The gravitational redshift

- all the theory behind it

Volker Perlick

ZARM – Center of Applied Space Technology and Microgravity,
U Bremen, Germany

1. Standard clocks
– Formal definition
– Operational characterisation

in general relativity (but also in more general theories of gravity)

2. Redshift
– General redshift formula
– Existence of a redshift potential

in general relativity (but also in more general theories of gravity)

ISSI workshop, Bern, 27 March 2019

1. Standard clocks

Standard clocks in general relativity

(M, g): Manifold with pseudo-Riemannian metric of Lorentzian signature

For arbitrarily parametrised timelike curve $\gamma(t)$ define proper time

$$\tau = \int_{t_0}^{t} \sqrt{-g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)}\, dt$$

Parametrisation with $t = \tau$ is characterised by

$$g\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big) = -1$$

Allow for another choice of (time) unit:

$$g\big(\dot{\gamma}(\tau), \dot{\gamma}(\tau)\big) = \text{const.}$$

$$g\big(\dot{\gamma}(\tau), \nabla_{\dot{\gamma}(\tau)}\dot{\gamma}(\tau)\big) = 0$$

Rigid rulers and standard clocks are not appropriate as fundamental objects

Better use freely falling particles and light signals

Basis of the Ehlers-Pirani-Schild axiomatics

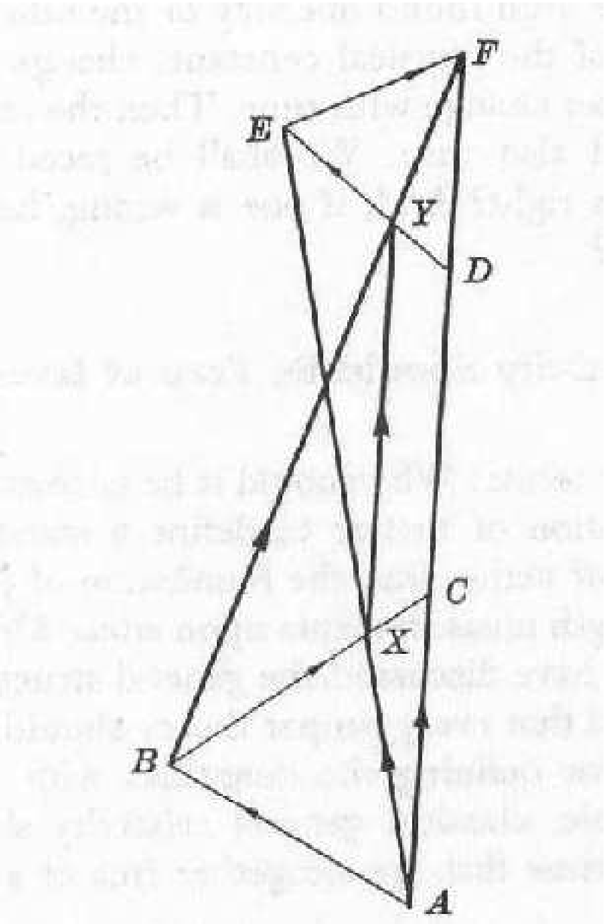
J. Ehlers, F. A. E. Pirani and A. Schild: "The geometry of free fall and light propagation" in: General Relativity, papers in honour of J. L. Synge. Edited by L. O'Raifeartaigh. Clarendon Press, Oxford (1972)

Axiomatic foundation for the result: Light signals are lightlike geodesics and freely falling particles are timelike geodesics of a Lorentzian metric

This motivates the goal: To characterise standard clocks with the help of light signals and freely falling particles

1st method:

R. F. Marzke and J. A. Wheeler: "Gravitation as geometry. I: The geometry of space-time and the geometrodynamical standard meter" In "Gravitation and relativity". Edited by H. Y. Chiu and W. F. Hoffmann. Benjamin, New York (1964)



Construct "infinitesimally neighbouring parallel" world-line

Let a light ray bounce back and forth

Prove that it arrives with the rhythm of a standard clock

2nd method:

W. Kundt and B. Hoffmann: "Determination of gravitational standard time". In "Recent developments in general relativity". Edited by ???. Pergamon, Oxford (1962)

Write metric as $ds^2 = e^{2U}\Big(\tilde{\gamma}_{\kappa\lambda}dx^\kappa dx^\lambda - (dx^0 + g_\mu dx^\mu)^2\Big)$. Want to determine e^{2U} along a chosen x^0-line.

