Experimental characterisation of standard clocks

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1. **Standard clocks in a general-relativistic spacetime**

   - **Characterising standard clocks with light rays and freely falling particles**
   - **Clock transport**
   - **Redshift**

2. **Standard clocks in a Weyl spacetime**

   - **Characterising standard clocks with light rays and freely falling particles**
   - **Clock transport**
   - **Redshift**

3. **Standard clocks in a Finsler spacetime**

   - **Characterising standard clocks with light rays and freely falling particles**
   - **Clock transport**
   - **Redshift**

Standard clocks in a general-relativistic spacetime

Definition: (M, g) is a general-relativistic spacetime if M is a 4-dimensional manifold and g is a pseudo-Riemannian metric of signature $(-+++)$.

Definition of proper time along a timelike curve $\gamma : I \subset \mathbb{R} \to M$:

$$\tau = \int_{t_0}^{t} \sqrt{-g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)}\, dt$$

Parametrisation with $t = \tau$ is characterised by

$$g\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big) = -1$$

A standard clock is a curve $\gamma : I \subset \mathbb{R} \to M$ with $g\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big) = -1$.

If we allow for another choice of (time) unit:

$$g\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big) = \text{const.}$$

$$g\big(\dot{\gamma}(\tau), \nabla_{\dot{\gamma}(\tau)}\dot{\gamma}(\tau)\big) = 0$$

Standard clocks (and rigid rulers) are not appropriate as fundamental objects in view of applications to astrophysics.

Better use light signals (lightlike geodesics) and freely falling particles (timelike geodesics).

Knowing the lightlike and timelike geodesics (as unparametrised curves) determines the metric up to a constant factor.

H. Weyl: "Raum. Zeit. Materie." 2^{nd} edition, Springer, Berlin (1919)

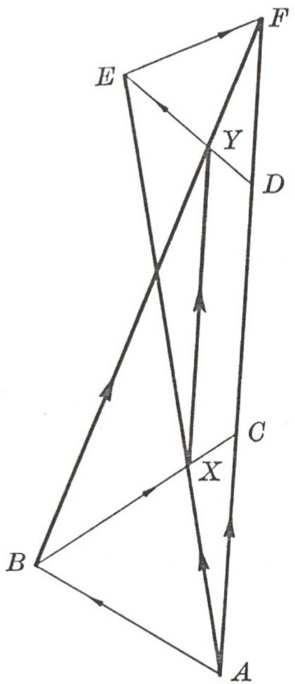
Light rays and freely falling particles are used as the primitive concepts in the Ehlers-Pirani-Schild axiomatics

J. Ehlers, F. A. E. Pirani and A. Schild: "The geometry of free fall and light propagation" in
L. O'Raifeartaigh (ed.): "General Relativity", papers in honour of J. L. Synge.
Clarendon Press, Oxford (1972)

This motivates the goal: To characterise standard clocks with the help of light signals and freely falling particles.

1st method:

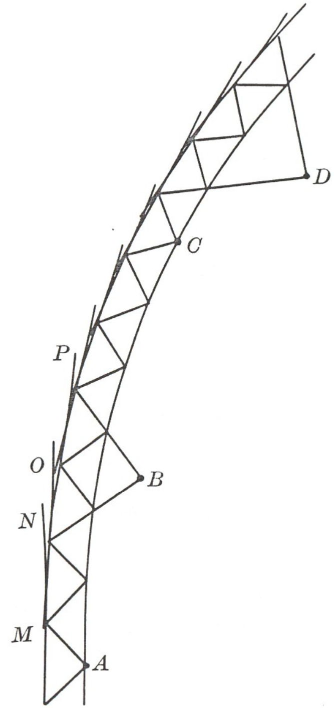
R. F. Marzke and J. A. Wheeler: "Gravitation as geometry. I: The geometry of space-time and the geometrodynamical standard meter" in H. Y. Chiu and W.F. Hoffmann (eds.): "Gravitation and relativity" Benjamin, New York (1964)



Construct "infinitesimally neighbouring parallel" worldline of a straight worldline in Minkowski spacetime.

Generalise to an "infinitesimally neighbouring parallel" worldline of a geodesic worldline in curved spacetime.

Let a light ray bounce back and forth between the two worldlines and prove that it arrives with the rhythm of a standard clock.



