Using clocks for probing the spacetime gemetry

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1. Standard clocks
– Formal definition
– Operational characterisation

2. Redshift
– General redshift formula
– Existence of a redshift potential

in general relativity (but also in Weyl geometry and in Finsler geometry)

309th Heraeus Seminar "Relativistic Geodesy", Bad Honnef, 18 March 2016

Standard clocks in general relativity

(M, g): Manifold with pseudo-Riemannian metric of Lorentzian signature

For arbitrarily parametrised timelike curce $\gamma(t)$ define proper time

$$\tau = \int_{t_0}^{t} \sqrt{-g\big(\dot\gamma(t), \dot\gamma(t)\big)}\, dt$$

Parametrisation with $t = \tau$ is characterised by

$$g\big(\dot\gamma(\tau), \dot\gamma(\tau)\big) = -1$$

Allow for another choice of (time) unit:

$$g\big(\dot\gamma(\tau), \dot\gamma(\tau)\big) = \text{const.}$$

$$g\big(\dot\gamma(\tau), \nabla_{\dot\gamma(\tau)}\dot\gamma(\tau)\big) = 0$$

Rigid rulers and standard clocks are not appropriate as fundamental objects

Better use freely falling particles and light signals

Basis of the Ehlers-Pirani-Schild axiomatics

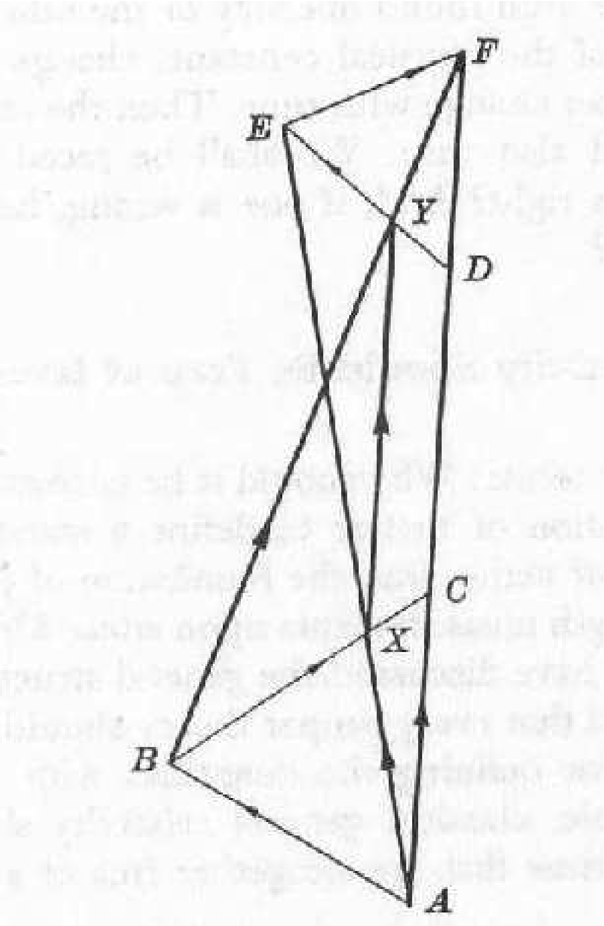
> J. Ehlers, F. A. E. Pirani and A. Schild: "The geometry of free fall and light propagation" in: General Relativity, papers in honour of J. L. Synge. Edited by L. ORaifeartaigh. Clarendon Press, Oxford (1972)

Axiomatic foundation for the result: Light signals are lightlike geodesics and freely falling particles are timelike geodesics of a Lorentzian metric

This motivates the goal: To characterise standard clocks with the help of light signals and freely falling particles

1st method:

R. F. Marzke and J. A. Wheeler: "Gravitation as geometry. I: The geometry of space-time and the geometrodynamical standard meter" In "Gravitation and relativity". Edited by H. Y. Chiu and W. F. Hoffmann. Benjamin, New York (1964)



Construct "infinitesimally neighbouring parallel" world-line

Let a light ray bounce back and forth

Prove that it arrives with the rhythm of a standard clock

2nd method:

W. Kundt and B. Hoffmann: “Determination of gravitational standard time”. In “Recent developments in general relativity”. Edited by ???. Pergamon, Oxford (1962)

Write metric as $ds^2 = e^{2U}\Big(\tilde{\gamma}_{\kappa\lambda}dx^{\kappa}dx^{\lambda} - (dx^0 + g_{\mu}dx^{\mu})^2\Big)$. Want to determine e^{2U} along a chosen x^0-line.

Choose three neighbouring x^0 lines and assume that all four observers can measure x^0 along their worldlines.

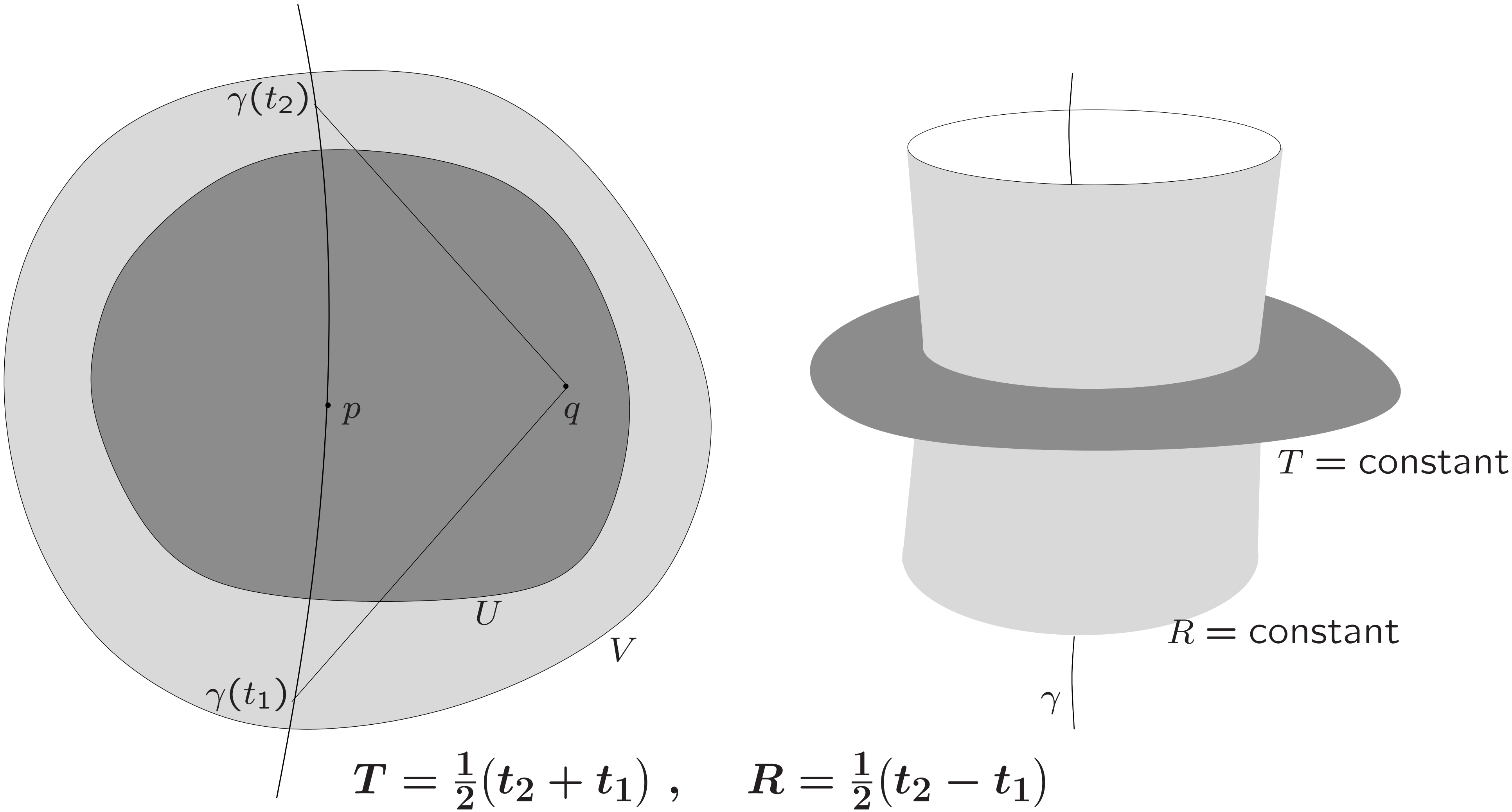
Let the four observers exchange light rays and freely falling particles and measure emission and reception x^0 time.

