The brachistochrone problem in general relativity

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Johann Bernoulli (1667-1748)

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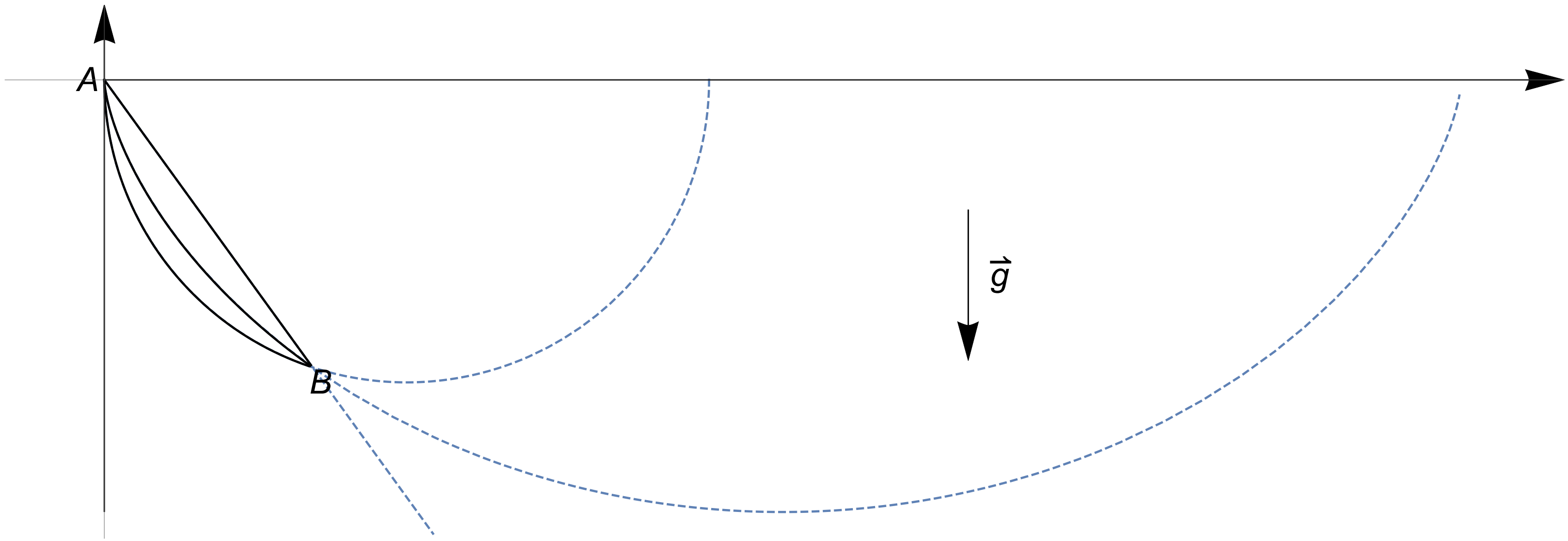
1. The Newtonian brachistochrone problem

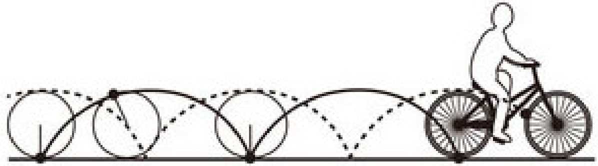
In 1696 Johann Bernoulli challenged the scientific community to solve the following problem:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?



Here "gravity" means a homogeneous Newtonian gravitational field $\vec{g}$.

The solution is a cycloid.



Correct solutions were supplied by Jakob Bernoulli, Isaac Newton, Guillaume de l'Hospital and others.

