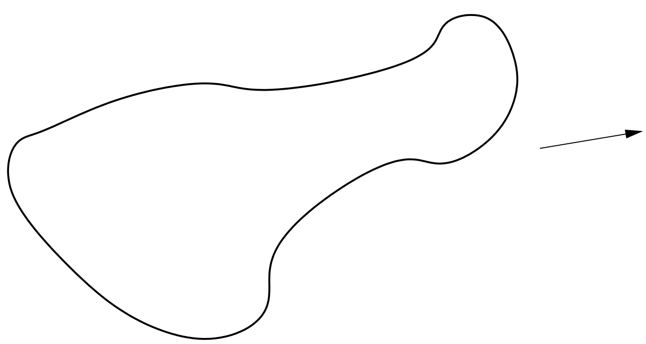
Note: The combined force onto a charged particle produced by an electric and a magnetic field, $\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$, is called the *Lorentz force*, after the Dutch physicist Hendrik Antoon Lorentz.

This expression for the Lorentz force is valid only as long as $|\boldsymbol{v}|$ is small in comparison to the speed of light. For relativistic motion, the formula for the Lorentz force must be corrected. There is a modified version of the cyclotron, called the *synchrotron*, which takes these corrections into account.

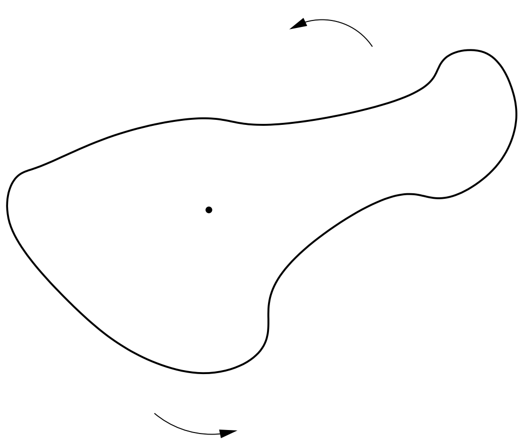
II.3 Extended bodies

Sofar we have considered kinematics and dynamics of point particles only. We now consider extended rigid bodies.

The motion of an extended rigid body can be decomposed into translational motion (a Cartesian basis fixed to the body remains parallel to itself) and rotational motion around a reference point (the reference point remains fixed).

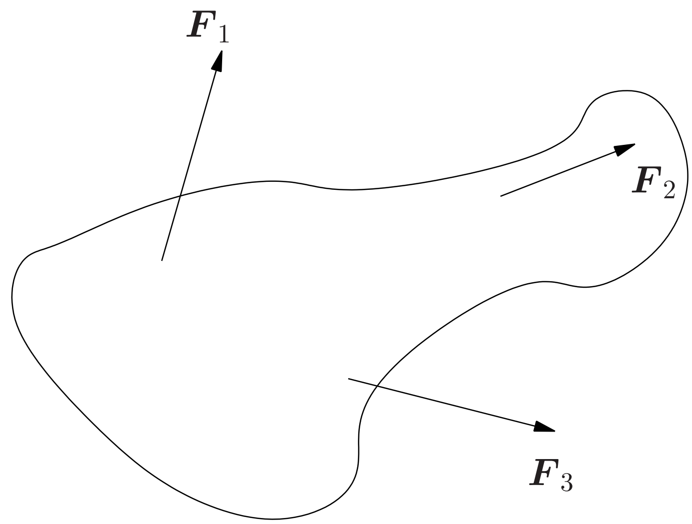


translational motion

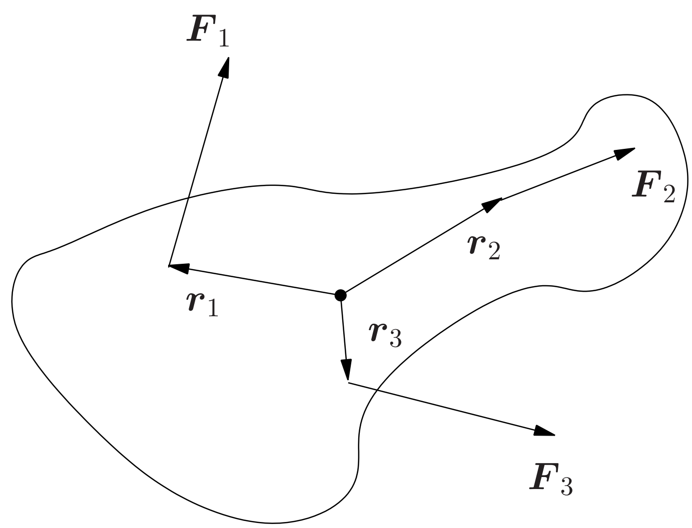


rotational motion

The total force $\boldsymbol{F} = \boldsymbol{F}_1 + \cdots + \boldsymbol{F}_N$ causes translational motion.



The total torque $\boldsymbol{\tau} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \cdots + \boldsymbol{r}_N \times \boldsymbol{F}_N$ causes rotational motion. Note that changing the reference point changes the torque.

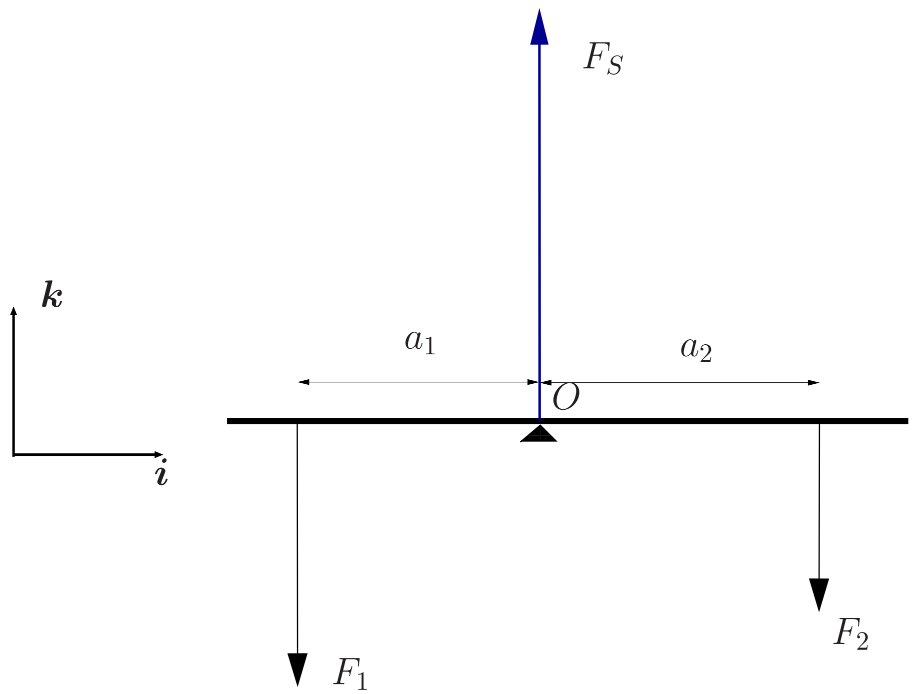


The torque is also called the "moment of force".