Get a system of 9 equations for 9 unknowns that determines e^{2U} and thus proper time along the chosen worldline.

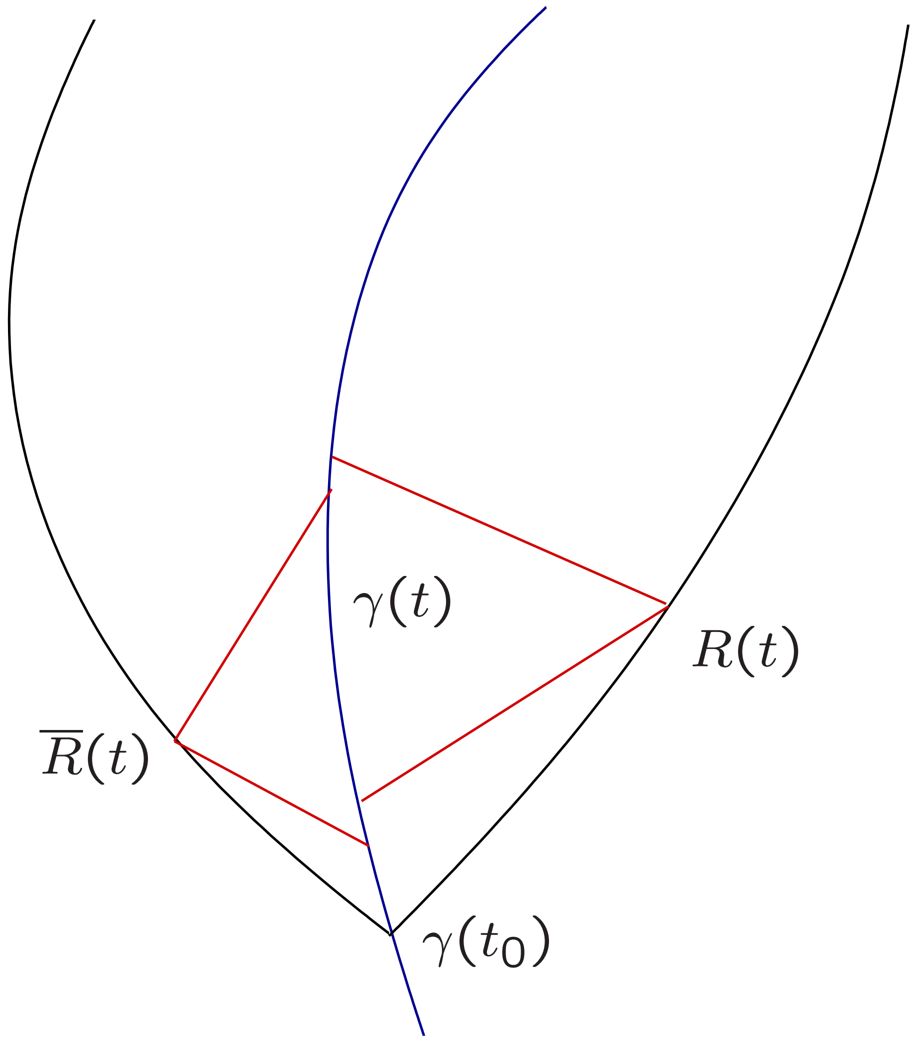
3rd method:

VP: "Characterization of standard clocks by means of light rays and freely falling particles". Gen. Rel. Grav. 19, 1059 (1987)

Uses radar time T and radar distance R



$$T = \frac{1}{2}(t_2 + t_1)\ , \qquad R = \frac{1}{2}(t_2 - t_1)$$



Want to test γ for being a standard clock

Emit two freely falling particles in opposite directions at $\gamma(t_0)$

Measure radar distances $R(t)$ and $\overline{R}(t)$ as functions of radar time $T(t) = \overline{T}(t) = t$

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} \frac{R''(t)}{(1 - R'(t)^2)} = - \lim_{t \to t_0} \frac{\overline{R}''(t)}{(1 - \overline{R}'(t)^2)}$$

If γ is freely falling:

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} R''(t) = 0$$

Standard clocks in Weyl geometry

$(M, \mathfrak{g}, \nabla)$: Manifold with a conformal class of pseudo-Riemannian metrics of Lorentzian signature and a compatible connection

Compatibility: For every g in $\mathfrak{g}$ there is a covector field φ such that $\nabla_X g = \varphi(X) g$.

Gauge transformation: $g \mapsto e^h g$, $\phi \mapsto \varphi + dh$

$F = d\varphi$ is gauge-invariant ("Streckenkrümmung" = length curvature)

Light signals ($\mathfrak{g}$-lightlike ∇-geodesics) and freely falling particles ($\mathfrak{g}$-timelike ∇-geodesics) are well defined

Standard clocks are well defined:

$$g\big(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}\big) = 0 \,, \quad g \in \mathfrak{g}$$

The third method of characterising standard clocks works.