2nd method:

W. Kundt and B. Hoffmann: "Determination of gravitational standard time" in ??? (ed.): "Recent developments in general relativity", Pergamon, Oxford (1962)

Write metric as

$$ds^2 = e^{2U}\Big(-(dx^0 + g_\mu dx^\mu)^2 + \tilde{\gamma}_{\kappa\lambda} dx^\kappa dx^\lambda\Big).$$

Want to determine e^{2U} along a chosen x^0-line.

Choose three neighbouring x^0 lines and assume that all four observers can measure x^0 along their worldlines.

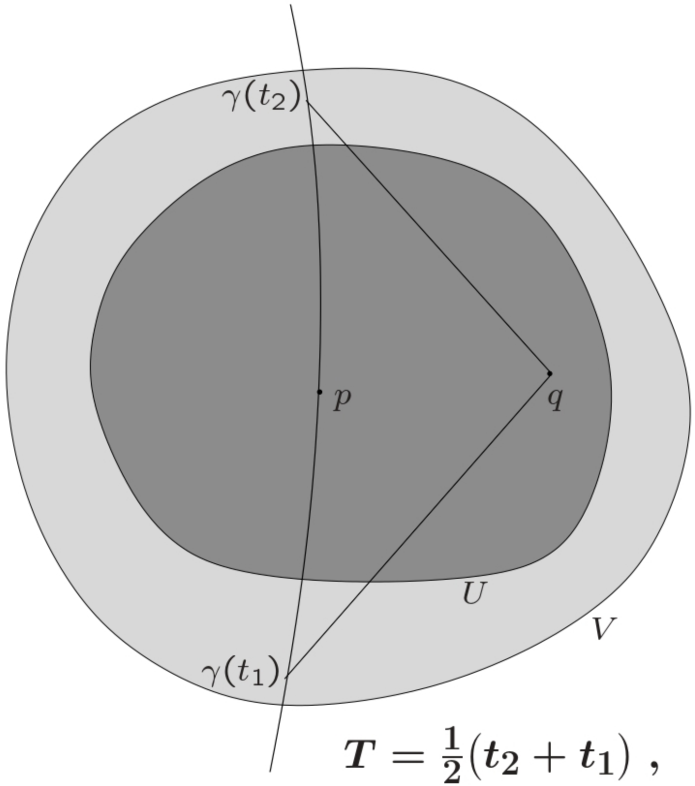
Let the four observers exchange light rays and freely falling particles and measure emission and reception x^0 time.

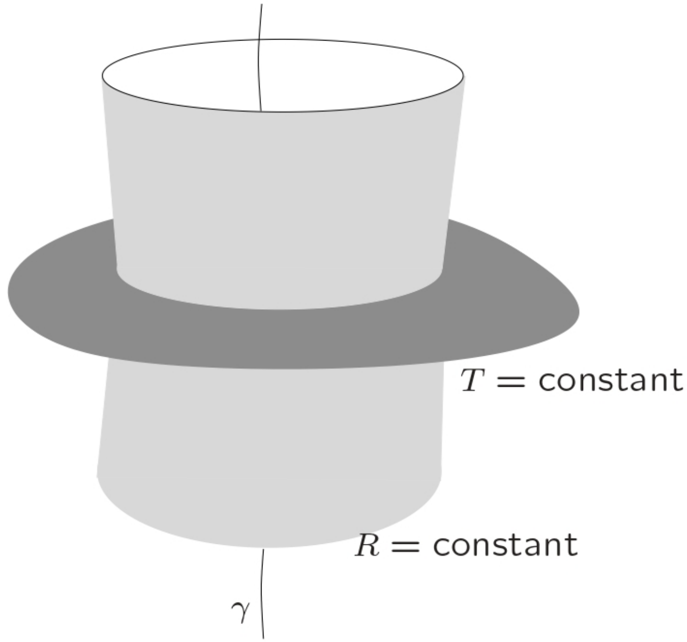
Get a system of 9 equations for 9 unknowns that determines e^{2U} and thus proper time along the chosen worldline.

3rd method:

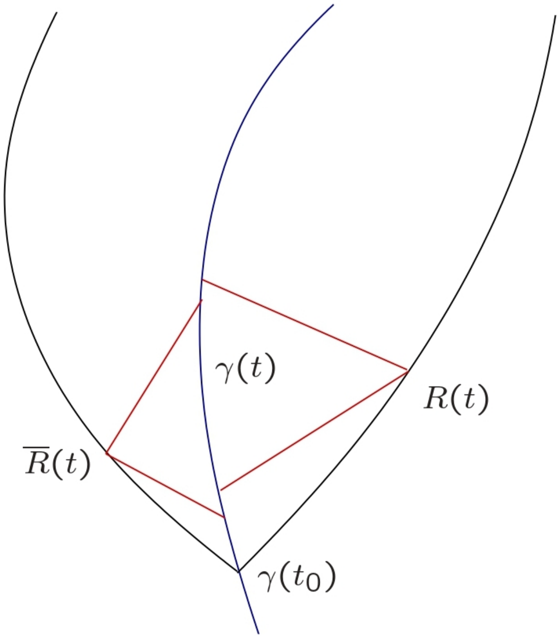
VP: "Characterization of standard clocks by means of light rays and freely falling particles", Gen. Rel. Grav. 19, 1059 (1987)

