The brachistochrone problem in general relativity

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Johann Bernoulli (1667-1748)

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1. The Newtonian brachistochrone problem

In 1696 Johann Bernoulli challenged the scientific community to solve the following problem:

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?

Here “gravity” means a homogeneous Newtonian gravitational field $\vec{g}$. 
The solution is a cycloid.

Correct solutions were supplied by Jakob Bernoulli, Isaac Newton, Guillaume de l’Hospital and others.

Solution in modern terminology:

\[ m \frac{d^2 \vec{r}}{dt^2} = -m \vec{\nabla}V(\vec{r}) + F_{\text{const}}, \quad V(\vec{r}) = \text{gravitational Potential}, \quad \frac{d\vec{r}}{dt} \cdot F_{\text{const}} = 0 \]

\[ 0 = \frac{d\vec{r}}{dt} \cdot \left( \frac{d^2 \vec{r}}{dt^2} + \vec{\nabla}V(\vec{r}) \right) = \frac{d}{dt} \left( \frac{1}{2} \left| \frac{d\vec{r}}{dt} \right|^2 + V(\vec{r}) \right) \]

\[ \frac{1}{2} \left| \frac{d\vec{r}}{dt} \right|^2 + V(\vec{r}) = C = \text{const.} \]

\[ dt = \frac{|d\vec{r}|}{\sqrt{2(C - V(\vec{r}))}} \]

Brachistochrones are geodesics of the Riemannian metric

\[ h_C = \frac{dx^2 + dy^2 + dz^2}{2(C - V(x, y, z))} \]
Example 1: Homogeneous gravitational field

\[ V(x, y, z) = gz \]

\[ h_C = \frac{dx^2 + dy^2 + dz^2}{2(C - gz)} \quad C = gz_0 \]
Example 2: Kepler potential

\[ V(x, y, z) = -\frac{GM}{\sqrt{x^2 + y^2 + z^2}} \]

\[ h_C = \frac{\left( dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right)}{2\left( C + \frac{GM}{r} \right)} , \quad C = -\frac{GM}{r_0} \]
Example 3: Centrifugal potential

\[ V(x, y) = -\omega^2(x^2 + y^2) \]

\[ h_C = \frac{dr^2 + r^2 d\phi^2}{2(C + \omega^2 r^2)}, \quad C = -\omega^2 r_0^2 \]
Compare with Maupertuis’ principle in the version of Jacobi:

The trajectories of (unconstrained) particles with specific energy $C$ in a potential $V(x, y, z)$ are geodesics of the “Jacobi metric”

$$\hat{h}_C = 2(C - V(x, y, z))(dx^2 + dy^2 + dz^2)$$

Hence:

Brachistochrones of specific energy $C$ in the potential $V(\vec{r})$ =

Free trajectories of specific energy $\overline{C}$ in the potential $\overline{V}$ where

$$4(\overline{C} - \overline{V}(\vec{r}))(C - V(\vec{r})) = 1$$
2. The brachistochrone problem in a stationary spacetime

Spacetime metric $g = -e^{2V}(dt + \psi_i dx^i)^2 + h_{ij} dx^i dx^j$

with $V, \psi_i, h_{ij}$ depending on $(x^1, x^2, x^3)$

Metric representation subject to gauge transformations: $t \mapsto t + u$, $\psi \mapsto \psi - du$

with $u$ depending on $(x^1, x^2, x^3)$

"Spatial path" = worldsheet spanned by $W = \partial_t$ and spacetime curve $\gamma$

\[ g(\nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) = 0 \quad \text{and} \quad g(\nabla_{\dot{\gamma}} \dot{\gamma}, W) = 0 \]

\[ g(\dot{\gamma}, \dot{\gamma}) = -1 \]

\[ -g(\dot{\gamma}, W) = e^{2V}(\dot{t} + \psi_i \dot{x}^i) = e^C = \text{const.} \]
Two types of brachistochrones: Extremising proper time $\tau$ or coordinate time $t$

$\tau$-brachistochrones ($= \text{travel time brachistochrones}$):

