PHYS113: Series and Differential Equations

Lent Term 2010

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Literature:

- FLAP
- K. Riley, M. Hobson, S. Bence: "Mathematical Methods for Physics and Engineering" Cambridge UP (1974)
- D. Jordan, P.Smith: "Mathematical Techniques" Oxford UP (1994)

I. Sequences and series of real numbers

(Cf. FLAP M1.7)

I.1. Sequences of real numbers

Here we consider only real numbers, complex numbers will be the subject of PHYS114.

Finite sequence (= ordered N-tuple): $\{a_n\}_{n=1}^{N} = \{a_1, a_2, a_3, \dots, a_N\}$

$$\text{e.g. } \{5, \sqrt{2}, -3\}\,,\ \{3, 3, 3, 7, 0, \frac{1}{2}\}\,,\ \dots$$

Infinite sequence: $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$

Of course, we cannot write down infinitely many numbers. There are two ways in which an infinite sequence can be unambiguously determined.

(i) Give the rule that assigns to each n the element a_n, e.g.

$$a_n = n + 3$$

$$\{a_n\}_{n=1}^{\infty} = \{4, 5, 6, 7, \dots\}$$

Graphical representation:

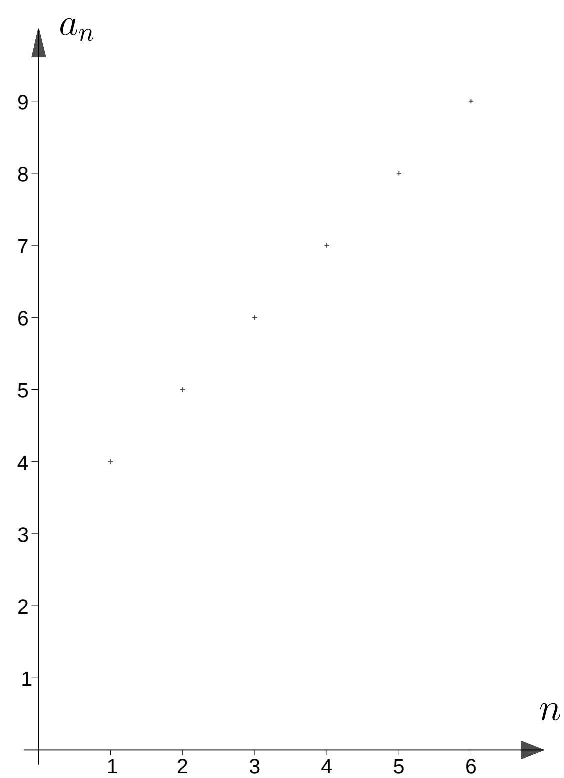
Compare with function of continuous variable $f(x)$.

A sequence is a function of a discrete variable n, with the variable written as index rather than as argument, a_n.

Analogy:

$$x \mapsto f(x)$$

$$n \mapsto a_n$$



(ii) ("Recursive" or "inductive" definition:) Give the first element, a_1, and the rule how a_{n+1} is constructed from a_n, e.g.

$$a_1 = 4\,, \quad a_{n+1} = a_n + 1$$

$$a_1 = 4\,,\ a_2 = 5\,,\ a_3 = 6\,,\ a_4 = 7\,, \dots$$

Note:

- The index can be renamed:

$$\{a_n\}_{n=1}^{\infty} = \{a_m\}_{m=1}^{\infty}$$

- One can rewrite the sequence in terms of a new index that starts with a value different from 1:

$$\{a_n\}_{n=1}^{\infty} = \{a_{k+1}\}_{k=0}^{\infty} = \{a_{\ell+2}\}_{\ell=-1}^{\infty} = \dots$$

It is often convenient to have the index starting at 0.

Two important types of sequences:

- Arithmetic sequence (also called "arithmetic progression"):

$$a_{n+1} - a_n = s \quad (\text{with } s \text{ independent of } n)$$

$$\{a_n\}_{n=0}^{\infty} \quad \text{with} \quad a_n = a_0 + n\,s$$

e.g. $\{a_n\}_{n=0}^{\infty} = \{4, 7, 10, 13, 16, \dots\}$, $s = 3$.

- Geometric sequence (also called "geometric progression"):

$$\frac{a_{n+1}}{a_n} = q \quad (\text{with } q \text{ independent of } n)$$

$$\{a_n\}_{n=0}^{\infty} \quad \text{with} \quad a_n = a_0\,q^n$$

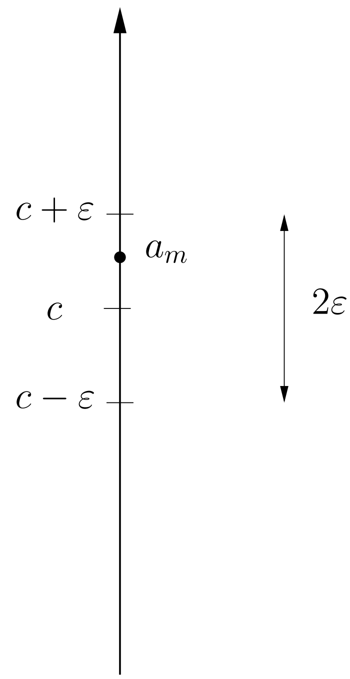
e.g. $\{a_n\}_{n=0}^{\infty} = \{4, 8, 16, 32, \dots\}$, $q = 2$.

Convergence of infinite sequences:

Rough idea: A sequence $\{a_n\}_{n=n_0}^{\infty}$ converges towards c if for all sufficiently large n the number a_n is arbitrarily close to c.

Precise definition: A sequence $\{a_n\}_{n=n_0}^{\infty}$ converges towards c if for every $\varepsilon > 0$ there is an n such that

$$|a_m - c| < \varepsilon \quad \text{for all}\, m > n\,.$$



If $\{a_n\}_{n=n_0}^{\infty}$ converges towards c we write

$$\lim_{n\to\infty} a_n = c$$

or equivalently

$$a_n \to c \quad \text{for} \quad n \to \infty\,.$$

The essential point is that the definition allows to choose ε as small as you like.

Note that for convergence it is irrelevant what the first 5, the first 100, or the first 10^{10} members of the sequence do. What matters is the behaviour of a_n if n becomes arbitrarily large.

Examples of convergent sequences:

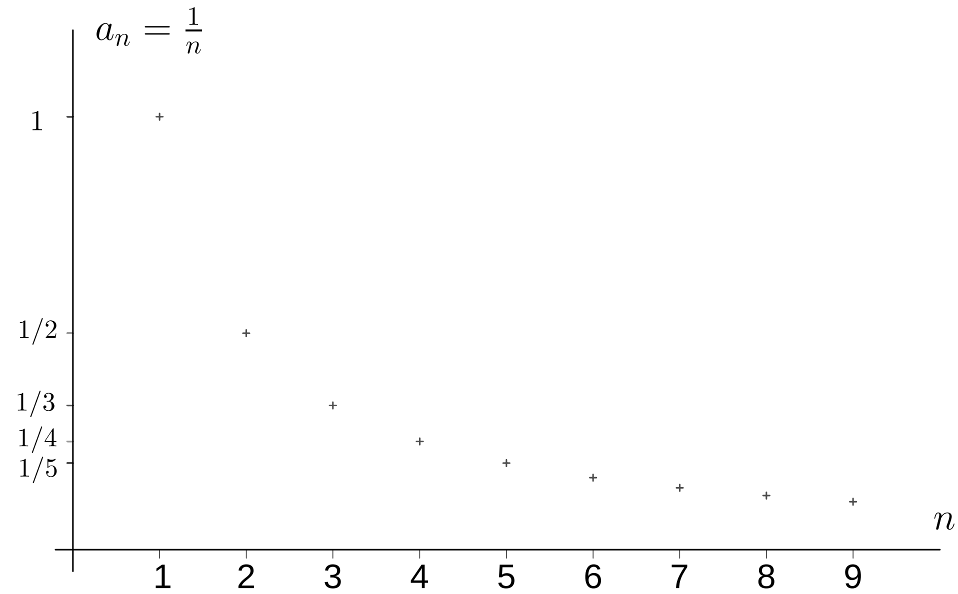
(i) $a_n = \dfrac{1}{n} \to 0$ for $n \to \infty$.

This is intuitively obvious. Here is a formal proof:

Choose any $\varepsilon > 0$. Let n be the smallest integer such that $n > 1/\varepsilon$. Then we have, for all $m > n$,

$$|a_m - 0| = |a_m| = \frac{1}{m} < \frac{1}{n} < \varepsilon\,.$$

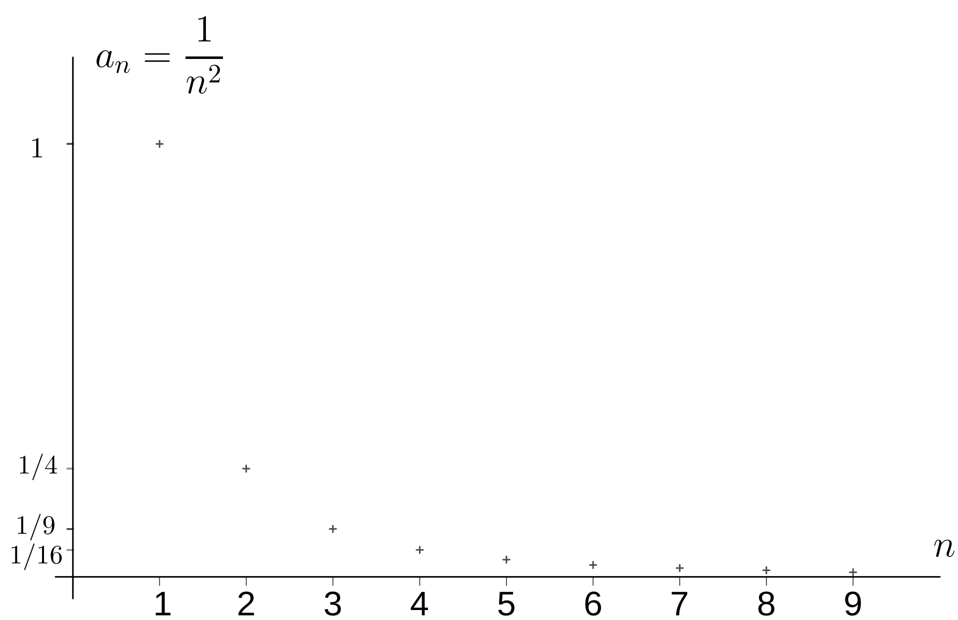
The convergence $a_n = \dfrac{1}{n} \to 0$ is illustrated by the following plot.



In the following two examples the convergence is again intuitively obvious. It can be formally proven in a way similar to (i).