Uses radar time T and radar distance R





$$\boldsymbol{T} = \frac{1}{2}(\boldsymbol{t}_2 + \boldsymbol{t}_1) \ , \qquad \boldsymbol{R} = \frac{1}{2}(\boldsymbol{t}_2 - \boldsymbol{t}_1)$$



Want to test γ for being a standard clock

Emit two freely falling particles in opposite directions at $\gamma(t_0)$

Measure radar distances $R(t)$ and $\overline{R}(t)$ as functions of radar time $T(t) = \overline{T}(t) = t$

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} \frac{R''(t)}{\left(1 - R'(t)^2\right)} = -\lim_{t \to t_0} \frac{\overline{R}''(t)}{\left(1 - \overline{R}'(t)^2\right)}$$

If γ is freely falling:

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} R''(t) = 0$$

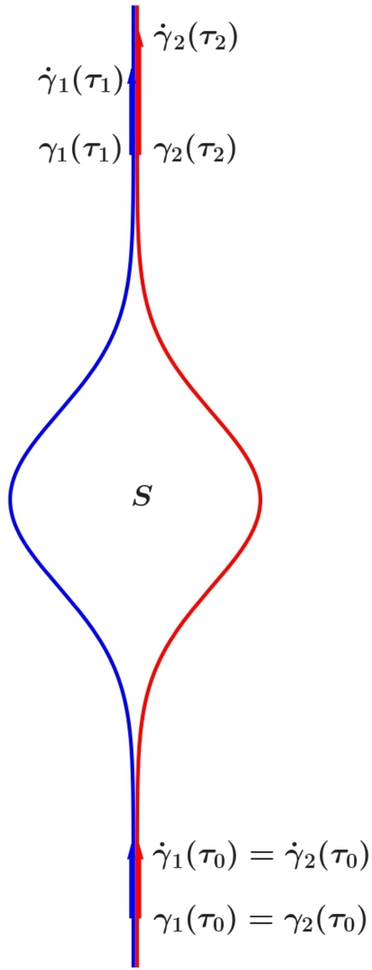
Properties of standard clocks in a general-relativistic spacetime:

(a) Clock transport

Let $\gamma_1 : \mathbb{R} \to M$ and $\gamma_2 : \mathbb{R} \to M$ be two standard clocks with

$$\gamma_1(\tau_0) = \gamma_2(\tau_0), \;\; \dot{\gamma}_1(\tau_0) = \dot{\gamma}_2(\tau_0)$$

$$\gamma_1(\tau_1) = \gamma_2(\tau_2), \;\; \dot{\gamma}_1(\tau_1) || \dot{\gamma}_2(\tau_2)$$



First clock effect:

$$\tau_1 \neq \tau_2$$

occurs already in Special Relativity ("twin paradox")

No second clock effect:

$$\dot{\gamma}_1(\tau_1) = \dot{\gamma}_2(\tau_2)$$

(b) Redshift

For comparing the ticking of two standard clocks γ and $\tilde{\gamma}$, we send light rays from one to the other.

