PHYS274 Classical Fields

Lent Term 2010

Volker Perlick

Email: v.perlick@lancaster.ac.uk

Office hour: Wed 16-17, C35

Recommended Reading:

- D. J. Griffiths: "Introduction to Electrodynamics"
  Prentice Hall, 1999
  (covers all the material of the lectures except plasmas)
- F. F. Chen: "Introduction to Plasma Physics"
  Plenum, 1974
  (parts of Chapters 3 and 4 cover relevant material on plasmas)

I. Maxwell's equations

We put Maxwell's equations axiomatically at the beginning. We will then *derive* all electromagnetic phenomena to be discussed from Maxwell's equations.

I.1 Maxwell's equations in differential form

In SI units, Maxwell's equations read

$$
\boxed{
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 && \text{(MI)} \\
\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} &= \boldsymbol{0} && \text{(MII)} \\
\boldsymbol{\nabla} \cdot \boldsymbol{D} &= \rho && \text{(MIII)} \\
\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\partial}{\partial t}\boldsymbol{D} &= \boldsymbol{J} && \text{(MIV)}
\end{aligned}
}
$$

Here
$\rho =$ electric charge density (scalar field, depending on $\boldsymbol{r}$ and t),
$\boldsymbol{J} =$ electric current density (vector field, depending on $\boldsymbol{r}$ and t).
$\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are vector fields, depending on $\boldsymbol{r}$ and t.

	traditional names	better names
$\boldsymbol{E}$	electric field strength	electric field strength
$\boldsymbol{B}$	magnetic induction	magnetic field strength
$\boldsymbol{D}$	electric displacement	electric excitation
$\boldsymbol{H}$	magnetic field strength	magnetic excitation

The names in the right-hand column are more systematic:

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ produce the force onto a charged particle, see the Lorentz force equation below, so they should be called the "field strengths".

The fields $\boldsymbol{D}$ and $\boldsymbol{H}$ couple to the sources ρ and $\boldsymbol{J}$, see (MIII) and (MIV), so they should be called the "excitations" (meaning the fields that are "excited" by the sources).

Also, in a relativistic formulation $\boldsymbol{E}$ and $\boldsymbol{B}$ are combined into one geometric quantity (the "field strength tensor"), and $\boldsymbol{D}$ and $\boldsymbol{H}$ are combined into one geometric quantity (the "excitation tensor").

If the sources ρ and $\boldsymbol{J}$ are given, Maxwell's equations give $1+3+1+3 = 8$ scalar differential equations for the $3+3+3+3 = 12$ components of $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$. So one needs additional equations to determine the fields. These additional equations are the socalled *constitutive equations.*

I.2 Constitutive equations

Constitutive equations characterise a particular medium. In the simplest case ("linear isotropic medium") they are of the form

$$\boxed{\begin{aligned} \boldsymbol{D}(\boldsymbol{r}\,,\,t\,) &= \varepsilon_0\,\varepsilon_r(\,\boldsymbol{r}\,,\,t\,)\,\boldsymbol{E}(\,\boldsymbol{r}\,,\,t\,) && \text{(CI)} \\ \boldsymbol{B}(\,\boldsymbol{r}\,,\,t\,) &= \mu_0\,\mu_r(\,\boldsymbol{r}\,,\,t\,)\,\boldsymbol{H}(\,\boldsymbol{r}\,,\,t\,) && \text{(CII)} \end{aligned}}$$

ε_0 and μ_0 are constants of nature. In SI units:

$\varepsilon_0 =$ absolute permittivity $= 8.85 \cdot 10^{-12}\,\frac{C^2}{N\,m^2}$

$\mu_0 =$ absolute permeability $= 1.26 \cdot 10^{-6}\,\frac{N}{A^2}$

$c = 1/\sqrt{\varepsilon_0\,\mu_0}$ has the dimension of a velocity and turns out to be the vacuum velocity of light (see discussion of electromagnetic waves below).

ε_r and μ_r are dimensionless scalar quantities that may depend on $\boldsymbol{r}$ and t. Their values (at $\boldsymbol{r}$ and t) characterise the medium (at $\boldsymbol{r}$ and t).

$\varepsilon_r =$ relative permittivity

$\mu_r =$ relative permeability.

Vacuum is characterised by constitutive equations (CI) and (CII) with $\varepsilon_r = 1$ and $\mu_r = 1$.

Note that (CI) and (CII) are valid only for the simplest kind of media.

In general, constitutive equations can be much more complicated, e.g.

- in *anisotropic* media, $\boldsymbol{D}$ is in general not parallel to $\boldsymbol{E}$ and $\boldsymbol{B}$ is in general not parallel to $\boldsymbol{H}$ (in crystals, for instance, the scalar quantity ε_r must be replaced by a matrix);
- $\boldsymbol{D}$ and $\boldsymbol{B}$ at some instant of time t may depend on $\boldsymbol{E}$ and $\boldsymbol{H}$ at earlier times (in a ferromagnet, for instance, $\boldsymbol{B}$ at some instant of time depends on what has been done to the ferromagnet in the past; such media are called "memory materials" or "hysteretic materials").

In the following we will restrict to simple media with constitutive equations of the form (CI) and (CII).
If ρ and $\boldsymbol{J}$ are given, and if the constitutive equations are known, Maxwell's equations give a system of differential equations for $\boldsymbol{E}$ and $\boldsymbol{B}$.

I.3 Lorentz force

An electromagnetic field with electric field strength $\boldsymbol{E}$ and magnetic field strength $\boldsymbol{B}$ exerts the *Lorentz force*

$$\boxed{\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)}$$

onto a charged particle with charge q and velocity $\boldsymbol{v}$. This can be used for measuring $\boldsymbol{E}$ and $\boldsymbol{B}$: Measure the force onto a charge with $\boldsymbol{v} = \boldsymbol{0}$ to determine $\boldsymbol{E}$ and measure the forces onto charges with different $\boldsymbol{v}$ to determine $\boldsymbol{B}$.
If the mass of the charged particle is denoted m, the Lorentz force gives a second-order differential equation

$$m\,\ddot{\boldsymbol{r}}(t) = q\Big(\boldsymbol{E}\big(\boldsymbol{r}(t),t\big) + \dot{\boldsymbol{r}}(t) \times \boldsymbol{B}\big(\boldsymbol{r}(t),t\big)\Big)$$

for the particle's trajectory

$$\boldsymbol{r}(t) = x(t)\,\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}} + z(t)\,\hat{\boldsymbol{z}}\,.$$

Example: Consider a charged particle in a constant electric field,

$$\boldsymbol{B} = \boldsymbol{0}\,, \qquad \boldsymbol{E} = E\,\hat{\boldsymbol{z}} \qquad \text{with} \quad E = \text{constant}\,,$$

with initial conditions

$$\boldsymbol{r}(0) = \boldsymbol{0}\,, \qquad \dot{\boldsymbol{r}}(0) = v_0\,\hat{\boldsymbol{x}}\,.$$

Then the three components of the Lorentz force equation

$$m\,\ddot{\boldsymbol{r}}(t) = q\,E\,\hat{\boldsymbol{z}}$$

read

$$\ddot{x}(t) = 0\,, \qquad \ddot{y}(t) = 0\,, \qquad \ddot{z}(t) = \frac{qE}{m}\,.$$

Upon integrating twice,

$$x(t) = \alpha\,t + \delta\,, \qquad y(t) = \beta\,t + \xi\,, \qquad z(t) = \frac{qE}{2m}t^2 + \gamma\,t + \eta$$

with integration constants $\alpha\,,\,\beta\,,\,\gamma\,,\,\delta\,,\,\xi\,,\,\eta$. Matching to the initial conditions yields

$$\alpha = v_0\,, \qquad \beta = \gamma = \delta = \xi = \eta = 0$$

and thus

$$\boldsymbol{r}(t) = v_0\,t\,\hat{\boldsymbol{x}} + \frac{qE}{2m}t^2\,\hat{\boldsymbol{z}}\,.$$

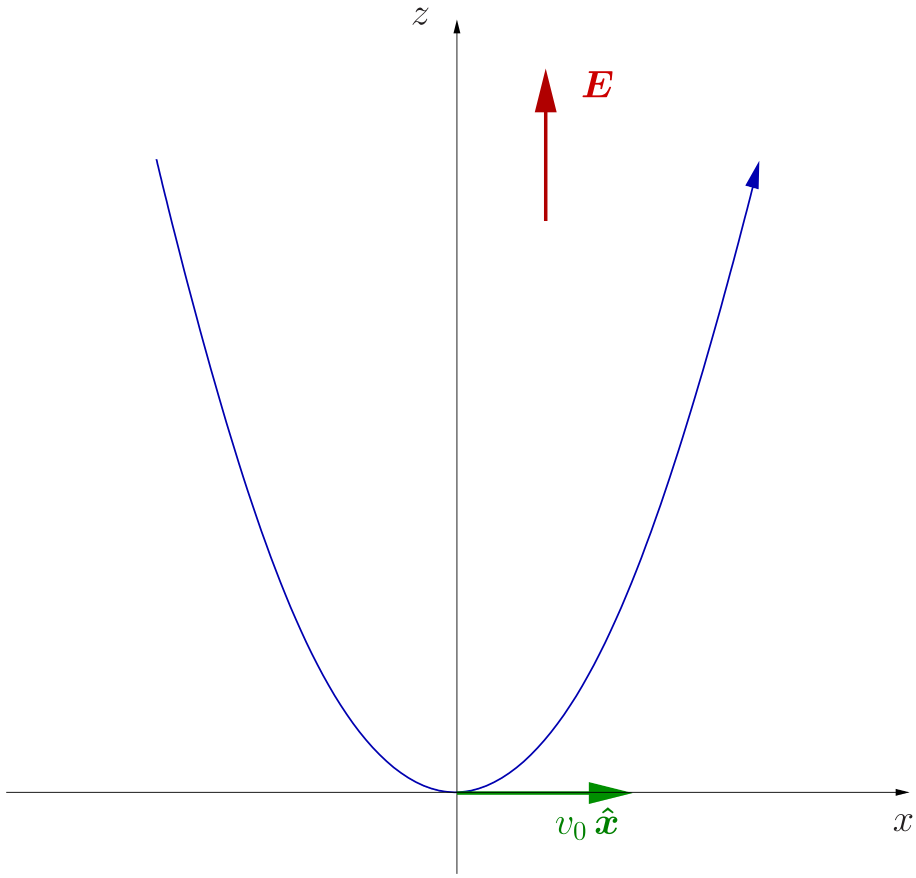
The particle's trajectory is a parabola, see next page.

For the motion of a charged particle in a magnetic field see 1st worksheet.

Note that in a magnetic field the Lorentz force $\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$ is perpendicular to the velocity. With $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ this implies

$$\frac{d}{dt}\,|\dot{\boldsymbol{r}}(t)|^2 = \frac{d}{dt}\big(\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t)\big) = 2\,\dot{\boldsymbol{r}}(t)\cdot\ddot{\boldsymbol{r}}(t) = 0\,.$$

In a magnetic field, the magnitude of the velocity of a charged particle is constant, i.e., the magnetic field does no work.



Parabolic trajecory of a charged particle in a constant electric field.

Note: The traditional Lorentz force equation considered here is actually only a non-relativistic approximation, valid as long as the particle's velocity is small in comparison to the velocity of light.

I.4 Maxwell's equations in integral form

For general theoretical investigations one usually considers Maxwell's equations in differential form, as given above. However, rewriting Maxwell's equations in integral form is sometimes helpful when investigating problems with symmetry. (Adapt the integration domain to the symmetry!) Also, experimentalist often prefer the integral form because measuring $\boldsymbol{E}$, $\boldsymbol{B}$ and other fields *at a point* is an idealisation: Any real measurement involves an integration over some finite domain, because every measuring device has a finite extension. We now rewrite the four Maxwell equations, one by one, in integral form and interpret the resulting equations.

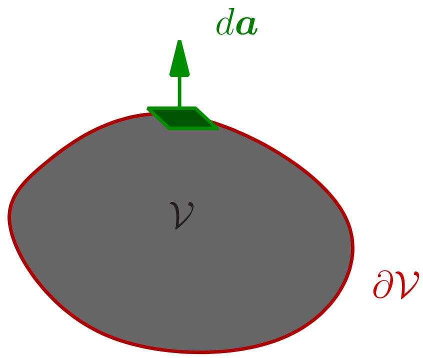
(MI) Integration of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ over a volume $\mathcal{V}$ yields

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{B} \, d\tau = 0$$

where $d\tau$ is the volume element. With the Gauss theorem we get the first Maxwell equation in integral form:

$$\boxed{\int_{\partial\mathcal{V}} \boldsymbol{B} \cdot d\boldsymbol{a} = 0}$$

where $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$ (a closed surface, see figure), and $d\boldsymbol{a}$ is the vectorial area element (by convention outward-pointing). This equation expresses the fact that the flow of $\boldsymbol{B}$ through any



closed surface is zero, i.e., that the magnetic field has no sources. (A modification of the first Maxwell equation has been suggested where the right-hand side of (MI) is not zero. This would correspond to the existence of "magnetic sources", called *magnetic monopoles*. However, up to now there is no experimental indication that they exist.)

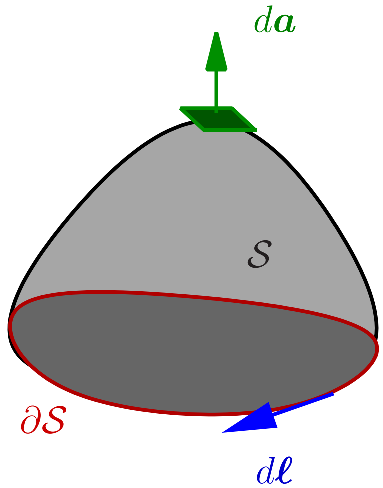
(MII) Integration of $\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = \boldsymbol{0}$ over a (non-closed) surface $\mathcal{S}$ yields

$$\int_{\mathcal{S}} \big(\boldsymbol{\nabla} \times \boldsymbol{E} \big) \cdot d\boldsymbol{a} = - \int_{\mathcal{S}} \Big(\frac{\partial}{\partial t} \boldsymbol{B} \Big) \cdot d\boldsymbol{a} \,,$$

where $d\boldsymbol{a}$ denotes, as before, the vectorial area element. On the left-hand side we use the Stokes theorem, on the right-hand side we commute differentiation with integration. This gives the second Maxwell equation in integral form:

$$\boxed{\int_{\partial\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{a} \,.}$$

Here $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$ (a closed curve), see picture. $d\boldsymbol{\ell}$ is



the vectorial length element along $\partial\mathcal{S}$. The usual sign conventions are: If the thumb of your right hand points in the direction of $d\boldsymbol{a}$, the remaining four fingers point in the direction of $d\boldsymbol{\ell}$. – This equation says that the temporal change of the flow of $\boldsymbol{B}$ through a non-closed surface equals, up to sign, the circulation of $\boldsymbol{E}$ in the boundary of this surface ("Faraday's law").

(MIII) Integration of $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ over a volume $\mathcal{V}$ yields

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{D} \, d\tau = \int_{\mathcal{V}} \rho \, d\tau \,.$$

Rewriting the left-hand side with the Gauss theorem yields the third Maxwell equation in integral form:

$$\boxed{\int_{\partial\mathcal{V}} \boldsymbol{D} \cdot d\boldsymbol{a} = \int_{\mathcal{V}} \rho \, d\tau \,.}$$

This equation expresses the fact that the flow of $\boldsymbol{D}$ through any *closed* surface is equal to the total charge surrounded by this surface ("Gauss law").

(MIV) Integration of $\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\partial}{\partial t}\boldsymbol{D} = \boldsymbol{J}$ over a (non-closed) surface $\mathcal{S}$ yields

$$\int_{\mathcal{S}} \big(\boldsymbol{\nabla} \times \boldsymbol{H} \big) \cdot d\boldsymbol{a} = \int_{\mathcal{S}} \big(\frac{\partial}{\partial t} \boldsymbol{D} \big) \cdot d\boldsymbol{a} + \int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{a} \, .$$

On the left-hand side we use the Stokes theorem, on the right-hand side we commute differentiation with integration. This gives the fourth Maxwell equation in integral form:

$$\boxed{\int_{\partial\mathcal{S}} \boldsymbol{H} \cdot d\boldsymbol{\ell} = \frac{d}{dt} \int_{\mathcal{S}} \boldsymbol{D} \cdot d\boldsymbol{a} + \int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{a} \, .}$$

This equation says that the flow of $\boldsymbol{J}$ through a non-closed surface plus the temporal change of the flux of $\boldsymbol{D}$ through this surface equals the circulation of $\boldsymbol{H}$ in the boundary of this surface ("Ampère-Maxwell law").

The integral form of Maxwell's equations is equivalent to the differential form: From the integral form the differential form can be recovered by choosing the integration domain arbitrarily small.

Summary: The basic equations of electrodynamics are

- Maxwell's equations (in differential or in integral form);
- the constitutive equations (depending on the medium);
- the Lorentz force equation.

From this basic set of equations all electrodynamic phenomena can be derived.

II. Conservation laws

In this section we derive two important conservation laws from Maxwell's equations: Charge conservation and energy conservation.

II.1 Charge conservation

The third and fourth Maxwell equation

$$\nabla \cdot \boldsymbol{D} = \rho \qquad \text{(MIII)}$$
$$\nabla \times \boldsymbol{H} - \frac{\partial}{\partial t}\boldsymbol{D} = \boldsymbol{J} \qquad \text{(MIV)}$$

imply

$$\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{D} = \frac{\partial}{\partial t}\rho\,,$$
$$\nabla \cdot (\nabla \times \boldsymbol{H}) - \nabla \cdot \left(\frac{\partial}{\partial t}\boldsymbol{D}\right) = \nabla \cdot \boldsymbol{J}\,.$$

Partial derivatives commute (if applied to continuously differentiable functions). This implies

$$\nabla \cdot \left(\frac{\partial}{\partial t}\boldsymbol{D}\right) = \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{D}$$

and, after a quick calculation,

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = 0\,.$$

We have thus found

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla \cdot \boldsymbol{J} = 0\,.}$$

This is the law of charge conservation in differential form (continuity equation). Integration over a volume $\mathcal{V}$ yields

$$\int_{\mathcal{V}} \frac{\partial}{\partial t}\rho\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \boldsymbol{J}\, d\tau$$

On the left-hand side we commute differentiation and integration, on the right-hand side we use the Gauss theorem:

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\tau \;=\; -\int_{\partial\mathcal{V}} \boldsymbol{J}\cdot d\boldsymbol{a}\,.}$$

This is the law of charge conservation in integral form: The temporal change of the total charge in a volume $\mathcal{V}$ equals the negative flow of $\boldsymbol{J}$ through the boundary of $\mathcal{V}$. (The minus sign ensures that the charge decreases if the flow is outward.)