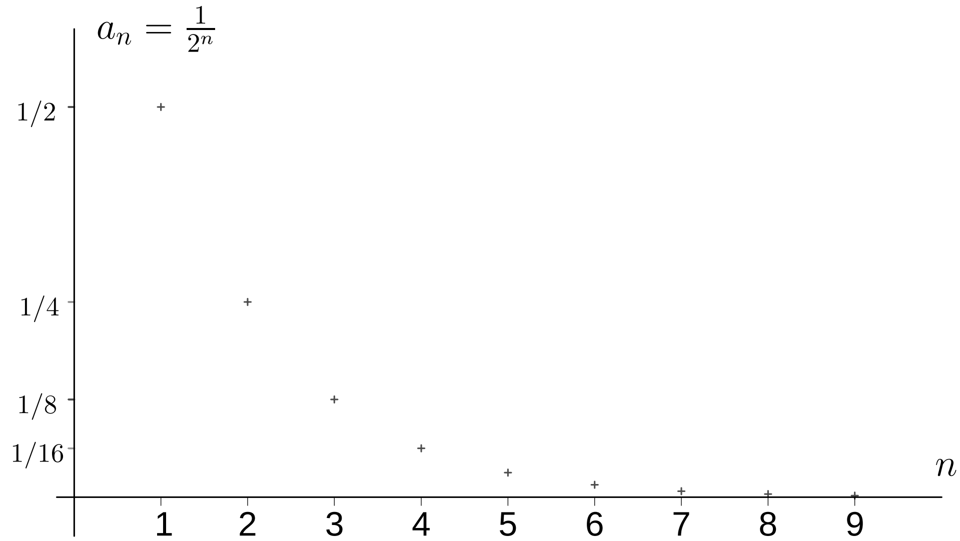
(ii) $a_n = \dfrac{1}{n^p} \to 0$ for $n \to \infty$ if $p > 0$.

Plot for $p = 2$:



(iii) $a_n = q^n \to 0$ for $n \to \infty$ if $0 < |q| < 1$.

Plot for
$q = \frac{1}{2}$:



Other limits can be reduced to these cases. Here are two examples.

(iv) $$a_n = \frac{n^3+3}{2n^3+5n} = \frac{1+3\frac{1}{n^3}}{2+5\frac{1}{n^2}} \to \frac{1}{2} \quad \text{for} \quad n \to \infty$$

(v) $$a_n = \sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \frac{\cancel{n} + 1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} \to 0 \quad \text{for} \quad n \to \infty .$$

Examples of non-convergent sequences:

(i) $a_n = q^n$ with $q > 1$, e.g. $\{a_n\}_{n=0}^{\infty} = \{1, 2, 4, 8, \ldots\}$, $q = 2$.

For any number c we can find an n such that $a_n = q^n$ is bigger than c, so the sequence cannot converge. In this case we say that a_n diverges to ∞ and we write

$$\lim_{n\to\infty} a_n = \infty$$

or

$$a_n \to \infty \quad \text{for} \quad n \to \infty .$$

Analogously, we say that a_n diverges to $-\infty$ if for any c we can find an n such that a_n is smaller than c.

(ii) $a_n = (-1)^n 2$, i.e. $\{a_n\}_{n=1}^{\infty} = \{-2, 2, -2, 2, -2, \ldots\}$.

It is obvious that this sequence does not converge, because it is not confined to an arbitrarily small interval for large n. Also, it does not diverge to plus or minus infinity because the members of the sequence are obviously bounded. In this case we speak of an "oscillatory sequence".

So a sequence may converge, diverge (to $+\infty$ or $-\infty$), or neither. A sequence with non-decreasing members, $a_{n+1} \geq a_n$, either converges or diverges to $+\infty$.

I.2. Series of real numbers

Finite series (= sum): $a_1 + \ldots + a_N = \sum_{n=1}^{N} a_n$

Σ = capital greek Sigma = summation sign.

Examples:

(a) $3+5+7+9 = \sum_{n=1}^{4} (2n+1)$

(b) $9+16+25+36 = \sum_{n=3}^{6} n^2$

Note:

- $$\sum_{n=n_0}^{N} a_n = \sum_{m=n_0}^{N} a_m \,.$$
- $$\sum_{n=n_0}^{N} a_n = \sum_{m=n_0-1}^{N-1} a_{m+1} = \sum_{\ell=n_0-2}^{N-2} a_{\ell+2} = \ldots$$
- $$\sum_{n=n_0}^{N} (s\,a_n + t\,b_n) = s \sum_{n=n_0}^{N} a_n + t \sum_{n=n_0}^{N} b_n \,.$$
- $$\sum_{n=n_0}^{N} \Big(\sum_{m=m_0}^{M} b_m \Big) a_n = \sum_{m=m_0}^{M} \Big(\sum_{n=n_0}^{N} a_n \Big) b_m \,,$$

  i.e., the brackets can be dropped and the order of the sums is irrelevant.

Infinite series : $\displaystyle\sum_{n=n_0}^{\infty} a_n$

This does *not* mean to sum up infinitely many numbers (which is impossible). Rather:

$$\sum_{n=n_0}^{\infty} a_n = \lim_{N\to\infty} S_N \quad \text{where} \quad S_N = \sum_{n=n_0}^{N} a_n \,.$$

Two important types of series:

- Arithmetic series

  Recall that an arithmetic sequence is a sequence of the form $a_n = a_0 + n\,s$. Summing up the members of an arithmetic sequence gives an arithmetic series:

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} (a_0 + ns) = (N+1)a_0 + s\sum_{n=1}^{N} n \,.$$

Claim:

$$\boxed{\sum_{n=1}^{N} n = \frac{N(N+1)}{2}}$$

Proof:

$$\begin{aligned}
\sum_{n=1}^{N} n &= \quad 1 \quad + \quad 2 \quad + \ldots + (N-1) + \quad N \\
\sum_{n=1}^{N} n &= \quad N \quad + (N-1) + \ldots + \quad 2 \quad + \quad 1 \\
\hline
2\sum_{n=1}^{N} n &= (N+1) + (N+1) + \ldots + (N+1) + (N+1) \\
&= N(N+1)
\end{aligned}$$

[This result dates back to Carl-Friedrich Gauss (1777 - 1855) who found it as a school boy when he was asked to sum up all integers from 1 to 100.]

- Geometric series

  Recall that a geometric sequence is a sequence of the form $a_n = a_0\, q^n$. Summing up the members of a geometric sequence gives a geometric series:

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} a_0\, q^n = a_0 \sum_{n=0}^{N} q^n \, .$$

Claim:

$$\boxed{\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}}$$

Proof:
$$(1-q)\sum_{n=0}^{N} q^n = \sum_{n=0}^{N} q^n - q \sum_{n=0}^{N} q^n =$$

$$\begin{aligned} = \sum_{n=0}^{N} q^n - \sum_{n=0}^{N} q^{n+1} &= 1 + q + \ldots + q^N \\ &\quad - \left(\qquad q + \ldots + q^N + q^{N+1} \right) \\ &= 1 - q^{N+1} \, . \end{aligned}$$

For $0 < |q| < 1$ we have $q^{N+1} \to 0$ for $N \to \infty$. This gives the

summation formula for the infinite geometric series

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \quad \text{for } 0 < |q| < 1 \, .$$

For $|q| \geq 1$ the geometric series does not converge.

Here are two examples where the summation formulas for arithmetic and geometric series are used.

(i) Check if $\sum_{n=1}^{\infty}\big(2n-\dfrac{1}{3}\big)$ converges.

We calculate the partial sums

$$S_N=\sum_{n=1}^{N}\big(2n-\frac{1}{3}\big)=2\sum_{n=1}^{N}n-\sum_{n=1}^{N}\frac{1}{3}=$$

$$=\;2\,\frac{N(N+1)}{2}-\frac{N}{3}\;=\;N^2+\frac{2}{3}N\,.$$

Hence

$$\sum_{n=1}^{\infty}(2n-\frac{1}{3})=\lim_{N\to\infty}S_N=\lim_{N\to\infty}\big(N^2+\frac{2}{3}N\big)=\infty\,,$$

so the series diverges.

(ii) Check if $\sum_{n=0}^{\infty}\dfrac{(2^n-1)}{3^n}$ converges.

We calculate the partial sums

$$S_N=\sum_{n=0}^{N}\frac{(2^n-1)}{3^n}=\sum_{n=0}^{N}\big(\frac{2}{3}\big)^n-\sum_{n=0}^{N}\big(\frac{1}{3}\big)^n=\frac{1-(\frac{2}{3})^{N+1}}{1-\frac{2}{3}}-\frac{1-(\frac{1}{3})^{N+1}}{1-\frac{1}{3}}\,.$$

Hence

$$\sum_{n=0}^{\infty}\frac{(2^n-1)}{3^n}\;=\;\lim_{N\to\infty}\Big(\frac{1-(\frac{2}{3})^{N+1}}{1-\frac{2}{3}}-\frac{1-(\frac{1}{3})^{N+1}}{1-\frac{1}{3}}\Big)\;=\;3-\frac{3}{2}=\frac{3}{2}$$

so the series converges towards $\dfrac{3}{2}$.

In the following we discuss several tests which can be used to determine whether or not a series converges. These tests are relevant, in particular, in cases where the partial sums cannot be calculated explicitly.

The ratio test for convergence (d'Alembert test):

Assume $a_n > 0$ for all n. Let $R = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

If $R < 1$, the series $\sum_{n=n_0}^{\infty} a_n$ converges.

If $R > 1$, the series $\sum_{n=n_0}^{\infty} a_n$ does not converge.

If $R = 1$, the series $\sum_{n=n_0}^{\infty} a_n$ may or may not converge.

Proof:

If $R < 1$: For convergence it is irrelevant which n_0 we choose. Let us choose n_0 sufficiently large. Then there is a number q with $0 < q < 1$ such that

$$\frac{a_{n+1}}{a_n} \leq q \quad \text{for all} \quad n \geq n_0\,,$$

see picture. As $a_n > 0$, the inequality is preserved if both sides are multiplied with a_n,

$$a_{n+1} \leq q\, a_n \quad \text{for all} \quad n \geq n_0\,.$$

We write this inequality for successive values of n:

$$\begin{aligned} a_{n_0} &= a_{n_0} \\ a_{n_0+1} &\leq q\, a_{n_0} \\ a_{n_0+2} &\leq q\, a_{n_0+1} \leq q^2\, a_{n_0} \\ &\dots \\ a_{n_0+N} &\leq \quad \dots \quad \leq q^N\, a_{n_0} \end{aligned}$$

$$\sum_{n=n_0}^{n_0+N} a_n \leq \sum_{m=0}^{N} q^m\, a_{n_0}$$

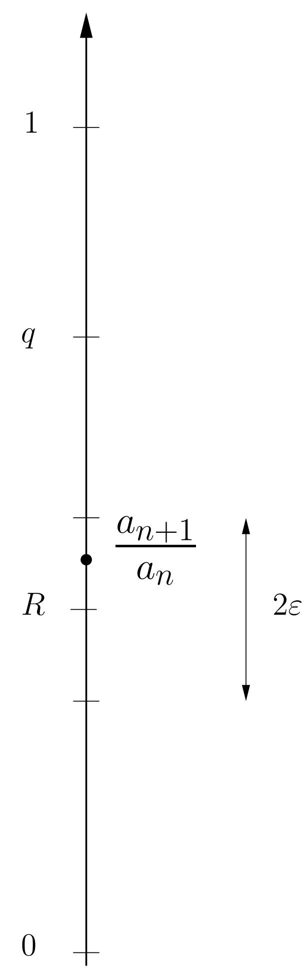
With the summation formula for the geometric series (p.9)

$$\sum_{n=n_0}^{n_0+N} a_n \leq a_{n_0}\, \frac{1-q^{N+1}}{1-q}\,.$$

Now let $N \to \infty$. As $0 < q < 1$, we have $q^{N+1} \to 0$ and thus

$$\sum_{n=n_0}^{\infty} a_n \leq \frac{a_{n_0}}{1-q}\,.$$

So $\sum_{n=n_0}^{\infty} a_n$ is bounded from above by a finite value. As $a_n > 0$, this implies that $\sum_{n=n_0}^{\infty} a_n$ converges. (Note that, if $a_n > 0$, the sequence with elements $S_N = \sum_{n=n_0}^{N} a_n$ is increasing. Such a sequence either converges or it diverges to $+\infty$.)