Equilibrium requires $\boldsymbol{F} = \boldsymbol{0}$ and $\boldsymbol{\tau} = \boldsymbol{0}$.

We consider now the six socalled "simple machines": lever, inclined plane, pulley, wheel and axle, screw and wedge.

1) Lever



The pivotal point (or fulcrum) O is fixed. The support of the lever must provide a force $\boldsymbol{F}_S$ to make the total force equal to zero. As $\boldsymbol{F}_S$ attacts at O, it gives no contribution to the torque $\boldsymbol{\tau}$ with respect to O. (Therefore it is usually omitted in pictures.)

The torque is

$$\boldsymbol{\tau} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 =$$

$$= (-a_1\,\boldsymbol{i}) \times (-F_1\,\boldsymbol{k}) + a_2\,\boldsymbol{i} \times (-F_2\,\boldsymbol{k}) =$$

$$= (a_1 F_1 - a_2 F_2)\,\boldsymbol{i} \times \boldsymbol{k} = (a_2 F_2 - a_1 F_1)\,\boldsymbol{j}$$

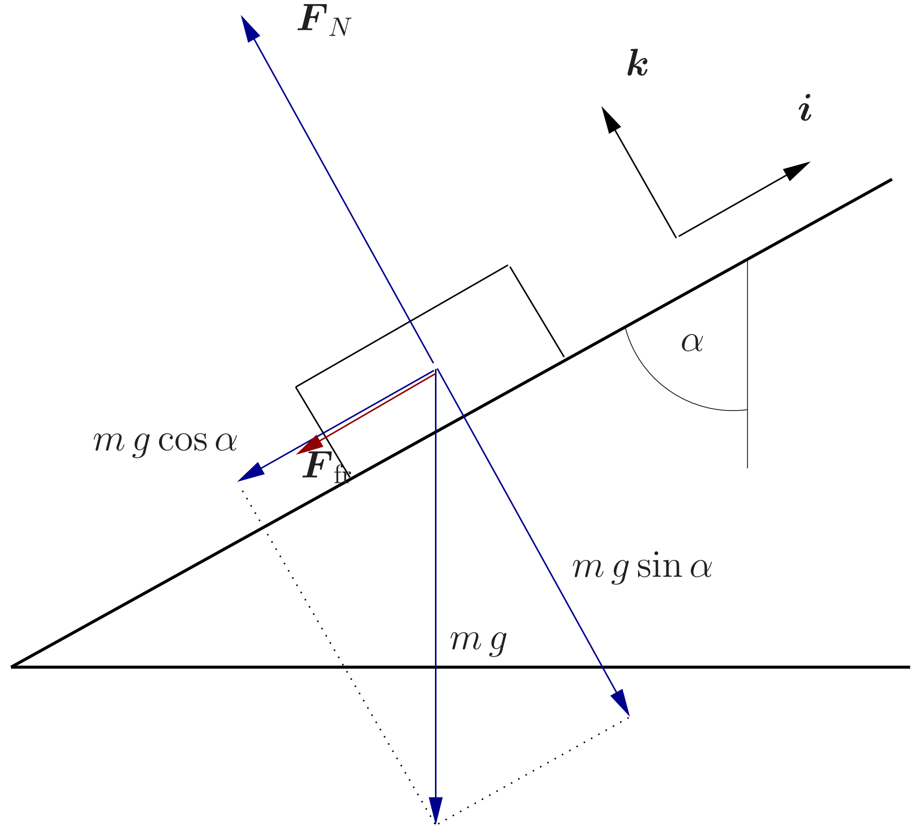
$a_2F_2 > a_1F_1$: left-hand side goes up
$a_2F_2 = a_1F_1$: equilibrium
$a_2F_2 < a_1F_1$: right-hand side goes up

If F_1 and a_1 are given, an arbitrarily small F_2 can yield equlibrium; one just has to choose a_2 sufficiently large.

Archimedes: "Give me a fixed point and I will lift the Earth."

For a lever with more than two forces see 4th worksheet.

2) Inclined plane



The plane provides a normal force F_N which compensates the normal component of the gravitational force, $F_N = m\,g\sin\alpha$. Thus, only the tangential component $m\,g\cos\alpha$ of the gravitational force has to be overcome if the body is to be dragged upwards. This is an advantage in comparison to lifting the body vertically which requires to overcome the force mg.

For a body moving on an inclined plane, friction cannot usually be ignored. The friction force is assumed proportional to the normal force F_N,

$$F_{\mathrm{fr}} = \mu\,m\,g\sin\alpha\,,$$

and directed opposite to the velocity. The proportionality factor μ depends on the materials and is called the "coefficient of kinetic friction". (There is also a "coefficient of static friction" which is relevant if the body is to start from rest.)