Standard clocks in Finsler geometry

(M, g): Manifold with metric that depends on position and velocity, $g(x, v)$ where $(x, v) \in TM$ and

$g(x, v)$ is of Lorentzian signature

$g(x, kv) = g(x, v)$, $k > 0$

$\dfrac{\partial g_{ab}(x, v)}{\partial v^c}$ is totally symmetric

Geodesics:

$$\frac{d}{ds}\frac{\partial \mathcal{L}\big(x(s), \dot{x}(s)\big)}{\partial \dot{x}^a(s)} = \frac{\partial \mathcal{L}\big(x(s), \dot{x}(s)\big)}{\partial x^a(s)}$$

$$\mathcal{L}(x, v) = g_{ab}(x, v) v^a v^b$$

Light signals (geodesics with $\mathcal{L} = 0$) and freely falling particles (geodesics with $\mathcal{L} < 0$) are well defined

Proper time is well defined

$$\tau = \int_{t_0}^{t} \sqrt{-\mathcal{L}\big(\gamma(t), \dot{\gamma}(t)\big)}\, dt$$

Multiple light cones possible; under certain additional conditions there is a unique light cone

E. Minguzzi: “Light cones in Finsler spacetime” Commun. Math. Phys. 334, 1529 (2015)

Radar method works, but synchroneous surfaces are not in general smooth

C. Pfeifer: “Radar orthogonality and radar length in Finsler and metric spacetime geometry” Phys. Rev. D 90, 064052 (2014)

Characterising standard clocks with light signals and freely falling particles .... (to be worked out)

Clock transport

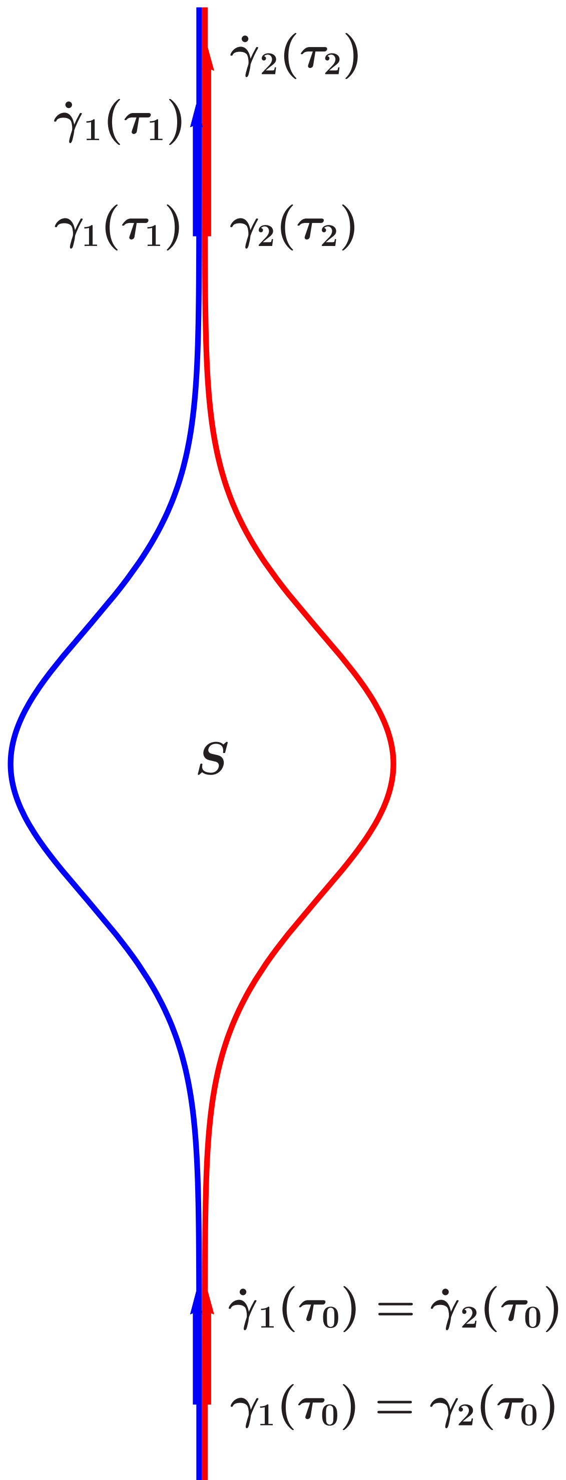
First clock effect: $\tau_1 \neq \tau_2$

Second clock effect: $\dot{\gamma}_1(\tau_1) \neq \dot{\gamma}_2(\tau_2)$

First clock effect occurs already in Specal Relativity

Second clock effect occurs only in non-reducible Weyl geometry and is proportional to

$$\int_S F = \oint \varphi$$



Redshift

For comparing the ticking of two standard clocks γ and $\tilde{\gamma}$, we send light rays from one to the other