If $R > 1$: Again we choose n_0 sufficiently large. Then there is a number $q > 1$ such that

$$\frac{a_{n+1}}{a_n} \geq q \quad \text{for all} \quad n \geq n_0\,,$$

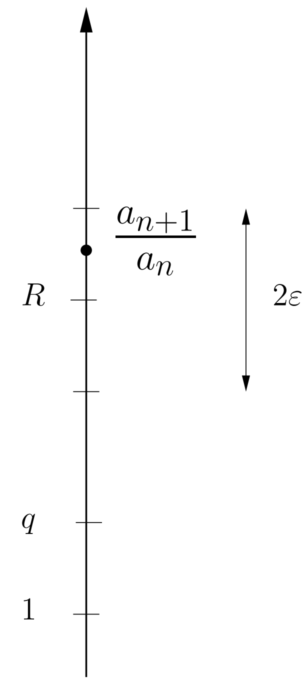
see picture next page. As $a_n > 0$,

$$a_{n+1} \geq q\, a_n \quad \text{for all} \quad n \geq n_0\,.$$

Again, we write this inequality for successive values of n:

$$\begin{aligned} a_{n_0} &= a_{n_0} \\ a_{n_0+1} &\geq q\, a_{n_0} \\ a_{n_0+2} &\geq q\, a_{n_0+1} \geq q^2\, a_{n_0} \\ &\dots \\ a_{n_0+N} &\geq \quad \dots \quad \geq q^N\, a_{n_0} \end{aligned}$$

$$\sum_{n=n_0}^{n_0+N} a_n \geq \sum_{m=0}^{N} q^m\, a_{n_0}$$



Again with the summation formula for the geometric series,

$$\sum_{n=n_0}^{n_0+N} a_n \geq a_{n_0}\, \frac{1 - q^{N+1}}{1 - q} = a_{n_0}\, \frac{q^{N+1} - 1}{q - 1}$$

Now we let $N \to \infty$. As $q > 1$, we have $q^{N+1} \to \infty$, so our inequlity implies that $\sum_{n=n_0}^{\infty} a_n$ diverges to ∞.

If $R = 1$, no information about convergence can be gained.

Example: $\quad a_n = \dfrac{1}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 \quad \text{for} \quad n \to \infty\,, \quad \text{so} \quad \sum_{n=n_0}^{\infty} \frac{1}{n!} \quad \text{converges.}$$

The ratio test does not tell towards which value the series converges. We will later see that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

where e is the Euler number,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.718\dots$$

We now turn to the second test for convergence of a series.

The root test for convergence (Cauchy test):

Assume $a_n > 0$ for all n. Let $Q = \lim_{n\to\infty} \sqrt[n]{a_n}$.

If $Q < 1$, the series $\sum_{n=n_0}^{\infty} a_n$ converges.

If $Q > 1$, the series $\sum_{n=n_0}^{\infty} a_n$ does not converge.

If $Q = 1$, the series $\sum_{n=n_0}^{\infty} a_n$ may or may not converge.

Proof (was not given in the lecture because it is very similar to the proof of the ratio test):

If $Q < 1$: Choose n_0 sufficiently large. Then there is a number q with $0 < q < 1$ such that

$$\sqrt[n]{a_n} \leq q \quad \text{for all} \quad n \geq n_0\,,$$

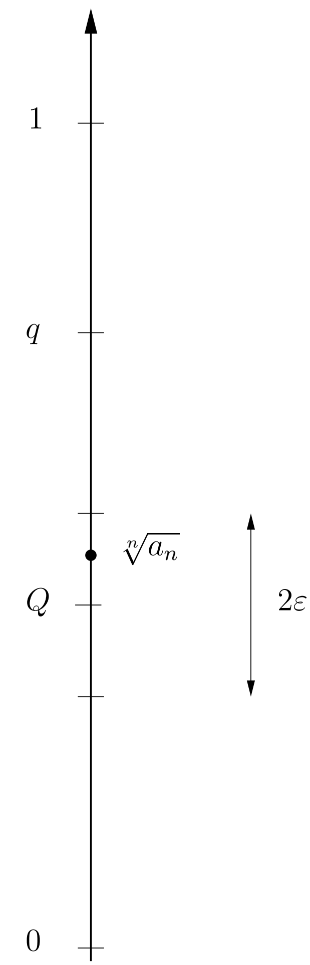
see picture. As the expressions are positive, the inequality is preserved if both sides are raised to the power n:

$$a_n \leq q^n \quad \text{for all} \quad n \geq n_0\,.$$

We write this inequality for successive values of n:

$$\begin{aligned} a_{n_0} &\leq q^{n_0} \\ a_{n_0+1} &\leq q^{n_0+1} = q\,q^{n_0} \\ a_{n_0+2} &\leq q^{n_0+2} = q^2\,q^{n_0} \\ &\dots \\ a_{n_0+N} &\leq q^{n_0+N} = q^N\,q^{n_0} \end{aligned}$$

$$\sum_{n=n_0}^{n_0+N} a_n \leq \sum_{m=0}^{\infty} q^m\,q^{n_0}$$



With the summation formula for the geometric series:

$$\sum_{n=n_0}^{n_0+N} a_n \leq q^{n_0}\,\frac{1-q^{N+1}}{1-q}$$

Now we let $N \to \infty$. As $0 < q < 1$, we have $q^{N+1} \to 0$ and thus

$$\sum_{n=n_0}^{\infty} a_n \leq \frac{q^{n_0}}{1-q}\,.$$

So $\sum_{n=n_0}^{\infty} a_n$ is bounded from above by a finite value.
As $a_n > 0$, this implies that $\sum_{n=n_0}^{\infty} a_n$ converges.

If $Q > 1$: Again we choose n_0 sufficiently large. Then there is a number $q > 1$ such that

$$\sqrt[n]{a_n} \geq q \quad \text{for all} \quad n \geq n_0 \,,$$

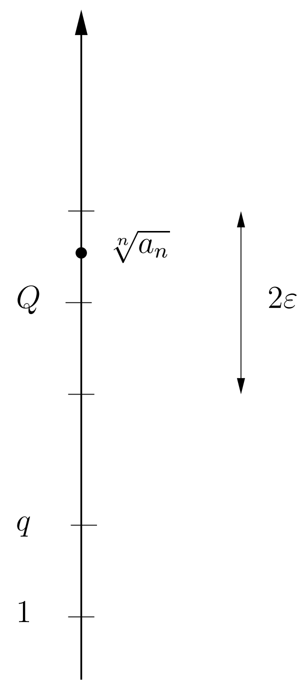
see picture. As the expressions are positive, the inequality is preserved if both sides are raised to the power n:

$$a_n \geq q^n \quad \text{for all} \quad n \geq n_0 \,.$$

Again, we write this inequality for successive values of n:

$$\begin{aligned} a_{n_0} &\geq q^{n_0} \\ a_{n_0+1} &\geq q^{n_0+1} = q\, q^{n_0} \\ a_{n_0+2} &\geq q^{n_0+2} = q^2\, q^{n_0} \\ &\dots \\ a_{n_0+N} &\geq q^{n_0+N} = q^N\, q^{n_0} \end{aligned}$$

$$\sum_{n=n_0}^{n_0+N} a_n \geq \sum_{m=0}^{N} q^m\, q^{n_0}$$



With the summation formula of the geometric series:

$$\sum_{n=n_0}^{n_0+N} a_n \geq q^{n_0} \frac{1 - q^{N+1}}{1-q} = q^{n_0} \frac{q^{N+1} - 1}{q-1}$$

Now we let $N \to \infty$. As $q > 1$, we have $q^{N+1} \to \infty$ and thus our inequlity implies that $\sum_{n=n_0}^{\infty} a_n$ diverges to ∞.

If $Q = 1$, no information about convergence can be gained.

Example: $\quad a_n = \dfrac{1}{n^n}$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \left(\frac{1}{n^n}\right)^{1/n} = \frac{1}{n} \to 0 \quad \text{for} \quad n \to \infty \,, \quad \text{so} \quad \sum_{n=n_0}^{\infty} \frac{1}{n^n} \quad \text{converges.}$$

This is a particularly simple example. Usually the nth square root leads to awkward expressions. Therefore, it is recommendable to try the ratio test first und to use the root test only in cases where the ratio test fails.

We now turn to the third test of convergence. As a preparation, we need some results about the relation of series and integrals. This will then result in the socalled ‘integral test’ for convergence.

Let $f(x)$ be a monotonically decreasing and positive function. [Recall that a function $f(x)$ is called ‘monotonically decreasing’ or simply ‘decreasing’ if $f(x+\varepsilon) \le f(x)$ for $\varepsilon > 0$. A differentiable function $f(x)$ is monotonically decreasing if and only if $f'(x) \le 0$.]

Define a_n, for integer n, by $a_n = f(n)$, see pictures on the right. In the first picture, clearly the shaded area is smaller than the area under the graph of $f(x)$. Thus, for any integer N,

$$\sum_{n=N+1}^{\infty} a_n \le \int_N^{\infty} f(x)\,dx\,,$$

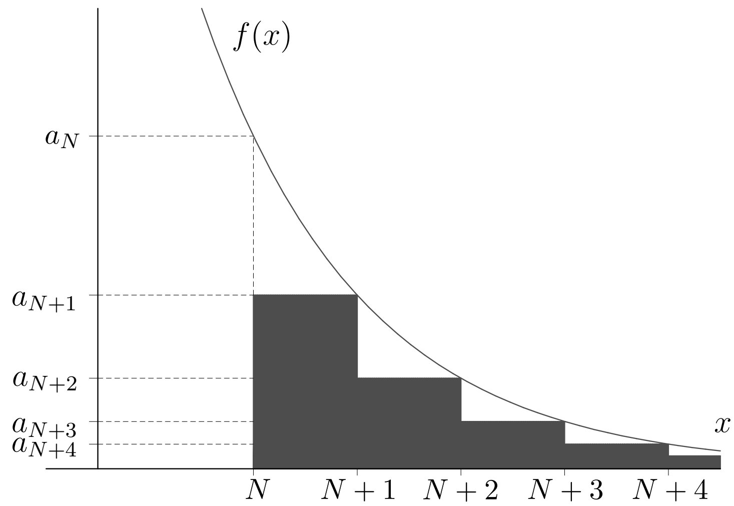
which can be rewritten as

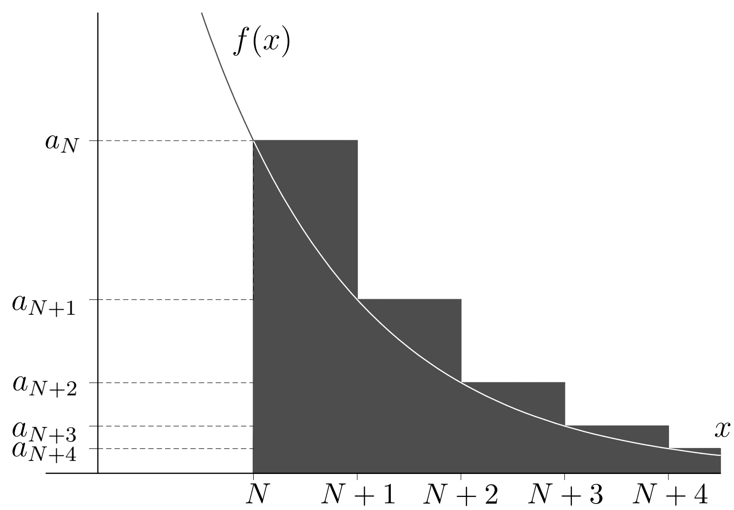
$$\sum_{n=N}^{\infty} a_n - a_N \le \int_N^{\infty} f(x)\,dx\,.$$

In the second picture, clearly the shaded area is bigger than the area under the graph of $f(x)$. This gives us the inequlity

$$\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} a_n\,.$$

We combine both results and we get the following result.