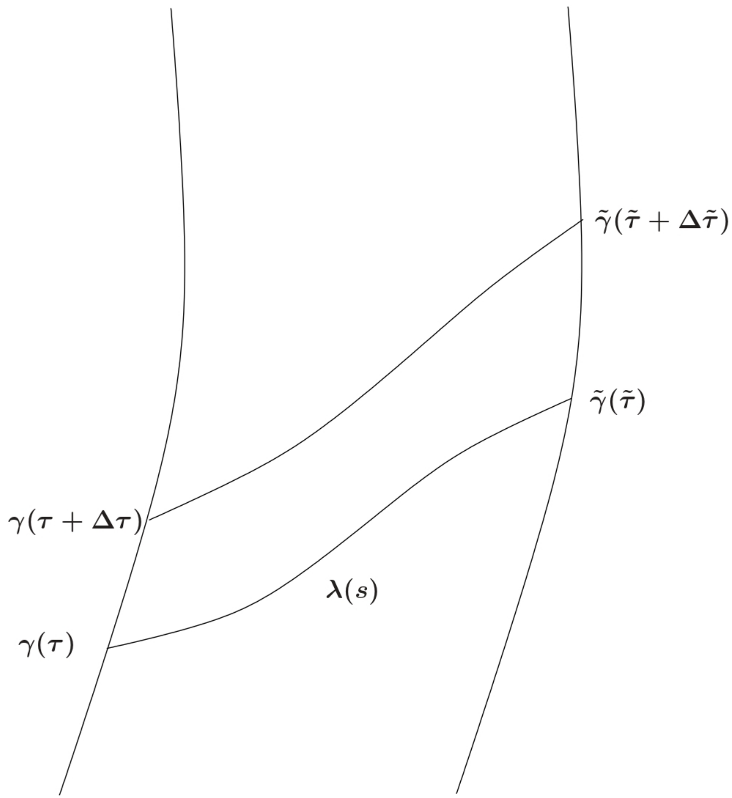
Introduce the frequency ratio

$$\frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta\tilde{\tau}}{\Delta\tau} =$$

$$= \frac{\omega_{\text{emitter}}}{\omega_{\text{receiver}}} = 1 + z$$

This defines the redshift

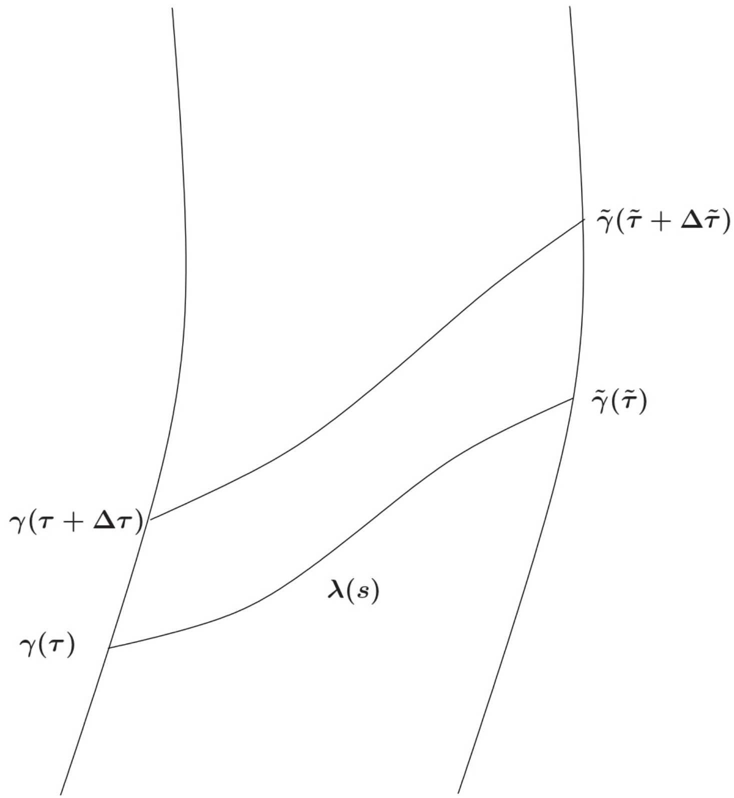
$$z = \frac{\omega_{\text{emitter}} - \omega_{\text{receiver}}}{\omega_{\text{receiver}}}$$



Universal redshift formula for standard clocks in general relativity:

$$1 + z = \frac{g_{ab}\big(\lambda(s_1)\big)\,\dfrac{d\lambda^a}{ds}\Big|_{s=s_1}\dfrac{d\gamma^b}{d\tau}}{g_{cd}\big(\lambda(s_2)\big)\,\dfrac{d\lambda^c}{ds}\Big|_{s=s_2}\dfrac{d\tilde{\gamma}^d}{d\tilde{\tau}}}$$

W.O. Kermack, W.H. McCrea, E.T. Whittacker: "On properties of null geodesics and their application to the theory of radiation", Proc. Roy. Soc. Edinburgh 53, 31 (1932)