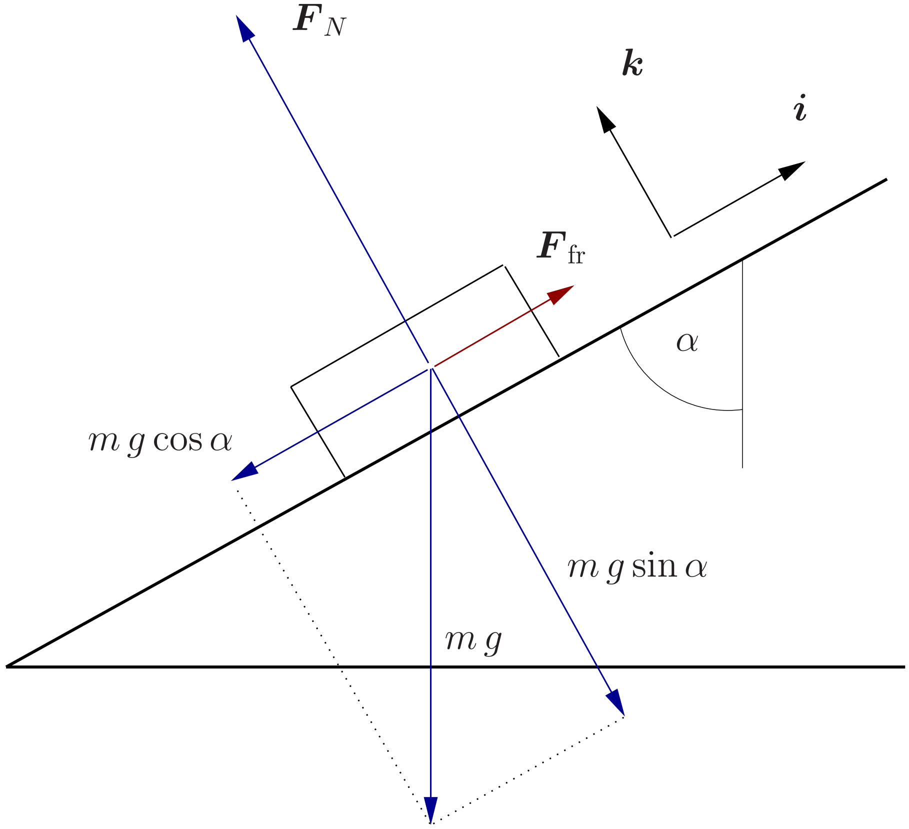
Thus, for upward motion the equation of motion reads

$$\cancel{m}\,\ddot{x}(t) \;=\; -\cancel{m}\,g\cos\alpha \;-\; \mu\cancel{m}\,g\sin\alpha$$

if no other forces are supplied. As both gravity and friction decelerate the body, it will come to rest if it starts with non-zero initial velocity. For calculating the time at which the body comes to rest see 4th worksheet.

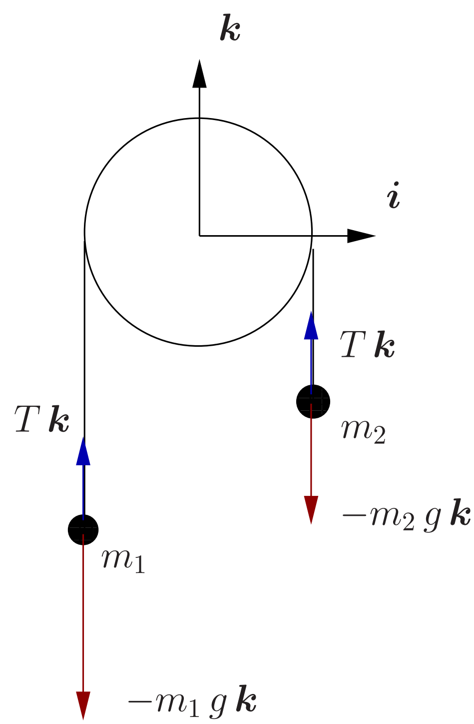
For downward motion the equation of motion reads

$$\cancel{m}\,\ddot{x}(t) \;=\; -\cancel{m}\,g\cos\alpha \;+\; \mu\cancel{m}\,g\sin\alpha\,.$$



The friction force now points into the positive $\boldsymbol{i}-$direction. The motion is decelerated if $\mu > \cot\alpha$ and accelerated if $\mu > \cot\alpha$. For $\mu = \cot\alpha$ the velocity is constant.

3) Pulley



The axle of the pulley is assumed to be fixed. Thus, the support of the pulley has to provide a force F_S that compensates for gravity. This force F_S is not shown in the picture because it is irrelevant for the upward or downward motion of the masses m_1 and m_2.

If friction can be ignored, the only forces acting on the two masses are gravity and the tension T of the string.

Newton's second law for the first particle:

$$m_1\,\ddot{z}_1 \;=\; T \;-\; m_1\,g\,. \qquad (1)$$

Newton's second law for the second particle:

$$m_2\,\ddot{z}_2 \;=\; T \;-\; m_2\,g\,. \qquad (2)$$

As the particles are connected by an inextensible string:

$$z_1 + z_2 \;=\; \text{constant}\,. \qquad (3)$$

From these three equations one can determine T, $\ddot{z}_1$ and $\ddot{z}_2$.

With (3), adding (1) and (2) yields

$$\frac{T - m_2 g}{m_2} + \frac{T - m_1 g}{m_1} = 0$$

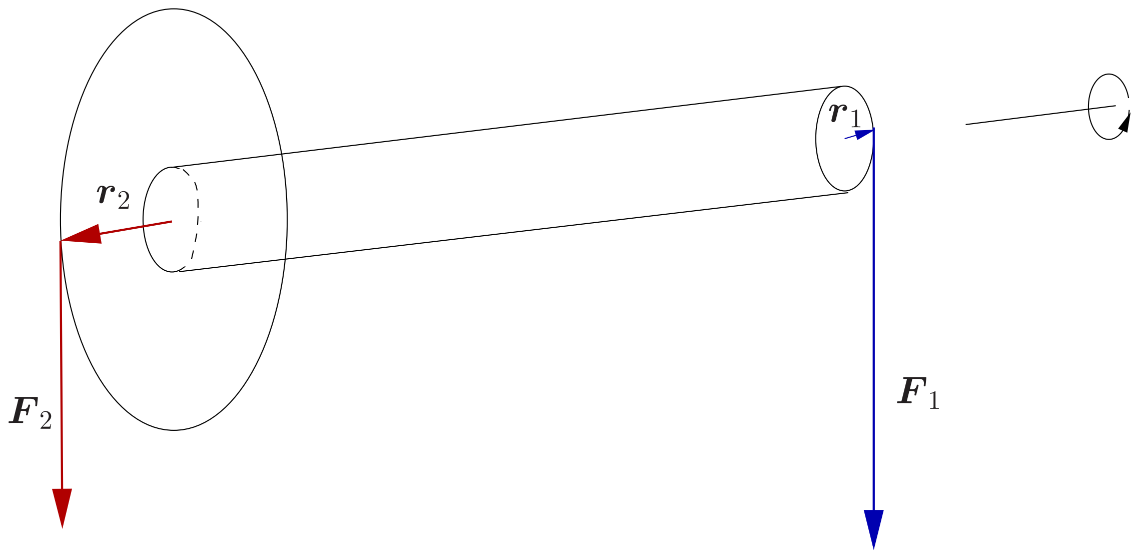
$$m_1 T - m_1 m_2 g + m_2 T - m_1 m_2 g = 0$$