The integral test inequality:

Assume that $f(x)$ is a monotonically decreasing and positive function on the interval $N \le x < \infty$, and let $a_n = f(n)$ for integer n. Then

$$\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} a_n \le a_N + \int_N^{\infty} f(x)\,dx\,.$$

This gives us the third test for convergence.

> The integral test for convergence:
>
> Assume that $f(x)$ is a monotonically decreasing and positive function on the interval $N \leq x < \infty$, and let $a_n = f(n)$ for integer n. Then the following is true.
>
> If $\int_N^\infty f(x)\,dx$ is finite, the series $\sum_{n=N}^\infty a_n$ converges.
>
> If $\int_N^\infty f(x)\,dx$ is infinite, the series $\sum_{n=N}^\infty a_n$ does not converge.

Proof: We use the integral test inequality. If $\int_N^\infty f(x)\,dx$ is finite, the inequality

$$\sum_{n=N}^{\infty} a_n \leq a_N + \int_N^\infty f(x)\,dx$$

demonstrates that $\sum_{n=N}^\infty a_n$ is bounded from above by a finite number. As the numbers $a_n = f(n)$ are positive, this implies that the series converges.

If $\int_N^\infty f(x)\,dx$ is infinite, the inequality

$$\int_N^\infty f(x)\,dx \leq \sum_{n=N}^{\infty} a_n$$

demonstrates that $\sum_{n=N}^\infty a_n$ is infinite.

Keep in mind:

- All three tests – ratio test, root test and integral test – are applicable only to series with positive members, $a_n > 0$.
- The integral test requires, in addition, that $a_{n+1} \leq a_n$.

We will now investigate the convergence of the socalled 'harmonic series'

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

with the help of our three tests.

- Ratio test:

$$a_n = \frac{1}{n}, \quad a_{n+1} = \frac{1}{n+1}, \quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \quad \text{for } n \to \infty\,,$$

so the ratio test is indecisive.

- Root test:

$$a_n = \frac{1}{n}, \quad \sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n}} = \frac{1}{n^{1/n}}$$

Claim: $n^{1/n} \to 1 \quad \text{for } n \to \infty$.
Proof: For all real numbers x we have $x^{1/x} = e^{\ln\left(x^{1/x}\right)} = e^{\frac{1}{x}\ln x}$.
Sending $x \to \infty$ yields in the exponent an 'undetermined expression',

$$\frac{\ln x}{x} \to \frac{\infty}{\infty} \quad \text{for } x \to \infty\,,$$

which can be calculated with the rule of l'Hospital (or l'Hôpital):

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0\,.$$

This proves that, indeed,

$$x^{1/x} = e^{\frac{1}{x}\ln x} \to e^0 = 1 \quad \text{for } x \to \infty\,.$$

As a consequence,

$$\sqrt[n]{a_n} = \frac{1}{n^{1/n}} \to 1 \quad \text{for } n \to \infty\,,$$

so the root test is indecisive.

- Integral test:

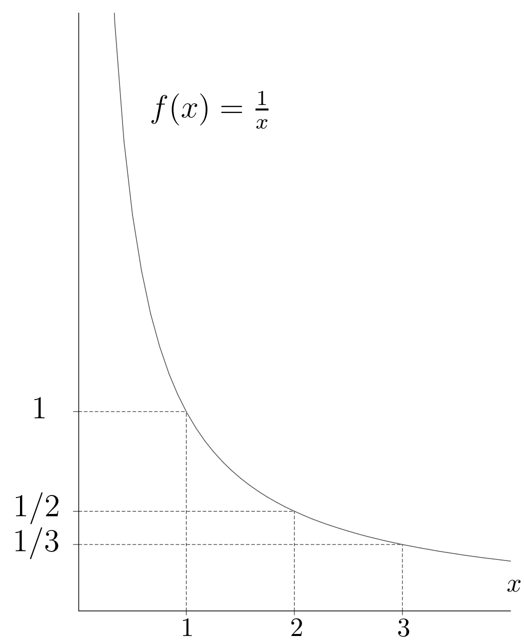
$$f(x) = \frac{1}{x}, \quad a_n = \frac{1}{n}$$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{x} = \big[\ln x\big]_1^\infty =$$

$$= \ln\infty - \ln 1 = \infty - 0 = \infty\,,$$

so the harmonic series diverges,

$$\boxed{\sum_{n=1}^\infty \frac{1}{n} = \infty\,.}$$



More generally, one can show with the integral test that

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \begin{cases} \text{converges if } p > 1\,, \\ \text{diverges if } p \le 1\,. \end{cases}$$

[For $p = 2$ see second worksheet.]

Remark: The *alternating* harmonic series converges,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 = 0.69\ldots$$

The harmonic and alternating harmonic series exemplify the following general statements:

- $\lim_{n\to\infty} a_n = 0$ is a necessary, but not a sufficient condition for convergence of $\sum_{n=n_0}^{\infty} a_n$.
- If $\sum_{n=n_0}^{\infty} |a_n|$ converges, so does $\sum_{n=n_0}^{\infty} a_n$. The converse is, in general, not true.

Applications:

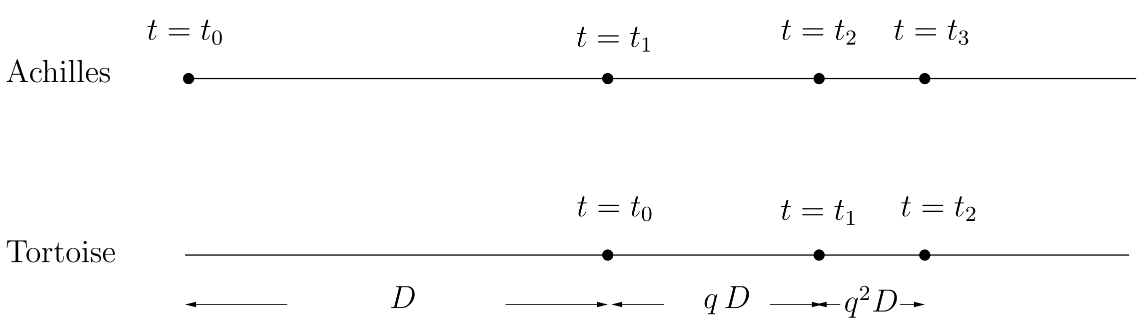
a) Paradox of Achilles and the tortoise [Zeno of Elea (ca. 490 – ca. 430 BC)]

(Probably the first occasion where an infinite series occurred in human thinking.)

Speed of Achilles: v

Speed of the tortoise: $q\,v\,, \quad 0 < q < 1$

The tortoise gets a head start of distance D:



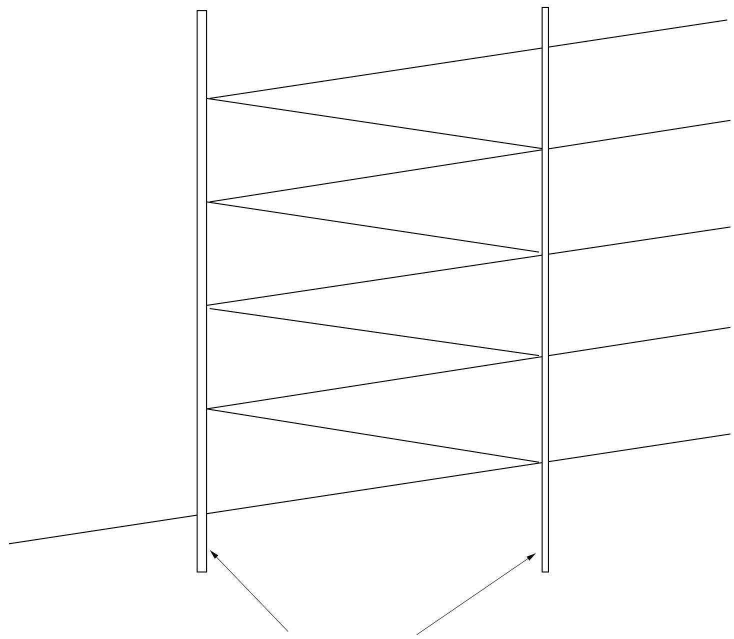
Whenever Achilles reaches the point where the tortoise was one time step before, the tortoise is ahead of him. From this (correct) observation Zeno (incorrectly) concludes that Achilles *never* reaches the tortoise.

Where is the error?

Zeno decomposes the interval until Achilles reaches the tortoise into infinitely many subintervals. He implicitly assumes that summing up these infinitely many subintervals gives infinity. This is wrong. An infinite series can have a finite sum. Here we get a geometric series. The time T at which Achilles reaches the tortoise can be calculated as

$$T = \frac{1}{v}\left(D + q\,D + q^2\,D + \ldots\right) = \frac{1}{v}\sum_{n=0}^{\infty} q^n\,D = \frac{D}{v(1-q)}\,.$$

b) Reflection at mirrors



At each reflection, a fraction r of the intensity is reflected, the rest is transmitted.

Intensity of first transmitted beam: I_1

Intensity of second transmitted beam: $I_2 = r^2 I_1$

Intensity of third transmitted beam: $I_3 = r^2 I_2 = r^4 I_1$

$$\dots$$

Intensity of n^{th} transmitted beam: $I_n = r^{2n} I_1$

$$\dots$$

so the intensities $I_1, I_2, \dots, I_n, \dots$ form a geometric sequence with common factor $q = r^2$.

[Note that summing up the resulting geometric series,

$$I = \sum_{n=1}^{\infty} I_n = \sum_{n=1}^{\infty} r^{2(n-1)} I_1 = \sum_{n=0}^{\infty} r^{2n} I_1 = \frac{I_1}{1 - r^2},$$

does *not* give the total intensity of all transmitted beams because of interference: Parts of the transmitted beams will cancel each other, where a wave crest of one beam falls on a wave trough of another.]

II. Taylor series

(Cf. FLAP M4.5)

II.1. Definition and basic properties of Taylor series

Goal:

- Approximate a function $f(x)$ by a polynomial

$$P_N(x) \;=\; c_0 \,+\, c_1\,x \,+\, c_2\,x^2 \,+\, \ldots \,+\, c_N x^N \;=\; \sum_{n=0}^{N} c_n x^n$$

  ('N^{th} order Taylor approximation').

- Then send $N \to \infty$ to get '(infinite) Taylor series' of $f(x)$.