Note that charge conservation holds in any medium; the constitutive equations have not been used.

Historical note: Before Maxwell the equation (MIV) was believed to read $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}$ ("Ampères law"). Maxwell modified this equation by introducing the socalled *displacement current* $-\frac{\partial}{\partial t}\boldsymbol{D}$ for the sole reason that only then would charge conservation be satisfied. (The displacement current is negligibly small for slowly varying fields. This is the reason why it was detected experimentally only after its theoretical introduction by Maxwell.)

II.2 Energy conservation

We will discuss energy conservation only for media with constitutive equations of the form

$$\boldsymbol{D}(\boldsymbol{r},t) \;=\; \varepsilon_r(\boldsymbol{r})\,\varepsilon_0\,\boldsymbol{E}(\boldsymbol{r},t)$$

$$\boldsymbol{B}(\boldsymbol{r},t) \;=\mu_r(\boldsymbol{r})\,\mu_0\,\boldsymbol{H}(\boldsymbol{r},t)$$

with ε_r and μ_r independent of time.

Define

$$\boxed{\begin{gathered} u(\boldsymbol{r},t) \;=\; \frac{1}{2}\big(\,\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{D}(\boldsymbol{r},t) \,+\, \boldsymbol{B}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t)\,\big)\,, \\ \boldsymbol{N}(\boldsymbol{r},t) \;=\; \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\,. \end{gathered}}$$

$\boldsymbol{N}$ is called the *Poynting vector*. We will derive a conservation law which suggests to interpret u as the *energy density* of the electromagnetic field and $\boldsymbol{N}$ as the *energy flow vector* of the electromagnetic field.

From the constitutive equations we find

$$\frac{\partial}{\partial t}u(\boldsymbol{r},t) \;=\; \varepsilon_r(\boldsymbol{r})\,\varepsilon_0\,\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r},t) + \mu_r(\boldsymbol{r})\,\mu_0\,\boldsymbol{H}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial t}\boldsymbol{H}(\boldsymbol{r},t)\,.$$

On the other hand, we find using the $\times$ product rule for the $\boldsymbol{\nabla}$ operator, Maxwell's equations and the constitutive equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{N}(\boldsymbol{r},t) \;&=\; \boldsymbol{\nabla}\cdot\big(\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\big) \;= \\
&=\; \big(\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)\big)\cdot\boldsymbol{H}(\boldsymbol{r},t) \;-\; \boldsymbol{E}(\boldsymbol{r},t)\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)\big) \;= \\
&=\; -\big(\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t)\big)\cdot\boldsymbol{H}(\boldsymbol{r},t) \;-\; \boldsymbol{E}(\boldsymbol{r},t)\cdot\big(\frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r},t) \;+\; \boldsymbol{J}(\boldsymbol{r},t)\big) \;= \\
&=\; -\,\mu_r(\boldsymbol{r})\,\mu_0\big(\frac{\partial}{\partial t}\boldsymbol{H}(\boldsymbol{r},t)\big)\cdot\boldsymbol{H}(\boldsymbol{r},t) \;-\; \varepsilon_r(\boldsymbol{r})\,\varepsilon_0\,\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r},t) \\
&\qquad -\,\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{J}(\boldsymbol{r},t)\,.
\end{aligned}$$

Comparing these two expressions yields

$$\boxed{\frac{\partial}{\partial t}u(\boldsymbol{r},t) + \boldsymbol{\nabla}\cdot\boldsymbol{N}(\boldsymbol{r},t) \;=\; -\,\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{J}(\boldsymbol{r},t)\,.}$$

For $\boldsymbol{J} = \boldsymbol{0}$ (or, more generally, if $\boldsymbol{E}\cdot\boldsymbol{J} = 0$), this is again a continuity equation, i.e., it says that the energy of the electromagnetic field is conserved.

For $\boldsymbol{E}\cdot\boldsymbol{J} \neq 0$, the energy of the electromagnetic field is partly converted into other forms of energy. In most cases this other form of energy is heat ("Joulean heat"); however, it could also be e.g. deformation energy.

In analogy to the charge conservation law, one can integrate the energy law over a volume $\mathcal{V}$ and use the Gauss theorem to find

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} u\,d\tau \;=\; -\int_{\partial\mathcal{V}}\boldsymbol{N}\cdot d\boldsymbol{a} \;-\; \int_{\mathcal{V}}\boldsymbol{E}\cdot\boldsymbol{J}\,d\tau\,.}$$

This is the energy law in integral form: The temporal change of the energy of the electromagnetic field in a volume $\mathcal{V}$ equals the negative flow of the Poynting vector over the boundary of $\mathcal{V}$ minus the amount of energy that is converted into other energy forms, typically Joulean heat.

Summary: Maxwell's equations imply two conservation laws.

- The charge is always conserved, in any medium.
- For simple media, one can derive an energy law. The energy of the electromagnetic field is conserved if $\boldsymbol{E} \cdot \boldsymbol{J} = 0$. Otherwise, the energy of the electromagnetic field is partly converted into some other form of energy.

III. The electromagnetic potentials

Recall that for vector fields $\boldsymbol{F}$ and $\boldsymbol{B}$ the folowing implications hold.

$$\boldsymbol{F} = \boldsymbol{\nabla} V \quad \text{for some } V \qquad \Longrightarrow \qquad \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0} \qquad \text{(SI)}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad \text{for some } \boldsymbol{A} \qquad \Longrightarrow \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \text{(SII)}$$

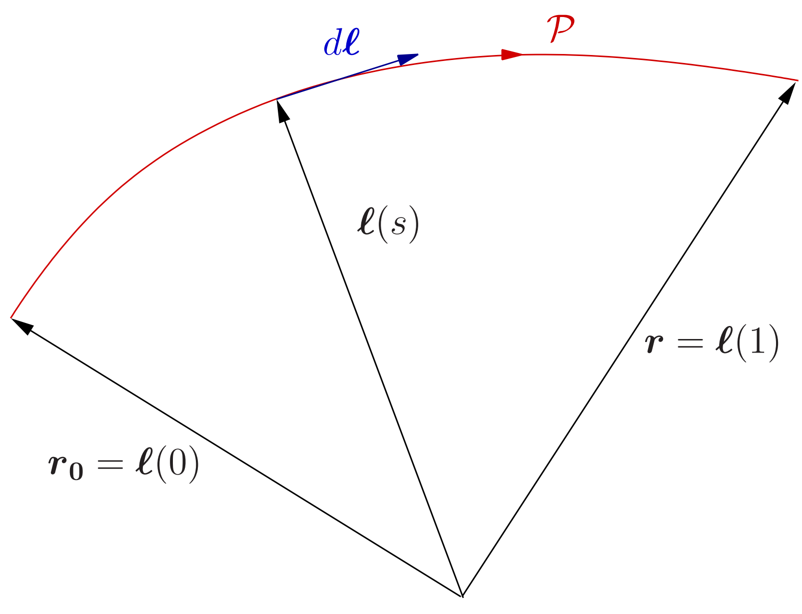
To prove this, write $\boldsymbol{\nabla} = \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z}$ and use the fact that partial derivatives commute.

We will now investigate whether the *converse* of (SI) and (SII) holds.

Converse of (SI):
Assume $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$. We want to find V such that $\boldsymbol{F} = \boldsymbol{\nabla} V$.
Fix a point $\boldsymbol{r_0}$ and a constant V_0. Define

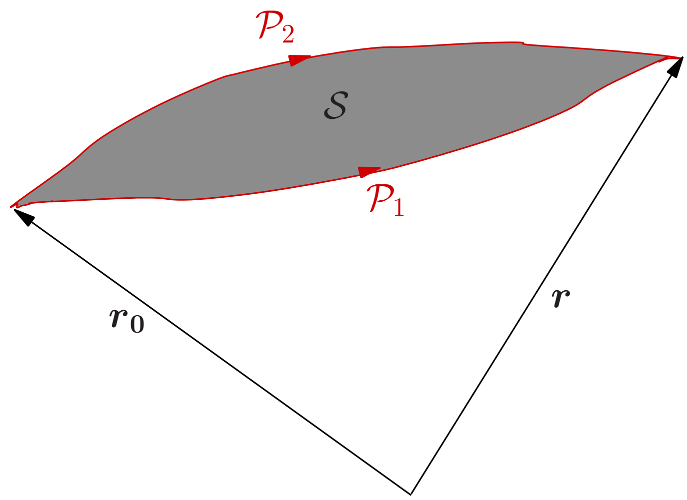
$$V(\boldsymbol{r}) = V_0 + \int_{\mathcal{P}} \boldsymbol{F} \cdot d\boldsymbol{\ell} = V_0 + \int_0^1 \boldsymbol{F}\big(\boldsymbol{\ell}(s)\big) \cdot \frac{d\boldsymbol{\ell}(s)}{ds}\, ds$$

where $\mathcal{P}$ is any path from the fixed chosen $\boldsymbol{r_0}$ to $\boldsymbol{r}$ and $\boldsymbol{\ell}(s)$ is a parametrisation of $\mathcal{P}$ with $\boldsymbol{\ell}(0) = \boldsymbol{r_0}$ and $\boldsymbol{\ell}(1) = \boldsymbol{r}$, see picture.



Then

(i) V is well-defined, i.e., independent of the path chosen.
Proof: Consider two different paths $\mathcal{P}_1$ and $\mathcal{P}_2$ from $\boldsymbol{r_0}$ to $\boldsymbol{r}$. Let $\mathcal{S}$ be the surface bounded by these two paths, see picture,



such that the boundary $\partial\mathcal{S}$ consists of $\mathcal{P}_1$ with positive orientation and $\mathcal{P}_2$ with negative orientation. Then, by Stokes theorem,

$$\int_{\mathcal{P}_1} \boldsymbol{F} \cdot d\boldsymbol{\ell} - \int_{\mathcal{P}_2} \boldsymbol{F} \cdot d\boldsymbol{\ell} = \int_{\partial\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{\ell} = \int_{\mathcal{S}} \big(\boldsymbol{\nabla} \times \boldsymbol{F} \big) \cdot d\boldsymbol{a} = \boldsymbol{0}$$

because $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ by assumption. $\square$

(ii) $\boldsymbol{F} = \boldsymbol{\nabla} V$.
Proof: Note that the way we defined V implies that $V(\boldsymbol{r_0}) = V_0$. Thus for any path $\mathcal{P}$ from $\boldsymbol{r_0}$ to $\boldsymbol{r}$:

$$\int_{\mathcal{P}} \boldsymbol{F}\cdot d\boldsymbol{\ell} = V(\boldsymbol{r}) - V_0 = V(\boldsymbol{r}) - V(\boldsymbol{r_0}) = V\big(\boldsymbol{\ell}(1)\big) - V\big(\boldsymbol{\ell}(0)\big) =$$

$$\int_0^1 \frac{d}{ds} V\big(\boldsymbol{\ell}(s)\big)\, ds = \int_0^1 \big(\boldsymbol{\nabla} V\big)\big(\boldsymbol{\ell}(s)\big) \cdot \frac{d\boldsymbol{\ell}(s)}{ds}\, ds = \int_{\mathcal{P}} (\boldsymbol{\nabla} V) \cdot d\boldsymbol{\ell}\,.$$

Since this holds for *any* path from $\boldsymbol{r_0}$ to *any* $\boldsymbol{r}$, the integrands must be equal, $\boldsymbol{F} = \boldsymbol{\nabla} V$. $\square$

We have thus shown that the converse of (SI) is also true:

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0} \qquad \Longleftrightarrow \qquad \boldsymbol{F} = \boldsymbol{\nabla} V \quad \text{for some } V}$$

Our proof shows that, if $\boldsymbol{F}$ with $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$ is given, the desired V is of the form $V(\boldsymbol{r}) = V_0 + \int_{\mathcal{P}} \boldsymbol{F} \cdot d\boldsymbol{\ell}$ where $\mathcal{P}$ is a path from a fixed chosen point $\boldsymbol{r_0}$ to $\boldsymbol{r}$ and V_0 is a chosen constant. Obviously, V is determined by $\boldsymbol{F}$ up to an additive constant.

Remarks:

(a) In physics, V is called a *potential* of $\boldsymbol{F}$ if $\boldsymbol{F} = -\boldsymbol{\nabla} V$. The reason for the minus sign comes from mechanics: If a particle moves in a force field $\boldsymbol{F}$, Newton's second law implies

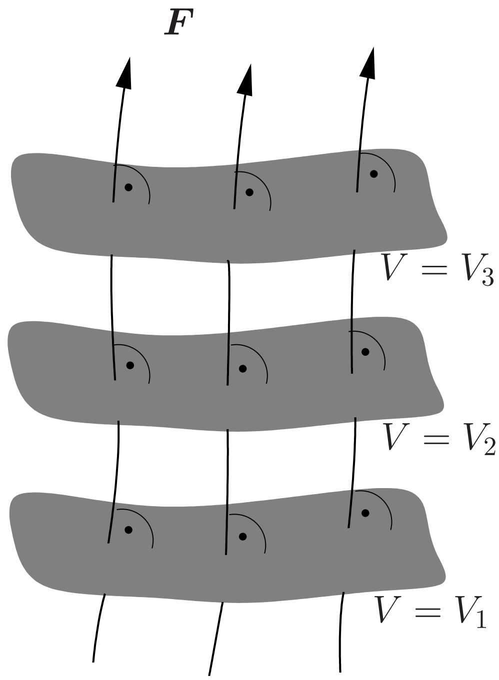
$$\boldsymbol{F}\big(\boldsymbol{r}(t)\big) = m\,\ddot{\boldsymbol{r}}(t)\,.$$

If $\boldsymbol{F} = -\boldsymbol{\nabla} V$, the potential V gives the potential energy of the particle, and the law of conservation of enegy reads

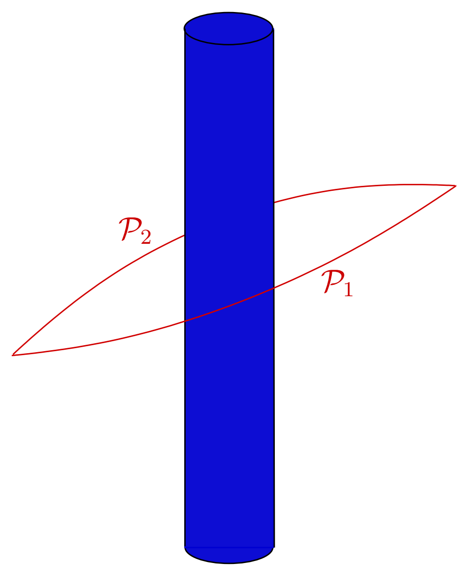
$$\frac{d}{dt}\left(\frac{m}{2}\,|\dot{\boldsymbol{r}}(t)|^2 + V\big(\boldsymbol{r}(t)\big)\right) = \dot{r} \cdot \Big(m\,\ddot{\boldsymbol{r}} + \boldsymbol{\nabla} V\big(\boldsymbol{r}(t)\big)\Big) = 0\,.$$

Our above argument shows that this law holds if and only if the force field $\boldsymbol{F}$ satisfies $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$. Such force fields are called *conservative.*

(b) Recall that the gradient of V is perpendicular to the surfaces $V =$constant. Thus, we have proven that a vector field $\boldsymbol{F}$ is curl free, $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$, if and only if its integral curves are everywhere perpendicular to a family of surfaces, see picture next page. For a vector field $\boldsymbol{F}$ with $\boldsymbol{\nabla} \times \boldsymbol{F} \neq \mathbf{0}$, the integral curves have a 'twist'; then it is impossible to find surfaces which are everywhere perpendicular to this integral curves.



(c) If the equation $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ holds only on part of 3-dimensional space, e.g. with an infinite cylinder removed, then we *cannot* conclude that $\boldsymbol{F} = \boldsymbol{\nabla} V$. Our proof fails because in this mutilated space it is not possible to write any two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ connecting two points as the boundary of a surface, see picture.



Converse of (SII):

Assume $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. We want to find $\boldsymbol{A}$ such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, i.e.

$$B_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \,,$$

$$B_y = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \,,$$

$$B_z = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \,.$$

Let us try to find a solution with $A_z = 0$. Then the first desired equation

$$B_x = - \frac{\partial}{\partial z} A_y$$

requires

$$A_y(x, y, z) = - \int_0^z B_x(x, y, z')\, dz' + g(x, y)$$

and the second desired equation

$$B_y = \frac{\partial}{\partial z} A_x$$

requires

$$A_x(x, y, z) = \int_0^z B_y(x, y, z')\, dz' + h(x, y) \,.$$

These two conditions imply

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x =$$

$$-\int_0^z \Big(\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y\Big)(x,y,z')\,dz' + \frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}h(x,y)\,.$$

As, by assumption, $\boldsymbol{\nabla}\cdot B = 0$, we can substitute

$$\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y = -\frac{\partial}{\partial z}B_z$$

to find

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x =$$

$$\int_0^z \frac{\partial}{\partial z}B_z(x,y,z')\,dz' + \frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}h(x,y) =$$

$$B_z(x,y,z) - B_z(x,y,0) + \frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}h(x,y)\,.$$

This demonstrates that the third desired equation is true if g and h satisfy

$$B_z(x,y,0) = \frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}h(x,y)\,.$$

Clearly, such functions g and h exist, e.g.

$$g(x,y) = \int_0^x B_z(x',y,0)\,dx' \qquad \text{and} \qquad h(x,y) = 0\,.$$

We have thus shown that the converse of (SII) is also true:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = O \qquad \Longleftrightarrow \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad \text{for some } \boldsymbol{A}$$

Our proof shows that, if $\boldsymbol{B}$ with $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is given, the desired $\boldsymbol{A}$ can be chosen, e.g., as

$$A_x(x,y,z) = \int_0^z B_y(x,y,z')\,dz'\,,$$

$$A_y(x,y,z) = -\int_0^z B_x(x,y,z')\,dz' + \int_0^x B_z(x',y,0)\,dx'\,,$$

$$A_z(x,y,z) = 0\,.$$

$\boldsymbol{A}$ is not uniquely determined by $\boldsymbol{B}$: If $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}_1$ and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}_2$, we must have $\boldsymbol{\nabla} \times \big(\boldsymbol{A}_2 - \boldsymbol{A}_1\big) = \boldsymbol{0}$. From our earlier result we know that this is true if $\boldsymbol{A}_2 - \boldsymbol{A}_1 = \boldsymbol{\nabla} V$. Thus, $\boldsymbol{B}$ determines $\boldsymbol{A}$ up to the freedom of adding a gradient.