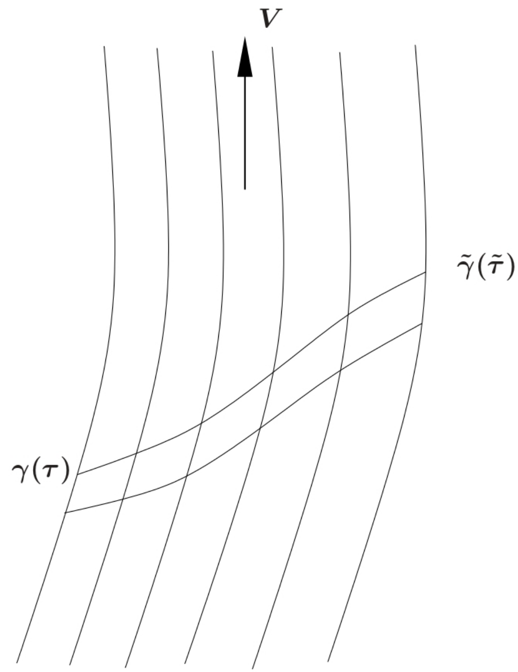
Let V be a standard observer field (=vector field with $g(V,V) = -1$):

Definition: $f : M \to \mathbb{R}$ is called a redshift potential for V if for any two integral curves γ and $\tilde{\gamma}$:

$$\ln(1+z) = f\big(\tilde{\gamma}(\tau)\big) - f\big(\gamma(\tau)\big)$$

Theorem: (i) f is a redshift potential for V if and only if $e^f V$ is a conformal Killing vector field.

(ii) f is a time-independent redshift potential for V if and only if $e^f V$ is a Killing vector field.

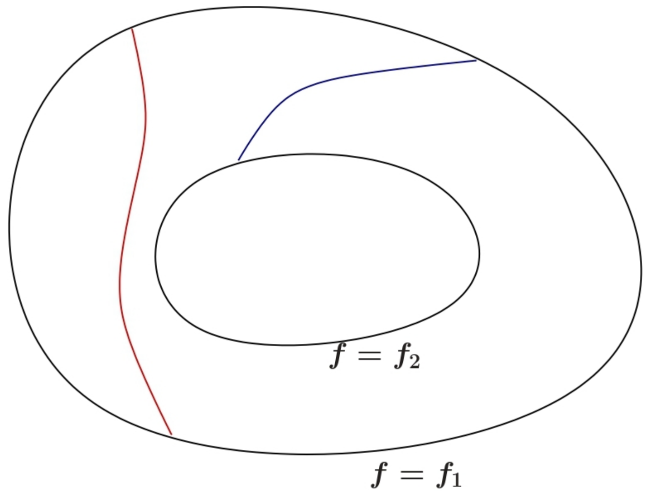


W. Hasse and VP: "Geometrical and kinematical characterization of parallax-free world models", J. Math. Phys. 29, 2064 (1988)

A time-independent redshift potential foliates the 3-space into surfaces $f = \mathrm{const.}$ ("isochronometric surfaces")

$$g_{ab}dx^a dx^b =$$

$$e^{2f}\Big(- \left(dt + \psi_\mu dx^\mu\right)^2 + h_{\mu\nu}dx^\mu dx^\nu\Big)$$



Coordinate travel time of signal with speed of light along spatial path:

$$t_2 - t_1 = \int \sqrt{h_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}ds$$

$$- \int \psi_\mu \frac{dx^\mu}{ds}ds$$

is independent of the emission time

$\Longrightarrow$ redshift potential gives correct redshift also for signals sent through optical fibers

Example: Kerr metric

$$g_{ab}dx^a dx^b = -\left(1 - \frac{2mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2$$

$$+\rho^2 d\vartheta^2 - \frac{4mra\sin^2\vartheta}{\rho^2}\,dt\,d\varphi$$

$$+\sin^2\vartheta\left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\rho^2}\right)d\varphi^2$$