First order Taylor approximation:

Approximate function $f(x)$ near $x = 0$ by first order polynomial

$$P_1(x) = c_0 + c_1 x\,.$$

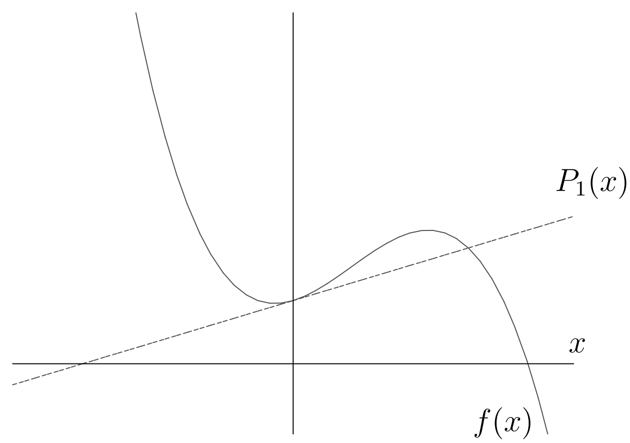
Graph of $P_1(x)$ is a straight line which is supposed to be tangent to the graph of $f(x)$ at $x = 0$:

$$P_1(x) = c_0 + c_1 x \;\Rightarrow\; P_1(0) = c_0 \overset{!}{=} f(0)$$

$$P_1'(x) = c_1 \;\Rightarrow\; P_1'(0) = c_1 \overset{!}{=} f'(0)$$

So the first order Taylor polynomial of $f(x)$ at $x = 0$ reads

$$\boxed{P_1(x) = f(0) + f'(0)x}$$

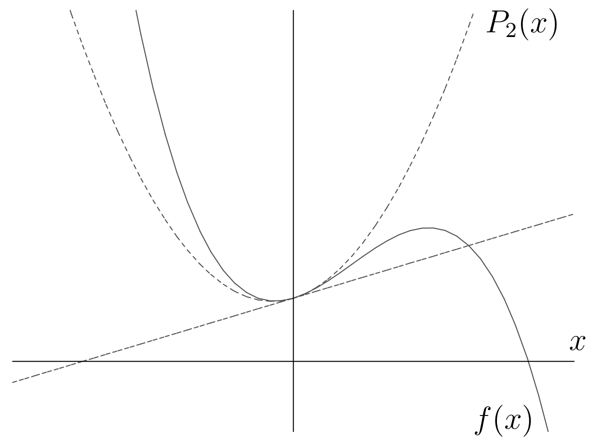


Second order Taylor approximation:

Approximate function $f(x)$ near $x = 0$ by second order polynomial

$$P_2(x) = c_0 + c_1 x + c_2 x^2 \,.$$

Graph of $P_2(x)$ is a parabola which is supposed to be as close to the graph of $f(x)$ near $x = 0$ as possible:



$$P_2(x) = c_0 + c_1 x + c_2 x^2 \;\Rightarrow\; P_2(0) = c_0 \stackrel{!}{=} f(0)$$

$$P_2'(x) = c_1 + 2c_2 x \;\Rightarrow\; P_2'(0) = c_1 \stackrel{!}{=} f'(0)$$

$$P_2''(x) = 2c_2 \;\Rightarrow\; P_2''(0) = 2c_2 \stackrel{!}{=} f''(0)$$

So the second order Taylor polynomial of $f(x)$ at $x = 0$ reads

$$\boxed{P_2(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2}}$$

By iterating this procedure we find that the N^{th} order Taylor polynomial of $f(x)$ at $x = 0$ reads

$$\boxed{P_N(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots + f^{(N)}(0)\frac{x^N}{N!} \;=\; \sum_{n=0}^{N} f^{(n)}(0)\frac{x^n}{n!}}$$

$f(x) \approx P_N(x)$ is a good approximation for small x. The higher N, the better the approximation.

Alternative notation for the nth derivative: $\quad f^{(n)}(0) \;=\; \left.\dfrac{d^n f(x)}{dx^n}\right|_{x\,=\,0}$.

Sending N to infinity gives the (infinite) Taylor series of $f(x)$ about $x = 0$:

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}\,.$$

For each value of x, this is a series of real numbers which can be studied with the methods of Section I. Convergence of this series can be investigated, e.g., with the ratio test, the root test or the integral test. Of course, the series may converge for some values of x and not converge for other values of x.

The idea behind Taylor series is that with increasing N the Nth order Taylor polynomial $P_N(x)$ becomes a better and better approximation of $f(x)$. So one wants to have that the limit of $P_N(x)$ for $N \to \infty$ gives the function $f(x)$. However, convergence of the Taylor series towards $f(x)$ is not guaranteed. The set of all x for which

$$f(x) \;=\; \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

is called the 'interval of convergence'. In Section II.2 we will give examples of functions where the interval of convergence is the whole real line and examples of functions where it is smaller.

Example:

Calculate the second order Taylor polynomial for $f(x) = \dfrac{1}{1+x}$ at $x = 0$:

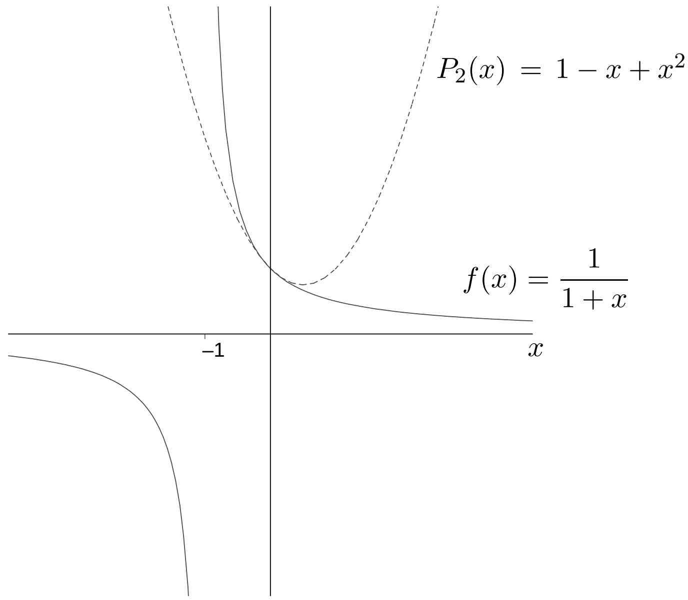
$$f(x) = \frac{1}{1+x} \;\Rightarrow\; f(0) = 1$$

$$f'(x) = \frac{-1}{(1+x)^2} \;\Rightarrow\; f'(0) = -1$$

$$f''(x) = \frac{2}{(1+x)^3} \;\Rightarrow\; f''(0) = 2$$

So the second order Taylor polynomial of $f(x) = \dfrac{1}{1+x}$ reads

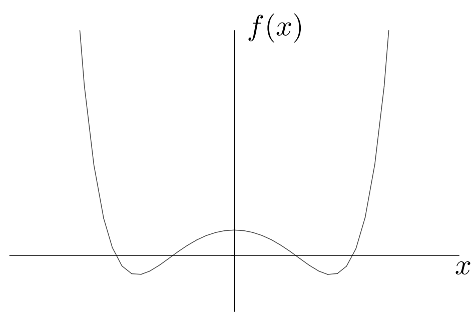
$$P_2(x) = 1 - x + x^2\,.$$



The Taylor series for even functions, $f(x) = f(-x)$ and for odd functions, $f(x) = -f(-x)$, have a simpler form than for arbitrary functions:

Even functions:

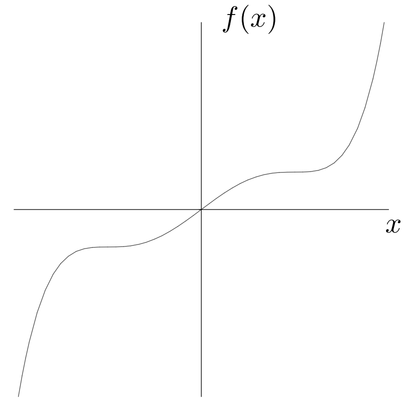
$$\begin{aligned} f(x) &= f(-x) \\ f'(x) &= -f'(-x) \;\Rightarrow\; \quad f'(0) = -f'(0) \;\Rightarrow\; f'(0) = 0 \\ f''(x) &= f''(-x) \\ f'''(x) &= -f'''(-x) \;\Rightarrow\; f'''(0) = -f'''(0) \;\Rightarrow\; f'''(0) = 0 \\ &\cdots \\ f^{(n)}(0) &= 0 \quad \text{for odd } n\,, \end{aligned}$$



so the Taylor series for an even function reduces to

$$f(0) + f''(0)\frac{x^2}{2} + f^{(4)}(0)\frac{x^4}{4!} + \ldots \;=\; \sum_{k=0}^{\infty} f^{(2k)}(0)\frac{x^{2k}}{(2k)!}\,.$$

Odd functions:



$$f(x) = -f(-x) \;\Rightarrow\; \quad f(0) = -f(0) \;\Rightarrow\; f(0) = 0$$
$$f'(x) = f'(-x)$$
$$f''(x) = -f''(-x) \;\Rightarrow\; f''(0) = -f''(0) \;\Rightarrow\; f''(0) = 0$$
$$f'''(x) = f'''(-x)$$
$$\dots$$
$$f^{(n)}(0) = 0 \quad \text{for even } n\,,$$

so the Taylor series for an odd function reduces to

$$f'(0)x + f'''(0)\frac{x^3}{3!} + f^{(5)}(0)\frac{x^5}{5!} + \dots = \sum_{k=0}^{\infty} f^{(2k+1)}(0)\frac{x^{2k+1}}{(2k+1)!}\,.$$

Taylor expansion of $f(x)$ about an arbitrary point $x = a$:

Define $y = x - a$ and $f(x) = g(y)$.
Then $f'(x) = g'(y)$, hence $f^{(n)}(x) = g^{(n)}(y)$ and thus $f^{(n)}(a) = g^{(n)}(0)$.
Write the Taylor expansion of $g(y)$ about $y = 0$:

$$g(y) \;=\; g(0) \;+\; g'(0)y \;+\; g''(0)\frac{y^2}{2} \;+\; g'''(0)\frac{y^3}{3!} \;+\; \dots \;=\; \sum_{n=0}^{\infty} g^{(n)}(0)\frac{y^n}{n!}\,.$$

After resubstitution, this gives the Taylor expansion of $f(x)$ about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!} + \dots = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}\,.$$

Example: Calculate the second order Taylor polynomial of $f(x) = \sin x$ about $x = \pi/4$:

$$f(x) = \sin x\,, \qquad f\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}$$
$$f'(x) = \cos x\,, \qquad f'\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}$$
$$f''(x) = -\sin x\,, \qquad f''\left(\tfrac{\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}$$

so

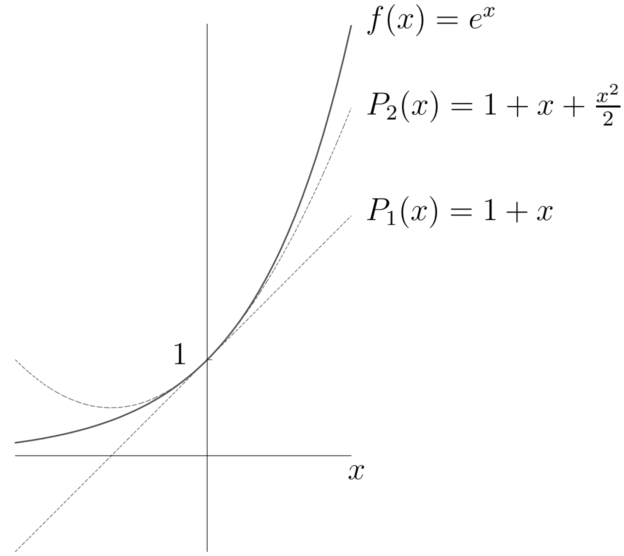
$$f(x) \;\approx\; \frac{1}{\sqrt{2}} \;+\; \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) \;-\; \frac{1}{\sqrt{2}}\,\frac{\left(x - \frac{\pi}{4}\right)^2}{2}$$

This is a good approximation if x is close to $\frac{\pi}{4}$.

II.2. Taylor series of some standard functions

a) Exponential function

$$
\begin{aligned}
f(x) &= e^x\,, & f(0) &= 1\\
f'(x) &= e^x\,, & f'(0) &= 1\\
f''(x) &= e^x\,, & f''(0) &= 1\\
&\dots & &\dots\\
f^{(n)}(x) &= e^x\,, & f^{(n)}(0) &= 1
\end{aligned}
$$



So the Taylor series of e^x reads

$$
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots
$$

One may use this Taylor series as the *definition* of the exponential function.