Choose three neighbouring x^0 lines and assume that all four observers can measure x^0 along their worldlines.

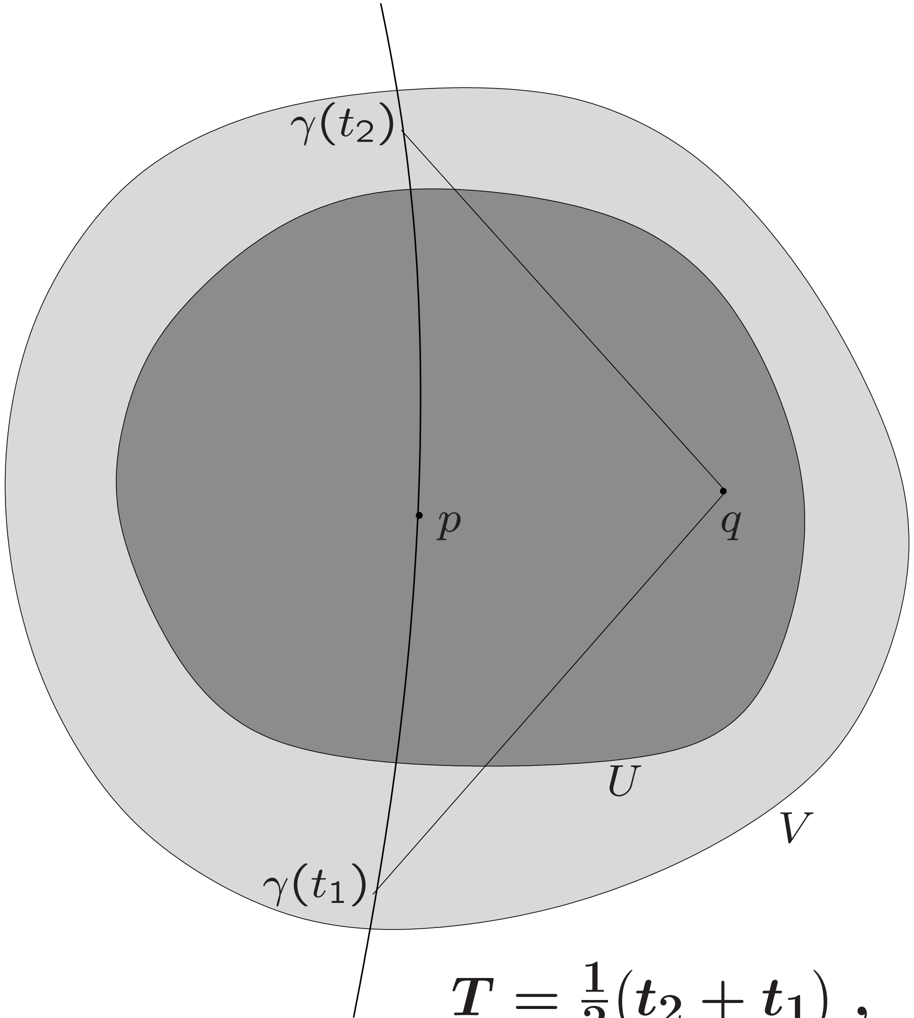
Let the four observers exchange light rays and freely falling particles and measure emission and reception x^0 time.