Remark:

As in the case of (SI), the converse of (SII) is not true if the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ holds only on part of the three-dimensional space. In this case, our proof fails because our calculation of integrals along the coordinate axes need not be true if the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ does not hold everywhere.

We are now ready to define the electromagnetic potentials.

The first Maxwell equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \text{(MI)}$$

implies that there is a vector field $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \, .$$

Then the second Maxwell equation

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = \boldsymbol{0} \qquad \text{(MII)}$$

takes the form

$$\boldsymbol{\nabla} \times \Big(\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A} \Big) = \boldsymbol{0} \, .$$

This equation implies that there is a scalar field V such that

$$\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A} = -\,\boldsymbol{\nabla} V \, .$$

We have thus expressed $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{A}$ and V:

$$\boxed{\begin{aligned} \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \, , && \text{(PI)} \\ \boldsymbol{E} &= -\,\boldsymbol{\nabla} V - \frac{\partial}{\partial t}\boldsymbol{A} \, . && \text{(PII)} \end{aligned}}$$

V is a scalar function of $\boldsymbol{r}$ and t, called the *scalar potential.*

$\boldsymbol{A}$ is a vector-valued function of $\boldsymbol{r}$ and t, called the *vector potential.*

V and $\boldsymbol{A}$ are unique up to *gauge transformations*

$$\boxed{\begin{aligned} \boldsymbol{A} &\longmapsto \boldsymbol{A} + \boldsymbol{\nabla} f \,, && \text{(GI)} \\ V &\longmapsto V - \frac{\partial}{\partial t} f \,. && \text{(GII)} \end{aligned}}$$

where f is an arbitrary scalar function of $\boldsymbol{r}$ and t, called the *gauge function.*

The fact that the potentials are defined only up to gauge transformations shows that they cannot be measurable. They are auxiliary mathematical quantities which are used to calculate the measurable fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

With $\boldsymbol{E}$ and $\boldsymbol{B}$ expressed in terms of the potentials V and $\boldsymbol{A}$, the first two Maxwell equations (MI) and (MII) are automatically satisfied. The remaining two Maxwell equations (MIII) and (MIV) become equations for V and $\boldsymbol{A}$. The special form of these equations depend on the constitutive equations.

We want to work this out for the simplest kind of constitutive equations,

$$\boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon_r \, \varepsilon_0 \, \boldsymbol{E}(\boldsymbol{r}, t) \,,$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mu_r \, \mu_0 \, \boldsymbol{H}(\boldsymbol{r}, t) \,,$$

with *constant* ε_r and μ_r. Then the third Maxwell equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \qquad \text{(MIII)}$$

takes the form

$$\rho \,=\, \varepsilon_r\,\varepsilon_0\,\boldsymbol{\nabla}\cdot\boldsymbol{E} \,=\, \varepsilon_r\,\varepsilon_0\,\boldsymbol{\nabla}\cdot\Big(-\,\boldsymbol{\nabla}V \,-\, \frac{\partial}{\partial t}\boldsymbol{A}\Big)$$

which, with the Laplace operator

$$\Delta \,=\, \boldsymbol{\nabla}^2 \,=\, \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} \,=\, \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,,$$

can be rewritten as

$$\Delta V \,+\, \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} \,=\, -\,\frac{\rho}{\varepsilon_r\,\varepsilon_0}\,. \qquad \text{(MIII}'\text{)}$$

On the other hand, the fourth Maxwell equation

$$\boldsymbol{\nabla}\times\boldsymbol{H} \,-\, \frac{\partial}{\partial t}\boldsymbol{D} \,=\, \boldsymbol{J} \qquad \text{(MIV)}$$

takes the form

$$\boldsymbol{J} \,=\, \frac{1}{\mu_r\,\mu_0}\,\boldsymbol{\nabla}\times\boldsymbol{B} \,-\, \varepsilon_r\,\varepsilon_0\,\frac{\partial}{\partial t}\boldsymbol{E} \,=$$

$$\frac{1}{\mu_r\,\mu_0}\,\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\boldsymbol{A}\big) \,+\, \varepsilon_r\,\varepsilon_0\,\frac{\partial}{\partial t}\Big(\boldsymbol{\nabla}V \,+\, \frac{\partial}{\partial t}\boldsymbol{A}\Big) \,=$$

$$\frac{1}{\mu_r\,\mu_0}\Big(\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{A}\big) \,-\, \big(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\big)\,\boldsymbol{A}\Big) \,+\, \varepsilon_r\,\varepsilon_0\,\frac{\partial}{\partial t}\Big(\boldsymbol{\nabla}V \,+\, \frac{\partial}{\partial t}\boldsymbol{A}\Big).$$

With the abbreviation

$$v \,=\, \frac{1}{\sqrt{\varepsilon_r\,\varepsilon_0\,\mu_r\,\mu_0}}$$

the last equation can be rewritten as

$$\Delta \boldsymbol{A} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{v^2}\frac{\partial}{\partial t}V\right) = -\,\mu_r\,\mu_0\,\boldsymbol{J} \qquad (\mathrm{MIV}')$$

Claim: We can always make a gauge transformation (GI), (GII) such that the new potentials satisfy the *Lorentz gauge condition*

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{v^2}\frac{\partial}{\partial t}V = 0\,. \qquad (\mathrm{LG})}$$

Proof: Under a gauge transformation, the left-hand side of (LG) transforms as

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{v^2}\frac{\partial}{\partial t}V \longmapsto \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f + \frac{1}{v^2}\frac{\partial}{\partial t}V - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}f\,.$$

The resulting expression can be made equal to zero by choosing f such that

$$\Delta f - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}f = -\boldsymbol{\nabla}\cdot\boldsymbol{A} - \frac{1}{v^2}\frac{\partial}{\partial t}V\,.$$

For any given $\boldsymbol{A}$ and V, solutions f to this inhomogeneous wave equation exist. (We will find a particular solution to the inhomogeneous wave equation when deriving the retarded potentials in the next chapter.) □

With the Lorentz gauge condition (LG), the equations (MIII') and (MIV') simplify to

$$\boxed{\begin{aligned} \Delta V - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}V &= -\frac{\rho}{\varepsilon_r\varepsilon_0} \qquad (\mathrm{WI}) \\ \Delta \boldsymbol{A} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A} &= -\,\mu_r\,\mu_0\,\boldsymbol{J} \qquad (\mathrm{WII}) \end{aligned}}$$

(WI) and (WII) are referred to as the *inhomogeneous wave equations for the potentials.* They give us four scalar equations for the four scalar variables V, A_x, A_y, A_z. Working with V and $\boldsymbol{A}$ instead of $\boldsymbol{E}$ and $\boldsymbol{B}$ is advantageous because it reduces the number of variables from six to four. Even more importantly, (WI) and (WII) are *uncoupled* equations for the potentials V and $\boldsymbol{A}$. Note, however, that the potentials are coupled by the Lorentz gauge condition. The wave equations (WI) and (WII) demonstrate that electromagnetic fields, expressed in terms of the potentials, propagate with velocity v in the considered medium. In vacuo, we have $\varepsilon_r = \mu_r = 1$ and $v = c$ where

$$c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}} \approx 300\,000\,\mathrm{km/s}\,.$$

Note: Even with the Lorentz gauge condition imposed, the potentials V and $\boldsymbol{A}$ are not unique. There is still the freedom of making gauge transformations (GI) and (GII) with a gauge function that satisfies the homogeneous wave equation $\Delta f - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} f = 0$.

Summary: In any kind of media, $\boldsymbol{E}$ and $\boldsymbol{B}$ can be expressed in terms of the potentials V and $\boldsymbol{A}$ according to (PI) and (PII). The first two Maxwell equations (MI) and (MII) are then automatically satisfied.

In media with constant ε_r and μ_r, the remaining two Maxwell equations reduce to the inhomogeneous wave equations (WI) and (WII) if we impose the Lorentz gauge condition (LG) on the potentials.

The solutions to Maxwell's equations, for given ρ and $\boldsymbol{J}$, in a medium with constant ε_r and μ_r can thus be found in four steps:

1. Find the general solution V and $\boldsymbol{A}$ to the wave equations (WI) and (WII).

2. Single out those solutions that satisfy the Lorentz gauge condition (LG).

3. Determine $\boldsymbol{E}$ and $\boldsymbol{B}$ from (PI) and (PII).

4. Determine $\boldsymbol{D}$ and $\boldsymbol{H}$ from the constitutive equations.

IV. Solving Maxwell's equations in free space

We restrict to media with constitutive equations of the form

$$\boldsymbol{D}(\boldsymbol{r},t) \,=\, \varepsilon_r\,\varepsilon_0\,\boldsymbol{E}(\boldsymbol{r},t)\,, \qquad \boldsymbol{B}(\boldsymbol{r},t) \,=\, \mu_r\,\mu_0\,\boldsymbol{H}(\boldsymbol{r},t)\,,$$

where ε_r and μ_r are constants.

In this chapter we want to solve Maxwell's equations for given sources $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ in free space. (For Maxwell's equations in bounded regions see Chapter V below.)

The following is clear from the outset:

- The solution can exist only if ρ and $\boldsymbol{J}$ satisfy the continuity equation

$$\frac{\partial}{\partial t}\rho \,+\, \boldsymbol{\nabla}\cdot\boldsymbol{J} \,=\, 0\,.$$

- The solution is unique only up to the freedom of adding a solution of the homogeneous equations (i.e., to the equations with $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$). Physical interpretation: We may add source-free electromagnetic waves that "come in from infinity" and "go out to infinity".

According to the results of Chapter III, we have to solve:

$$\Delta V \,-\, \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}V \,=\, -\frac{\rho}{\varepsilon_r\varepsilon_0}\,, \qquad \text{(WI)}$$

$$\Delta\boldsymbol{A} \,-\, \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}\boldsymbol{A} \,=\, -\,\mu_r\,\mu_0\,\boldsymbol{J}\,, \qquad \text{(WII)}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} \,+\, \frac{1}{v^2}\,\frac{\partial}{\partial t}V \,=\, 0\,, \qquad \text{(LG)}$$

where $v = 1/\sqrt{\varepsilon_r \, \varepsilon_0 \, \mu_r \, \mu_0}$.

The linearity of the left-hand sides implies:

general solution of the inhomogeneous equations
$=$
general solution of the homogeneous equations
$+$
one particular solution of the inhomogeneous equations

We shall discuss the general solution of the homogeneous equations in Section IV.1 and one particular solution of the inhomogeneous equations in Section IV.2 below.

IV.1 General solution of the homogeneous equations

We want to find the general solution of

$$\Delta V \, - \, \frac{1}{v^2} \frac{\partial^2}{\partial t^2} V \, = \, 0 \, ,$$

$$\Delta \boldsymbol{A} \, - \, \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \boldsymbol{A} \, = \, \boldsymbol{0} \, ,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} \, + \, \frac{1}{v^2} \frac{\partial}{\partial t} V \, = \, 0 \, ,$$

in the class of functions that admit an expansion with respect to plane harmonic waves (discrete and/or continuous Fourier expansion). Thus, we have to determine all solutions of the form

$$V(\boldsymbol{r}, t) \, = \, V_0 \cos\big(\boldsymbol{k} \cdot \boldsymbol{r} - \omega \, t \, + \, \alpha \big) \, ,$$

$$\boldsymbol{A}(\boldsymbol{r}, t) \, = \, \boldsymbol{A}_0 \cos\big(\boldsymbol{k} \cdot \boldsymbol{r} - \omega \, t \, + \, \alpha \big) \, .$$

The general solution is then a superposition of such solutions.

(One could try, more generally, a wave vector $\boldsymbol{k}_1$, a frequency ω_1 and a phase shift α_1 for $V(\boldsymbol{r},t)$ and a wave vector $\boldsymbol{k}_2$, a frequency ω_2 and a phase shift α_2 for $\boldsymbol{A}(\boldsymbol{r},t)$. Then one would find that the Lorentz gauge condition cannot be satisfied unless $\boldsymbol{k}_1 = \boldsymbol{k}_2$, $\omega_1 = \omega_2$ and $\alpha_1 = \alpha_2$.)

The equation $\quad \Delta V - \dfrac{1}{v^2}\dfrac{\partial^2}{\partial t^2}V = 0 \quad$ requires

$$-V_0\left(|\boldsymbol{k}|^2 - \frac{\omega^2}{v^2}\right)\cos\big(\boldsymbol{k}\cdot\boldsymbol{r} - \omega\, t + \alpha\big) = 0$$

and thus the dispersion relation $\quad \boxed{\omega = v\,|\boldsymbol{k}|\,.}$

So the waves have phase velocity and group velocity both equal to v,

$$v_{\mathrm{ph}} = \frac{\omega}{|\boldsymbol{k}|} = v \qquad \text{and} \qquad v_{\mathrm{gr}} = \frac{d\omega}{d|\boldsymbol{k}|} = v\,.$$

The equation $\quad \Delta\boldsymbol{A} - \dfrac{1}{v^2}\dfrac{\partial^2}{\partial t^2}\boldsymbol{A} = \boldsymbol{0} \quad$ requires the same condition.

The Lorentz gauge condition $\;\boldsymbol{\nabla}\cdot\boldsymbol{A} + \dfrac{1}{v^2}\dfrac{\partial}{\partial t}V = 0\;$ requires

$$\left(-\boldsymbol{A}_0\cdot\boldsymbol{k} + V_0\frac{\omega}{v^2}\right)\sin\big(\boldsymbol{k}\cdot\boldsymbol{r} - \omega\, t + \alpha\big) = 0$$

and thus

$$\boldsymbol{A}_0\cdot\boldsymbol{k} = V_0\frac{\omega}{v^2}\,.$$

We are free to make gauge transformations,

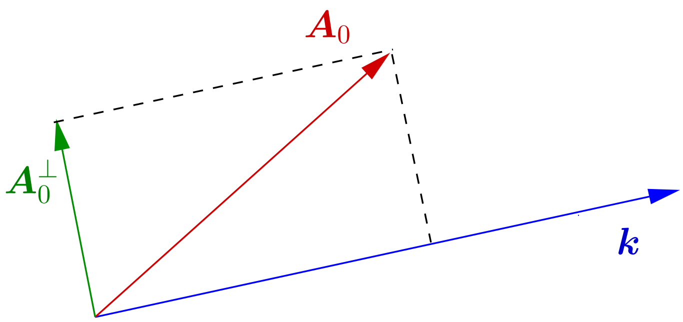
$$\boldsymbol{A} \longmapsto \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}f\,, \qquad V \longmapsto V' = V - \frac{\partial}{\partial t}f\,,$$

that leave the Lorentz gauge condition unchanged. Choosing

$$f(\boldsymbol{r},t) \;=\; -\,\frac{V_0}{\omega}\sin\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-\omega\,t\,+\,\alpha\,\big) \;=\; -\,\frac{\boldsymbol{A}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\alpha)$$

transforms the potentials into

$$\boldsymbol{A}'(\boldsymbol{r},t) \;=\; \boldsymbol{A}_0^{\perp}\cos\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-\omega\,t\,+\,\alpha\,\big)\,, \qquad V'(\boldsymbol{r},t) \;=\; 0\,,$$



where

$$\boldsymbol{A}_0^{\perp} \;=\; \boldsymbol{A}_0 \;-\; \frac{\boldsymbol{A}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\;\boldsymbol{k}\;\;.$$

Note that $\boldsymbol{\nabla}\cdot\boldsymbol{A}' \,=\, 0$, so the primed potentials satisfy, indeed, the Lorentz gauge condition.

Dropping the primes on $\boldsymbol{A}'$ and V', we can thus write the general plane harmonic wave solution of the homogeneous wave equations and the Lorentz gauge condition as

$$\boxed{\boldsymbol{A}(\boldsymbol{r},t) \;=\; \boldsymbol{A}_0^{\perp}\cos\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-|\boldsymbol{k}|\,v\,t\,+\,\alpha\,\big)\,, \qquad V(\boldsymbol{r},t) \;=\; 0}$$

where $\boldsymbol{A}_0^{\perp}$ is any vector with $\boldsymbol{A}_0^{\perp}\cdot\boldsymbol{k} \,=\, 0$.

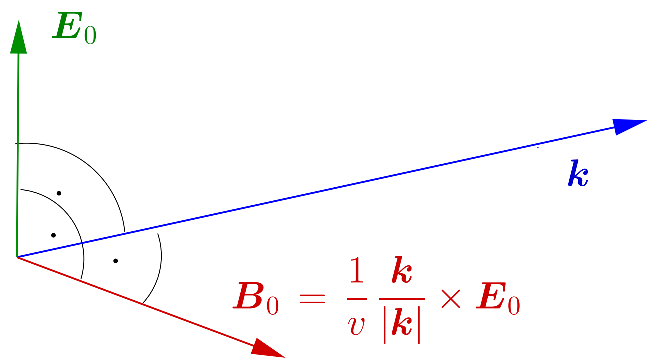
Thus, $\boldsymbol{E} = -\boldsymbol{\nabla}V - \dfrac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} \,=\, \boldsymbol{\nabla}\times\boldsymbol{A}$ take the form

$$\boxed{\begin{aligned}\boldsymbol{E}\,(\boldsymbol{r},t) \;&=\; \boldsymbol{E}_0\sin\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-|\boldsymbol{k}|\,v\,t\,+\,\alpha\,\big)\,,\\ \boldsymbol{B}(\boldsymbol{r},t) \;&=\; \boldsymbol{B}_0\sin\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-|\boldsymbol{k}|\,v\,t\,+\,\alpha\,\big)\,,\end{aligned}}$$

where $\boldsymbol{E}_0 \;=\; -\,|\boldsymbol{k}|\,v\,\boldsymbol{A}_0^{\perp}$ and $\boldsymbol{B}_0 \;=\; -\,\boldsymbol{k}\times\boldsymbol{A}_0^{\perp} \;=\; \dfrac{1}{v}\,\dfrac{\boldsymbol{k}}{|\boldsymbol{k}|}\times\boldsymbol{E}_0$.