To make sure that the exponential function is well defined this way, one has to verify that the Taylor series converges for all real numbers x.

To demonstrate convergence of the series

$$
\sum_{n=0}^{\infty} a_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,,
$$

recall that convergence of $\sum_{n=0}^{\infty} |a_n(x)|$ implies convergence of $\sum_{n=0}^{\infty} a_n(x)$ (though not the other way round!). Hence, it suffices to demonstrate convergence for positive x. In this case the ratio test is applicable, and it gives convergence of the series, see third worksheet.

Hence, the Taylor series of the exponential function converges for all x.

Setting $x = 1$ in the Taylor series for e^x gives a series formula for e:

$$
e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots
$$

This can be used for calculating $e = 2.71828\dots$ numerically. One may use this series as the definition of the number e. There are several alternative ways of defining e, all of which are equivalent, e.g.

$$
e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n
$$

or

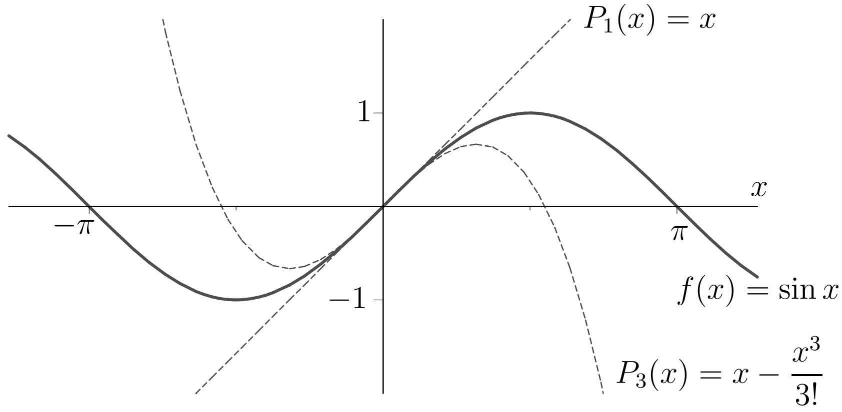
$$
e = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}\,.
$$

b) Sine function

$f(x) = \sin x$ is an odd function,

$$\sin(x) = -\sin(-x)\,,$$

so it is clear from the outset that only odd powers of x will occur in the Taylor series.



$$\begin{aligned}
f(x) &= \sin x\,, & f(0) &= 0\\
f'(x) &= \cos x\,, & f'(0) &= 1\\
f''(x) &= -\sin x\,, & f''(0) &= 0\\
f'''(x) &= -\cos x\,, & f'''(0) &= -1\\
f^{(4)}(x) &= \sin x\,, & f^{(4)}(0) &= 0\\
&\dots & &\dots\\
f^{(n+4)}(x) &= f^{(n)}(x)\,, & f^{(n+4)}(0) &= f^{(n)}(0)
\end{aligned}$$

Hence

$$f^{(n)}(0) \;=\; \begin{cases} 0 & \text{if} \quad n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if} \quad n = 1, 5, 9, 13, 17, \dots \\ -1 & \text{if} \quad n = 3, 7, 11, 15, 19, \dots \end{cases}$$

and the Taylor series for the sine function reads

$$\boxed{\sin x \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

This series converges for all x. [The proof is similar to that for the exponential function.]

The first order Taylor approximation

$$\boxed{\sin x \;\approx\; x}$$

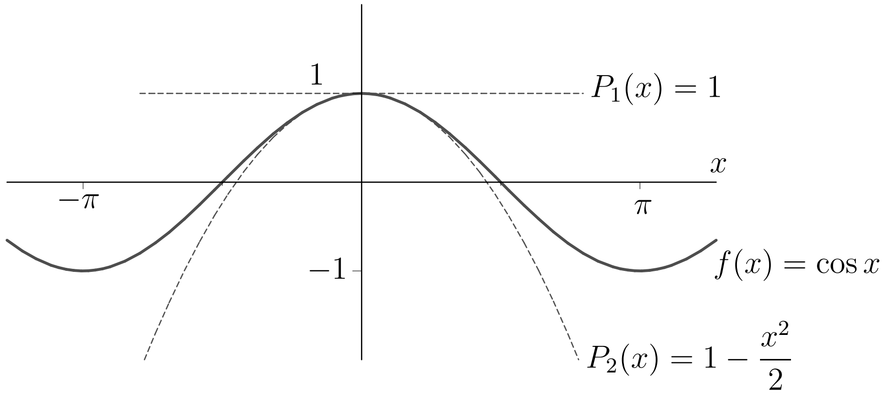
is often used in physics. It is a good approximation if $|x|$ is small.

c) Cosine function

$f(x) = \cos x$ is an even function,

$$\cos(x) = \cos(-x)\,,$$

so it is clear from the outset that only even powers of x will occur in the Taylor series.



For calculating the Taylor series for the cosine function we could use the same method as for the sine function. However, with the result for $\sin x$ already at hand, there is a quicker way:

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k)!} =$$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{d}{dx}\frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!}$$

where we used that the differentiation can be carried under the summation sign. [For finite sums this is obvious; for infinite series it is true if the series converges sufficiently nicely.]

From the last expression we find the Taylor series for the cosine function:

$$\boxed{\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots}$$

The series converges for all x. [Again, the proof is very similar to that for the exponential function.]

The second order Taylor approximation

$$\boxed{\cos x \approx 1 - \frac{x^2}{2}}$$

is often used in physics. It is a good approximation if $|x|$ is small.

d) Hyperbolic functions

For the Taylor series of $f(x) = \sinh x$ and $f(x) = \cosh x$ see third worksheet.

e) Binomial series

The Taylor series of the functions $f(x) = (1+x)^r$, where r is any real number, are called the 'binomial series'. (For the case that r is a positive integer they reduce to the standard binomial formula.)

$$
\begin{aligned}
f(x) &= (1+x)^r & f(0) &= 1 \\
f'(x) &= r\,(1+x)^{r-1} & f'(0) &= r \\
f''(x) &= r\,(r-1)\,(1+x)^{r-2} & f''(0) &= r\,(r-1) \\
f''(x) &= r\,(r-1)\,(r-2)\,(1+x)^{r-3} & f''(0) &= r\,(r-1)\,(r-2) \\
&\cdots & &\cdots \\
f^{(n)}(x) &= r\,(r-1)\,\cdots\,(r-n+1)\,(1+x)^{r-n} & f^{(n)}(0) &= r\,(r-1)\,\cdots\,(r-n+1)
\end{aligned}
$$

We define the binomial coefficients

$$\binom{r}{n} = \frac{r\,(r-1)\,\cdots\,(r-n+1)}{n!}$$

which some authors (e.g. FLAP) denote by ${}^rC_n$ instead.

Then the Taylor series for $(1+x)^r$ becomes

$$\boxed{(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + r\,x + r\,(r-1)\,\frac{x^2}{2!} + r\,(r-1)\,(r-2)\,\frac{x^3}{3!} + \ldots}$$

If r is a non-negative integer, this is a finite sum (i.e., a polynomial) because then

$$\binom{r}{n} = 0 \qquad \text{for } n \geq r+1\,.$$

For all other values of r it is an infinite series. This series converges for $-1 < x < 1$. [This can be proven with the ratio test, quite similarly as for the exponential function.]

The first order approximation

$$\boxed{(1+x)^r \approx 1 + r\,x}$$

is often used in physics. It is a good approximation if $|x|$ is small. E.g., one has

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{1}{2}\,x\,,$$

$$\frac{1}{\sqrt{1+x}^{\,3}} = (1+x)^{-3/2} \approx 1 - \frac{3}{2}\,x\,.$$

Applications:

a) Relativistic energy

In special relativity the energy of a particle with rest mass m that moves at speed v is

$$E = \frac{m\,c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where c is the vacuum speed of light. We want to get an approximation for E by Taylor expansion up to second order with respect to v/c.

We use the binomial series approximation $(1 + x)^r = 1 + r\,x + \ldots$. Setting $x = v^2/c^2$ we have

$$E = m\,c^2\left(1 - x\right)^{-1/2} = m\,c^2\big(1 + \frac{1}{2}\,x + \ldots\big) = m\,c^2 + m\,c^2\,\frac{1}{2}\,\frac{v^2}{c^2} + \ldots = m\,c^2 + \frac{m}{2}\,v^2 + \ldots$$

The first term is the rest energy, the second term is the non-relativistic kinetic energy. Neglecting the terms indicated by $\ldots$ gives a good approximation for E if v/c is sufficiently small.

b)Planck's radiation formula

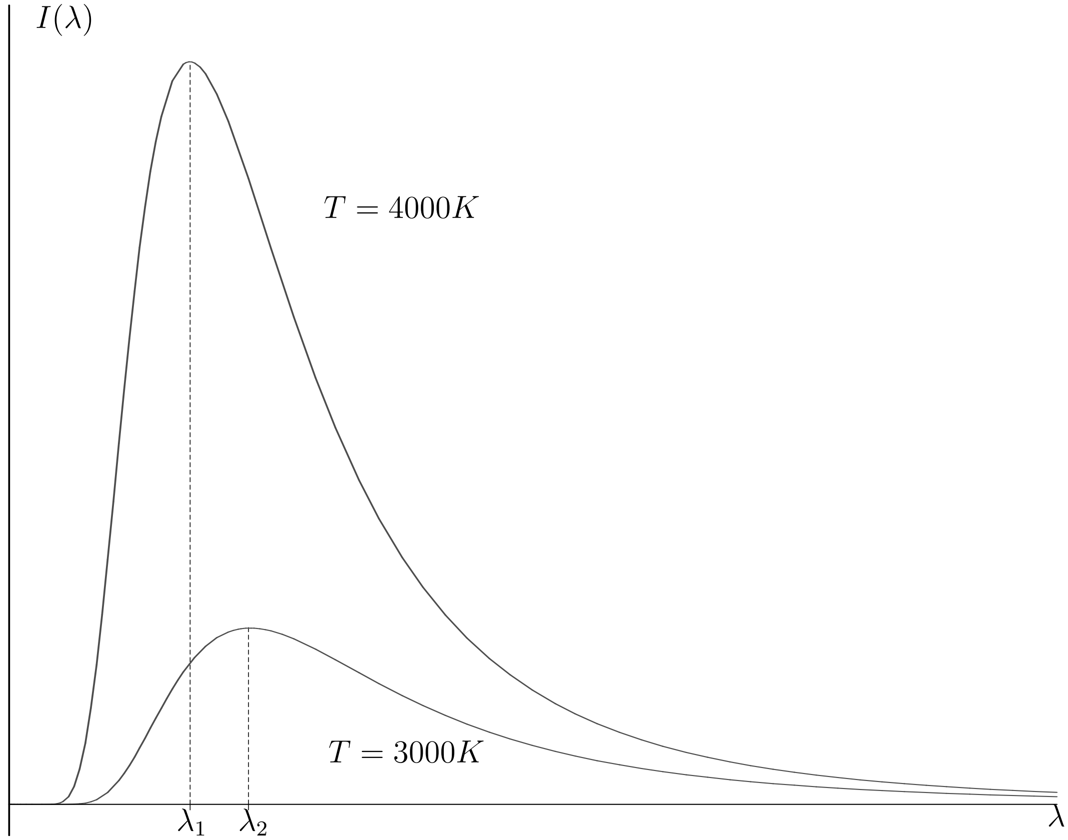
If a non-reflecting ("black") body is heated to temperature T, it emits electromagnetic radiation whose intensity is given, as a function of the wave length λ, by Planck's formula

$$I(\lambda) = \frac{2\,h\,c^2}{\lambda^5}\,\frac{1}{\left(e^{\frac{hc}{kT\lambda}} - 1\right)}$$

where h, c and k are constants of nature:

$h =$ Planck's constant,
$c =$ vacuum velocity of light,
$k =$ Boltzmann's constant.