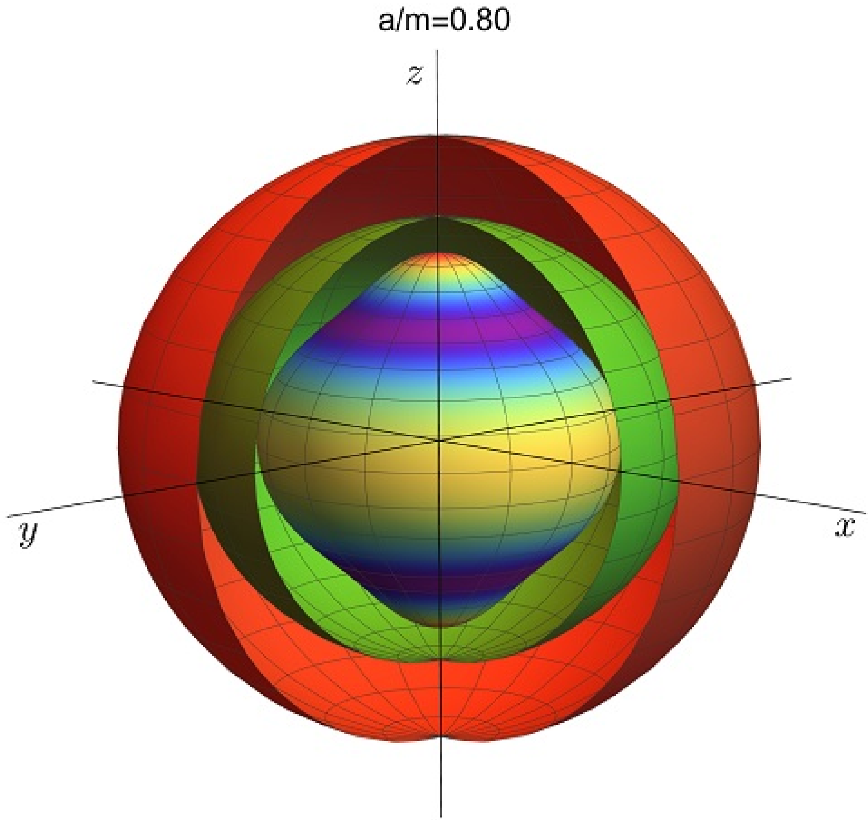
where

$\rho^2 = r^2 + a^2\cos^2\vartheta$, $\Delta = r^2 + a^2 - 2mr$

∂_t is a Killing vector field.

$V = (-g_{tt})^{-1/2}\partial_t$ is a standard observer field.

f is a redshift potential for V, where $e^{2f} = -g_{tt} = 1 - \dfrac{2mr}{\rho^2}$



Isochronometric surfaces
$f = \text{const.}$

Define the geoid (and the generalisation to other celestial bodies) with the help of isochronometric surfaces.

A.Bjerhammer (1985): “The relativistic geoid is the surface where precise clocks run with the same speed and the surface is nearest to mean sea level.”

Interpretation:

“Precise clocks” means “standard clocks”.

“Running with the same speed” does NOT refer to being (Einstein) synchronous but rather to a surface of constant redshift potential.

This makes sense as long as the spacetime geometry around the Earth can be viewed as stationary.

D. Philipp, VP, D. Puetzfeld, E. Hackmann, C. Lämmerzahl: “Definition of the relativistic geoid in terms of isochronometric surfaces” Phys. Rev. D 95, 104037 (2017)

Standard clocks in a Weyl spacetime

Definition: $(M, \mathfrak{g}, \nabla)$ is a Weyl spacetime if M is a 4-dimensional manifold, $\mathfrak{g}$ is a conformal equivalence class of metrics of signature $(-+++)$ and ∇ is a compatible torsion-free connection.

Compatibility: For every g in $\mathfrak{g}$ there is a covector field φ such that $\nabla_X g = \varphi(X) g$.

Gauge transformations: $g \mapsto e^h g$, $\phi \mapsto \varphi + dh$

$F = d\varphi$ is gauge-invariant ("Streckenkrümmung" = length curvature)

Definition of standard clocks: $g(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) = 0$ for all $g \in \mathfrak{g}$. Unit cannot be fixed.

Light signals ($\mathfrak{g}$-lightlike ∇-geodesics) and freely falling particles ($\mathfrak{g}$-timelike ∇-geodesics) determine $\mathfrak{g}$ and ∇ uniquely.

Characterisation of standard clocks with light rays and freely falling particles carries over into Weyl geometry.

VP: "Characterization of standard clocks by means of light rays and freely falling particles" Gen. Rel. Grav. 19, 1059 (1987)

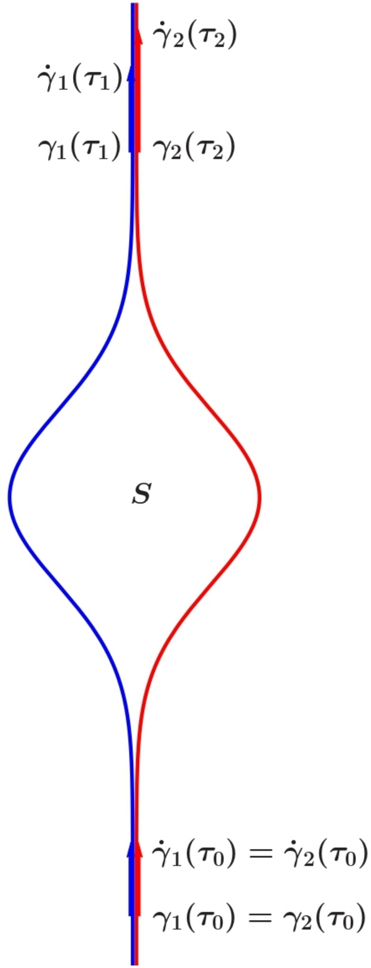
Properties of standard clocks in a Weyl spacetime:

(a) Clock transport

Let $\gamma_1 : \mathbb{R} \to M$ and $\gamma_2 : \mathbb{R} \to M$ be two standard clocks with

$$\gamma_1(\tau_0) = \gamma_2(\tau_0), \ \dot{\gamma}_1(\tau_0) = \dot{\gamma}_2(\tau_0)$$

$$\gamma_1(\tau_1) = \gamma_2(\tau_2), \ \dot{\gamma}_1(\tau_1) || \dot{\gamma}_2(\tau_2)$$



First clock effect:

$$\tau_1 \neq \tau_2$$

Second clock effect:

$$\dot{\gamma}_1(\tau_1) \neq \dot{\gamma}_2(\tau_2)$$

unless $\int_S F = \oint_{\partial S} \varphi = 0$.

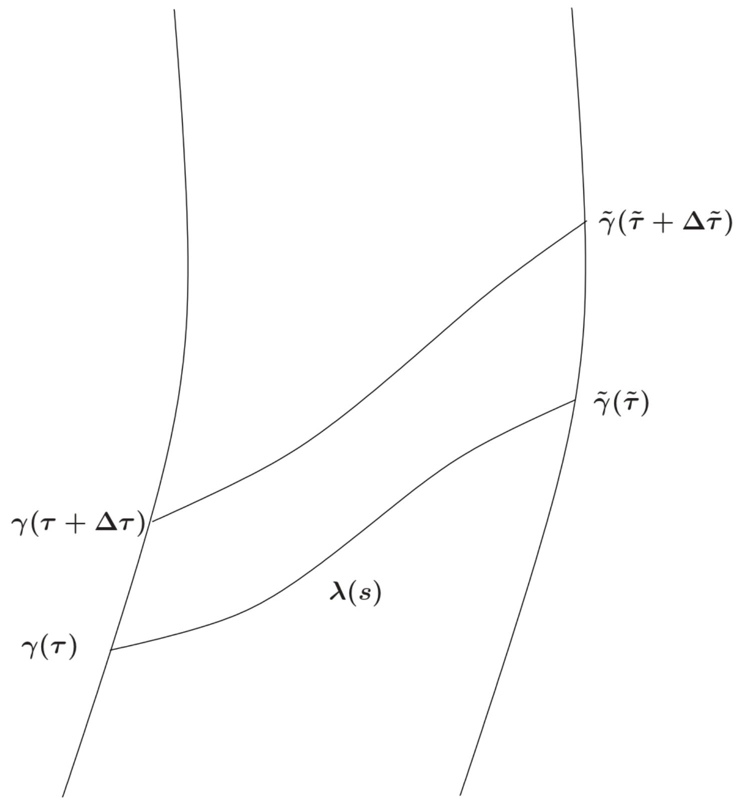
(b) Redshift

Definition of redshift carries over into Weyl geometry without changes.

Universal redshift formula for standard clocks in Weyl spacetime:

$$\Big(1+z\Big)\exp\Big(-\int_{s_1}^{s_2}\varphi(\lambda(s))ds\Big)=$$

$$\frac{g_{ab}(\lambda(s_1))\,\frac{d\lambda^a}{ds}\Big|_{s=s_1}\frac{d\gamma^b}{d\tau}}{g_{cd}(\lambda(s_2))\,\frac{d\lambda^c}{ds}\Big|_{s=s_2}\frac{d\tilde{\gamma}^d}{d\tilde{\tau}}}$$



VP: PhD Thesis (1989)

Standard clocks in a Finsler spacetime

Definition: (M, L) is a Finsler spacetime if M is a 4-dimensional manifold and $L : \overset{o}{T}M \to \mathbb{R}$ is a function that satisfies

(i) $L(x, kv) = k^2 L(x, v)$ for all $(x, v) \in \overset{o}{T}M$ and $k > 0$.