Solution in modern terminology:

$$m\frac{d^2\vec{r}}{dt^2} = -m\vec{\nabla}V(\vec{r}) + \vec{F}_{\mathrm{const}}\,, \qquad V(\vec{r}) = \text{gravitational Potential}\,, \qquad \frac{d\vec{r}}{dt}\cdot\vec{F}_{\mathrm{const}} = 0$$

$$0 = \frac{d\vec{r}}{dt}\cdot\left(\frac{d^2\vec{r}}{dt^2} + \vec{\nabla}V(\vec{r})\right) = \frac{d}{dt}\left(\frac{1}{2}\left|\frac{d\vec{r}}{dt}\right|^2 + V(\vec{r})\right)$$

$$\frac{1}{2}\left|\frac{d\vec{r}}{dt}\right|^2 + V(\vec{r}) = C = \text{const.}$$

$$dt = \frac{|d\vec{r}|}{\sqrt{2\big(C - V(\vec{r})\big)}}$$

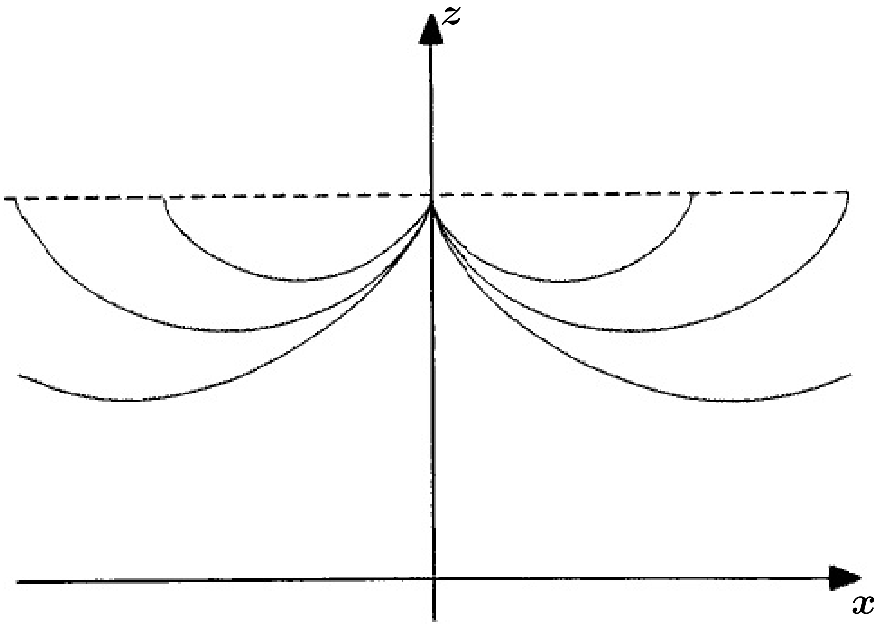
Brachistochrones are geodesics of the Riemannian metric

$$h_C = \frac{dx^2 + dy^2 + dz^2}{2\big(C - V(x,y,z)\big)}$$

Example 1: Homogeneous gravitational field

$V(x, y, x) = gz$

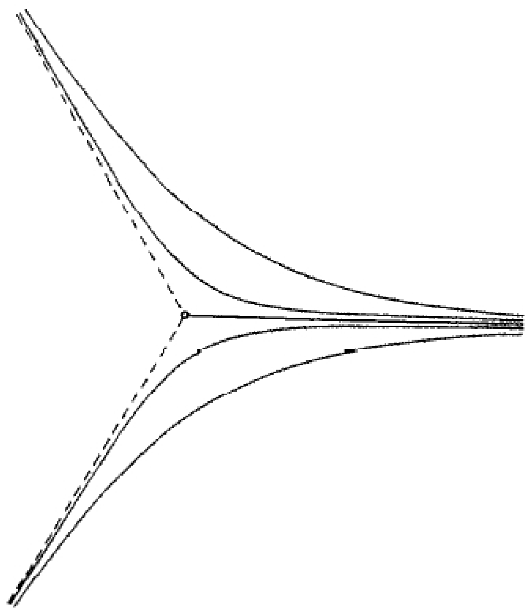
$$h_C = \frac{dx^2 + dy^2 + dz^2}{2(C - gz)}, \qquad C = gz_0$$