$$T = \frac{2 m_1 m_2 g}{m_1 + m_2}$$

Inserting this into (2), and using again (3), yields

$$\ddot{z}_2 = -\ddot{z}_1 = \frac{(m_1 - m_2) g}{m_1 + m_2}$$

4) Wheel and axle

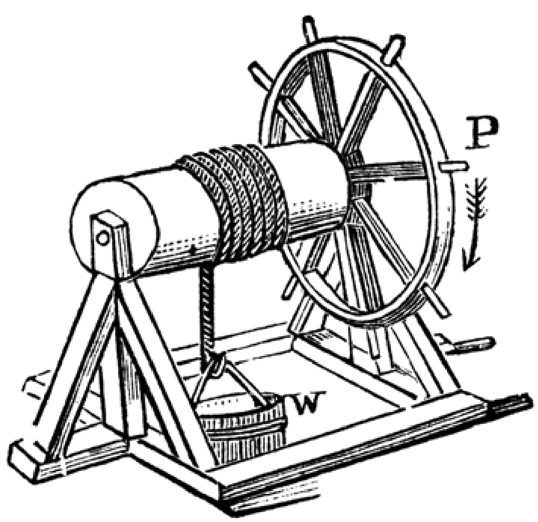


The bearing allows only rotational motion around the axis of symmetry.

There is equilibrium if the two torques about the axis of symmetry add up to zero, $\boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 = \boldsymbol{0}$.

For any given $\boldsymbol{F}_1$, an arbitrarily small force $\boldsymbol{F}_2$ can provide equilibrium if $|\boldsymbol{r}_2|/|\boldsymbol{r}_1|$ is chosen sufficiently large.

Application: Lifting a pail of water from a well.



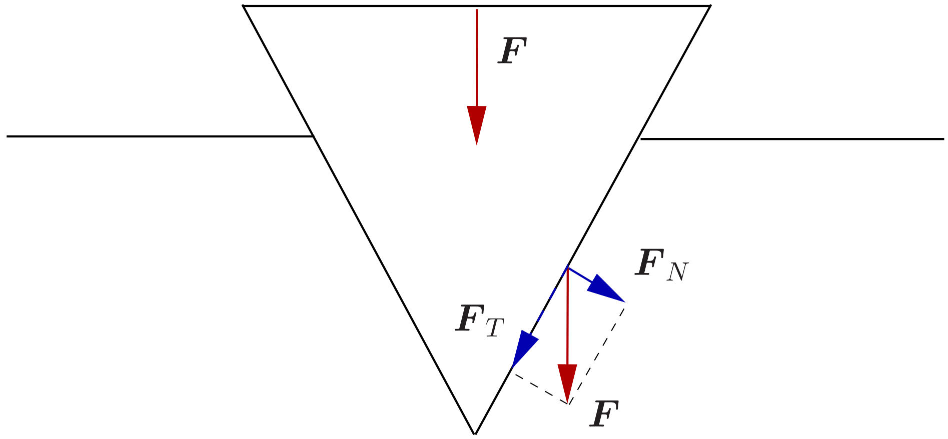
Picture from en.wikipedia.org

5) Wedge

The force $\boldsymbol{F}$ is decomposed into a tangential and a normal component,

$$\boldsymbol{F} = \boldsymbol{F}_T + \boldsymbol{F}_N ,$$

similarly to the inclined plane.



$\boldsymbol{F}_T$ has to overcome friction.

$\boldsymbol{F}_N$, if large enough, splits the material.

6) Screw

The screw transforms rotational motion into (vertical) translational motion.

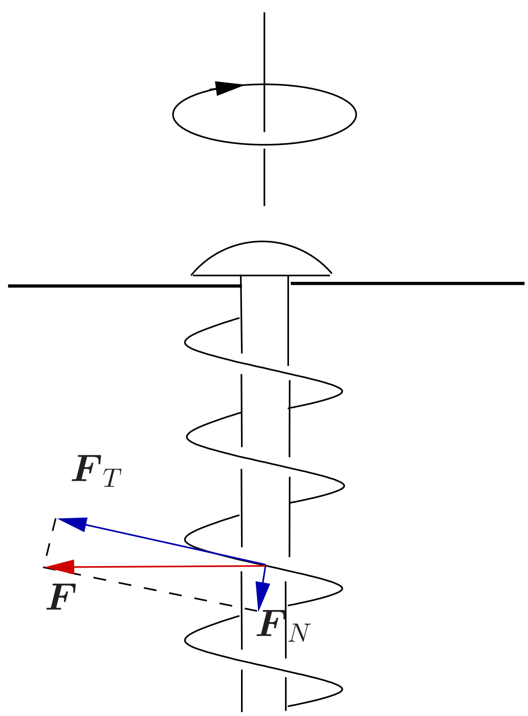
Again, the force $\boldsymbol{F}$ is decomposed into tangential and normal components,

$$\boldsymbol{F} = \boldsymbol{F}_T + \boldsymbol{F}_N ,$$

similarly to the inclined plane.

$\boldsymbol{F}_T$ has to overcome friction.

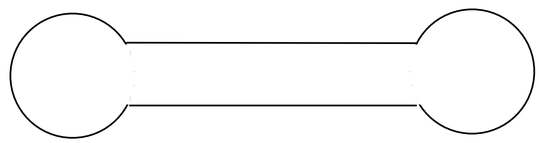
$\boldsymbol{F}_N$ must be balanced by the surrounding material.

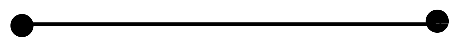


II.4 Systems of particles

Relation to preceding section: A system of point particles can be used as an approximation for a rigid body.

Example: A dumbbell can be approximated by a system of two (point) particles, connected by a weightless rod.