(ii) $g_{ab}(x, v) = \dfrac{1}{2} \dfrac{\partial L(x, v)}{\partial v^a \partial v^b}$ has signature $(- + + +)$ for all $(x, v) \in \overset{o}{T}M$.

Then $L(x, v) = g_{ab}(x, v)$.

A curve $x(s)$ is timelike if $L(x(s), \dot{x}(s) < 0$ and lightlike if $L(x(s), \dot{x}(s) = 0$.

Geodesics are solutions of the Euler-Lagrange equations

$$\frac{d}{ds} \frac{\partial L\big(x(s), \dot{x}(s)\big)}{\partial \dot{x}^a(s)} = \frac{\partial L\big(x(s), \dot{x}(s)\big)}{\partial x^a(s)}$$

Light signals (geodesics with $\mathcal{L} = 0$) and freely falling particles (geodesics with $\mathcal{L} < 0$) are well defined.

Definition of proper time: $\tau = \displaystyle\int_{t_0}^{t} \sqrt{-\mathcal{L}\big(\gamma(t), \dot{\gamma}(t)\big)}\, dt$

Problems for characterising standard clocks with light rays and freely falling particles:

Problem 1: Multiple light cones are possible.

E. Minguzzi: "Light cones in Finsler spacetime" Commun. Math. Phys. 334, 1529 (2015)

Problem 2: Radar method works if light cones are unique, but even then synchronous surfaces are not in general smooth.

C. Pfeifer: "Radar orthogonality and radar length in Finsler and metric spacetime geometry" Phys. Rev. D 90, 064052 (2014)

Problem 3: Timelike and lightlike geodesics do not in general characterise a Finsler spacetime uniquely.

R. K. Tavakol, N. Van den Bergh: "Viability criteria for the theories of gravity and Finsler spaces" Gen. Rel. Grav. 18, 849 (1986)

Properties of standard clocks in a Finsler spacetime:

(a) Clock transport

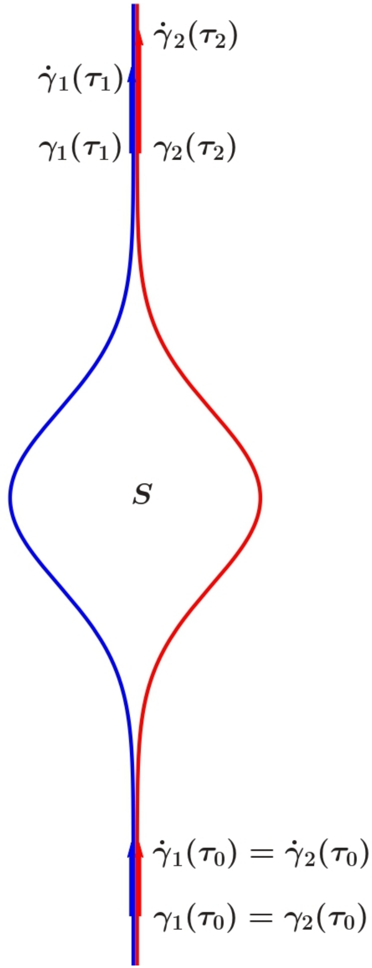
Let $\gamma_1 : \mathbb{R} \to M$ **and** $\gamma_2 : \mathbb{R} \to M$ **be two standard clocks with**

$$\gamma_1(\tau_0) = \gamma_2(\tau_0), \; \dot{\gamma}_1(\tau_0) = \dot{\gamma}_2(\tau_0)$$

$$\gamma_1(\tau_1) = \gamma_2(\tau_2), \; \dot{\gamma}_1(\tau_1) || \dot{\gamma}_2(\tau_2)$$

First clock effect: $\tau_1 \neq \tau_2$

No second clock effect: $\dot{\gamma}_1(\tau_1) = \dot{\gamma}_2(\tau_2)$



(b) Redshift

Definition of redshift carries over into Finsler spacetimes without modification.

Universal redshift formula for standard clocks in Finsler spacetime:

$$1 + z = \frac{g_{ab}\big(\lambda(s_1), d\lambda/ds\big)\,\dfrac{d\lambda^a}{ds}\Big|_{s=s_1}\dfrac{d\gamma^b}{d\tau}}{g_{cd}\big(\lambda(s_2), d\lambda/ds\big)\,\dfrac{d\lambda^c}{ds}\Big|_{s=s_2}\dfrac{d\tilde{\gamma}^d}{d\tilde{\tau}}}$$

W. Hasse and VP: "Redshift in Finsler spacetimes"
Phys. Rev. D 100, 024033 (2019)