Any such solution describes a transverse linearly polarised plane harmonic wave.

Note that $\boldsymbol{B}_0$ is determined by $\boldsymbol{E}_0$ and the wave vector $\boldsymbol{k}$.



The general solution of the homogeneous wave equation is an arbitrary superposition of such waves. This can be a finite sum, an infinite series (Fourier series), or an integral (Fourier integral). A circularly polarised wave, e.g., is a superposition of just two such linearly polarised waves, see 2nd worksheet. The general form of a Fourier integral solution is given by the potentials $V(\boldsymbol{r},t)=0$ and

$$\boldsymbol{A}(\boldsymbol{r},t) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{A}_0^{\perp}(\boldsymbol{k})\cos\big(\,\boldsymbol{k}\cdot\boldsymbol{r}-|\boldsymbol{k}|\,v\,t+\alpha(\boldsymbol{k})\,\big)\,dk_x dk_y dk_z$$

where $\boldsymbol{A}_0^{\perp}(\boldsymbol{k})\cdot\boldsymbol{k} \;=\; 0\,$; otherwise $\boldsymbol{A}_0^{\perp}(\boldsymbol{k})$ and $\alpha(\boldsymbol{k})$ are arbitrary.

Note: We have worked here with the potentials. As an alternative, one can show that $\boldsymbol{E}$ and $\boldsymbol{B}$ themselves satisfy the wave equation, see 3rd worksheet. For the homogeneous case (i.e., $\rho=0$ and $\boldsymbol{J}=\boldsymbol{0}$), one finds

$$\boxed{\Delta\boldsymbol{E}-\frac{1}{v^2}\frac{\partial}{\partial t^2}\boldsymbol{E} \;=\; \boldsymbol{0} \qquad \text{and} \qquad \Delta\boldsymbol{B}-\frac{1}{v^2}\frac{\partial}{\partial t^2}\boldsymbol{B} \;=\; \boldsymbol{0}\,.}$$

So one could discuss the general solution of the homogeneous Maxwell equations directly in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$, without using the potentials. However, using the potentials is of great advantage if we now turn to the inhomogeneous equations.

IV.2 The retarded potentials

We want to find one particular solution of the inhomogeneous wave equations for the potentials. We consider the equation for the scalar potential

$$\Delta V \,-\, \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2} V \,=\, -\frac{\rho}{\varepsilon_r\,\varepsilon_0} \qquad \text{(WI)}$$

with a given ρ that depends on $\boldsymbol{r}$ and t. We will *guess* a solution and then *prove* that it does the job.

Heuristic consideration: A *static* charge Q at $\boldsymbol{r}_0$ produces the time-independent Coulomb potential

$$V(\boldsymbol{r}) \,=\, \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\,\frac{Q}{|\boldsymbol{r}-\boldsymbol{r}_0|}\,.$$

By superposition, a *static* charge density ρ produces the time-independent potential

$$V(\boldsymbol{r}) \,=\, \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\int_{\mathcal{R}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,,$$

where $\mathcal{R}$ denotes all of 3-dimensional space.

For a *time-dependent* charge density, one has to take into account that the action in our medium travels with speed v, thus needs time $|\boldsymbol{r}-\boldsymbol{r}'|/v$ from $\boldsymbol{r}'$ to $\boldsymbol{r}$. So it seems reasonable to *guess* that, for a time-dependent charge density, the potential is

$$V(\boldsymbol{r},t) \,=\, \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\int_{\mathcal{R}} \frac{\rho\big(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big)\,d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,. \qquad \text{(RI)}$$

Claim: (RI) is a solution of (WI).

Proof: We need the 3-dimensional Dirac delta “function”, defined by

$$\delta^{(3)}(\boldsymbol{r}) \,=\, 0 \quad \text{for} \quad \boldsymbol{r} \neq \boldsymbol{0}\,, \qquad \int_{\mathcal{R}} \delta^{(3)}(\boldsymbol{r})\,d\tau \,=\, 1\,.$$

Obviously, this can be true only if "$\delta^{(3)}(\mathbf{0}) = \infty$", i.e., $\delta^{(3)}$ cannot be a function in the usual sense. (Actually, it is a socalled *generalised function* or *distribution.*) The defining properties of the Dirac delta function imply that for any ordinary function $f(\boldsymbol{r})$ we must have

$$f(\boldsymbol{r})\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a}) \;=\; f(\boldsymbol{a})\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a})$$

(check this for $\boldsymbol{r} = \boldsymbol{a}$ and for $\boldsymbol{r} \neq \boldsymbol{a}$), thus

$$\int_{\mathcal{R}} f(\boldsymbol{r})\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a})\,d\tau \;=\; f(\boldsymbol{a})\,.$$

A particular representation of the Dirac delta function is given by

$$\delta^{(3)}(\boldsymbol{r}) \;=\; \frac{1}{4\,\pi}\,\boldsymbol{\nabla}\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\,,$$

see 3rd worksheet.

To calculate $\Delta V(\boldsymbol{r},t)$, we now proceed step by step. (Note that $\boldsymbol{\nabla}$ acts on $\boldsymbol{r}$, i.e. $\boldsymbol{r}'$ and t are constants with respect to this differentiation!)

With the abbreviations $\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}'$, $R = |\boldsymbol{R}|$, $t_r = t - R/v$:

- $$\boldsymbol{\nabla} R \;=\; \boldsymbol{\nabla}|\boldsymbol{r}-\boldsymbol{r}'| \;=\; \boldsymbol{\nabla}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \;=$$
$$=\frac{2\,(x-x')\,\hat{\boldsymbol{x}} \,+\, 2\,(y-y')\,\hat{\boldsymbol{y}} \,+\, 2\,(z-z')\,\hat{\boldsymbol{z}}}{2\,\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \;=\; \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \;=\; \frac{\boldsymbol{R}}{R}\,.$$

- $$\boldsymbol{\nabla} t_r \;=\; -\frac{1}{v}\,\frac{\boldsymbol{R}}{R}\,,$$

- $$\boldsymbol{\nabla}\rho(\boldsymbol{r}',t_r) \;=\; \frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\,\boldsymbol{\nabla} t_r \;=\; -\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\,\frac{\boldsymbol{R}}{v\,R}\,,$$

- $$\boldsymbol{\nabla}\,\frac{\rho(\boldsymbol{r}',t_r)}{R} \;=\; \frac{\boldsymbol{\nabla}\rho(\boldsymbol{r}',t_r)}{R} - \frac{\rho(\boldsymbol{r}',t_r)}{R^2}\,\boldsymbol{\nabla}R \;=$$

$$=\;-\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\,\frac{\boldsymbol{R}}{v\,R^2}\;-\;\rho(\boldsymbol{r}',t_r)\,\frac{\boldsymbol{R}}{R^3}\;=$$

$$=\;-\Big(\,\frac{R}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\;+\;\rho\big(\boldsymbol{r}',t_r\big)\,\Big)\,\frac{\boldsymbol{R}}{R^3}$$

- $$\Delta\,\frac{\rho(\boldsymbol{r}',t_r)}{R} \;=\; -\Big(\,\frac{\boldsymbol{\nabla}R}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)+\frac{R}{v}\,\frac{\partial}{\partial t}\boldsymbol{\nabla}\rho(\boldsymbol{r}',t_r)+\boldsymbol{\nabla}\rho(\boldsymbol{r}',t_r)\,\Big)\cdot\frac{\boldsymbol{R}}{R^3}$$

$$-\Big(\,\frac{R}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\;+\;\rho(\boldsymbol{r}',t_r)\,\Big)\,\boldsymbol{\nabla}\cdot\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\;=$$

$$=\;-\Big(\,\cancel{\frac{\boldsymbol{R}}{vR}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)}\;-\;\frac{R}{v^2}\,\frac{\partial^2\rho}{\partial t^2}\big(\boldsymbol{r}',t_r\big)\,\frac{\boldsymbol{R}}{R}\;+\;\cancel{\boldsymbol{\nabla}\rho(\boldsymbol{r}',t_r)}\,\Big)\cdot\frac{\boldsymbol{R}}{R^3}$$

$$-\Big(\,\frac{R}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\;+\;\rho(\boldsymbol{r}',t_r)\,\Big)\,4\,\pi\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\;=$$

$$=\;\frac{1}{v^2}\,\frac{\partial^2\rho}{\partial t^2}\big(\boldsymbol{r}',t_r\big)\,\frac{1}{R}\;-\;\Big(\,\frac{R}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}',t_r\big)\;+\;\rho(\boldsymbol{r}',t_r)\,\Big)\,4\,\pi\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\;.$$

With $\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\;=\;\delta^{(3)}(\boldsymbol{r}'-\boldsymbol{r})$:

$$\Big(\,\Delta\;-\;\frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}\,\Big)\,\frac{\rho\big(\boldsymbol{r}'\,,\;t\;-\;\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\,\big)}{|\boldsymbol{r}-\boldsymbol{r}'|}\;=$$

$$=\;-\Big(\,\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\,\frac{\partial\rho}{\partial t}\big(\boldsymbol{r}'\,,\;t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\,\big)+\rho\big(\boldsymbol{r}'\,,\;t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\,\big)\,\Big)\,4\,\pi\,\delta^{(3)}(\boldsymbol{r}'-\boldsymbol{r})\,.$$

Dividing by $4\pi\varepsilon_r\varepsilon_0$ and integrating $\boldsymbol{r}'$ over $\mathcal{R}$ yields the desired result

$$\left(\Delta - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{4\pi\varepsilon_r\varepsilon_0}\int_{\mathcal{R}}\frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\right)d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{1}{\varepsilon_r\varepsilon_0}\rho(\boldsymbol{r},t) \ .$$

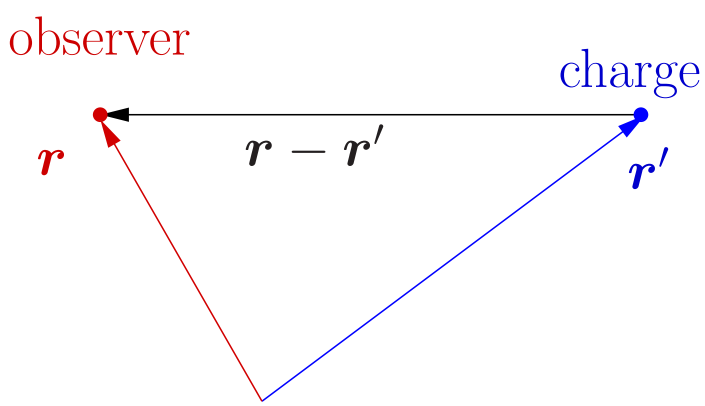
$\square$

An analogous calculation gives a solution $\boldsymbol{A}$ of (WII). These solutions

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_r\varepsilon_0}\int_{\mathcal{R}}\frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\right)d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad \text{(RI)}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_r\mu_0}{4\pi}\int_{\mathcal{R}}\frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\right)d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad \text{(RII)}$$

are called the *retarded potentials*. The name refers to the fact that ρ and $\boldsymbol{J}$ are taken at the *retarded* time $t_r = t - |\boldsymbol{r}-\boldsymbol{r}'|/v$. The action travels from the charge to the observer at speed v:



The observer at $\boldsymbol{r}$ notices an action at time t if the charge at $\boldsymbol{r}'$ was moved at the retarded time $t_r = t - |\boldsymbol{r}-\boldsymbol{r}'|/v$.

Claim: The retarded potentials satisfy the Lorentz gauge condition if ρ and $\boldsymbol{J}$ satisfy the continuity equation.

Proof: See 4th worksheet.

For *time-independent* ρ and $\boldsymbol{J}$, the retarded potentials reduce to the electro- and magnetostatic potentials:

$$V(\boldsymbol{r}) = \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0} \int_{\mathcal{R}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,,$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_r\,\mu_0}{4\,\pi} \int_{\mathcal{R}} \frac{\boldsymbol{J}(\boldsymbol{r}')\,d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,.$$

If one knows these formulas for the static case by heart, it is easy to reproduce the formulas for the time-dependent case: Just replace

$V(\boldsymbol{r})$ by $V(\boldsymbol{r},t)$

$\boldsymbol{A}(\boldsymbol{r})$ by $\boldsymbol{A}(\boldsymbol{r},t)$

$\rho(\boldsymbol{r}')$ by $\rho(\boldsymbol{r}',t_r)$

$\boldsymbol{J}(\boldsymbol{r}')$ by $\boldsymbol{J}(\boldsymbol{r}',t_r)$

where $t_r = t - |\boldsymbol{r}-\boldsymbol{r}'|/v$.

The general solution of the inhomogeneous equations is given by the retarded potentials plus the general solution of the homogeneous equations.

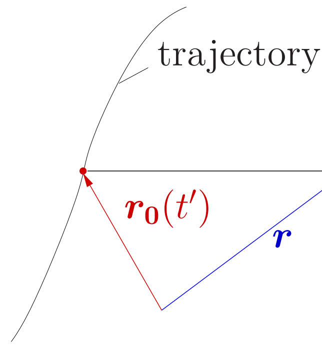
Recall: Solutions to the homogeneous equation are (source-free) electromagnetic waves.

So there are many different solutions for any given ρ and $\boldsymbol{J}$. It depends on the physical situation which of them is realised in nature. If no electromagnetic waves come in from infinity, the retarded potentials give the right solution, otherwise one has to add a solution of the homogeneous equations.

IV.3 The retarded potentials for a moving point charge

We want to evaluate the retarded potentials (RI) and (RII) for the case of a point particle with charge q that moves along a given trajectory $\boldsymbol{r_0}(t')$.

We write t' for the time parameter along the particle's trajectory because we want to leave the symbol t for the time where the field is measured at position $\boldsymbol{r}$.



Then the charge density and the current are given at position $\boldsymbol{r}'$ and time t' as

$$\rho(\boldsymbol{r}', t') \;=\; q\, \delta^{(3)}\big(\boldsymbol{r}' - \boldsymbol{r_0}(t')\big)\,,$$

$$\boldsymbol{J}(\boldsymbol{r}', t') \;=\; q\, \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t')\, \delta^{(3)}\big(\boldsymbol{r}' - \boldsymbol{r_0}(t')\big)\,.$$

To calculate $V(\boldsymbol{r}, t)$, we rewrite $\rho\big(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big)$ with the help of the (one-dimensional) Dirac delta function as

$$\rho\big(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big) \;=\; \int_{-\infty}^{\infty} \delta\big(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big)\, \rho(\boldsymbol{r}', t')\, dt'\,.$$

Then the retarded potential (RI) takes the form

$$V(\boldsymbol{r}, t) \;=\; \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0} \int_{\mathcal{R}} \int_{-\infty}^{\infty} \frac{\rho(\boldsymbol{r}', t')\, \delta\big(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big)}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dt'\, d\tau' \;=$$

$$=\; \frac{1}{4\,\pi\,\varepsilon_r\,\varepsilon_0} \int_{-\infty}^{\infty} \int_{\mathcal{R}} \frac{q\, \delta^{(3)}\big(\boldsymbol{r}' - \boldsymbol{r_0}(t')\big)\, \delta\big(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{v}\big)}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dt'\, d\tau' \;=$$

$$=\; \frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0} \int_{-\infty}^{\infty} \frac{\delta\big(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r_0}(t')|}{v}\big)}{|\boldsymbol{r}-\boldsymbol{r_0}(t')|}\, dt'\,.$$

We substitute, for fixed $\boldsymbol{r}$ and t, the integration variable t' into a new integration variable s via

$$s = t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r_0}(t')|}{v} = t' - t + \frac{\sqrt{\big(\boldsymbol{r} - \boldsymbol{r_0}(t')\big) \cdot \big(\boldsymbol{r} - \boldsymbol{r_0}(t')\big)}}{v} \, ,$$

$$ds = dt' - \frac{\cancel{2} \big(\boldsymbol{r} - \boldsymbol{r_0}(t')\big) \cdot \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t')}{\cancel{2} \, v \, |\boldsymbol{r} - \boldsymbol{r_0}(t')|} \, dt' \, .$$

This results in

$$V(\boldsymbol{r}, t) \; = \; \frac{q}{4 \, \pi \, \varepsilon_r \, \varepsilon_0} \int_{-\infty}^{\infty} \frac{\delta(s) \, ds}{\Big\{ |\boldsymbol{r} - \boldsymbol{r_0}(t')| \; - \; \frac{1}{v} \big(\boldsymbol{r} - \boldsymbol{r_0}(t')\big) \cdot \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t') \Big\}} \; =$$

$$= \; \frac{q}{4 \, \pi \, \varepsilon_r \, \varepsilon_0} \, \frac{1}{\Big\{ |\boldsymbol{r} - \boldsymbol{r_0}(t_r)| \; - \; \frac{1}{v} \big(\boldsymbol{r} - \boldsymbol{r_0}(t_r)\big) \cdot \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r) \Big\}}$$

where $t' = t_r$ if $s = 0$, i.e., t_r is determined by the equation

$$t_r - t + \frac{|\boldsymbol{r} - \boldsymbol{r_0}(t_r)|}{v} \; = \; 0 \, . \qquad (*)$$

After an analogous calculation for $\boldsymbol{A}(\boldsymbol{r}, t)$ we find the retarded potentials for a point charge as

$$V(\boldsymbol{r}, t) \; = \; \frac{q}{4 \, \pi \, \varepsilon_r \, \varepsilon_0} \, \frac{1}{\Big\{ |\boldsymbol{r} - \boldsymbol{r_0}(t_r)| \; - \; \frac{1}{v} \big(\boldsymbol{r} - \boldsymbol{r_0}(t_r)\big) \cdot \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r) \Big\}} \, ,$$

$$\boldsymbol{A}(\boldsymbol{r}, t) \; = \; \frac{q \, \mu_r \, \mu_0}{4 \, \pi} \, \frac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{\Big\{ |\boldsymbol{r} - \boldsymbol{r_0}(t_r)| \; - \; \frac{1}{v} \big(\boldsymbol{r} - \boldsymbol{r_0}(t_r)\big) \cdot \dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r) \Big\}} \, .$$

These expressions were found independently by Liénard and Wiechert in 1899 and are known as the "Liénard-Wiechert potentials". Here the retarded time t_r is implicitly given, as a function of $\boldsymbol{r}$ and t, by equation $(*)$.