Example 2: Kepler potential

$$V(x,y,z) = -\frac{GM}{\sqrt{x^2+y^2+z^2}}$$

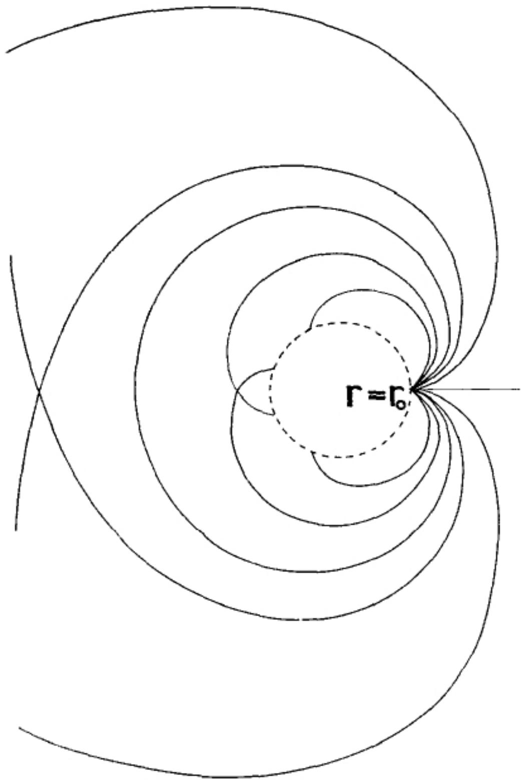
$$h_C = \frac{\Big(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\Big)}{2\Big(C + \frac{GM}{r}\Big)}, \qquad C = -\frac{GM}{r_0}$$



Example 3: Centrifugal potential

$$V(x, y) = -\omega^2(x^2 + y^2)$$

$$h_C = \frac{dr^2 + r^2 d\varphi^2}{2\Big(C + \omega^2 r^2\Big)}, \qquad C = -\omega^2 r_0^2$$



Compare with Maupertuis' principle in the version of Jacobi:

The trajectories of (unconstrained) particles with specific energy C in a potential $V(x,y,z)$ are geodesics of the "Jacobi metric"

$$\hat{h}_C = 2\big(C - V(x,y,z)\big)(dx^2 + dy^2 + dz^2)$$

Hence:

Brachistochrones of specific energy C in the potential $V(\vec{r})$

=

Free trajectories of specific energy $\overline{C}$ in the potential $\overline{V}$ where

$$4\big(\overline{C} - \overline{V}(\vec{r})\big)\big(C - V(\vec{r})\big) = 1$$

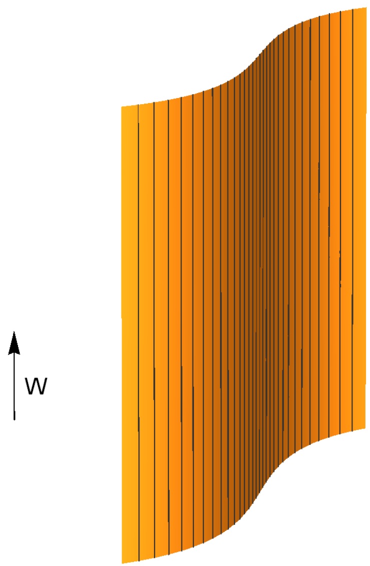
2. The brachistochrone problem in a stationary spacetime

Spacetime metric $g = -e^{2V}(dt + \psi_i dx^i)^2 + h_{ij} dx^i dx^j$

with V, ψ_i, h_{ij} depending on (x^1, x^2, x^3)

Metric representation subject to gauge transformations: $t \mapsto t + u$, $\psi \mapsto \psi - du$

with u depending on (x^1, x^2, x^3)



"Spatial path" = worldsheet spanned by $W = \partial_t$ and spacetime curve γ

$$g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = 0 \quad \text{and} \quad g(\nabla_{\dot\gamma}\dot\gamma, W) = 0$$

$$\overset{\Longleftrightarrow}{\scriptstyle g(\dot\gamma,\dot\gamma)=-1}$$

$$-g(\dot\gamma, W) = e^{2V}(\dot t + \psi_i \dot x^i) = e^C = \text{const.}$$

Two types of brachistochrones: Extremising proper time τ or coordinate time t

τ-brachistochrones (= travel time brachistochrones):