In the following we consider N particles with

- masses $m_1, \ldots, m_N$
- position vectors $\boldsymbol{r}_1(t), \ldots, \boldsymbol{r}_N(t)$
- velocity vectors $\boldsymbol{v}_1(t) = \dot{\boldsymbol{r}}_1(t), \ldots, \boldsymbol{v}_N(t) = \dot{\boldsymbol{r}}_N(t)$
- acceleration vectors $\boldsymbol{a}_1(t) = \ddot{\boldsymbol{r}}_1(t), \ldots, \boldsymbol{a}_N(t) = \ddot{\boldsymbol{r}}_N(t)$
- (linear) momentum vectors $\boldsymbol{p}_1(t) = m_1\boldsymbol{v}_1(t), \ldots, \boldsymbol{p}_N(t) = m_N\boldsymbol{v}_N(t)$
- angular momentum vectors $\boldsymbol{L}_1(t) = \boldsymbol{r}_1(t) \times \boldsymbol{p}_1(t), \ldots, \boldsymbol{L}_N(t) = \boldsymbol{r}_N(t) \times \boldsymbol{p}_N(t)$
- forces $\boldsymbol{F}_1(t) = \dfrac{d}{dt}\boldsymbol{p}_1(t), \ldots, \boldsymbol{F}_N(t) = \dfrac{d}{dt}\boldsymbol{p}_N(t)$

  $\boldsymbol{F}_1(t) = m_1\boldsymbol{a}_1(t), \ldots, \boldsymbol{F}_N(t) = m_N\boldsymbol{a}_N(t)$
- torques $\boldsymbol{\tau}_1(t) = \boldsymbol{r}_1(t) \times \boldsymbol{F}_1(t), \ldots, \boldsymbol{\tau}_N(t) = \boldsymbol{r}_N(t) \times \boldsymbol{F}_N(t)$

We now define several quantities which characterise the total system:

- the total mass $\quad m_{\text{tot}} = m_1 + \ldots + m_N = \sum_{n=1}^{N} m_n$

- the position vector of the centre of mass

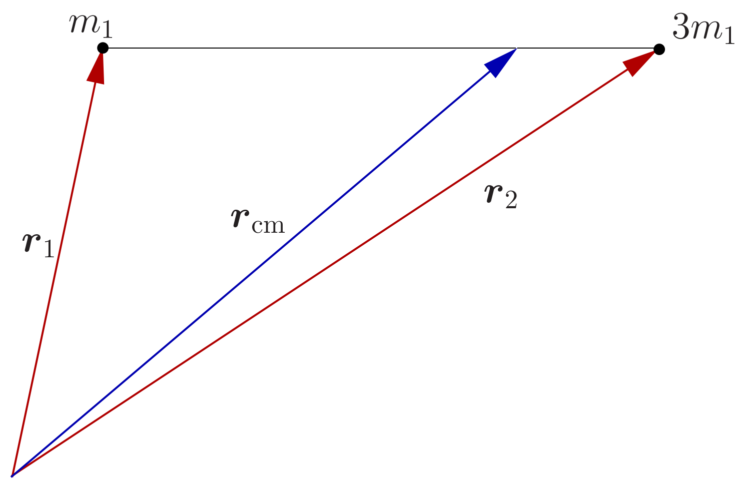
$$\boldsymbol{r}_{\text{cm}} = \frac{1}{m_{\text{tot}}}\big(m_1\boldsymbol{r}_1 + \ldots + m_N\boldsymbol{r}_N\big) = \frac{1}{m_{\text{tot}}}\sum_{n=1}^{N} m_n\boldsymbol{r}_n$$

  e.g.

$$m_2 = 3m_1$$

$$\Rightarrow$$

$$\boldsymbol{r}_{\text{cm}} = \boldsymbol{r}_1 + \frac{3}{4}\big(\boldsymbol{r}_2 - \boldsymbol{r}_1\big)$$



- the total momentum

$$\boldsymbol{p}_{\text{tot}} = \boldsymbol{p}_1 + \ldots + \boldsymbol{p}_N = \sum_{n=1}^{N} \boldsymbol{p}_n$$

- the total angular momentum

$$\boldsymbol{L}_{\text{tot}} = \boldsymbol{L}_1 + \ldots + \boldsymbol{L}_N = \sum_{n=1}^{N} \boldsymbol{L}_n$$

- the total force

$$\boldsymbol{F}_{\text{tot}} = \boldsymbol{F}_1 + \ldots + \boldsymbol{F}_N = \sum_{n=1}^{N} \boldsymbol{F}_n$$

- the total torque

$$\boldsymbol{\tau}_{\text{tot}} = \boldsymbol{\tau}_1 + \ldots + \boldsymbol{\tau}_N = \sum_{n=1}^{N} \boldsymbol{\tau}_n$$

Now we calculate

- the acceleration of the centre of mass

$$\ddot{\boldsymbol{r}}_{\mathrm{cm}} = \frac{1}{m_{\mathrm{tot}}}\big(m_1\ddot{\boldsymbol{r}}_1 + \ldots + m_1\ddot{\boldsymbol{r}}_N\big) = \frac{1}{m_{\mathrm{tot}}}\big(\boldsymbol{F}_1 + \ldots + \boldsymbol{F}_N\big) = \frac{1}{m_{\mathrm{tot}}}\boldsymbol{F}_{\mathrm{tot}}$$

$$\boldsymbol{F}_{\mathrm{tot}} = m_{\mathrm{tot}}\,\ddot{\boldsymbol{r}}_{\mathrm{cm}}$$

  i.e., the system as a whole moves as if its total mass would be concentrated in the centre of mass; in particular:

$$\boldsymbol{F}_{\mathrm{tot}} = \boldsymbol{0} \quad\Longrightarrow\quad \ddot{\boldsymbol{r}}_{\mathrm{cm}} = \boldsymbol{0}$$

  i.e., if there is no total force the centre of mass moves uniformly in a straight line;

- the time derivative of the total momentum

$$\dot{\boldsymbol{p}}_{\mathrm{tot}} = \frac{d}{dt}\sum_{n=1}^{N} m_n\dot{\boldsymbol{r}}_n = \sum_{n=1}^{N} m_n\ddot{\boldsymbol{r}}_n = \sum_{n=1}^{N}\boldsymbol{F}_n = \boldsymbol{F}_{\mathrm{tot}}$$

  in particular

$$\boldsymbol{F}_{\mathrm{tot}} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{p}_{\mathrm{tot}} = \textbf{constant}$$

  i.e., if there is no total force the total momentum is conserved;