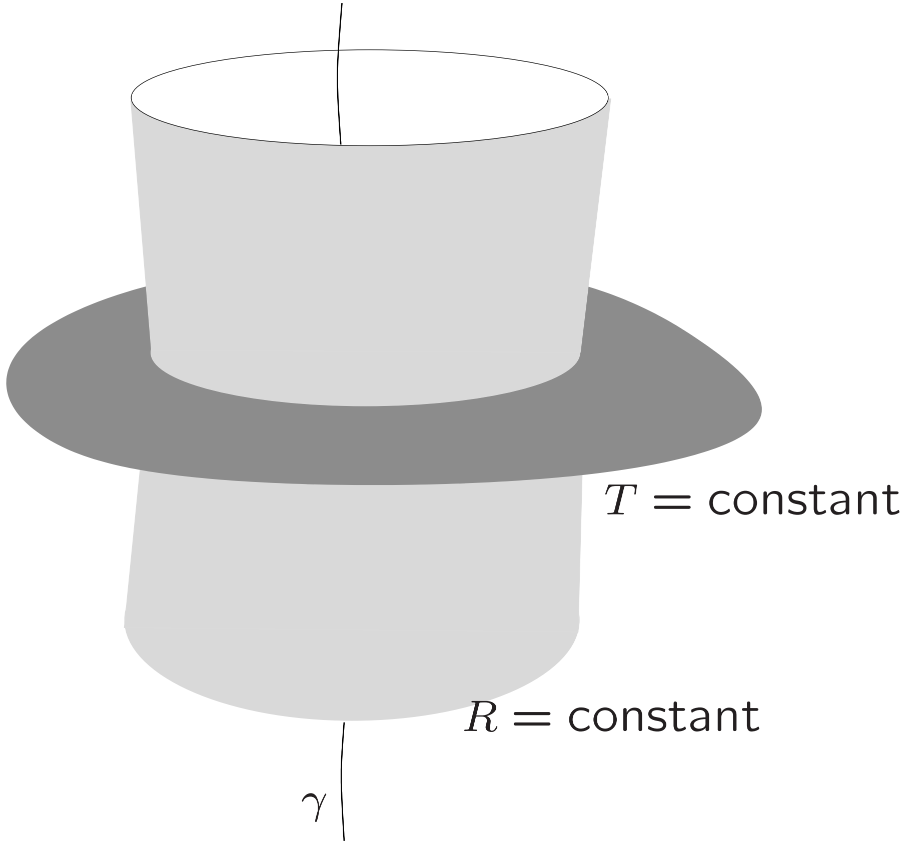
Get a system of 9 equations for 9 unknowns that determines e^{2U} and thus proper time along the chosen worldline.

3rd method:

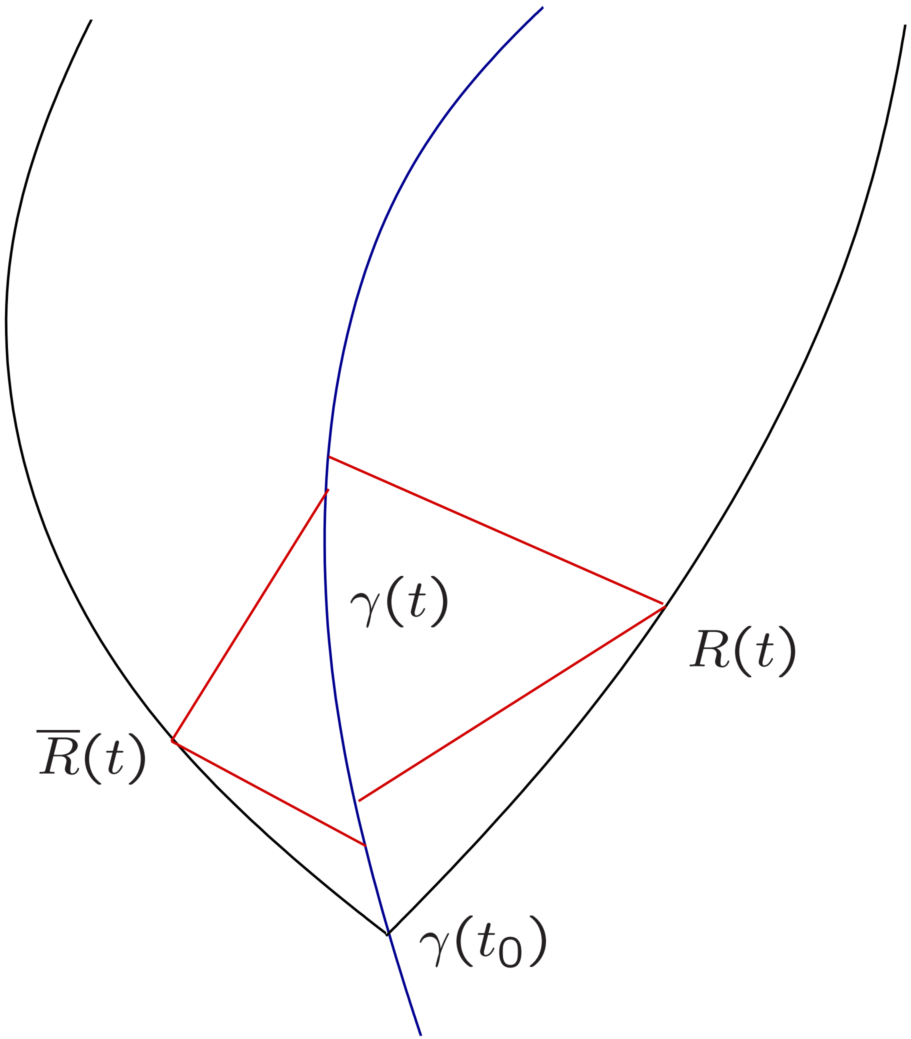
VP: "Characterization of standard clocks by means of light rays and freely falling particles". Gen. Rel. Grav. 19, 1059 (1987)