The first picture on the next page shows the graph of $I(\lambda)$ for two different temperatures T. For the higher temperature the intensity is bigger over the whole wave length range, and the maximum is at a smaller wave length ("bluer").



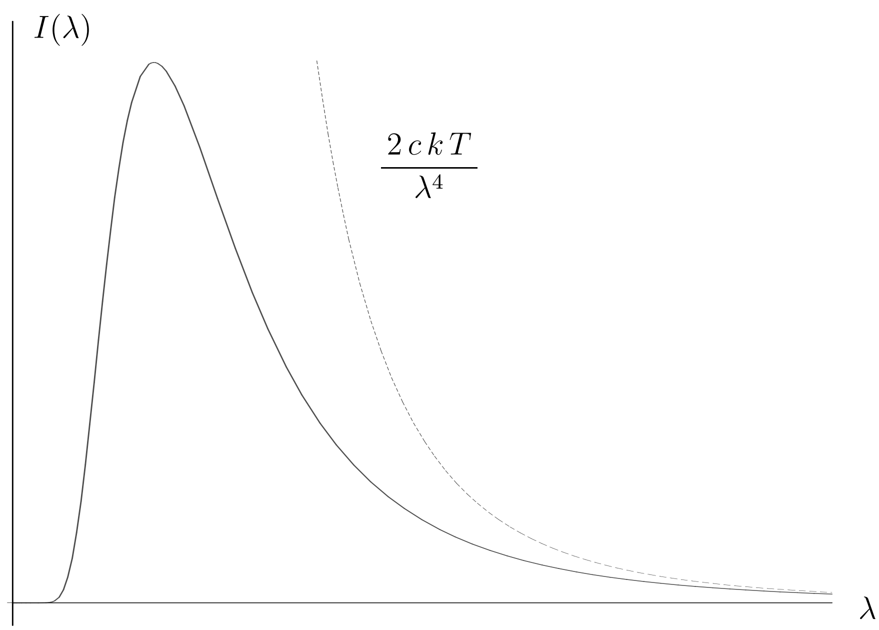
For large wave lengths λ, the dimensionless quantity $\dfrac{hc}{kT\lambda}$ is small.

We can then use a first order Taylor approximation:

$$e^{\frac{hc}{kT\lambda}} \approx 1 + \frac{hc}{kT\lambda},$$

$$I(\lambda) \approx \frac{2\,h\,c^2}{\lambda^5}\,\frac{k\,T\,\lambda}{h\,c} =$$

$$= \frac{2\,c\,k\,T}{\lambda^4}.$$



III. Ordinary differential equations

(Cf. FLAP M6.1, M6.2, M6.3)

III.1. Introducing differential equations

A differential equation is an equation for an unknown function which involves derivatives of the unknown function, e.g.

$$\frac{dy}{dx} + x\,y \;=\; x^2$$

is a differential equation for the unknown function $y(x)$;

$$u^5\,\frac{d^2u}{dt^2} + \cos u \;=\; 0$$

is a differential equation for the unknown function $u(t)$.

When dealing with differential equations it is recommendable to use the notation $\frac{d}{dx}$ for derivatives, rather than $(\,\cdot\,)'$ or $(\,\cdot\,)^{\cdot}$.

Distinguish:

- ordinary differential equation (ODE) $\leftrightarrow$ partial differential equation (PDE)

  In an ODE the unknown function depends on one variable only, e.g. $y(x)$; an ODE involves *ordinary* derivatives, e.g. $\frac{dy}{dx}$.

  In a PDE the unknown function depends on an least two variables, e.g. $y(x_1, x_2)$; a PDE involves *partial* derivatives, e.g. $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$.

- (single) differential equation $\leftrightarrow$ system of differential equations

  A (single) differential equation is one equation for one unknown function, e.g. $y(x)$.

  A system of differential equations consists of several equations for several unknown functions, e.g. $y_1(x), y_2(x), \ldots$

In PHYS113 we will be concerned only with (single) ordinary differential equations, i.e., we will not deal with systems of differential equations and not with partial differential equations.

The *order* of a differential equation is the order of the highest occurring derivative, e.g

$$\frac{dy}{dx} + x\,y \;=\; x^2$$

is a first order ODE for the unknown function $y(x)$;

$$u^5\,\frac{d^2u}{dt^2} + \cos u \;=\; 0$$

is a second order ODE for the unknown function $u(t)$.

Differential equations in physics:

- Newton's second law for a particle on the x-axis

  $$m\,\frac{d^2x}{dt^2} \;=\; F\left(x\,,\,\frac{dx}{dt}\,,\,t\right)$$

  is a second order ODE for $x(t)$.

- Newton's second law for a particle in 3-dimensional space

  $$m\,\frac{d^2}{dt^2}\boldsymbol{r} \;=\; \boldsymbol{F}\left(\boldsymbol{r}\,,\,\frac{d}{dt}\boldsymbol{r}\,,\,t\right)$$

  is a system of second order ODE for the three unknown functions $\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$.

- Schroedinger's equation in quantum mechanics and Maxwell's equations in electrodynamics are examples of partial differential equations.

III.2 Solving first order ODE

There is no universal method (no 'algorithm') for solving differential equations. In this section we discuss two methods of how to solve particular types of first order ODE.

First method: "Separation of variables"

Consider the first order ODE $\dfrac{dy}{dx} = x\,y$:

"Separate the variables" (i.e., y to the left-hand side and x to the right-hand side)

$$\frac{dy}{y} \;=\; x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx \qquad \text{(indefinite integrals)}$$

$$\ln y + C_1 = \frac{x^2}{2} + C_2$$

$$\text{With } C = C_2 - C_1 : \qquad \ln y = \frac{x^2}{2} + C$$

$$y(x) = e^{\frac{x^2}{2}+C} .$$

The *general* solution to a first order ODE involves an arbitrary constant C.

A particular choice for C gives a *particular* solution.

C can be fixed by prescribing an *initial condition*

$$y(0) = y_0 .$$

In our example the initial condition fixes C according to

$$y_0 = e^{0+C} = e^C ,$$

so

$$y(x) = y_0\, e^{\frac{x^2}{2}} .$$

Separation of variables is possible for a first order ODE whenever it can be brought into the form

$$\frac{dy}{dx} = q(x)\,p(y) .$$

In this case the first order ODE is called “separable”.

Two more examples:

- $\frac{dw}{ds} + w \sin s = 0$ is separable:

$$\frac{dw}{w} = -\sin s\, ds$$

$$\int \frac{dw}{w} = -\int \sin s\, ds$$

$$\ln w + C_1 = \cos s + C_2$$

$$\text{With } C = C_2 - C_1 : \qquad w(s) = e^{\cos s + C}$$

Check that the differential equation is, indeed, identically satisfied if this $w(s)$ is inserted!

- $\dfrac{dy}{dx} + x\,y = x^2$ is not separable; so we need another method.

Second method: "Method of integrating factor"

Consider a first order ODE of the form

$$\frac{dy}{dx} + g(x)\,y = h(x)\,.$$

This is called a *linear* first order ODE, because it is linear with respect to the unknown function y and its derivative dy/dx. (Note that the attribute 'linear' only refers to the dependent variable, here y, and its derivatives. The independent variable, here x, may enter into the differential equation in a non-linear way.)

We multiply the differential equation with a factor $I(x)$:

$$I(x)\,\frac{dy}{dx} + I(x)\,g(x)\,y = I(x)\,h(x)\,. \qquad (*)$$

This is true for any $I(x)$. If we choose $I(x)$ such that

$$\frac{d}{dx}I(x) = I(x)\,g(x)\,, \qquad (**)$$

$(*)$ can be rewritten as

$$\frac{d}{dx}\big(I(x)\,y\big) = I(x)\,h(x)\,,$$

so the left-hand side can be integrated,

$$I(x)\,y = \int I(x)\,h(x)\,dx\,.$$

For this reason, a function $I(x)$ that satisfies $(**)$ is called an "integrating factor" for our linear ODE. Solving the last equation for y gives the general solution

$$y(x) = \frac{1}{I(x)} \int I(x)\,h(x)\,dx\,.$$

Note that there is an arbitrary constant hidden in the indefinite integral.

Example: $\dfrac{dy}{dx} + \underbrace{x}_{g(x)}\, y = \underbrace{e^{-\frac{x^2}{2}}}_{h(x)}.$

The integrating factor is determined by (**) which, in our case, reads

$$\frac{dI(x)}{dx} = I(x)\, x\,,$$

$$\int \frac{dI}{I} = \int x\, dx\,,$$

$$\ln I = \frac{x^2}{2} + A\,.$$

We need just *one* function which does the job, so we can set the integration constant A equal to zero. This gives us the integrating factor

$$I(x) = e^{\frac{x^2}{2}}\,.$$

Upon multiplying our differential equation with this $I(x)$ we get

$$e^{\frac{x^2}{2}} \frac{dy}{dx} + e^{\frac{x^2}{2}}\, x\, y = e^{\frac{x^2}{2}}\, e^{-\frac{x^2}{2}}\,,$$

$$\frac{d}{dx}\big(e^{\frac{x^2}{2}}\, y \big) = 1\,,$$

$$e^{\frac{x^2}{2}}\, y = \int dx = x + C\,.$$

So the general solution to the differential equation is

$$y(x) = e^{-\frac{x^2}{2}}\,(\, x + C\,)\,.$$

Check that the differential equation is, indeed, identically satisfied if this $y(x)$ is inserted!

Summary of first order ODE, $F\big(\, x\,,\, y\,, \frac{dy}{dx}\big) = 0$:

- The general solution $y(x)$ involves one arbitrary constant.
- This constant can be fixed by choosing an initial condition $y(0) = y_0$.
- A first order ODE of the form
$$\frac{dy}{dx} = q(x)\, p(y)$$
can be solved by separation of variables.

- A first order ODE of the form

$$\frac{dy}{dx} + g(x)\, y = h(x)$$

can be solved by multiplying with an integrating factor $I(x)$, determined by $\frac{dI(x)}{dx} = I(x)\, g(x)$, which allows to directly integrate the left-hand side.

- Other types of first order ODE require more sophisticated methods not to be treated here.

Application:

Radioactive decay: $\frac{dN}{dt} = -\lambda\, N\,.$

$N =$ number of nuclei of the decaying substance

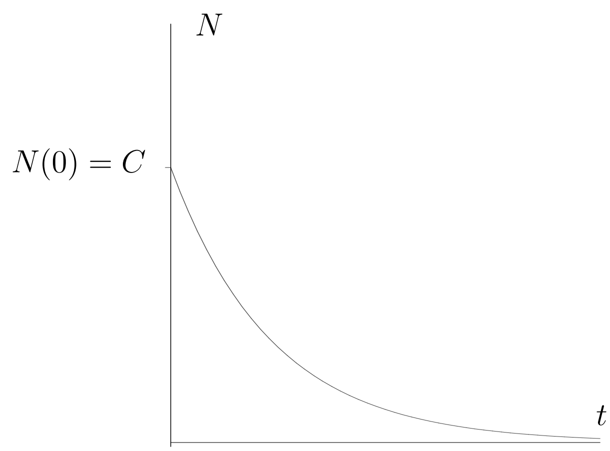
$\lambda =$ decay rate (= positive constant characteristic of the radioactive substance)

$$\frac{dN}{N} = -\lambda\, dt\,,$$

$$\int \frac{dN}{N} = -\lambda \int dt\,,$$

$$\ln N = -\lambda\, t + \ln C\,,$$

$$N(t) = C\, e^{-\lambda t}\,.$$



III.3 Solving second order ODE

General form of a second order ODE: $F\big(x\,,\,y\,,\,\dfrac{dy}{dx}\,,\,\dfrac{d^2y}{dx^2}\big) \;=\; 0\,.$

Here we will discuss two special cases of second order ODE.