Curve parametrised by proper time satisfies

$$-1 = -e^{2V} (\dot{t} + \psi_i \dot{x}^i)^2 + h_{ij} \dot{x}^i \dot{x}^j$$

With $e^{2V} (\dot{t} + \psi_i \dot{x}^i) = e^C$:

$$-1 = -e^{-2V} e^{2C} + h_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}$$

$$d\tau = \sqrt{\frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}}$$

$\tau$-brachistochrones are geodesics of the Riemannian metric

$$h_C = \frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}$$
$t$-brachistochrones ($= \text{arrival time brachistochrones}$):

$$
\frac{dt}{d\tau} = e^{C-2V} - \psi_i \frac{dx^i}{d\tau}
$$

Inserting our previous result

$$
d\tau = \sqrt{(h_C)_{ij} dx^i dx^j} = \sqrt{\frac{h_{ij} dx^i dx^j}{e^{2(C-V)} - 1}}
$$

yields

$$
dt = \sqrt{\tilde{h}_C \ dx^i dx^j} - \psi_i dx^i
$$

with

$$
\tilde{h}_C = e^{2C-4V} (h_C)_{ij} dx^i dx^j = \frac{e^{-2V}}{1 - e^{2V-2C}} h_{ij} dx^i dx^j
$$

$t$ is arclength with respect to a Finsler metric of Randers type.

The $t$-brachistochrones are Finsler geodesics.
Example 4: Rindler metric

\[ g = -x^2 dt^2 + dx^2 + dy^2 + dz^2 \]

\( \tau \)-brachistochrones

\[ h_C = \frac{x^2}{e^{2C} - x^2} (dx^2 + dy^2 + dz^2), \quad e^{2C} = x_0^2 \]
$t$-brachistochrones

$$\tilde{h}_C = \frac{e^{2C}}{x^2(e^{2C} - x^2)} \left( dx^2 + dy^2 + dz^2 \right), \quad e^{2C} = x_0^2$$
Example 5: Schwarzschild metric

\[ g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \]

τ-brachistochrones:

\[ h_C = \frac{\left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)}{e^{2C} - \left(1 - \frac{2M}{r}\right)} \]

\[ e^{2C} = \left(1 - \frac{2M}{r_0}\right) \]
t-brachistochrones:

$$\tilde{h}_C = \frac{e^{2C} \left( \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \right)}{(1 - \frac{2M}{r}) \left( e^{2C} - \left( 1 - \frac{2M}{r} \right) \right)}, \quad e^{2C} = \left( 1 - \frac{2M}{r_0} \right)$$
Example 6: Goedel spacetime

\[ g = -\left( dt + \frac{dy}{\omega x} \right)^2 + \frac{dx^2 + dy^2}{2\omega^2 x^2} + dz^2 \]

\( \tau \)-brachistochrones

\[ h_C = (e^{2C} - 1)^{-1} \left( \frac{dx^2 + dy^2}{2\omega^2 x^2} + dz^2 \right) \]
$t$-brachistochrones

$$\tilde{h}_C = \frac{e^{2C}}{e^{2C} - 1} \left( \frac{dx^2 + dy^2}{2\omega^2 x^2} + dz^2 \right), \quad \psi_i dx^i = -\frac{dy}{\omega x}$$
Compare with (unconstrained) timelike geodesics in a stationary spacetime:

The timelike geodesics with specific energy $C$ in a stationary spacetime project to geodesics of the Finsler metric

$$
\sqrt{(\hat{h}_C)_{ij} dx^i dx^j - \psi_i dx^i}, \quad (\hat{h}_C)_{ij} = (e^{-2V} - e^{-2C}) h_{ij}
$$

$t$-brachistochrones of specific energy $C$ in a stationary spacetime with $(V, \psi, h)$ =

Spatial paths of timelike geodesics of specific energy $\bar{C}$ in a stationary spacetime with $(\bar{V}, \psi, h)$ where \( \left( e^{-2V} - e^{-2\bar{C}} \right) \left( e^{2\bar{C} - 2V} - 1 \right) = 1 \)

In the limit $C \to \infty$ : $\tilde{h}_\infty = \hat{h}_\infty = e^{-2V} h$: Comparison with Fermat’s principle for stationary spacetimes (Levi-Civita, 1918) shows that $t$-brachistochrones approach the spatial paths of lightlike geodesics.