- the time derivative of the total angular momentum

$$\dot{\boldsymbol{L}}_{\mathrm{tot}} = \frac{d}{dt}\sum_{n=1}^{N}\boldsymbol{L}_n = \frac{d}{dt}\sum_{n=1}^{N}\boldsymbol{r}_n\times\boldsymbol{p}_n = \frac{d}{dt}\sum_{n=1}^{N}\boldsymbol{r}_n\times m_n\,\dot{\boldsymbol{r}}_n =$$

$$= \sum_{n=1}^{N}\big(\dot{\boldsymbol{r}}_n\times m_n\,\dot{\boldsymbol{r}}_n + \boldsymbol{r}_n\times m_n\,\ddot{\boldsymbol{r}}_n\big) = \sum_{n=1}^{N}\big(\boldsymbol{0} + \boldsymbol{r}_n\times\boldsymbol{F}_n\big) = \sum_{n=1}^{N}\boldsymbol{\tau}_n = \boldsymbol{\tau}_{\mathrm{tot}}$$

  in particular

$$\boldsymbol{\tau}_{\mathrm{tot}} = \boldsymbol{0} \quad\Longrightarrow\quad \boldsymbol{L}_{\mathrm{tot}} = \textbf{constant}$$

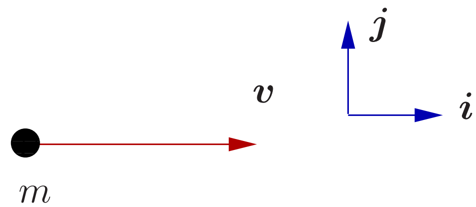
  i.e., if there is no total torque the total angular momentum is conserved.

Summary: In a closed system (i.e., if $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$ and $\boldsymbol{\tau}_{\text{tot}} = \boldsymbol{0}$) the total momentum and the total angular momentum are conserved.

This statement remains true if the masses $m_1, \ldots, m_N$ are time-dependent, as long as the total mass $m_{\text{tot}} = m_1(t) + \ldots + m_N(t)$ is time-independent.

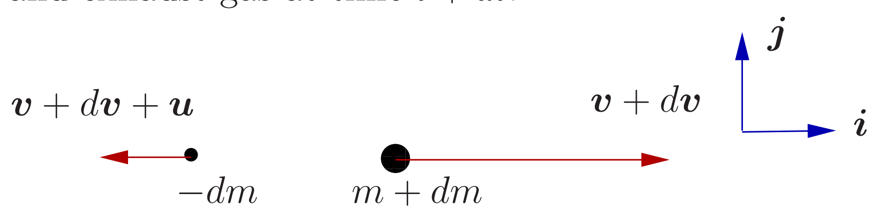
Example: Rocket equation

Rocket at time t:



total momentum $\boldsymbol{p}(t) = m\,\boldsymbol{v} = m\,v\,\boldsymbol{i}$

Rocket and exhaust gas at time $t + dt$:



$\boldsymbol{u} =$ exhaust velocity relative to rocket, $dm < 0$.

$$\text{total momentum } \boldsymbol{p}(t+dt) = (m+dm)\,(\boldsymbol{v}+d\boldsymbol{v}) - dm\,(\boldsymbol{v}+d\boldsymbol{v}+\boldsymbol{u}) = $$
$$= \big(m\,(v+dv) + dm\,u\big)\,\boldsymbol{i}$$

Momentum conservation: $\boldsymbol{p}(t+dt) = \boldsymbol{p}(t)$

$$m\,(v+dv) + u\,dm = m\,v$$

$$m\,dv = -\,u\,dm \quad \Longrightarrow \quad dv = -u\,\frac{dm}{m}$$

If u ($=$ exhaust speed of gas relative to rocket) is constant, we can integrate:

$$\int_{v_1}^{v_2} dv = -\,u \int_{m_1}^{m_2} \frac{dm}{m}$$

$$v_2 - v_1 = -u \ln m \Big|_{m_1}^{m_2} = -u \left(\ln m_2 - \ln m_1 \right) = u \left(\ln m_1 - \ln m_2 \right)$$

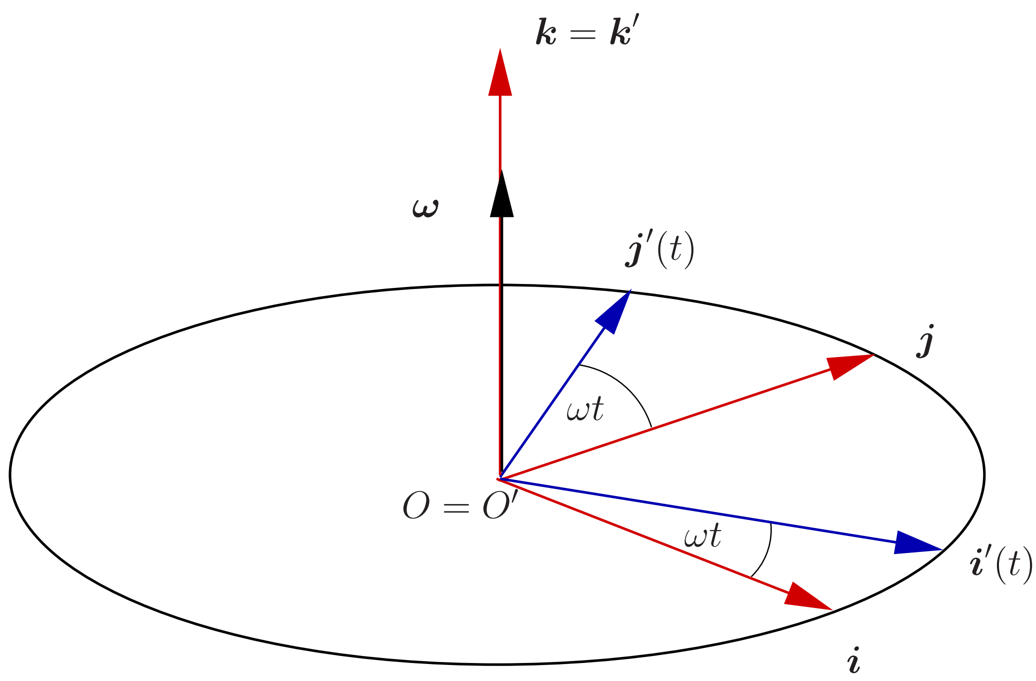
which gives the "rocket equation"

$$v_2 - v_1 = u \ln \frac{m_1}{m_2}$$

The bigger u and the bigger the ratio m_1/m_2, the bigger the gain in rocket speed.