From the Liénard-Wiechert potentials we can calculate the fields $\boldsymbol{E} = -\nabla V - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, for any given particle trajectory $\boldsymbol{r_0}(t')$. Several computer programs have been written, based on these formulas, that interactively visualise the electromagnetic fields of a moving point charge, see e.g. `http://www.cco.caltech.edu/~phys1/java/phys1/MovingCharge/MovingCharge.html` .

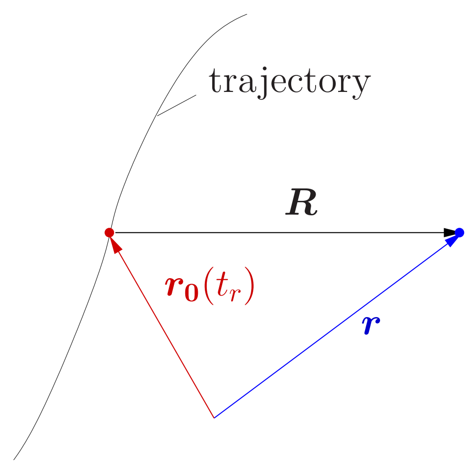
In the following we want to derive, from the Liénard-Wiechert potentials, the important fact that an accelerated point charge emits electromagnetic radiation. To that end we have to calculate the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and, thereupon, the Poynting vector $\boldsymbol{N}$.

To simplify notation, we write $\boldsymbol{R}$ for the vector from the position of the charge at the retarded time t_r to the position $\boldsymbol{r}$ where the field is measured at time t,

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_0}(t_r)\,,$$

$$R = |\boldsymbol{R}|\,,$$

see picture.



Then the Liénard-Wiechert potentials read

$$V(\boldsymbol{r},t) \;=\; \frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\;\frac{1}{\left\{\,R\;-\;\boldsymbol{R}\cdot\dfrac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{v}\,\right\}}\,,$$

$$\boldsymbol{A}(\boldsymbol{r},t) \;=\; \frac{q\,\mu_r\,\mu_0}{4\,\pi}\;\frac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{\left\{\,R\;-\;\boldsymbol{R}\cdot\dfrac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{v}\,\right\}}\,.$$

Calculating the fields $\boldsymbol{E} = -\nabla V - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ requires to know $\frac{\partial}{\partial t}t_r$ and ∇t_r. From differentiating the equation $(*)$ we find

$$\frac{\partial t_r}{\partial t} \;=\; \left(1\;-\;\frac{\boldsymbol{R}}{R}\cdot\frac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{v}\right)^{-1},$$

$$\nabla t_r \;=\; \frac{-\boldsymbol{R}}{v\,R}\Big(1\;-\;\frac{\boldsymbol{R}}{R}\cdot\frac{\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{v}\Big)^{-1}.$$

Also, we need the gradient of $R = |\boldsymbol{r} - \boldsymbol{r}_{\mathbf{0}}(t_r)|$:

$$\boldsymbol{\nabla} R \;=\; \frac{1}{2\,R}\,\boldsymbol{\nabla} R^2 \;=\; \frac{1}{2\,R}\,\boldsymbol{\nabla}\Big(\big(\boldsymbol{r}-\boldsymbol{r}_{\mathbf{0}}(t_r)\big)\cdot\big(\boldsymbol{r}-\boldsymbol{r}_{\mathbf{0}}(t_r)\big)\Big) \;=$$

$$=\;\frac{1}{2\,R}\,\boldsymbol{\nabla}\big(\boldsymbol{r}\cdot\boldsymbol{r}-2\,\boldsymbol{r}\cdot\boldsymbol{r}_{\mathbf{0}}(t_r)+\boldsymbol{r}_{\mathbf{0}}(t_r)\cdot\boldsymbol{r}_{\mathbf{0}}(t_r)\big) \;=$$

$$=\;\frac{1}{2\,R}\Big(2\boldsymbol{r}-2\boldsymbol{r}_{\mathbf{0}}(t_r)-2\big(\boldsymbol{r}\cdot\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\big)\,\boldsymbol{\nabla} t_r+2\big(\boldsymbol{r}_{\mathbf{0}}(t_r)\cdot\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\big)\boldsymbol{\nabla} t_r\Big) \;=$$

$$=\;\frac{\boldsymbol{R}}{R}\;-\;\frac{\boldsymbol{R}\cdot\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{R}\,\boldsymbol{\nabla} t_r$$

We want to determine $\boldsymbol{E}$ and $\boldsymbol{B}$ only for the case that the speed of the charged particle is small in comparison to the speed of light in the medium, $|\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)| \;\ll\; v$. We indicate terms which are of linear or higher order with respect to $|\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)|/v$ by "…" in the following.

$$\boldsymbol{E}(\boldsymbol{r},t) \;=\; -\boldsymbol{\nabla} V(\boldsymbol{r},t)-\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) \;=$$

$$=\;\frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\,\frac{\boldsymbol{\nabla} R-\Big(\boldsymbol{R}\cdot\dfrac{\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{v}\Big)\boldsymbol{\nabla} t_r+\ldots}{\Big(R-\boldsymbol{R}\cdot\dfrac{\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{v}\Big)^2}-\frac{q\,\mu_r\,\mu_0}{4\,\pi}\,\frac{\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\,\dfrac{\partial t_r}{\partial t}\;+\;\ldots}{\Big(R-\boldsymbol{R}\cdot\dfrac{\dot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{v}\Big)} \;=$$

$$=\;\frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\left(\frac{\dfrac{\boldsymbol{R}}{R}\;+\;\Big(\boldsymbol{R}\cdot\dfrac{\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)}{v}\Big)\dfrac{\boldsymbol{R}}{R\,v}\;+\;\ldots}{R^2\,\big(\,1\;+\;\ldots\,\big)}\;-\;\frac{\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\,(\,1\;+\;\ldots)}{v^2\,R\,(\,1\;+\;\ldots)}\right) \;=$$

$$=\;\frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\;\left(\frac{\boldsymbol{R}}{R^3}\;+\;\frac{\big(\boldsymbol{R}\cdot\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\big)\,\boldsymbol{R}-\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\,R^2}{v^2R^3}\;+\;\ldots\right)$$

$$\boldsymbol{B}(\boldsymbol{r},t) \;=\; \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) \;=\; \frac{q\,\mu_0\,\mu_+r}{4\,\pi}\,\frac{\boldsymbol{\nabla}t_r\times\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\;+\;\ldots}{\Big(R-\boldsymbol{R}\cdot\dfrac{\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{v}\Big)} \;=$$

$$=\;\frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\,\frac{\boldsymbol{R}\times\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)}{v^3R^2}\;+\;\ldots$$

If we neglect all terms of linear or higher order with respect to $|\dot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)|/v$ and use the bac-cab rule, we find

$$\boldsymbol{E}(\boldsymbol{r},t) \;=\; \frac{q}{4\,\pi\,\varepsilon_r\,\varepsilon_0}\,\left\{\frac{\boldsymbol{R}}{R^3}\;+\;\frac{\boldsymbol{R}\times\big(\boldsymbol{R}\times\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big)}{R^3\,v^2}\right\}\;,$$

$$\boldsymbol{B}(\boldsymbol{r},t) \;=\; \frac{1}{v}\,\frac{\boldsymbol{R}}{R}\times\boldsymbol{E}(\boldsymbol{r},t)\;.$$

To calculate the Poynting vector, we write $\boldsymbol{E} = \boldsymbol{E}_{||} + \boldsymbol{E}_{\perp}$ where

$$\boldsymbol{E}_{||} \;=\; \frac{q\,\boldsymbol{R}}{4\,\pi\,\varepsilon_r\,\varepsilon_0\,R}$$

is parallel to $\boldsymbol{R}$ and

$$\boldsymbol{E}_{\perp} \;=\; \frac{q\,\boldsymbol{R}\times\big(\boldsymbol{R}\times\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big)}{4\,\pi\,\varepsilon_r\,\varepsilon_0\,R^3\,v^2}$$

is perpendicular to $\boldsymbol{R}$.

Then the Poynting vector reads

$$\boldsymbol{N} \;=\; \boldsymbol{E}\times\boldsymbol{H} \;=\; \frac{1}{\mu_r\,\mu_0}\,\boldsymbol{E}\times\boldsymbol{B} \;=\; \frac{1}{\mu_r\,\mu_0}\,\big(\,\boldsymbol{E}_{||}+\boldsymbol{E}_{\perp}\,\big)\times\Big(\frac{1}{v}\,\frac{\boldsymbol{R}}{R}\times\boldsymbol{E}_{\perp}\Big) \;=$$

$$=\;\frac{1}{\mu_r\,\mu_0\,v}\,\left(\,\frac{\boldsymbol{R}}{R}\,\big|\boldsymbol{E}_{\perp}\big|^2\;-\;\big|\boldsymbol{E}_{||}\big|\,\boldsymbol{E}_{\perp}\,\right)$$

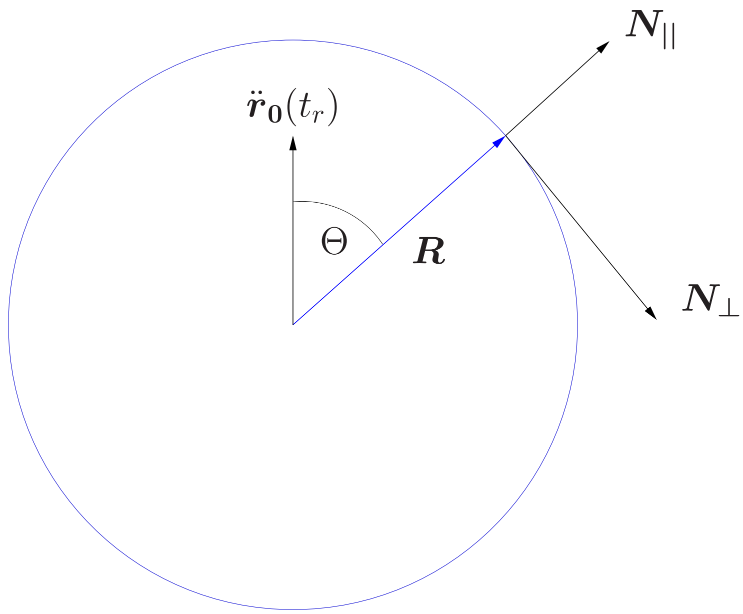
The first term is parallel to $\boldsymbol{R}$,

$$\boldsymbol{N}_{||} \;=\; \frac{1}{\mu_r\,\mu_0\,v}\,\frac{\boldsymbol{R}}{R}\left|\boldsymbol{E}_\perp\right|^2 ,$$

whereas the second term is perpendicular to $\boldsymbol{R}$,

$$\boldsymbol{N}_\perp \;=\; \frac{1}{\mu_r\,\mu_0\,v}\left|\boldsymbol{E}_{||}\right|\boldsymbol{E}_\perp \,.$$

Recall that $\boldsymbol{N}$ gives the energy flux of the electromagnetic field. If we consider a sphere of radius R around the charge, the energy flux through the surface of the sphere is given by $\boldsymbol{N}_{||}$, see picture.



$$\boldsymbol{N}_{||} \;=\; \frac{1}{\mu_r\,\mu_0\,v}\,\frac{\boldsymbol{R}}{R}\left|\boldsymbol{E}_\perp\right|^2 \;=\; \frac{1}{\mu_0\,\mu_r\,v}\,\frac{q^2}{16\,\pi^2\varepsilon_0^2\,\varepsilon_r^2 R^6 v^4}\,\frac{\boldsymbol{R}}{R}\left|\boldsymbol{R}\times\big(\boldsymbol{R}\times\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big)\right|^2 \;=$$

$$=\; \frac{q^2\,\mu_0\,\mu_r\,\boldsymbol{R}}{16\,\pi^2 v\,R^7}\left|\,\boldsymbol{R}\left(\boldsymbol{R}\cdot\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\right)\;-\;\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\,R^2\,\right|^2 \;=$$

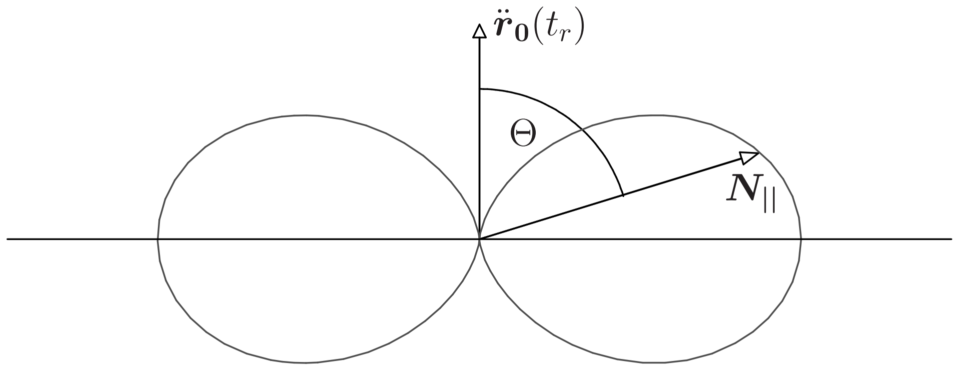
$$=\; \frac{q^2\,\mu_0\,\mu_r\,\boldsymbol{R}}{16\,\pi^2 v\,R^7}\left\{\,\cancel{R^2\big(\boldsymbol{R}\cdot\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big)^2}\;-\;\cancel{2}\left(\boldsymbol{R}\cdot\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\right)^2 R^2\;+\;\left|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\right|^2 R^4\,\right\}.$$

With $\Theta = \measuredangle\left(\boldsymbol{R}, \ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\right)$:

$$\boldsymbol{N}_{||} = \frac{q^2\,\mu_0\,\mu_r\,\boldsymbol{R}}{16\,\pi^2 v\,R^7}\,R^4\,(\,1 - \cos^2\Theta\,)\,\left|\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\right|^2 =$$

$$= \frac{q^2\,\mu_0\,\mu_r\,\left|\ddot{\boldsymbol{r}}_{\mathbf{0}}(t_r)\right|^2}{16\,\pi^2 v\,R^3}\sin^2\Theta\,\boldsymbol{R}\,.$$

So the radiated energy flux is maximal in the directions perpendicular to the acceleration ($\Theta = \pi/2$) and it is zero in the directions parallel to the acceleration ($\Theta = 0, \pi$).

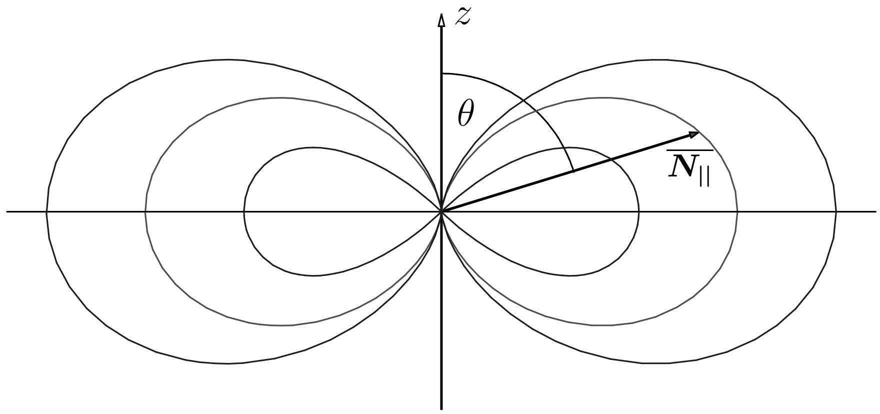


The picture shows how the "tip of the arrow" $\boldsymbol{N}_{||}$ varies if Θ varies from zero to π. This is known as the "radiation pattern" of an accelerated charge.

Examples:

- If a charged particle moves with constant speed in a circle, the radiation is zero in the directions towards the centre and away from the centre.

- An oscillating electric dipole does not radiate along the dipole axis; the radiation is maximal in directions perpendicular to the dipole axis. In this case the acceleration $|\ddot{\boldsymbol{r}}_{\mathbf{0}}|$ is proportional to ω^2, so $\boldsymbol{N}_{||}$ is proportional to ω^4.



Radiation pattern of oscillating electric dipole

Finally, we want to calculate the total power $P(t_r)$ radiated by the particle at time t_r.

Let $\mathcal{S}$ be the sphere of radius R, centered at the position of the charge at time t_r. Then

$$P(t_r) \;=\; \int_{\mathcal{S}} \boldsymbol{N}_{||} \cdot d\boldsymbol{a} \;=$$

$$= \int_{\Theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{q^2\,\mu_0\,\mu_r\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big|^2}{16\,\pi^2 v} \sin^2\Theta\,\frac{\boldsymbol{R}}{R^3} \cdot \underbrace{\frac{\boldsymbol{R}}{R}\,R^2 \sin\Theta\,d\Theta\,d\phi}_{=d\boldsymbol{a}} \;=$$

$$= \frac{q^2\,\mu_0\,\mu_r\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big|^2}{16\,\pi^2 v} \underbrace{\int_{\Theta=0}^{\pi} \sin^3\Theta\,d\Theta}_{=4/3} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{=2\pi} \;=\; \frac{q^2\,\mu_0\,\mu_r\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t_r)\big|^2}{6\,\pi\,v}\,.$$

After renaming t_r into t, we get the "Larmor formula"

$$P(t) \;=\; \frac{q^2\,\mu_0\,\mu_r\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t)\big|^2}{6\,\pi\,v} \;=\; \frac{q^2\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t)\big|^2}{6\,\pi\,\varepsilon_0\,\varepsilon_r\,v^3}\,.$$

In vacuum ($\mu_r = \varepsilon_r = 1$, $v = c$) the Larmor formula simplifies to

$$P(t) \;=\; \frac{q^2 \mu_0\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t)\big|^2}{6\,\pi\,c} \;=\; \frac{q^2\,\big|\ddot{\boldsymbol{r}}_{\boldsymbol{0}}(t)\big|^2}{6\,\pi\,\varepsilon_0\,c^3}\,.$$

Keep in mind that we have derived the Larmor formula under the assumption that the charged particle's speed is small in comparison to the speed of light. For particles moving nearly at the speed of light (e.g. electrons in a synchrotron) one has to use a modified Larmor formula which is not treated here.