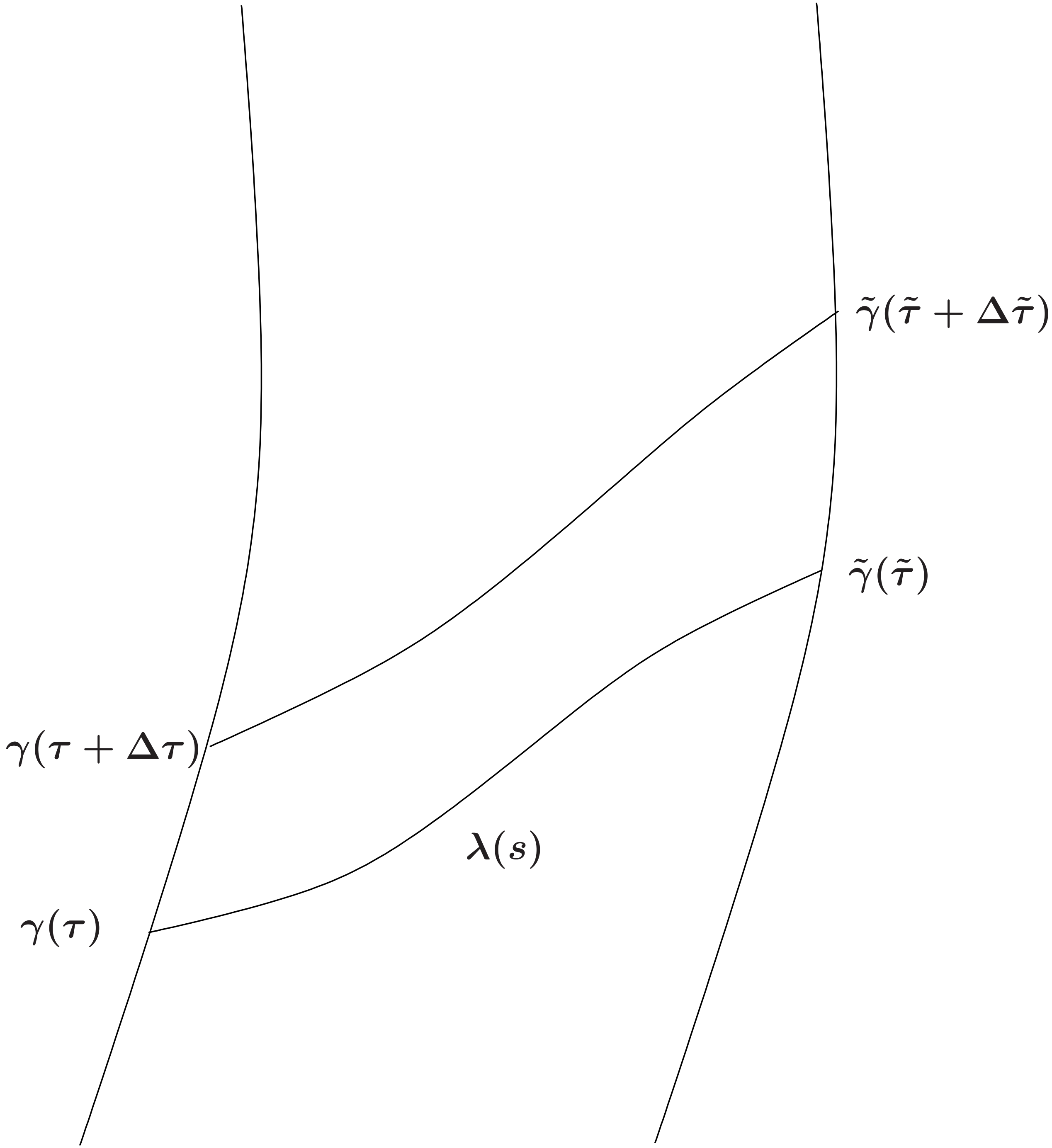
Introduce the frequency ratio

$$\frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau\to 0}\frac{\Delta\tilde{\tau}}{\Delta\tau} =$$

$$= \frac{\omega_{\text{emitter}}}{\omega_{\text{receiver}}} = 1 + z$$

This defines the redshift

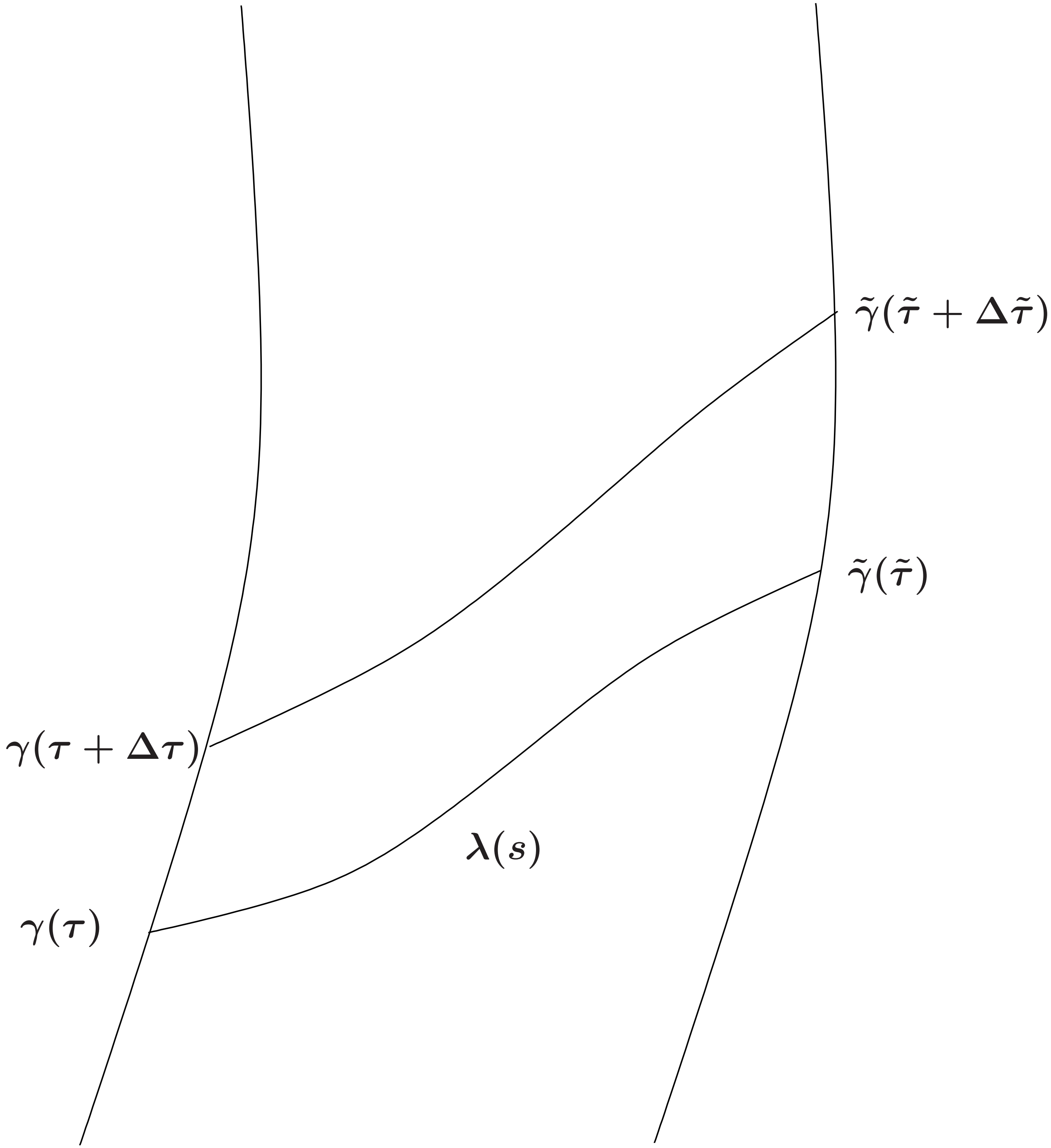
$$z = \frac{\omega_{\text{emitter}} - \omega_{\text{receiver}}}{\omega_{\text{receiver}}}$$



Universal redshift formula for standard clocks in general relativity:

$$1 + z = \frac{g_{ab}(\lambda(s_1))\, \dfrac{d\lambda^a}{ds}\Big|_{s=s_1} \dfrac{d\gamma^b}{d\tau}}{g_{cd}(\lambda(s_2))\, \dfrac{d\lambda^c}{ds}\Big|_{s=s_2} \dfrac{d\tilde{\gamma}^d}{d\tilde{\tau}}}$$

W. O. Kermack, W. H. McCrea and E. T. Whittacker: "On properties of null geodesics and their application to the theory of radiation", Proc. Roy. Soc. Edinburgh 53, 31 (1932)