II.5 Inertial forces

Until now we have worked in a fixed Cartesian basis, with the vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ independent of time. We will now calculate what happens if we change to a time-dependent Cartesian basis $\boldsymbol{i}'(t)$, $\boldsymbol{j}'(t)$, $\boldsymbol{k}'(t)$. We restrict to the special case that the second basis rotates with constant angular velocity with respect to the first.



From the picture we read that the primed and unprimed basis vectors are related by

$$\boldsymbol{i}'(t) \;=\; \cos(\omega t)\,\boldsymbol{i} \,+\, \sin(\omega t)\,\boldsymbol{j}\,,$$

$$\boldsymbol{j}'(t) \;=\; -\sin(\omega t)\,\boldsymbol{i} \,+\, \cos(\omega t)\,\boldsymbol{j}\,,$$

The first time derivative yields

$$\frac{d}{dt}\boldsymbol{i}' \;=\; -\,\omega\sin(\omega t)\,\boldsymbol{i} \,+\, \omega\cos(\omega t)\,\boldsymbol{j} \;=\; \omega\,\boldsymbol{j}'(t) \;=\; \omega\,\boldsymbol{k}\times\boldsymbol{i}'(t) \;=\; \boldsymbol{\omega}\times\boldsymbol{i}'(t)\;,$$

$$\frac{d}{dt}\boldsymbol{j}' \;=\; -\,\omega\cos(\omega t)\,\boldsymbol{i} \,-\, \omega\sin(\omega t)\,\boldsymbol{j} \;=\; -\,\omega\,\boldsymbol{i}'(t) \;=\; \omega\,\boldsymbol{k}\times\boldsymbol{j}'(t) \;=\; \boldsymbol{\omega}\times\boldsymbol{j}'(t)\;,$$

and the second time derivative yields

$$\frac{d^2}{dt^2}\boldsymbol{i}'(t) \;=\; \boldsymbol{\omega}\times\frac{d}{dt}\boldsymbol{i}'(t) \;=\; \boldsymbol{\omega}\times\big(\omega\times\boldsymbol{i}'(t)\big)\,,$$

$$\frac{d^2}{dt^2}\boldsymbol{j}'(t) \;=\; \boldsymbol{\omega}\times\frac{d}{dt}\boldsymbol{j}'(t) \;=\; \boldsymbol{\omega}\times\big(\omega\times\boldsymbol{j}'(t)\big)\,.$$

The position vector of a particle can be expressed with respect to the unprimed and with respect to the primed basis, $\boldsymbol{r}(t) \,=\, \boldsymbol{r}'(t)$,

$$x(t)\,\boldsymbol{i} \,+\, y(t)\,\boldsymbol{j} \,+\, z(t)\,\boldsymbol{k} \;=\; x'(t)\,\boldsymbol{i}'(t) \,+\, y'(t)\,\boldsymbol{j}'(t) \,+\, z'(t)\,\boldsymbol{k}'\,.$$

The first time derivative of this equation yields

$$\frac{dx(t)}{dt}\,\boldsymbol{i} \,+\, \frac{dy(t)}{dt}\,\boldsymbol{j} \,+\, \frac{dz(t)}{dt}\,\boldsymbol{k} \;=$$

$$=\; \frac{dx'(t)}{dt}\,\boldsymbol{i}'(t) \,+\, \frac{dy'(t)}{dt}\,\boldsymbol{j}'(t) \,+\, \frac{dz'(t)}{dt}\,\boldsymbol{k}' \,+\, x'(t)\frac{d}{dt}\boldsymbol{i}'(t) \,+\, y'(t)\frac{d}{dt}\boldsymbol{j}'(t)\,,$$

and the second time derivative yields

$$\frac{dx(t)}{dt^2}\boldsymbol{i} \,+\, \frac{dy(t)}{dt^2}\boldsymbol{j} \,+\, \frac{dz(t)}{dt^2}\boldsymbol{k} \;=\; \frac{d^2x'(t)}{dt^2}\,\boldsymbol{i}'(t) \,+\, \frac{d^2y'(t)}{dt^2}\,\boldsymbol{j}'(t) \,+\, \frac{d^2z'(t)}{dt^2}\,\boldsymbol{k}'$$

$$+\,2\,\frac{dx'(t)}{dt}\frac{d}{dt}\boldsymbol{i}'(t) \,+\, 2\,\frac{dy'(t)}{dt}\frac{d}{dt}\boldsymbol{j}'(t) \,+\, x'(t)\frac{d^2}{dt^2}\boldsymbol{i}'(t) \,+\, y'(t)\frac{d^2}{dt^2}\boldsymbol{j}'(t)$$

So the acceleration vectors with respect to the unprimed and primed bases are related by

$$\begin{aligned}\boldsymbol{a}(t) \;=\; \boldsymbol{a}'(t) \,+\, 2\,\frac{dx'(t)}{dt}\big(\boldsymbol{\omega}\times\boldsymbol{i}'(t)\big) \,+\, 2\,\frac{dy'(t)}{dt}\big(\boldsymbol{\omega}\times\boldsymbol{j}'(t)\big)\\
+\,x'(t)\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{i}'(t)\big) \,+\, y'(t)\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{j}'(t)\big)\\
=\; \boldsymbol{a}'(t) \,+\, 2\,\boldsymbol{\omega}\times\Big(\frac{dx'(t)}{dt}\boldsymbol{i}'(t) \,+\, \frac{dy'(t)}{dt}\boldsymbol{j}'(t)\Big)\\
+\,\boldsymbol{\omega}\times\Big(\boldsymbol{\omega}\times\big(x'(t)\boldsymbol{i}'(t) \,+\, y'(t)\boldsymbol{j}'(t)\big)\Big)\,.\end{aligned}$$