Note that in a medium the speed $|\dot{\boldsymbol{r}}_{\mathbf{0}}(t)|$ of a charged particle may even be bigger than the speed of light v in this medium. If this is the case, the emitted radiation is known as *Cherenkov radiation*.

The Larmor formula demonstrates that.

$$\boxed{P(t) \sim |\ddot{\boldsymbol{r}}_{\mathbf{0}}(t)|^2 \, .}$$

Whenever (the magnitude or direction of) the velocity vector $\dot{\boldsymbol{r}}_{\mathbf{0}}(t)$ of a charged particle changes, electromagnetic radiation is emitted. Thereby the particle loses kinetic energy. This recoil effect is known as "radiation reaction".

> As a consequence, a classical charged particle on a circular path loses kinetic energy. So a classical electron, circling around the nucleus of an atom, would quickly spiral into the nucleus. This was a fundamental problem of Rutherford's model of the atom when interpreted in terms of classical electrodynamics. One needs quantum mechanics to explain the stability of atoms.

Here is an important special situation to which the Larmor formula applies.

> Whenever a charged particle is decelerated, it radiates. The emitted radiation is known as "bremsstrahlung" (german for "deceleration radiation"). Bremsstrahlung occurs, e.g., when electrons are sent into a block of matter. It was first observed in the 1890s by N. Tesla.

Summary: In a medium with constant ε_r and μ_r, the general solution to the homogeneous Maxwell equations is a superposition of plane harmonic waves. These waves are transverse, with dispersion relation $\omega = v|\boldsymbol{k}|$. Electric and magnetic field are related by

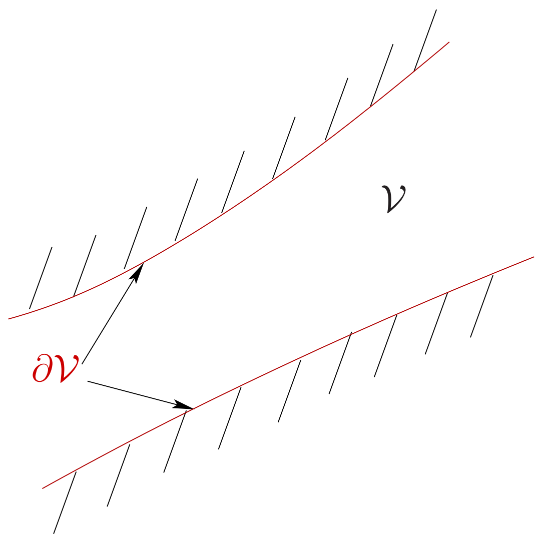
$$\boldsymbol{B} = \frac{1}{v}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \times \boldsymbol{E}\,.$$

A particular solution to the inhomogeneous equations is given by the retarded potentials. The general solution to the inhomogeneous equations is given by the retarded potentials plus the general solution to the homogeneous equations.

For a point particle, the retarded potentials take the form of the Liénard-Wiechert potentials. From these one can derive the Poynting vector and, in particular, the Larmor formula. It demonstrates that an accelerated charged particle emits electromagnetic radiation.

V. Solving Maxwell's equations in bounded space

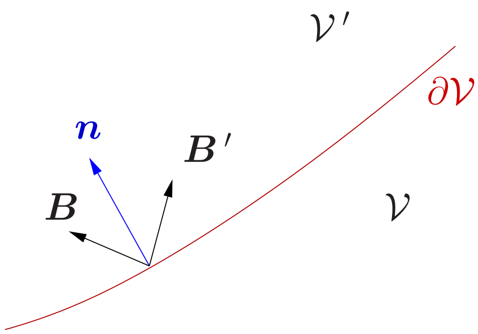
In Chapter IV we have discussed solutions to Maxwell's equations in free space. We will now investigate how these results have to be modified if Maxwell's equations are to hold only in a certain region $\mathcal{V}$, with prescribed boundary conditions on $\partial\mathcal{V}$. The special form of the boundary conditions depends on the physical nature of the material on either side of $\partial\mathcal{V}$.

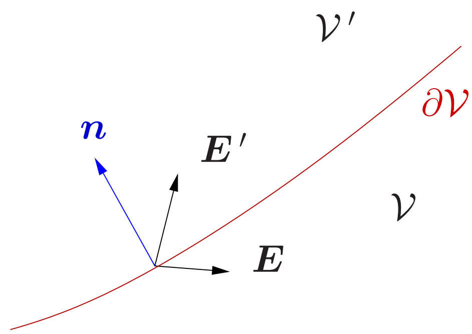


We will concentrate on the case that there is a simple medium, with constant permittivity ε_r and constant permeability μ_r, in $\mathcal{V}$ and a perfect conductor outside $\mathcal{V}$. As a preparation, we discuss the general jump conditions for electromagnetic fields at the interface between two media.

V.1 Jump conditions (discontinuity conditions) at the interface between two media

At an interface between two media, where the material properties behave discontinuously, electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are, in general, discontinuous as well. However, we will now derive that, for *any* kind of media, Maxwell's equations require that some components of $\boldsymbol{E}$ and $\boldsymbol{B}$ must be continuous.





Let $\boldsymbol{E}$ and $\boldsymbol{B}$ be the limit values of electric and magnetic fields at a point of the boundary surface $\partial\mathcal{V}$ if we approach it from $\mathcal{V}$. Similarly, let $\boldsymbol{E}'$ and $\boldsymbol{B}'$ be the limit values of electric and magnetic fields at a point of the boundary surface $\partial\mathcal{V}$ if we approach it from $\mathcal{V}'$. Let $\boldsymbol{n}$ be the unit vector normal to $\partial\mathcal{V}$.

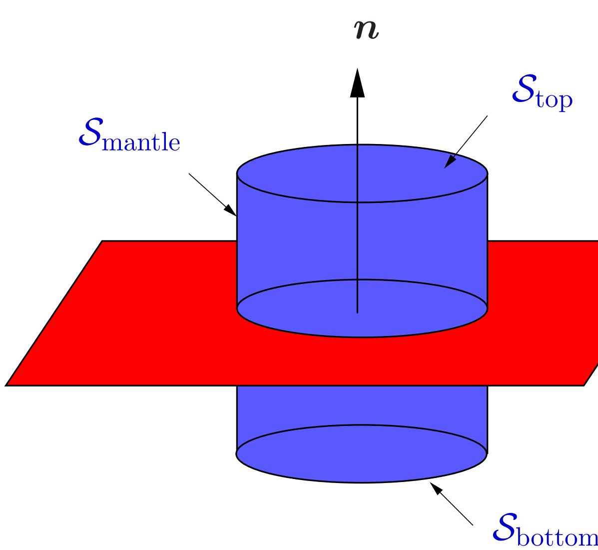
Claim:

$$(\boldsymbol{B} - \boldsymbol{B}') \cdot \boldsymbol{n} = 0$$

$$(\boldsymbol{E} - \boldsymbol{E}') \times \boldsymbol{n} = \boldsymbol{0}$$

i.e., the normal component of $\boldsymbol{B}$ and the tangential component of $\boldsymbol{E}$ are continuous.

Proof: Apply Maxwell's equation (MI) in integral form to a "pill-box" centered around a point of $\partial\mathcal{V}$. The surface of the pill-box consists of three parts: $\mathcal{S}_{\text{bottom}}$, $\mathcal{S}_{\text{top}}$, and $\mathcal{S}_{\text{mantle}}$.



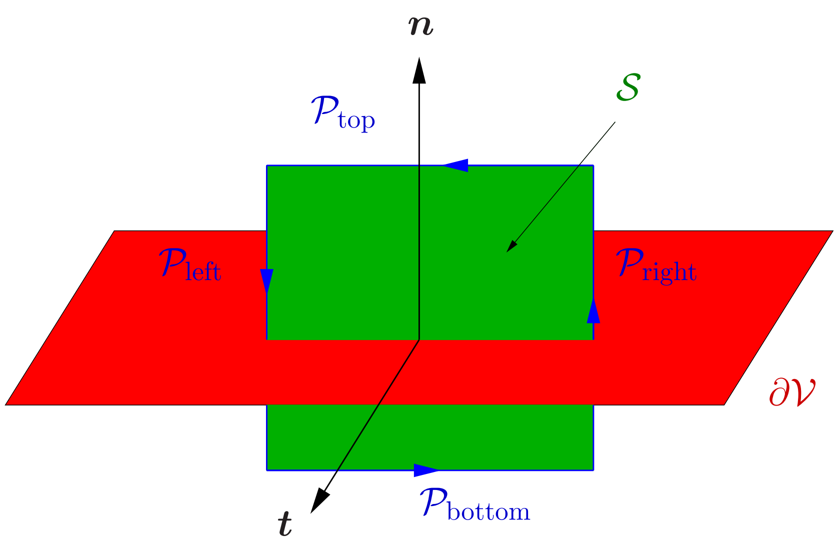
$$\int_{\mathcal{S}_{\text{bottom}}} \boldsymbol{B} \cdot d\boldsymbol{a} \; + \int_{\mathcal{S}_{\text{top}}} \boldsymbol{B} \cdot d\boldsymbol{a} \; + \int_{\mathcal{S}_{\text{mantle}}} \boldsymbol{B} \cdot d\boldsymbol{a} \; = \; 0 \, .$$

Now let the height of the pill-box go to zero and make the diameter of the pill-box infinitesimally small:

$$\boldsymbol{B} \cdot (\, -\, da\, \boldsymbol{n}\,) \, + \, \boldsymbol{B}' \cdot (\, da\, \boldsymbol{n}\,) \, + \, 0 \, = \, 0 \, .$$

As $da \neq 0$, this proves $(\, \boldsymbol{B} - \boldsymbol{B}'\,) \cdot \boldsymbol{n} \, = \, 0\,$.

To prove the boundary condition for $\boldsymbol{E}$, we apply Maxwell's equation (MII) in integral form to a rectangular loop centered around a point of $\partial\mathcal{V}$. The loop consists of four parts, $\mathcal{P}_{\text{bottom}}$, $\mathcal{P}_{\text{right}}$, $\mathcal{P}_{\text{top}}$, and $\mathcal{P}_{\text{left}}$.



$$\int_{\mathcal{P}_{\text{bottom}}} \boldsymbol{E} \cdot d\boldsymbol{\ell} + \int_{\mathcal{P}_{\text{right}}} \boldsymbol{E} \cdot d\boldsymbol{\ell} + \int_{\mathcal{P}_{\text{top}}} \boldsymbol{E} \cdot d\boldsymbol{\ell} + \int_{\mathcal{P}_{\text{left}}} \boldsymbol{E} \cdot d\boldsymbol{\ell} \; = \; - \frac{\mathrm{d}}{dt} \int_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{a} \, .$$

We have chosen the surface $\mathcal{S}$ spanned by the loop perpendicular to the boundary. Now we send $\mathcal{P}_{\text{right}}$ and $\mathcal{P}_{\text{left}}$ to zero length and we make $\mathcal{P}_{\text{bottom}}$ and $\mathcal{P}_{\text{top}}$ infinitesimally short. If $\boldsymbol{t}$ is the unit vector perpendicular to $\mathcal{S}$, we find

$$\boldsymbol{E} \cdot (\, \boldsymbol{n} \times \boldsymbol{t}\, d\ell\,) \, + \, 0 \, + \, \boldsymbol{E}' \cdot (\, -\, \boldsymbol{n} \times \boldsymbol{t}\, d\ell\,) \, + \, 0 \, = \, 0 \, .$$

As $d\ell \neq 0$, this proves that $(\, \boldsymbol{E} - \boldsymbol{E}'\,) \cdot (\, \boldsymbol{n} \times \boldsymbol{t}\,) \, = \, 0$. With the identity $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$:

$$\boldsymbol{t} \cdot \Big((\, \boldsymbol{E} - \boldsymbol{E}'\,) \times \boldsymbol{n} \Big) \, = \, 0 \, .$$

As we can choose such a loop for *any* vector $\boldsymbol{t}$ tangent to $\partial\mathcal{V}$, this proves that $(\boldsymbol{E}-\boldsymbol{E}')\times\boldsymbol{n} \parallel \boldsymbol{n}$. On the other hand, we know from the definition of the cross product that $(\boldsymbol{E}-\boldsymbol{E}')\times\boldsymbol{n}\perp\boldsymbol{n}$, so $(\boldsymbol{E}-\boldsymbol{E}')\times\boldsymbol{n}=\boldsymbol{0}$.

□

By definition, a *perfect conductor* is a medium in which charges can freely move without any time delay. As a consequence, the charges will always arrange themselves in such a position that the electric and magnetic fields are zero. Thus, if in the previous argument the region $\mathcal{V}'$ is occupied by a perfect conductor, we have $\boldsymbol{E}'=\boldsymbol{0}$ and $\boldsymbol{B}'=\boldsymbol{0}$. From the discontinuity conditions derived we thus find:

If the region $\mathcal{V}$ is surrounded by a perfect conductor, the electric and magnetic fields inside $\mathcal{V}$ satisfy the boundary conditions

$$\boldsymbol{B}\cdot\boldsymbol{n}=0 \quad \text{on} \quad \partial\mathcal{V}$$

$$\boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{0} \quad \text{on} \quad \partial\mathcal{V}$$

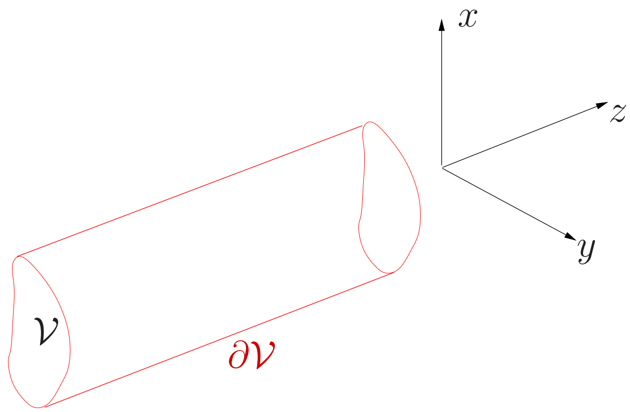
where $\boldsymbol{n}$ is the unit vector normal to $\partial\mathcal{V}$.

This is true for *any* medium inside $\mathcal{V}$ and for arbitrarily time-dependent fields. For time-independent fields, $\boldsymbol{E}=-\boldsymbol{\nabla}V$ and our boundary condition $\boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{0}$ requires the scalar potential V to be constant on $\partial\mathcal{V}$; in other words, $\partial\mathcal{V}$ is to be an equipotential. This is *not* true for time-dependent fields because then $\boldsymbol{E}=-\boldsymbol{\nabla}V-\frac{\partial}{\partial t}\boldsymbol{A}$.

V.2. Electromagnetic fields in a wave guide

As an example for solving Maxwell's equations in bounded regions, let $\mathcal{V}$ be a cylindrical hollow with arbitrarily shaped but constant cross-section, surrounded by a perfect conductor. We assume that the cylinder is infinitely long. It is then called a *wave guide*. (A cylinder of finite length, with end surfaces, is called a *cavity*.)

We choose Cartesian coordinates such that the axis of the wave guide coincides with the z-direction.



We assume that inside $\mathcal{V}$ there is a medium of constant relative permittivity ε_r and constant relative permeablity μ_r, and that in this region there are no charges and no currents, so:

$$\boldsymbol{D}(\boldsymbol{r},t) \;=\; \varepsilon_r\,\varepsilon_0\,\boldsymbol{E}(\boldsymbol{r},t)\;, \quad \boldsymbol{B}(\boldsymbol{r},t) \;=\; \mu_r\,\mu_0\,\boldsymbol{H}(\boldsymbol{r},t)\;,$$

$$\rho(\boldsymbol{r},t) \;=\; 0\;, \qquad \boldsymbol{J}(\boldsymbol{r},t) \;=\; \boldsymbol{0}\;.$$

Thus, with our usual abbreviation $v \;=\; (\,\varepsilon_r\,\mu_r\,\varepsilon_0\,\mu_0\,)^{-1/2}$, Maxwell's equations require that inside $\mathcal{V}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} \;=\; 0 \qquad \text{(MI)}$$

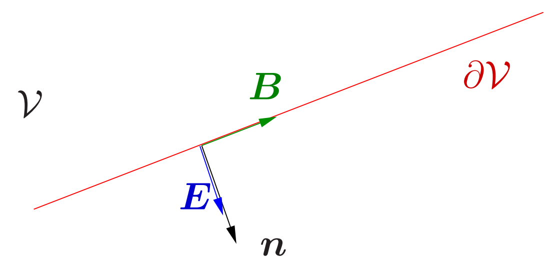
$$\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} \;=\; \boldsymbol{0} \qquad \text{(MII)}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} \;=\; 0 \qquad \text{(MIII)}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{v^2}\frac{\partial}{\partial t}\boldsymbol{E} \;=\; \boldsymbol{0} \qquad \text{(MIV)}$$

Outside $\mathcal{V}$ we assume a perfect conductor, so our boundary conditions are

$$\boldsymbol{B} \cdot \boldsymbol{n} \,=\, 0 \quad \text{and} \quad \boldsymbol{E} \times \boldsymbol{n} \,=\, \boldsymbol{0} \quad \text{on} \quad \partial\mathcal{V} \,.$$



We want to find the solution to this set of differential equations with boundary conditions in the form of harmonic waves,

$$\boldsymbol{E}(x,y,z,t) \,=\, \mathrm{Re}\big\{\boldsymbol{\mathcal{E}}(x,y)\, e^{i(kz-\omega t)}\big\} \,, \qquad \text{(HWI)}$$

$$\boldsymbol{B}(x,y,z,t) \,=\, \mathrm{Re}\big\{\boldsymbol{\mathcal{B}}(x,y)\, e^{i(kz-\omega t)}\big\} \,, \qquad \text{(HWII)}$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are complex-valued vector fields. The general solution is then a superposition of such waves.

Note that here we seek the general solution by a Fourier expansion with respect to z, but not with respect to x and y. This is different from what we have done for waves in free space. Correspondingly, here k is the wave number in z-direction and *not* the modulus of a wave vector $\boldsymbol{k}$. The ansatz (HWI) and (HWII) is convenient here because the geometry of the wave guide distinguishes the z coordinate.