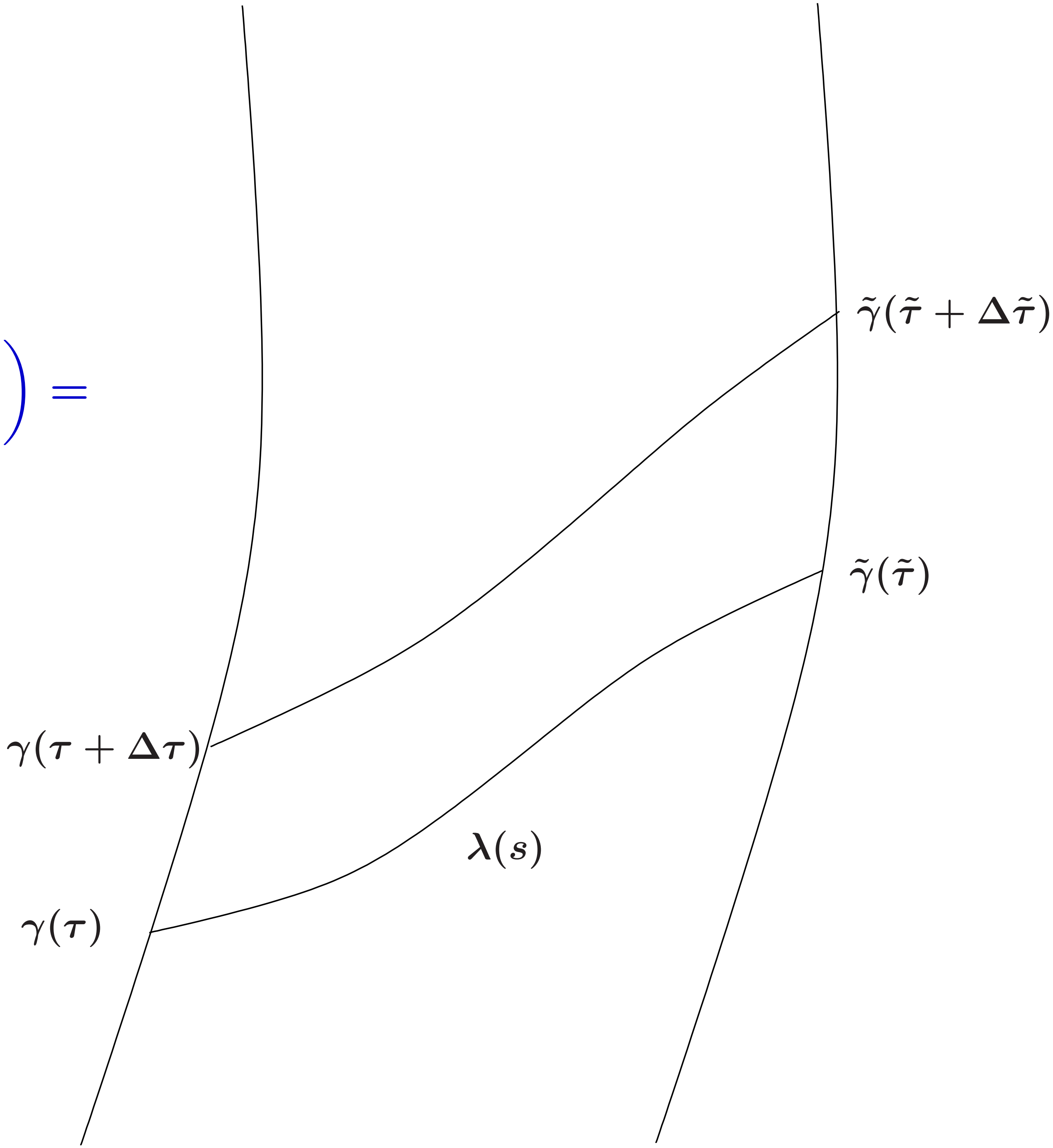


Universal redshift formula for standard clocks in Weyl spacetime:

$$\left(1+z\right)\exp\left(-\int_{s_1}^{s_2}\varphi(\boldsymbol{\lambda}(s))ds\right)=$$

$$\frac{g_{\mu\nu}(\boldsymbol{\lambda}(s_1))\,\frac{d\lambda^{\mu}}{ds}\Big|_{s=s_1}\frac{d\gamma^{\nu}}{d\tau}}{g_{\rho\sigma}(\boldsymbol{\lambda}(s_2))\,\frac{d\lambda^{\rho}}{ds}\Big|_{s=s_2}\frac{d\tilde{\gamma}^{\sigma}}{d\tilde{\tau}}}$$

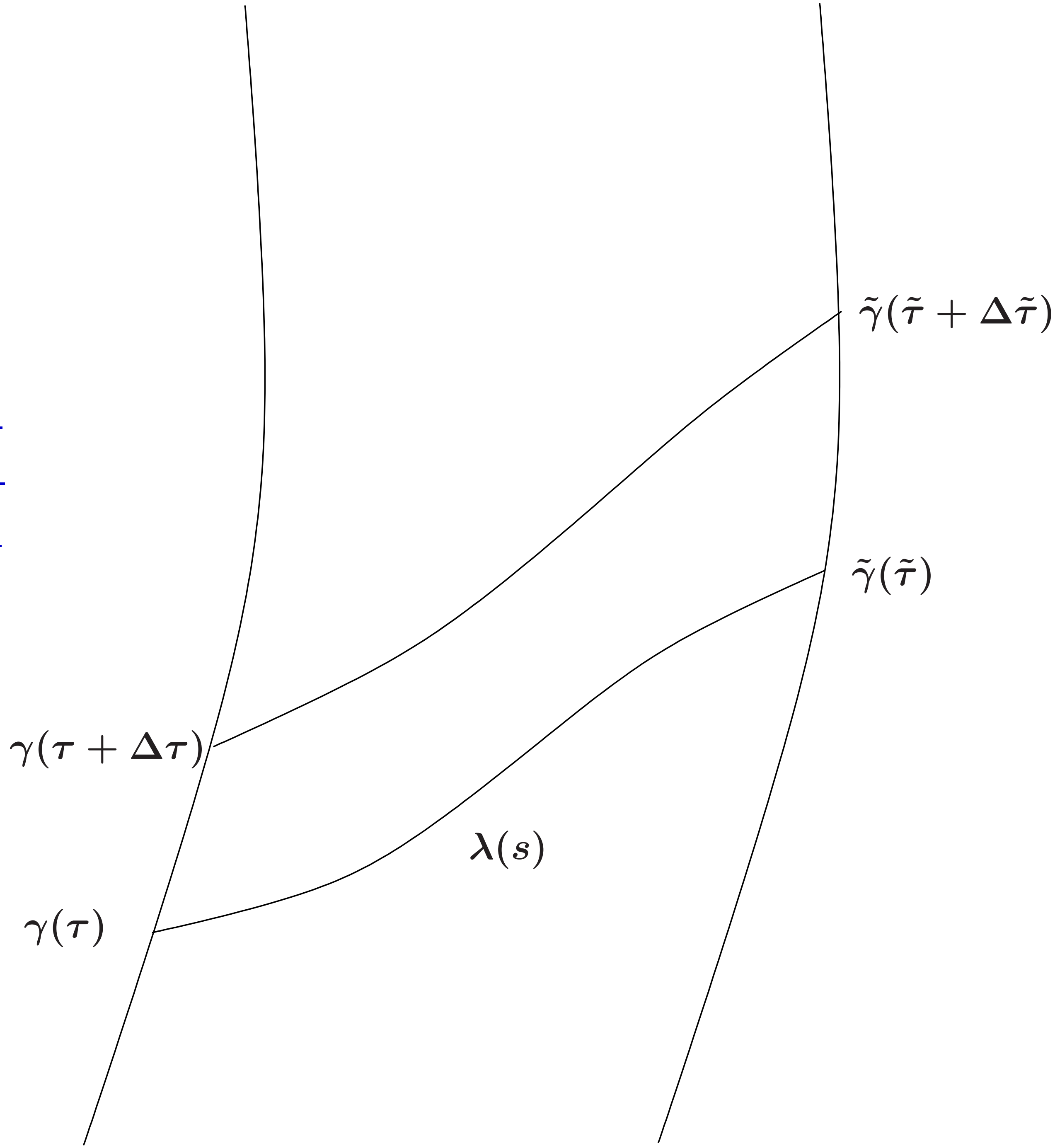
VP: PhD Thesis (1989)