As $\boldsymbol{\omega}\times\boldsymbol{k} \;=\; \boldsymbol{0}$, the last equation can be written as

$$\boldsymbol{a}(t) \;=\; \boldsymbol{a}'(t) \,+\, 2\,\boldsymbol{\omega}\times\boldsymbol{v}'(t) \,+\, \boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}'(t)\big)$$

where

$$\boldsymbol{v}'(t) \;=\; \frac{dx'(t)}{dt}\boldsymbol{i}'(t) \,+\, \frac{dy'(t)}{dt}\boldsymbol{j}'(t) \,+\, \frac{dz'(t)}{dt}\boldsymbol{k}\,.$$

Multiplication with the particle's mass m gives the relation between $\boldsymbol{F}(t)$ and $\boldsymbol{F}'(t)$,

$$\boldsymbol{F}(t) \;=\; \boldsymbol{F}'(t) \,+\, 2\,m\,\boldsymbol{\omega}\times\boldsymbol{v}'(t) \,+\, m\,\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}'(t)\big)\,.$$

Hence

$$\boldsymbol{F}(t)' \;=\; \boldsymbol{F}(t) \,+\, \boldsymbol{F}_{\text{cent}} \,+\, \boldsymbol{F}_{\text{Cor}}$$

where

centrifugal force: $\quad \boldsymbol{F}_{\text{cent}} \;=\; -\,m\,\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}'(t)\big)\,,$

Coriolis force: $\quad \boldsymbol{F}_{\text{Cor}} \;=\; -\,2\,m\,\boldsymbol{\omega}\times\boldsymbol{v}'(t)\,.$

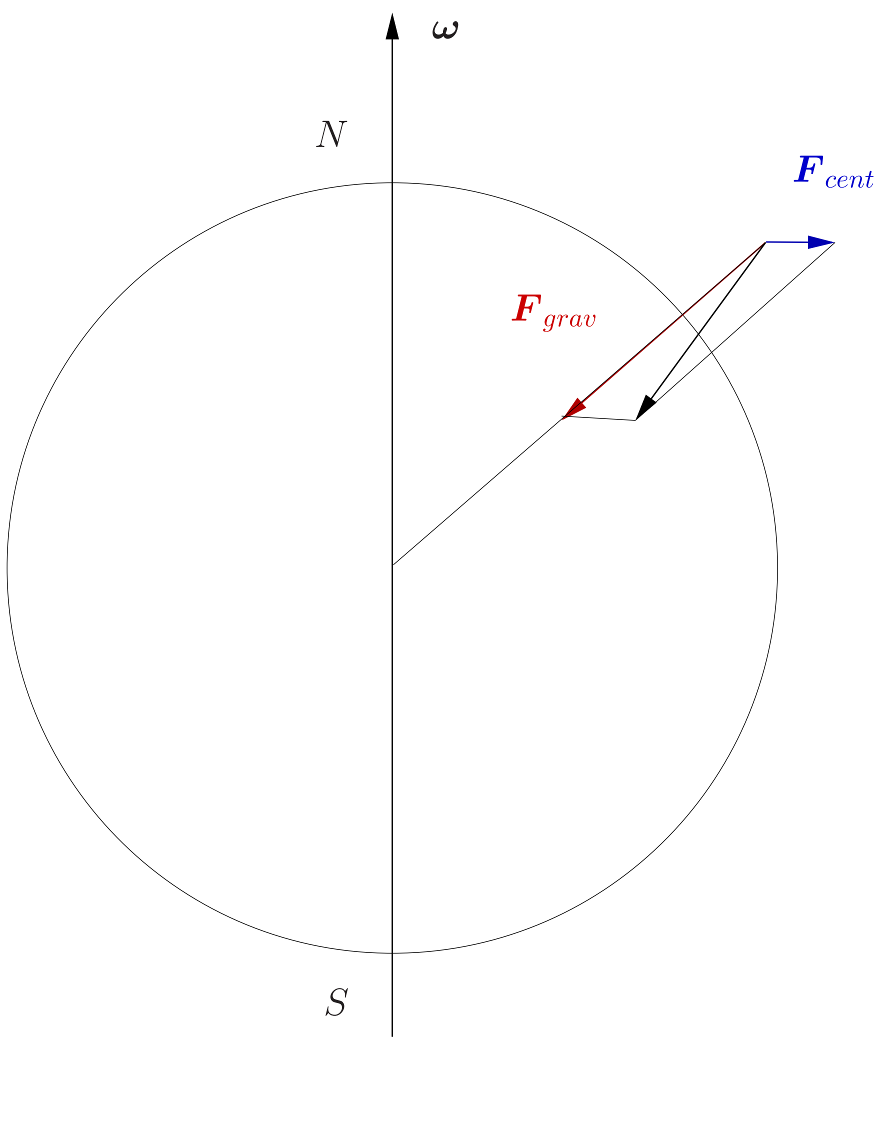
These forces reflect the inertia of the particle, i.e., the desire to stay in uniform rectilinear motion, therefore they are called "inertial forces". As an alternative, they are called "fictitious forces" to indicate that they result only from the motion of the used basis. However, any measurable effect is the same as for any "real force". Therefore, "inertial force" is a better name than "fictitious force".

The centrifugal force is well known from every-day experience. E.g., when riding a carousel your stomach wants to stay in uniform rectilinear motion, so it undergoes an acceleration relative to your body frame which is strapped to the seat. This results in an uneasy (or thrilling) feeling.

Actually, we are living on a carousel: The Earth rotates once in 24 hours. The resulting centrifugal force

$$\boldsymbol{F}_{\text{cent}} = -m\,\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{r}'(t)\big)$$

has the effect that, e.g., a plummet deviates a little bit from the direction towards the centre of the Earth.



The Coriolis force

$$\boldsymbol{F}_{\mathrm{Cor}} = -2\, m\, \boldsymbol{\omega} \times \boldsymbol{v}'(t)$$

resulting from the rotation of the Earth has several observable effects:

- The plane of a pendulum rotates once in 24 hours. This was demonstrated in 1851 by Léon Foucault in a famous experiment in Paris.
- The air does not flow radially into a low pressure area but rather in a spiral, counter-clockwise in the Northern hemisphere and clockwise in the Southern hemisphere.
- A stone falling into a deep well bounces against the wall.

  Wrong explanation: The Earth "turns under the falling stone", so the stone, so the stone bounces against the backward (western) wall.

  Correct explanation: When released the stone has a tangential velocity; when falling towards the centre of the Earth its tangential velocity is too big to keep the stone on a radial line, so it bounces against the forward (eastern) wall.