Feeding the harmonic wave ansatz (HWI) and (HWII) into Maxwell's equations yields a system of partial differential equations for the components of the complex amplitude vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$. We give the derivation in detail for the first Maxwell equation (MI):

$$0 \,=\, \boldsymbol{\nabla} \cdot \boldsymbol{B} \,=\, \mathrm{Re}\big\{\boldsymbol{\nabla} \cdot \big(\boldsymbol{\mathcal{B}}\, e^{i(kz-\omega t)}\big)\big\} \,=$$

$$=\, \mathrm{Re}\Big\{ \underbrace{\Big(\frac{\partial}{\partial x}\mathcal{B}_x \,+\, \frac{\partial}{\partial y}\mathcal{B}_y \,+\, ik\mathcal{B}_z\Big)}_{Z} e^{i(kz-\omega t)} \Big\}$$

After decomposing Z into modulus and phase, $Z = |Z|e^{i\alpha}$, we find

$$0 \,=\, \mathrm{Re}\Big\{\big(|Z|\, e^{i(\alpha+kz-\omega t)}\Big\} \,=\, |Z|\cos(\alpha + kz - \omega t)\,.$$

This equation can hold for all t only if $Z = 0$ which gives the desired differential equation for $\boldsymbol{\mathcal{B}}$ resulting from (MI).

By applying an analogous argument to all Maxwell equations, we get the following set of partial differential equations for $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$.

(MI)
$$\frac{\partial}{\partial x}\mathcal{B}_x \,+\, \frac{\partial}{\partial y}\mathcal{B}_y \,+\, ik\mathcal{B}_z \,=\, 0\,,$$

(MII)
$$\frac{\partial}{\partial y}\mathcal{E}_z \,-\, ik\mathcal{E}_y \,-\, i\omega\mathcal{B}_x \,=\, 0\,,$$

$$-\frac{\partial}{\partial x}\mathcal{E}_z \,+\, ik\mathcal{E}_x \,-\, i\omega\mathcal{B}_y \,=\, 0\,,$$

$$\frac{\partial}{\partial x}\mathcal{E}_y \,-\, \frac{\partial}{\partial y}\mathcal{E}_x \,-\, i\omega\mathcal{B}_z \,=\, 0\,,$$

(MIII)
$$\frac{\partial}{\partial x}\mathcal{E}_x \,+\, \frac{\partial}{\partial y}\mathcal{E}_y \,+\, ik\mathcal{E}_z \,=\, 0\,,$$

(MIV)
$$\frac{\partial}{\partial y}\mathcal{B}_z \,-\, ik\mathcal{B}_y \,+\, \frac{i\omega}{v^2}\mathcal{E}_x \,=\, 0\,,$$

$$-\frac{\partial}{\partial x}\mathcal{B}_z \,+\, ik\mathcal{B}_x \,+\, \frac{i\omega}{v^2}\mathcal{E}_y \,=\, 0\,,$$

$$\frac{\partial}{\partial x}\mathcal{B}_y \,-\, \frac{\partial}{\partial y}\mathcal{B}_x \,+\, \frac{i\omega}{v^2}\mathcal{E}_z \,=\, 0\,.$$

One usually refers to $\mathcal{E}_z$ and $\mathcal{B}_z$ as to the "longitudinal" components of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$. Correspondingly $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{B}_x$ and $\mathcal{B}_y$ are called the "transverse" components of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$. This terminology should not be confused with the terminology for waves in free space that was used in Section III. There "transverse" meant that the amplitude vectors are perpendicular to the wave vector $\boldsymbol{k}$ and we have seen that electromagnetic waves in a medium with constant ε_r and constant μ_r are always transverse in this sense. Here "transverse" means that the amplitude vectors are perpendicular to the axis of the wave guide, and we will see that, in general, this is not the case, i.e., that the longitudinal components $\mathcal{E}_z$ and $\mathcal{B}_z$ are non-zero.
One introduces the following terminology.

$$\boxed{\begin{array}{ll} \text{TEM (transverse electromagnetic) mode} & \text{if } \mathcal{E}_z = 0 \text{ and } \mathcal{B}_z = 0\,, \\ \text{TE (transverse electric) mode} & \text{if } \mathcal{E}_z = 0 \text{ and } \mathcal{B}_z \neq 0\,, \\ \text{TM (transverse magnetic) mode} & \text{if } \mathcal{E}_z \neq 0 \text{ and } \mathcal{B}_z = 0\,. \end{array}}$$

We first discuss TEM modes. It is our goal to show that they cannot exist in a hollow cylinder.

For a TEM mode we have (MII) and (MIII) yield

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = \begin{pmatrix} \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} \\ \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} \\ \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial 0}{\partial y} - 0 \\ 0 - \frac{\partial 0}{\partial x} \\ i\,\omega\,\mathcal{B}_z \end{pmatrix} = \mathbf{0}$$

which means that $\boldsymbol{\mathcal{E}}$ can be written as the gradient of a complex scalar field Φ. (To prove this, decompose $\boldsymbol{\mathcal{E}}$ into real and imaginary parts, $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_1 + i\boldsymbol{\mathcal{E}}_2$. Then $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = \mathbf{0}$ implies $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}_1 = \mathbf{0}$ and $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}_2 = \mathbf{0}$, hence $\boldsymbol{\mathcal{E}}_1 = \boldsymbol{\nabla}\Phi_1$ and $\boldsymbol{\mathcal{E}}_2 = \boldsymbol{\nabla}\Phi_2$ and the statement holds with $\Phi = \Phi_1 + i\Phi_2$.) From $\boldsymbol{\mathcal{E}} = \boldsymbol{\nabla}\Phi$ we calculate

$$\Delta\Phi = \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} = -\,i\,k\,\mathcal{E}_z = \mathbf{0}\,.$$

We now use the boundary condition

$$\boldsymbol{\mathcal{E}} \parallel \boldsymbol{n} \qquad \text{on} \quad \partial\mathcal{V} \,.$$

As $\boldsymbol{\mathcal{E}} = \boldsymbol{\nabla}\Phi$, this implies that $\Phi =$ constant on $\partial\mathcal{V}$. As we are free to add a constant to Φ, we may assume that

$$\Phi \,=\, 0 \qquad \text{on} \quad \partial\mathcal{V} \,.$$

We now apply the Gauss theorem to the vector field $\Phi^* \, \boldsymbol{\nabla}\Phi$ where the star means complex conjugation. (To prove that the Gauss theorem holds for complex vector fields, decompose into real and imaginary parts.)

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \big(\Phi^* \, \boldsymbol{\nabla}\Phi \big) \, d\tau \;=\; \int_{\partial\mathcal{V}} \Phi^* \, \boldsymbol{\nabla}\Phi \cdot d\boldsymbol{a} \,.$$

The right-hand side vanishes because Φ is zero on $\partial\mathcal{V}$. On the left-hand side we use the product rule:

$$\int_{\mathcal{V}} \Big(\Phi^* \, \Delta\Phi \,+\, \boldsymbol{\nabla}\Phi^* \cdot \boldsymbol{\nabla}\Phi \Big) \, d\tau \;=\; 0 \,.$$

As $\Delta\Phi = 0$ in $\mathcal{V}$, and $\boldsymbol{\nabla}\Phi^* \cdot \boldsymbol{\nabla}\Phi = |\boldsymbol{\nabla}\Phi|^2$, we have

$$\int_{\mathcal{V}} \big| \boldsymbol{\nabla}\Phi \big|^2 \, d\tau \;=\; 0 \,.$$

As $\big|\boldsymbol{\nabla}\Phi\big|^2 \geq 0$, this implies that $\boldsymbol{\mathcal{E}} \,=\, \boldsymbol{\nabla}\Phi \,=\, \boldsymbol{0}$ in $\mathcal{V}$. From (MII) we then find that $\mathcal{B}_y = \mathcal{B}_x = 0$ in $\mathcal{V}$, i.e., all components of the magnetic field vanish as well. Thus, our assumption that we have a TEM mode has led us to the conclusion that both the electric and the magnetic field must vanish:

> In a hollow cylinder of arbitrary cross-section, a TEM mode cannot exist.

Note that in a hollow between two cylinders (e.g., in a coaxial cable), a TEM mode can exist. Our proof does not work in this case because the equation $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} \,=\, \boldsymbol{0}$ does not imply $\boldsymbol{\mathcal{E}} \,=\, \boldsymbol{\nabla}\Phi$ if it holds only on a domain with "holes". (In mathematical terms, the region must be *simply connected.*) Also, we cannot set $\Phi = 0$ on the whole boundary if the boundary consists of two connected components, an inner one and an outer one.

We now consider the TE modes in detail. TM modes can be treated analogously. For TE modes Maxwell's equations become

(MI)
$$\frac{\partial}{\partial x}\mathcal{B}_x + \frac{\partial}{\partial y}\mathcal{B}_y + ik\mathcal{B}_z = 0 ,$$

(MII)
$$k\mathcal{E}_y + \omega\mathcal{B}_x = 0 ,$$

$$k\mathcal{E}_x - \omega\mathcal{B}_y = 0 ,$$

$$\frac{\partial}{\partial x}\mathcal{E}_y - \frac{\partial}{\partial y}\mathcal{E}_x - i\omega\mathcal{B}_z = 0 ,$$

(MIII)
$$\frac{\partial}{\partial x}\mathcal{E}_x + \frac{\partial}{\partial y}\mathcal{E}_y = 0 ,$$

(MIV)
$$\frac{\partial}{\partial y}\mathcal{B}_z - ik\mathcal{B}_y + \frac{i\omega}{v^2}\mathcal{E}_x = 0 ,$$

$$-\frac{\partial}{\partial x}\mathcal{B}_z + ik\mathcal{B}_x + \frac{i\omega}{v^2}\mathcal{E}_y = 0 ,$$

$$\frac{\partial}{\partial x}\mathcal{B}_y - \frac{\partial}{\partial y}\mathcal{B}_x = 0 ,$$

(MII) and (MIV) allow to express $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{B}_x$ and $\mathcal{B}_y$ in terms of $\mathcal{B}_z$:

$$\mathcal{E}_y = -\frac{\omega}{k}\,\mathcal{B}_x = \frac{-i\,\omega\,v^2}{\big(\omega^2 - v^2\,k^2\big)}\,\frac{\partial}{\partial x}\mathcal{B}_z ,$$

$$\mathcal{E}_x = \frac{\omega}{k}\,\mathcal{B}_y = \frac{i\,\omega\,v^2}{\big(\omega^2 - v^2\,k^2\big)}\,\frac{\partial}{\partial y}\mathcal{B}_z .$$

Thus, the solution is known if $\mathcal{B}_z$ is known. (Here one has to make sure that $\omega^2 \neq v^2k^2$ because otherwise we would divide by zero. One can show that, indeed, TE modes with $\omega^2 = v^2k^2$ cannot exist. The proof is similar to our proof that TEM modes cannot exist.)

To determine $\mathcal{B}_z$, we start from (MIV) which implies

$$\frac{\partial^2}{\partial x^2}\mathcal{B}_z = ik\frac{\partial}{\partial x}\mathcal{B}_x + \frac{i\omega}{v^2}\frac{\partial}{\partial x}\mathcal{E}_y \,,$$

$$\frac{\partial^2}{\partial y^2}\mathcal{B}_z = ik\frac{\partial}{\partial y}\mathcal{B}_y - \frac{i\omega}{v^2}\frac{\partial}{\partial y}\mathcal{E}_x \,.$$

Adding these two equations and using (MI) and (MII) results in

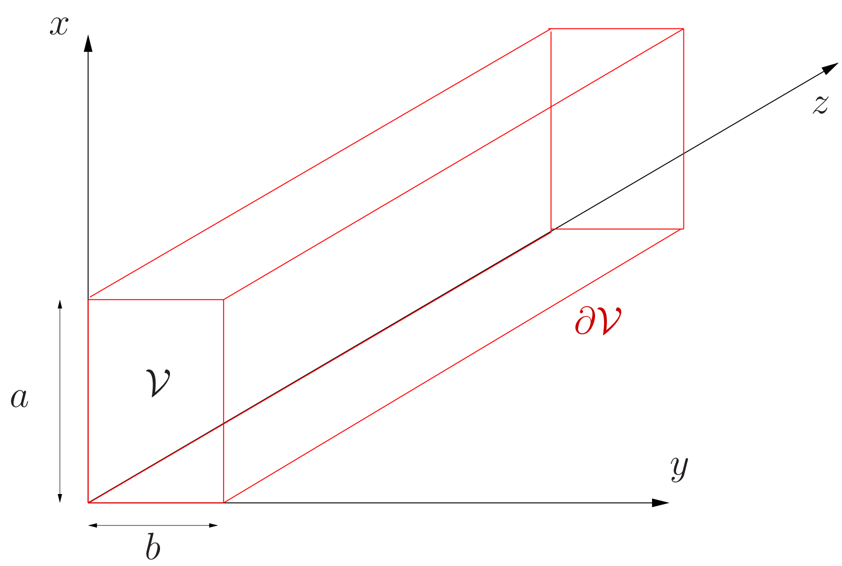
$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\mathcal{B}_z = \big(k^2 - \frac{\omega^2}{v^2}\big)\mathcal{B}_z \,. \qquad \text{(HE)}$$

Thus, $\mathcal{B}_z$ satisfies the *eigenvalue equation* of the two-dimensional Laplace operator. [The eigenvalue equation of the (N-dimensional) Laplace operator is known as the (N-dimensional) *Helmholtz equation*.] The boundary condition $\boldsymbol{\mathcal{B}} \cdot \boldsymbol{n} = 0$ requires

$$\begin{pmatrix} \dfrac{-\,i\,k\,v^2}{\big(\omega^2 - v^2\,k^2\big)}\,\dfrac{\partial\mathcal{B}_z}{\partial_x} \\[2ex] \dfrac{i\,k\,v^2}{\big(\omega^2 - v^2\,k^2\big)}\,\dfrac{\partial\mathcal{B}_z}{\partial_y} \\[2ex] \mathcal{B}_z \end{pmatrix} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\mathcal{V} \qquad \text{(BC)}$$

where we have written $\boldsymbol{\mathcal{B}}$ as a column vector with components $\mathcal{B}_x$, $\mathcal{B}_y$ and $\mathcal{B}_z$.

Up to here everything is true for cylinders of arbitrary cross-sectional shape. We will now work out the solution for the case that the cross-section is rectangular, with width a in the x-direction and width b in the y-direction, see picture.



To solve the Helmholtz equation (HE) with the boundary condition (BC) for this case, we make a separation ansatz

$$\mathcal{B}_z(x, y) \,=\, X(x)\, Y(y)\;.$$

This puts (HE) into the form

$$X''(x)\, Y(y) \,+\, X(x)\, Y''(y) \,+\, \Big(\frac{\omega^2}{v^2} \,-\, k^2\Big)\, X(x)\, Y(y) \,=\, 0\;.$$

After dividing by $X(x)\,Y(y)$:

$$-\,\frac{X''(x)}{X(x)} \,=\, \frac{Y''(y)}{Y(y)} \,+\, \frac{\omega^2}{v^2} \,-\, k^2 \,=\, k_x^2\,.$$

As the first expression is independent of y and the second is independent of x, the quantity we called k_x^2 must be a constant. The differential equation

$$X''(x) \,+\, k_x^2\, X(x) \,=\, 0$$

has the general solution

$$X(x) \,=\, \alpha \sin(k_x x) \,+\, \beta \cos(k_x x)\,,$$

$$X'(x) \,=\, \alpha\, k_x \cos(k_x x) \,-\, \beta\, k_x \sin(k_x x)\,.$$

Our boundary condition (BC) requires

$$\text{at } x = 0 : \qquad X'(0) \;=\; \alpha\, k_x \;=\; 0 \qquad \Rightarrow \qquad \alpha = 0\,;$$

$$\text{at } x = a : \qquad X'(a) \;=\; -\,\beta\, k_x \sin(k_x a) \;=\; 0 \qquad \Rightarrow \qquad k_x a = n\pi$$

where n is a non-negative integer. Thus, the solution is

$$X(x) \;=\; \beta \cos(k_x x) \qquad \text{with} \qquad k_x \;=\; \frac{n\pi}{a}\,.$$

Similarly, the differential equation $Y''(y) \,+\, k_y^2\, Y(y) \;=\; 0\,$, where

$$k_y^2 \;=\; -\,k_x^2 \,+\, \frac{\omega^2}{v^2} \,-\, k^2\,,$$

leads to the solution

$$Y(y) \;=\; \gamma \cos(k_y y) \qquad \text{with} \qquad k_y \;=\; \frac{m\pi}{b}\,.$$

If we write $\beta\,\gamma \;=\; B_0$, the general TE mode is, thus, of the form

$$\mathcal{B}_z(x,y) \;=\; B_0 \cos\bigl(\frac{n\pi x}{a}\bigr) \cos\bigl(\frac{m\pi y}{b}\bigr)$$

$$\mathcal{E}_y(x,y) \;=\; -\,\frac{\omega}{k}\,\mathcal{B}_x(x,y) \;=\; \frac{-\,i\,\omega\,v^2\,n\,\pi\,B_0}{\bigl(\,\omega^2 \,-\, v^2\,k^2\,\bigr)\,a} \sin\bigl(\frac{n\pi x}{a}\bigr) \cos\bigl(\frac{m\pi y}{b}\bigr)$$

$$\mathcal{E}_x(x,y) \;=\; \frac{\omega}{k}\,\mathcal{B}_y(x,y) \;=\; \frac{-\,i\,\omega\,v^2\,m\,\pi\,B_0}{\bigl(\,\omega^2 \,-\, v^2\,k^2\,\bigr)\,b} \cos\bigl(\frac{n\pi x}{a}\bigr) \sin\bigl(\frac{m\pi y}{b}\bigr)\,,$$

where n and m are non-negative integers. The dispersion relation ($\omega - k-$relation) reads

$$\boxed{\bigl(\frac{n\pi}{a}\bigr)^2 \,+\, \bigl(\frac{m\pi}{b}\bigr)^2 \;=\; \frac{\omega^2}{v^2} \,-\, k^2\,.}$$

One refers to a TE mode labeled by integers n and m as to a TE$_{nm}$ mode. The usual convention is that the first index refers to the larger dimension of the wave guide, i.e., our notation is valid if $a \geq b$.