Universal redshift formula for standard clocks in Finsler spacetime:

$$1 + z = \frac{g_{\mu\nu}(\lambda(s_1), d\lambda/ds)\, \dfrac{d\lambda^\mu}{ds}\Big|_{s=s_1} \dfrac{d\gamma^\nu}{d\tau}}{g_{\rho\sigma}(\lambda(s_2), d\lambda/ds)\, \dfrac{d\lambda^\rho}{ds}\Big|_{s=s_2} \dfrac{d\tilde{\gamma}^\sigma}{d\tilde{\tau}}}$$

W. Hasse and VP (in preparation)



Existence of a redshift potential for standard observer field V

$$\ln(1+z) = f\big(\tilde{\gamma}(\tau)\big) - f\big(\gamma(\tau)\big)$$

in general relativity:

f is a redshift potential if and only if $e^f V$ is a conformal Killing vector field.

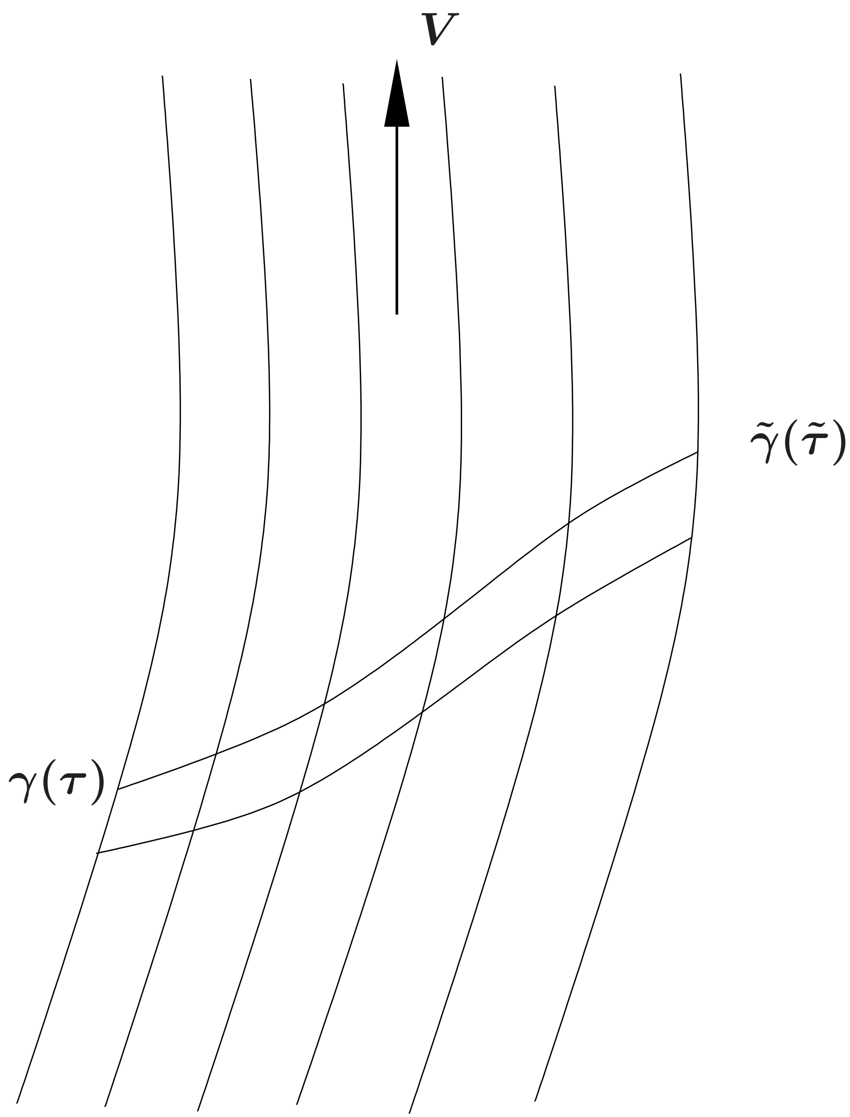
In coordinates $(x^0 = t, x^1, x^2, x^3)$ with $\partial_t = e^f V$ the metric reads

$$g_{ab} dx^a dx^b =$$

$$e^{2f}\Big(- \big(dt + \psi_\mu dx^\mu\big)^2 + h_{\mu\nu} dx^\mu dx^\nu \Big)$$

with $\partial_t \psi_\mu = \partial_t h_{\mu\nu} = 0$

W. Hasse and VP: "Geometrical and kinematical characterization of parallax-free world models", J. Math. Phys. 29, 2064 (1988)



Existence of a time-independent redshift potential for standard observer field V

$$\ln\big(1+z\big) = f\big(\tilde{\gamma}(\tau)\big) - f\big(\gamma(\tau)\big)$$

$$df(V) = 0$$

in general relativity:

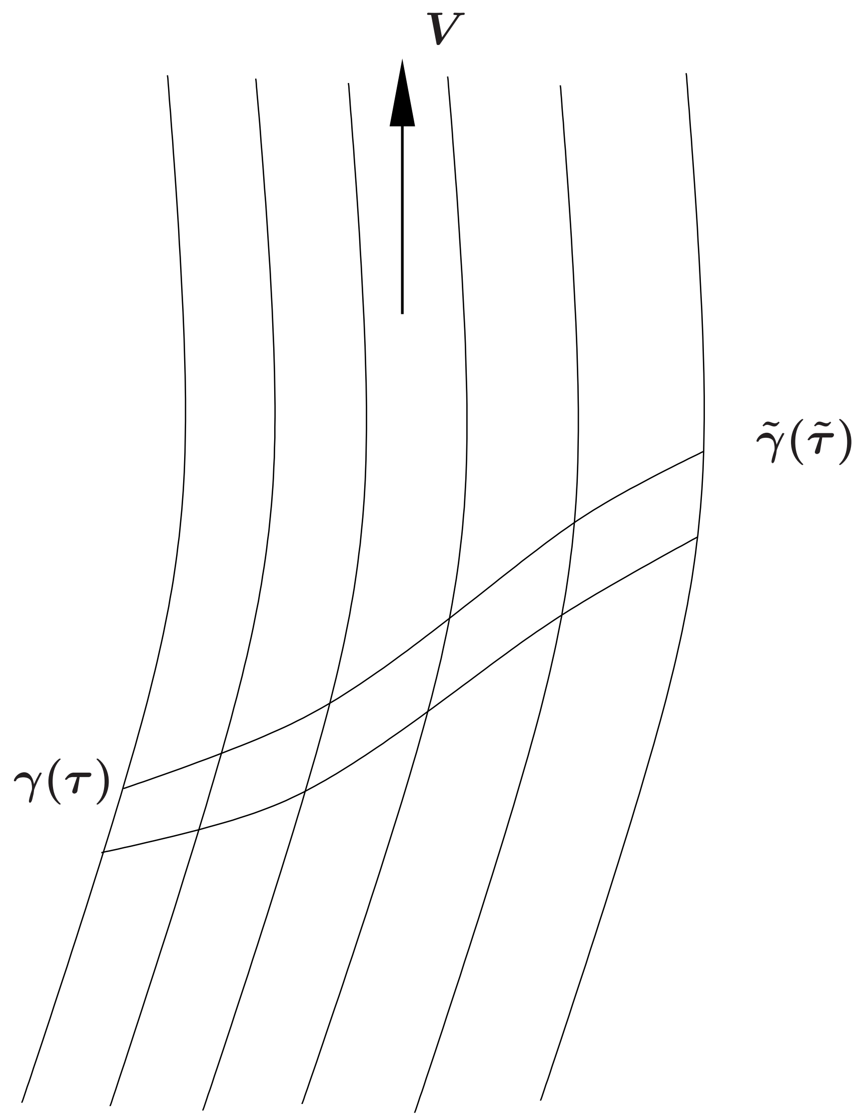
f is a time-independent redshift potential if and only if $e^f V$ is a Killing vector field.

In coordinates $(x^0 = t, x^1, x^2, x^3)$ with $\partial_t = e^f V$ the metric reads

$$g_{ab} dx^a dx^b =$$

$$e^{2f}\Big(-\big(dt + \psi_\mu dx^\mu\big)^2 + h_{\mu\nu} dx^\mu dx^\nu \Big)$$

with $\partial_t \psi_\mu = \partial_t h_{\mu\nu} = \partial_t f = 0$

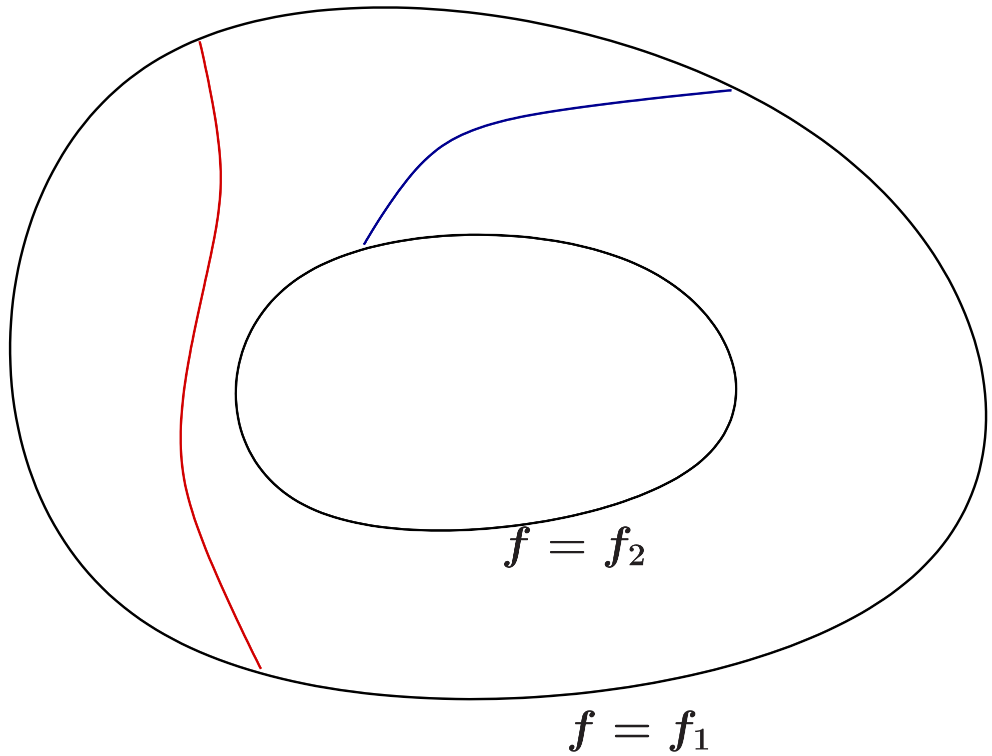


A time-independent redshift potential foliates the 3-space into surfaces $f = \mathrm{const.}$ ("isochronometric surfaces")

$$g_{ab}dx^a dx^b =$$
$$e^{2f}\Big(- (dt + \psi_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu \Big)$$