Curve parametrised by proper time satisfies

$$-1 = -e^{2V}(\dot{t} + \psi_i \dot{x}^i)^2 + h_{ij}\dot{x}^i \dot{x}^j$$

With $e^{2V}\left(\dot{t} + \psi_i \dot{x}^i\right) = e^C$:

$$-1 = -e^{-2V} e^{2C} + h_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}$$

$$d\tau = \sqrt{\frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}}$$

τ-brachistochrones are geodesics of the Riemannian metric

$$h_C = \frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}$$

t-brachistochrones (=arrival time brachistochrones):

$$\frac{dt}{d\tau} = e^{C-2V} - \psi_i \frac{dx^i}{d\tau}$$

Inserting our previous result

$$d\tau = \sqrt{(h_C)_{ij} dx^i dx^j} = \sqrt{\frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}}$$

yields

$$dt = \sqrt{\left(\tilde{h}_C\right)_{ij} dx^i dx^j} - \psi_i dx^i$$

with

$$\tilde{h}_C = e^{2C-4V} (h_C)_{ij} dx^i dx^j = \frac{e^{-2V}}{1 - e^{2V-2C}} h_{ij} dx^i dx^j$$

t is arclength with respect to a Finsler metric of Randers type.

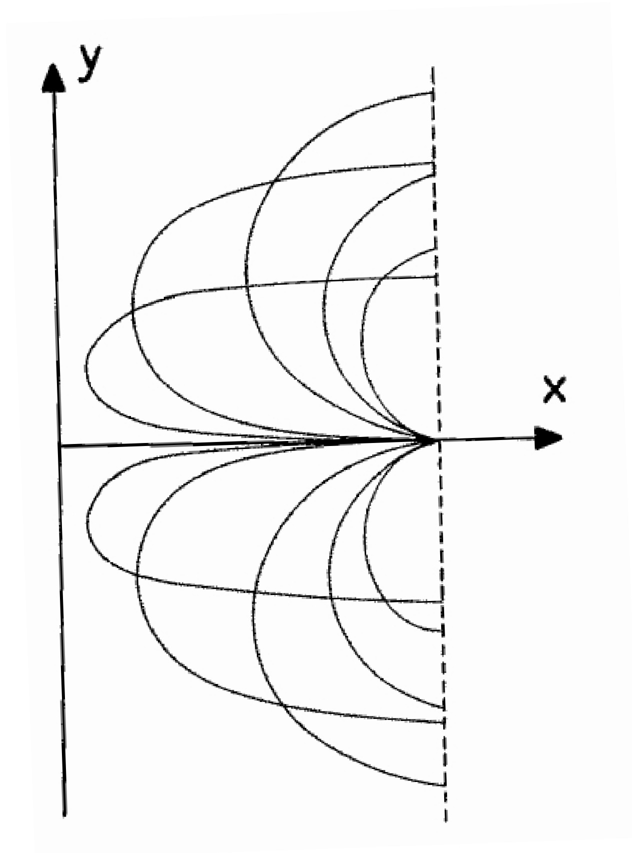
The t-brachistochrones are Finsler geodesics.

Example 4: Rindler metric

$$g = -x^2 dt^2 + dx^2 + dy^2 + dz^2$$

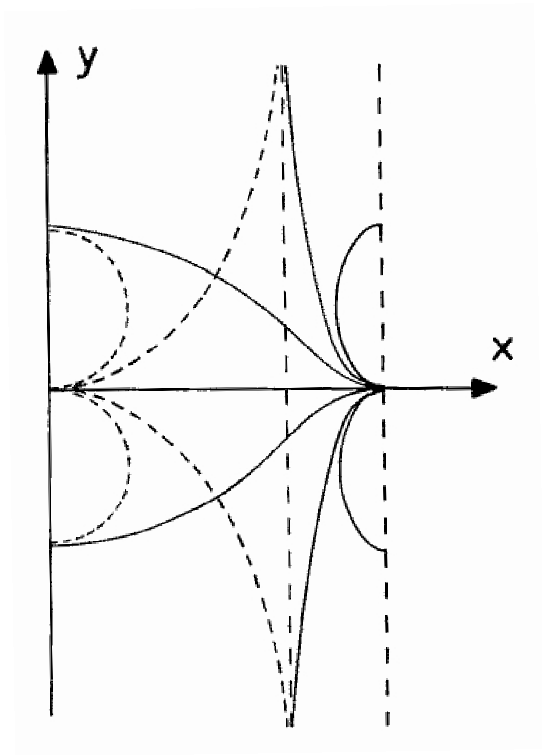
τ-brachistochrones

$$h_C = \frac{x^2}{e^{2C} - x^2}\left(dx^2 + dy^2 + dz^2\right), \quad e^{2C} = x_0^2$$