Note that ω and k are *not* related by the usual dispersion relation $\omega = kv$. The reason is that we made a Fourier expansion only with respect to the z coordinate. The dispersion relation for TE$_{nm}$ modes,

$$\omega = v \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + k^2}$$

requires that, for real k, we must have

$$\omega \geq \omega_{n,m}$$

where the *cut-off* frequency ω_{nm} is given by

$$\omega_{nm} = v \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} .$$

Quite generally, for any kind of waves, one uses the following terminology.

A value of ω where the wave length becomes infinite ($k = 2\pi/\lambda \to 0$) is called a "cut-off frequency" and a value of ω where the wave length becomes zero ($k = 2\pi/\lambda \to \infty$) is called a "resonance frequency".

Thus, any TE$_{nm}$ mode must have a frequency bigger than the cut-off frequency ω_{nm}. One can show that a TE$_{00}$ mode cannot exist. (The proof is similar to the proof that TEM modes cannot exist.) Hence, ω_{10} is the minimum frequency for *all* TE modes.

Solutions with imaginary k are not harmonic waves but rather exponentially decaying fields. They are refered to as *evanescent modes.*

From the dispersion relation we find the phase velocity and the group velocity of the TE$_{nm}$ modes:

$$v_p = \frac{\omega}{k} = v\sqrt{1 + \frac{\omega_{n,m}^2}{v^2k^2}} \geq v \ ,$$

$$v_g = \frac{d\omega}{dk} = \frac{v}{\sqrt{1 + \frac{\omega_{n,m}^2}{v^2k^2}}} \leq v \ .$$

If inside the wave guide there is vacuum, we have $\varepsilon_r = \mu_r = 1$ and v is the vacuum velocity of light, $v = c = (\varepsilon_0\,\mu_0)^{-1/2}$. Thus, in this case v_p is bigger than the vacuum velocity of light. However, this is no reason to worry because signals do not travel at the phase velocity.

VI. Plasmas

There are four states of matter: solid, liquid, gas and plasma.

Difference between gas and plasma:

> A gas consists of neutral atoms or molecules.
>
> A plasma consists of (negatively charged) electrons and (positively charged) ions.

The transition from a gas to a plasma is called *ionisation*.

The name “plasma” was introduced in 1928 by the US American chemist and physicist (Nobel laureate in chemistry) Irwin Langmuir. It derives from a greek word meaning “moldable” or “deformable”.

99 % of the visible matter in the Universe is a plasma. On Earth, plasmas are rare.

Plasmas in nature: stars, interstellar matter, lightning, ionosphere (aurora) ...

Manmade plasmas: fluorescent tubes, fusion reactors (tokamak, Iter), plasma screens ...

In the following we introduce a simple two-fluid model for a plasma, within the framework of Maxwell's equations, and discuss some basic features of plasma waves.

VI.1. Two-fluid model

We want to model a plasma as a mixture of two charged fluids. One fluid models the electrons, the other fluid models the ions. We will further restrict to a *simply* ionised plasma. This means that we assume that each ion results from a neutral atom by stripping *one* electron. Hence the charge of each ion is equal, up to sign, to the electron charge.

The plasma is modeled in terms of the sources ρ and $\boldsymbol{J}$ of Maxwell's equations. We assume that there is no further medium. So we consider Maxwell's equations in vacuo, i.e. with constitutive equations

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} \ .$$

We assume that the sources are of the form

$$\rho = \rho_e + \rho_i \quad \text{and} \quad \boldsymbol{J} = \boldsymbol{J}_e + \boldsymbol{J}_i \ .$$

The index e stands for "electron", the index i stands for "ion".

$$\rho_e = q_e \, n_e \quad \text{and} \quad \boldsymbol{J}_e = q_e \, n_e \, \boldsymbol{v}_e \ ,$$

$$\rho_i = q_i \, n_i \quad \text{and} \quad \boldsymbol{J}_i = q_i \, n_i \, \boldsymbol{v}_i \ ,$$

where

q_e	electron charge	(negative constant)
$q_i = -q_e$	ion charge	(positive constant)
n_e	electron number density	(scalar function of $\boldsymbol{r}$ and t)
n_i	ion number density	(scalar function of $\boldsymbol{r}$ and t)
$\boldsymbol{v}_e$	electron velocity field	(vector function of $\boldsymbol{r}$ and t)
$\boldsymbol{v}_i$	ion velocity field	(vector function of $\boldsymbol{r}$ and t)

Then Maxwell's equations take the following form.

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{MI}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial}{\partial t} \boldsymbol{B} = \boldsymbol{0} \tag{MII}$$

$$\varepsilon_0 \, \boldsymbol{\nabla} \cdot \boldsymbol{E} = q_e \big(n_e - n_i \big) \tag{MIII}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \varepsilon_0 \, \mu_0 \frac{\partial}{\partial t} \boldsymbol{E} = \mu_0 \, q_e \big(n_e \, \boldsymbol{v}_e - n_i \, \boldsymbol{v}_i \big) \tag{MIV}$$

If n_e, n_i, $\boldsymbol{v}_e$ and $\boldsymbol{v}_i$ are known, this is a system of differential equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. It describes the electromagnetic field generated by a plasma if the motion of the plasma is known. However, for a moving plasma we do not know n_e, n_i, $\boldsymbol{v}_e$ and $\boldsymbol{v}_i$ beforehand. So we have to treat n_e, n_i, $\boldsymbol{v}_e$ and $\boldsymbol{v}_i$ as *unknowns* which together with $\boldsymbol{E}$ and $\boldsymbol{B}$ form the total set of dynamical variables. It order to get a determined system for our dynamical variables, we have to supplement Maxwell's equations with the Lorentz force equation for the electrons and for the ions:

$$\frac{d}{dt}\boldsymbol{v}_e = \frac{q_e}{m_e}\big(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}\big)\,, \qquad \text{(LI)}$$

$$\frac{d}{dt}\boldsymbol{v}_i = \frac{q_i}{m_i}\big(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}\big)\,. \qquad \text{(LII)}$$

Here d/dt is the *total* time derivative and

m_e	electron mass	(positive constant)
m_i	ion mass	(positive constant)

We will later use the fact that $m_i \gg m_e$.

Maxwell's equations (MI), (MII), (MIII), (MIV) together with the Lorentz force equations (LI) and (LII) give a system of (non-linear) partial differential equations for the dynamical variables $\boldsymbol{E}$, $\boldsymbol{B}$, n_e, n_i, $\boldsymbol{v}_e$ and $\boldsymbol{v}_i$. This mathematical system is refered to as the "two-fluid model" of a plasma.

[More precisely, what we consider here is called the "*cold* two-fluid model" because collisions between the particles (electrons and ions) are ignored. For a hot plasma one has to add pressure terms to the Lorentz force equations (LI) and (LII) in order to take collisions into account. Actually, the cold two-fluid model has many important applications. E.g., it is a good model even for the solar corona, although its temperature is some $10^6\,$K, because its density is so low that collisions play no important role.]

VI.2. Plasma waves

We consider plasma waves as small perturbations of a plasma at rest.

Unperturbed state:

$$\boldsymbol{E} = \boldsymbol{0}\,, \quad \boldsymbol{B} = \boldsymbol{0}\,, \quad \boldsymbol{v}_e = \boldsymbol{0}\,, \quad \boldsymbol{v}_i = \boldsymbol{0}\,, \quad n_e = n_e^0\,, \quad n_i = n_i^0\,.$$

(MIII) requires that $n_e^0 = n_i^0$.

Perturbed state:

$$\boldsymbol{E} = \mathrm{Re}\big\{ \boldsymbol{E}^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

$$\boldsymbol{B} = \mathrm{Re}\big\{ \boldsymbol{B}^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

$$\boldsymbol{v}_e = \mathrm{Re}\big\{ \boldsymbol{v}_e^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

$$\boldsymbol{v}_i = \mathrm{Re}\big\{ \boldsymbol{v}_i^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

$$n_e = n_e^0 + \mathrm{Re}\big\{ n_e^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

$$n_i = n_e^0 + \mathrm{Re}\big\{ n_i^1 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \big\}\,,$$

where $\boldsymbol{E}^1$, $\boldsymbol{B}^1$, $\boldsymbol{v}_e^1$, $\boldsymbol{v}_i^1$, n_e^1 and n_i^1 are complex amplitudes which are so small that we have to take them into account only to within *linear* order. In other words, in the following we will neglect products of any two of such terms.

It is our goal to derive the dispersion relation for plasma waves, i.e., the relation between $k = |\boldsymbol{k}|$ and ω.

To that end we insert our fields in the perturbed state into Maxwell's equations (MII) and (MIV) and into the Lorentz force equations (LI) and (LII). Keeping only terms of linear order, we find

(MII): $$\boldsymbol{k} \times \boldsymbol{E}^1 - \omega \, \boldsymbol{B}^1 = \boldsymbol{0} \, ,$$

i.e., $\boldsymbol{B}^1$ is perpendicular to $\boldsymbol{k}$ and to $\boldsymbol{E}^1$;

(MIV): $$i \, \boldsymbol{k} \times \boldsymbol{B}^1 + i \, \varepsilon_0 \, \mu_0 \, \omega \, \boldsymbol{E}^1 = \mu_0 \, q_e \, n_e^0 \left(\boldsymbol{v}_e^1 - \boldsymbol{v}_i^1 \right) ;$$

(LI): $$- i \, \omega \, \boldsymbol{v}_e^1 = \frac{q_e}{m_e} \boldsymbol{E}^1 \quad \Rightarrow \quad \boldsymbol{v}_e^1 = \frac{i \, q_e}{\omega \, m_e} \boldsymbol{E}^1 \, ;$$

(LII): $$- i \, \omega \, \boldsymbol{v}_i^1 = - \frac{q_e}{m_i} \boldsymbol{E}^1 \quad \Rightarrow \quad \boldsymbol{v}_i^1 = - \frac{i \, q_e}{\omega \, m_i} \boldsymbol{E}^1 \, .$$

The Maxwell equations (MI) and (MIII) need not be considered because they give no additional information in view of the dispersion relation.

We insert (LI) and (LII) into (MIV):

$$i \, \boldsymbol{k} \times \boldsymbol{B}^1 + i \, \varepsilon_0 \, \mu_0 \, \omega \, \boldsymbol{E}^1 = \mu_0 \, q_e \, n_e^0 \left(\frac{i \, q_e}{\omega \, m_e} \boldsymbol{E}^1 + \frac{i \, q_e}{\omega \, m_i} \boldsymbol{E}^1 \right) .$$

$$\boldsymbol{k} \times \boldsymbol{B}^1 + \varepsilon_0 \, \mu_0 \, \omega \, \boldsymbol{E}^1 = \frac{\mu_0 \, q_e^2 \, n_e^0}{\omega \, m_e} \left(1 + \frac{m_e}{m_i} \right) \boldsymbol{E}^1 \, .$$

As m_e / m_i is small, it can be neglected in comparison to 1. (Even for a hydrogen nucleus, which is the lightest ion, we have $m_i \approx 1836 \, m_e$.) So

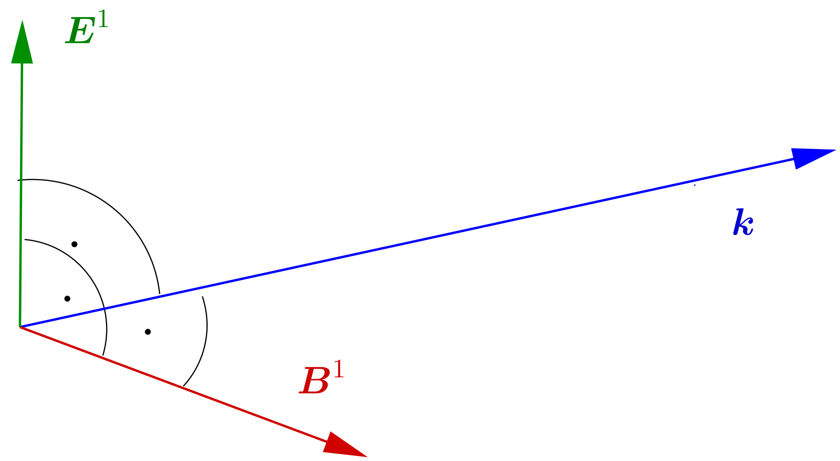
$$\boldsymbol{k} \times \boldsymbol{B}^1 = \frac{\varepsilon_0 \, \mu_0}{\omega} \left(\frac{q_e^2 \, n_e^0}{\varepsilon_0 \, m_e} - \omega^2 \right) \boldsymbol{E}^1 \, .$$

We will now assume that

$$\frac{q_e^2 \, n_e^0}{\varepsilon_0 \, m_e} \neq \omega^2 \, . \qquad (*)$$

$\Big($We will briefly comment on the case that $(*)$ is violated at the end of this chapter.$\Big)$ Then we can conclude that $\boldsymbol{E}^1$ is a multiple of $\boldsymbol{k} \times \boldsymbol{B}^1$ and, thus, perpendicular to $\boldsymbol{k}$. We already knew that $\boldsymbol{B}^1$ is perpendicular to $\boldsymbol{k}$ and to $\boldsymbol{E}^1$, so $\boldsymbol{k}$, $\boldsymbol{B}^1$ and $\boldsymbol{E}^1$ are mutually perpendicular.

Thus, to within our approximations electromagnetic waves in a plasma are transverse, as in vacuo. Also, as the relations between $\boldsymbol{B}^1$ and $\boldsymbol{E}^1$ are real, the electric and the magnetic fields are in phase. Hence $\boldsymbol{B}^1$ and $\boldsymbol{E}^1$ can be chosen real.



From (MII) we find

$$\boldsymbol{k} \times \left(\boldsymbol{k} \times \boldsymbol{E}^1 \right) = \omega \, \boldsymbol{k} \times \boldsymbol{B}^1 \, .$$

If we insert on the right-hand side the expression for $\boldsymbol{k} \times \boldsymbol{B}^1$ found above, this results in

$$\boldsymbol{k} \, (\underbrace{\boldsymbol{k} \cdot \boldsymbol{E}^1}_{=0}) - \boldsymbol{E}^1 k^2 = \varepsilon_0 \, \mu_0 \left(\frac{q_e^2 \, n_e^0}{\varepsilon_0 \, m_e} - \omega^2 \right) \boldsymbol{E}^1 \, .$$

As $\boldsymbol{E}^1$ is supposed to be different from $\boldsymbol{0}$, we can compare the coefficients on both sides and find, with $\varepsilon_0 \, \mu_0 = 1/c^2$:

$$\omega^2 - \frac{q_e^2 \, n_e^0}{\varepsilon_0 \, m_e} = c^2 \, k^2 \, .$$

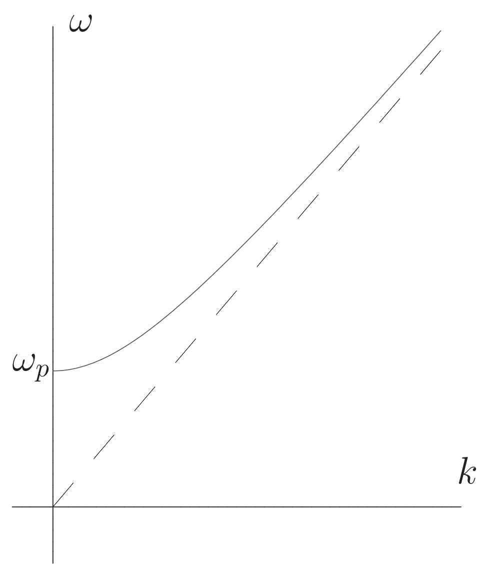
This is the desired dispersion relation for plasma waves. If the electron density n_e^0 is sent to zero, we recover the vacuum dispersion relation $\omega = c \, k$.

If we introduce the *plasma frequency* ω_p,

$$\boxed{\omega_p^2 = \frac{q_e^2\, n_e^0}{\varepsilon_0\, m_e}\,,}$$

the dispersion relation for plasma waves takes the form

$$\boxed{\omega = \sqrt{c^2\, k^2 + \omega_p^2}\,.}$$



Note that the electron density n_e^0 and, thus, the plasma frequency ω_p is, in general, a function of $\boldsymbol{r}$ and t.

From the dispersion relation we can calculate the phase velocity v_p and the group velocity v_g:

$$v_p = \frac{\omega}{k} = c\sqrt{1 + \frac{\omega_p^2}{c^2\, k^2}} \geq c\,,$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{\sqrt{1 + \frac{\omega_p^2}{c^2\, k^2}}} \leq c\,.$$

The fact that $v_p \geq c$ is no reason to worry because signals do not travel with the phase velocity.

For $k \to \infty$, the dispersion relation for plasma waves approaches the vacuum dispersion relation $\omega = c\,k$ which is indicated as a dashed line in the picture. In particular, for $k \to \infty$ we have $v_p \to c$ and $v_g \to c$.

For $k \to 0$, the frequency ω does *not* go to zero but rather $\omega \to \omega_p$ and $v_g \to 0$.

Recall the following general terminology which was introduced already when we discussed wave guides.

A value of ω where the wave length becomes infinite ($k = 2\pi/\lambda \to 0$) is called a "cut-off frequency" and a value of ω where the wave length becomes zero ($k = 2\pi/\lambda \to \infty$) is called a "resonance frequency".

For the case of waves in a non-magnetised plasma considered here, we have a cut-off frequency at $\omega = \omega_p$ and we have no resonance frequency. For waves in a magnetised plasma ($\boldsymbol{B}^0 \neq \boldsymbol{0}$), e.g., one finds both cut-off and resonance frequencies.

Finally, we have to remind ourselves that for our derivation we had to assume that $(*)$ holds. If this inequlity is violated, i.e., if $\omega = \omega_p$, we cannot conclude that $\boldsymbol{E}^1$ must be orthogonal to $\boldsymbol{k}$. Indeed, solutions with $\omega = \omega_p$ and $\boldsymbol{E}^1 \parallel \boldsymbol{k}$ exist. They are refered to as *plasma oscillations*.

Suggestions for exam preparation:

- Study these notes. Important formulas and statements, which are very likely to come up in exam questions, are given in red boxes. (Of course, it would not be sufficient just to learn all red boxes by heart; it is also necessary to *understand* this material and to know how to apply it.)
- Look through the exam papers of the preceding three years (on the departmental web page). Exam papers of earlier years will not be useful because the syllabus of PHYS274 has considerably changed since then.
- Study again the four worksheets we have done. Solutions are available on LUVLE.