First case: $F\big(x\,,\dfrac{dy}{dx}\,,\,\dfrac{d^2y}{dx^2}\big) \;=\; 0\,,$ i.e., y does not enter into the ODE; e.g.

$$\frac{d^2y}{dx^2} \;+\; \frac{1}{x}\frac{dy}{dx} \;=\; 0\,.$$

Then we can substitute

$$v \;=\; \frac{dy}{dx}\,, \qquad \frac{dv}{dx} \;=\; \frac{d^2y}{dx^2}\,.$$

This makes the second order ODE for $y(x)$ into a first order ODE for $v(x)$,

$$\frac{dv}{dx} \;+\; \frac{1}{x}\,v \;=\; 0\,,$$

which can be solved by separation of variables,

$$\int\frac{dv}{v} \;=\; -\int\frac{dx}{x}\,, \qquad \ln v \;=\; -\ln x \;+\; \ln C_1\,,$$

$$v \;=\; \frac{C_1}{x} \;=\; \frac{dy}{dx}\,, \qquad C_1\int\frac{dx}{x} \;=\; \int dy\,, \qquad C_1 \ln x \;=\; y \;+\; C_2\,.$$

So the general solution is

$$y(x) \;=\; C_1\,\mathrm{ln}x \;-\; C_2$$

where C_1 and C_2 are arbitrary constants. These constants can be fixed by choosing initial conditions $y\big|_{x=x_0} = y_0$ and $\dfrac{dy}{dx}\Big|_{x=x_0} = v_0$.

Reduction to a first order ODE by substituting $v = dy/dx$ is possible for any second order ODE of the form $F\big(x\,,\dfrac{dy}{dx}\,,\,\dfrac{d^2y}{dx^2}\big) \;=\; 0\,.$

It is a characteristic of all second order ODE that the general solution involves two arbitrary constants.

Second case: Linear second order ODE with constant coefficients,

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = h(x)\,, \qquad a \neq 0\,.$$

"linear" refers to the fact that the dependent variable and its derivatives (here y, dy/dx and d^2y/dx^2) enter linearly;

"with constant coefficients" refers to the fact that a, b and c are constants;

if $h(x) = 0$, the differential equation is called "homogeneous";

if $h(x) \neq 0$, the differential equation is called "inhomogeneous";

We consider the homogeneous equation first,

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = 0\,. \qquad (*)$$

Observation: If $y_1(x)$ and $y_2(x)$ are solutions to $(*)$, so is $y(x) = C_1\,y_1(x) + C_2\,y_2(x)$, where C_1 and C_2 are arbitrary constants. [This follows immediately by inserting this expression into the differential equation and using the linearity with respect to y and its derivatives.] Hence, if $y_1(x)$ and $y_2(x)$ are linearly independent (which means that one isn't just a constant multiple of the other), we get the general solution by taking linear combinations of them. So our goal is to find two linearly independent solutions to $(*)$.

Ansatz: $y(x) = e^{px}$. Then $(*)$ yields $a\,p^2\,\cancel{e^{px}} + b\,p\,\cancel{e^{px}} + c\,\cancel{e^{px}} = 0$, hence

$$p^2 + \frac{b}{a}\,p + \frac{c}{a} = 0\,.$$

This quadratic equation is called the "characteristic equation" or the "auxiliary equation" of $(*)$. It has two solutions,

$$p_{1/2} = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\,.$$

Case A : $\dfrac{b^2}{4a^2} - \dfrac{c}{a} > 0\,.$

In this case p_1 and p_2 are real, and $p_1 \neq p_2$.

Our ansatz gives us two linearly independent solutions

$$y_1(x) = e^{p_1 x} \qquad \text{and} \qquad y_2(x) = e^{p_2 x}\,.$$

So in this case the general solution to $(*)$ is

$$y(x) = C_1\,y_1(x) + C_2\,y_2(x) = C_1\,e^{p_1 x} + C_2\,e^{p_2 x}\,.$$

Case B : $\dfrac{b^2}{4a^2} - \dfrac{c}{a} = 0$.

In this case $p_1 = p_2 = -\dfrac{b}{2a}$.

Our ansatz gives us only one solution,

$$y_1(x) = e^{-\frac{bx}{2a}} .$$

It is easy to verify that a second, linearly independent, solution is

$$y_2(x) = x\, e^{-\frac{bx}{2a}} .$$

So in this case the general solution to $(*)$ is

$$y(x) = \big(C_1 + x\, C_2 \big)\, e^{-\frac{bx}{2a}} .$$

Case C : $\dfrac{b^2}{4a^2} - \dfrac{c}{a} < 0$.

In this case p_1 and p_2 are complex. [For complex numbers see PHYS114 .]

So our ansatz does not give any (real) solution.

In this case the general solution turns out to be

$$y(x) = e^{-\frac{bx}{2a}} \Big(C_1 \cos(\Omega x) + C_2 \sin(\Omega x) \Big) \quad \text{where } \Omega^2 = \frac{c}{a} - \frac{b^2}{4a^2} \; (> 0)$$

[Verify that this expression, indeed, satisfies $(*)$ identically !]

Examples:

- $\dfrac{d^2u}{dt^2} + 6\,\dfrac{du}{dt} - 7\,u = 0$.

  The ansatz $u(t) = e^{pt}$ gives the characteristic (or auxiliary) equation

  $$p^2 + 6\,p - 7 = 0 , \qquad p_{1/2} = -3 \pm \sqrt{9 + 7} = \begin{cases} +1 \\ -7 \end{cases}$$

  So the general solution is

  $$u(t) = C_1\, e^{t} + C_2\, e^{-7t} .$$

- $\dfrac{d^2x}{dt^2} + \omega^2\, x = 0$, (“harmonic oscillator”, “simple harmonic motion”)

  This is of the form of $(*)$ (where y has to be replaced by x and x has to be replaced by t), with $a = 1$, $b = 0$, $c = \omega^2$.

As $\omega^2 > 0$, we are in case C, so the general solution is

$$x(t) = e^{-\frac{bt}{2a}}\Big(C_1\cos(\Omega t) + C_2\sin(\Omega t)\Big) \quad \text{where } \Omega^2 = \frac{c}{a} - \frac{b^2}{4a^2}\,.$$

Inserting the above values for a, b and c we find that the general solution for the harmonic oscillator equation is

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)\,.$$

We now turn to the inhomogeneous equation

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = h(x)\,. \qquad (**)$$

Let

- $y_c(x)$ be the general solution to the homogeneous equation $(*)$ [in this situation $y_c(x)$ is sometimes called the "complementary function" of $(**)$; note that it involves two arbitrary constants];
- $y_p(x)$ be one particular solution to the inhomogeneous equation $(**)$.

Then

$$y(x) = y_c(x) + y_p(x)$$

is the general solution to the inhomogeneous equation $(**)$. [This is true because $y(x)$ obviously satisfies $(**)$ identically and involves two arbitrary constants, hidden in $y_c(x)$.]

We have learned above how to find $y_c(x)$. By contrast, there is no methodical way of how to determine $y_p(x)$. So $y_p(x)$ must be found for each case by "intelligent guesswork".

Example: $\dfrac{d^2u}{dt^2} + 6\,\dfrac{du}{dt} - 7\,u = 7\,.$

Here the general solution to the homogeneous equation is known from above,

$$u_c(t) = C_1\,e^t + C_2\,e^{-7t}\,.$$

A particular solution to the inhomogeneous equation can be found by guessing (and checking that it does the job!),

$$u_p(t) = -1\,.$$

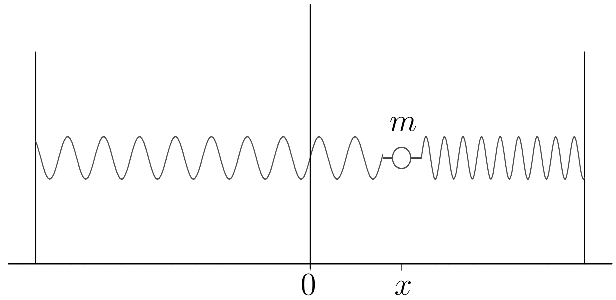
So the general solution is

$$u(t) = u_c(t) + u_p(t) = C_1\,e^t + C_2\,e^{-7t} - 1\,.$$

Application:

Consider a mass between two springs, subject to an additional driving force.
Newton's second law:

$$m\,\frac{d^2x}{dt^2} \;=\; \underbrace{-\,k\,x}_{\text{force of springs}} \;+\; \underbrace{f_0\cos(\Omega_0 t)}_{\text{external driving force}}$$



where m is the particle's mass, k is the spring constant, and f_0 and Ω_0 are constants.

If rewritten as

$$\frac{d^2x}{dt^2} \;+\; \frac{k}{m}\,x \;=\; \frac{f_0}{m}\cos(\Omega_0 t)\,,$$

the differential equation is of the form (∗∗) (where y has to be replaced by x and x has to be replaced by t) with

$$a \;=\; 1\,, \quad b \;=\; 0\,, \quad c \;=\; \frac{k}{m}\,, \quad h(t) \;=\; \frac{f_0}{m}\cos(\Omega_0 t)\,.$$

The general solution to the homogeneous equation (harmonic oscillator equation) is known from above,

$$x_c(t) \;=\; C_1\cos(\Omega t) \;+\; C_2\sin(\Omega t) \quad \text{where } \Omega^2 \;=\; \frac{k}{m}\,.$$

A particular solution to the inhomogeneous equation can be found with the ansatz $x_p(t) = \alpha\cos(\Omega_0 t)$. Upon inserting this into the inhomogeneous equation we get

$$\alpha\left(\frac{k}{m} \;-\; \Omega_0^2\right) \;=\; \frac{f_0}{m}\,,$$

hence

$$x_p(t) \;=\; \frac{f_0}{m\left(\frac{k}{m} \;-\; \Omega_0^2\right)}\cos(\Omega_0 t)\,.$$

So the general solution for the driven oscillator is

$$x(t) \;=\; x_c(t) \;+\; x_p(t) \;=\; C_1\cos(\Omega t) \;+\; C_2\sin(\Omega t) \;+\; \frac{f_0}{m\left(\frac{k}{m} \;-\; \Omega_0^2\right)}\cos(\Omega_0 t)\,.$$

The constants C_1 and C_2 can be fixed by choosing initial conditions $x(0) = x_0$ and $\frac{dx}{dt}(0) = v_0$.

In the "resonance case" $\Omega_0^2 \;=\; k/m$ the amplitude becomes infinite. This is known as the "resonance catastrophe". If friction is taken into account, we get a damping term into the differential equation (i.e., a term proportional to dx/dt). Then even in the resonance case the amplitude does not become infinite, although it may become very large. If you drive an oscillator with the resonance frequency, $\Omega_0 \;=\; \sqrt{k/m}$, the amplitude can become so large that it may even destroy the system.