t-brachistochrones

$$\tilde{h}_C = \frac{e^{2C}}{x^2\left(e^{2C} - x^2\right)}\left(dx^2 + dy^2 + dz^2\right), \quad e^{2C} = x_0^2$$

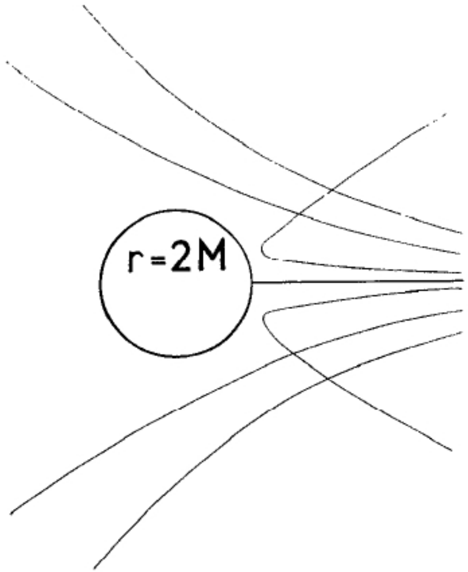


Example 5: Schwarzschild metric

$$g = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\Big)$$

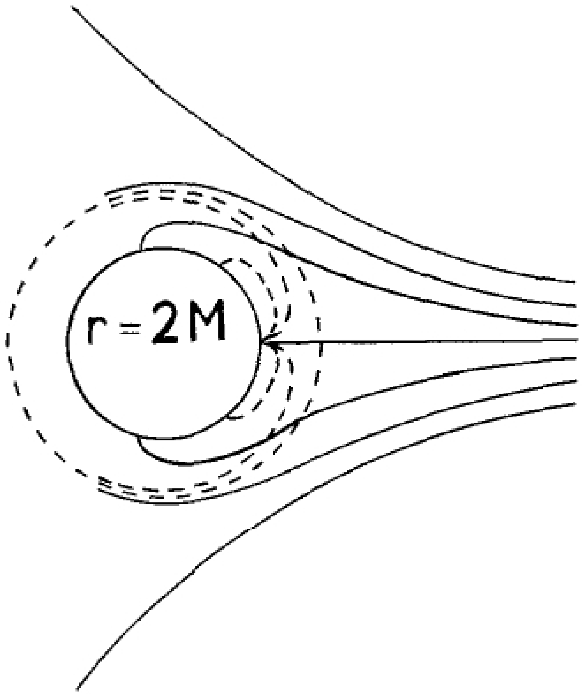
τ-brachistochrones:

$$h_C = \frac{\Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\Big)}{e^{2C} - \Big(1 - \frac{2M}{r}\Big)}\,, \qquad e^{2C} = \Big(1 - \frac{2M}{r_0}\Big)$$



t-brachistochrones:

$$\tilde{h}_C = \frac{e^{2C}\left(\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right)}{\left(1-\frac{2M}{r}\right)\left(e^{2C} - \left(1-\frac{2M}{r}\right)\right)}, \quad e^{2C} = \left(1-\frac{2M}{r_0}\right)$$