Coordinate travel time of signal with speed of light along spatial path:

$$t_2 - t_1 = \int \sqrt{h_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}ds$$
$$- \int \psi_\mu \frac{dx^\mu}{ds}ds$$



is independent of the emission time

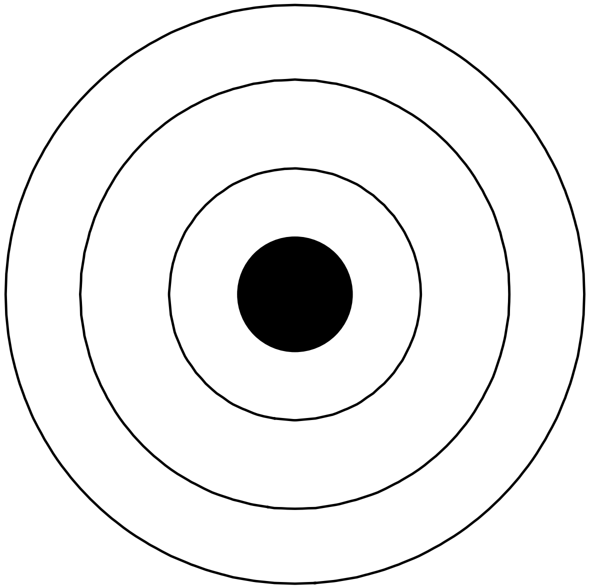
$\Longrightarrow$ redshift potential gives correct redshift also for signals sent through optical fibers

Schwarzschild:

$$g_{ab}dx^a dx^b = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

Killing vector field ∂_t

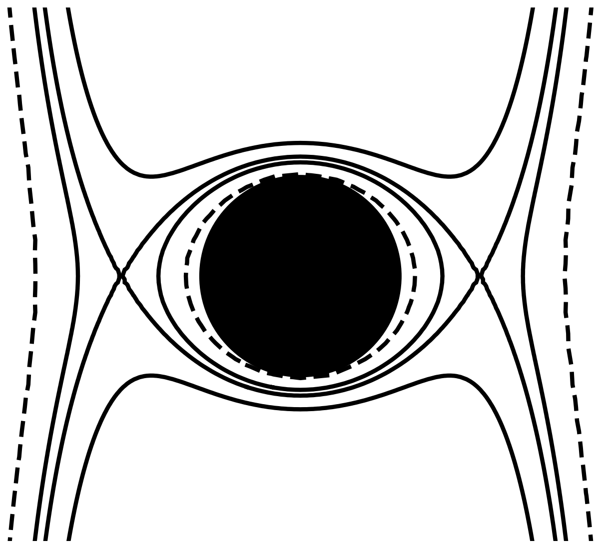
Redshift potential $e^{2f} = -g_{tt} = 1 - \frac{2m}{r}$



Coordinate transformation $\tilde{t} = t$, $\tilde{\varphi} = \varphi + \Omega t$, $\tilde{r} = r$, $\tilde{\vartheta} = \vartheta$

Killing vector field $\partial_{\tilde{t}} = \partial_t - \Omega \partial_\varphi$

Redshift potential $e^{2\tilde{f}} = -g_{\tilde{t}\tilde{t}} = -g_{tt} - \Omega^2 g_{\varphi\varphi} = 1 - \dfrac{2m}{r} - \Omega^2 r^2 \sin^2\vartheta$



Kerr:

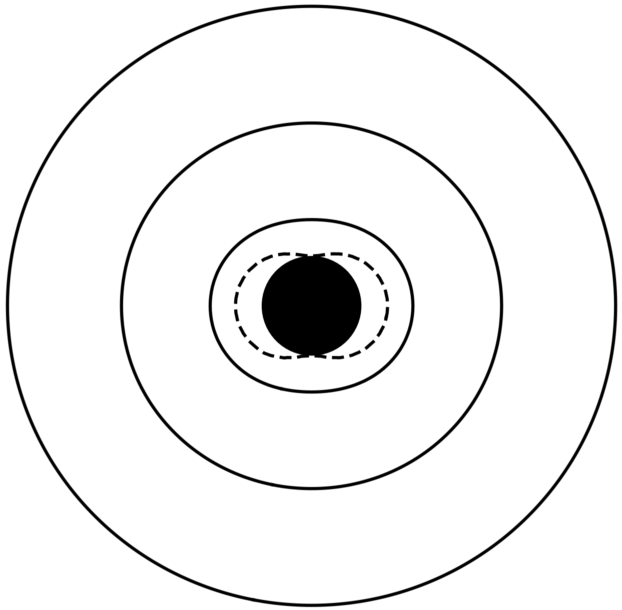
$$g_{ab}dx^a dx^b = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 - \frac{4mra\sin^2\vartheta}{\rho^2}\, dt\, d\varphi$$

$$+\sin^2\vartheta \left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\rho^2}\right) d\varphi^2$$

$$\rho^2 = r^2 + a^2\cos^2\vartheta,\ \Delta = r^2 + a^2 - 2mr$$

Killing vector field ∂_t

Redshift potential $e^{2f} = -g_{tt} = 1 - \dfrac{2mr}{\rho^2}$

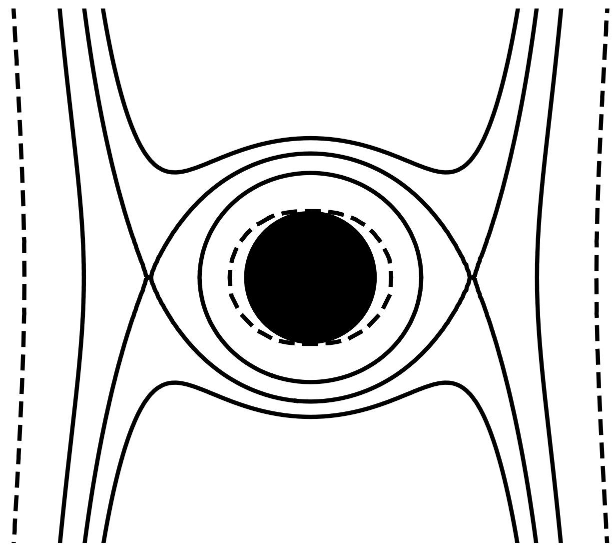


Coordinate transformation $\tilde{t} = t$, $\tilde{\varphi} = \varphi + \Omega t$, $\tilde{r} = r$, $\tilde{\vartheta} = \vartheta$

Killing vector field $\partial_{\tilde{t}} = \partial_t - \Omega \partial_\varphi$

Redshift potential $e^{2\tilde{f}} = -g_{\tilde{t}\tilde{t}} = -g_{tt} + 2\Omega g_{t\varphi} - \Omega^2 g_{\varphi\varphi}$

$$= 1 - \frac{2mr}{\rho^2} + 4\Omega \frac{mra\sin^2\vartheta}{\rho^2} - \Omega^2 \sin^2\vartheta \Big(r^2 + a^2 + \frac{2ma^2\sin^2\vartheta}{\rho^2} \Big)$$



A.Bjerhammer (1985): "The relativistic geoid is the surface where precise clocks run with the same speed and the surface is nearest to mean sea level."

Interpretation:

"Precise clocks" means "standard clocks".

"Running with the same speed" does NOT refer to being (Einstein) synchroneous but rather to a surface of constant redshift potential.

This makes sense as long as the spacetime geometry around the Earth can be viewed as stationary.