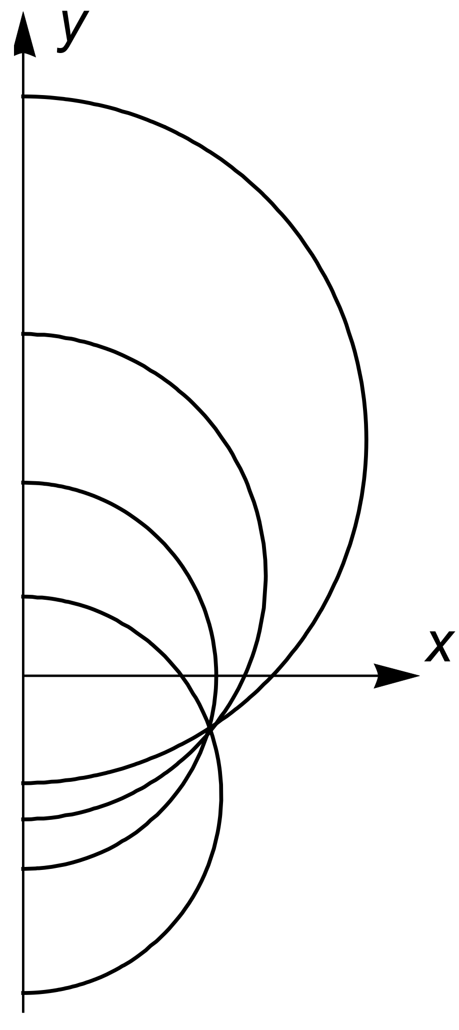


Example 6: Goedel spacetime

$$g = -\left(dt + \frac{dy}{\omega x}\right)^2 + \frac{dx^2 + dy^2}{2\omega^2 x^2} + dz^2$$

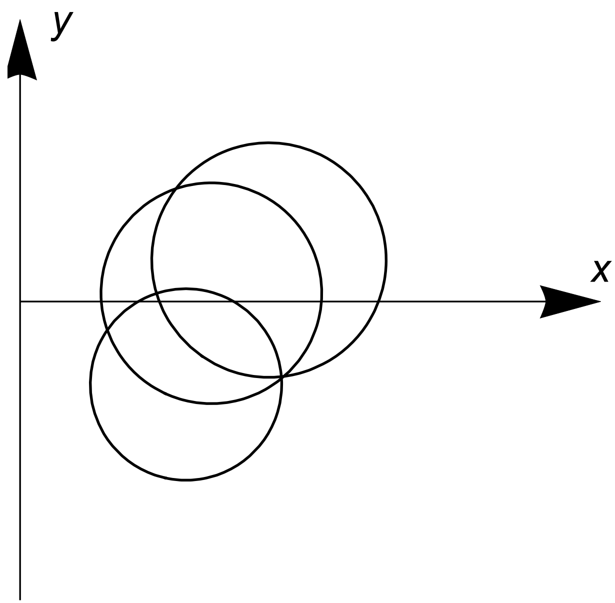
τ-brachistochrones

$$h_C = (e^{2C} - 1)^{-1}\left(\frac{dx^2 + dy^2}{2\omega^2 x^2} + dz^2\right)$$



t-brachistochrones

$$\tilde{h}_C = \frac{e^{2C}}{e^{2C}-1}\left(\frac{dx^2+dy^2}{2\omega^2x^2}+dz^2\right), \quad \psi_i dx^i = -\frac{dy}{\omega x}$$



Compare with (unconstrained) timelike geodesics in a stationary spacetime:

> The timelike geodesics with specific energy C in a stationary space-time project to geodesics of the Finsler metric
>
> $$\sqrt{(\hat{h}_C)_{ij}dx^i dx^j} - \psi_i dx^i\,, \qquad (\hat{h}_C)_{ij} = (e^{-2V} - e^{-2C})h_{ij}$$

t-brachistochrones of specific energy C in a stationary spacetime with (V, ψ, h)
=
Spatial paths of timelike geodesics of specific energy $\overline{C}$ in a stationary spacetime with $(\overline{V}, \psi, h)$ where $\left(e^{-2\overline{V}} - e^{-2\overline{C}}\right)\left(e^{2C-2V} - 1\right) = 1$

In the limit $C \to \infty : \tilde{h}_\infty = \hat{h}_\infty = e^{-2V}h$: Comparison with Fermat's principle for stationary spacetimes (Levi-Civita, 1918) shows that t-brachistochrones approach the spatial paths of lightlike geodesics.

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