Uses radar time T and radar distance R





$$T = \frac{1}{2}(t_2 + t_1)\ , \qquad R = \frac{1}{2}(t_2 - t_1)$$



Want to test γ for being a standard clock

Emit two freely falling particles in opposite directions at $\gamma(t_0)$

Measure radar distances $R(t)$ and $\overline{R}(t)$ as functions of radar time $T(t) = \overline{T}(t) = t$

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} \frac{R''(t)}{(1 - R'(t)^2)} = - \lim_{t \to t_0} \frac{\overline{R}''(t)}{(1 - \overline{R}'(t)^2)}$$

If γ is freely falling:

γ is a standard clock at $\gamma(t_0)$ if and only if

$$\lim_{t \to t_0} R''(t) = 0$$

Existence of special observer fields V, with $g(V,V) = -1$, in general relativity:

- All clocks (= integral curves of V) are Einstein synchronous.

  $\Leftrightarrow$ V is irrotational Killing vector field

- Any pair of clocks (= integral curves of V) has temporally constant radar distance

  $\Leftrightarrow$ V is proportional to a Killing vector field

VP: "On the radar method in general-relativistic spacetimes" In "Lasers, clocks, and drag-free control. Exploration of relativistic gravity in space" Edited by H. Dittus, C. Laemmerzahl and S. G. Turyshev. Springer (2007)

Standard clocks in Weyl geometry

$(M, \mathfrak{g}, \nabla)$: Manifold with a conformal class of pseudo-Riemannian metrics of Lorentzian signature and a compatible connection

Compatibility: For every g in $\mathfrak{g}$ there is a covector field φ such that $\nabla_X g = \varphi(X) g$.

Gauge transformation: $g \mapsto e^h g$, $\phi \mapsto \varphi + dh$

$F = d\varphi$ is gauge-invariant ("Streckenkrümmung" = length curvature)

Light signals ($\mathfrak{g}$-lightlike ∇-geodesics) and freely falling particles ($\mathfrak{g}$-timelike ∇-geodesics) are well defined

Standard clocks are well defined:

$$g(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) = 0 \,, \quad g \in \mathfrak{g}$$

The third method of characterising standard clocks works.

Standard clocks in Finsler geometry

(M, g): Manifold with metric that depends on position and velocity, $g(x, v)$ where $(x, v) \in TM$ and

$g(x, v)$ is of Lorentzian signature

$g(x, kv) = g(x, v)$, $k > 0$

$\dfrac{\partial g_{ab}(x, v)}{\partial v^c}$ is totally symmetric

Geodesics:

$$\frac{d}{ds}\frac{\partial \mathcal{L}\big(x(s), \dot{x}(s)\big)}{\partial \dot{x}^a(s)} = \frac{\partial \mathcal{L}\big(x(s), \dot{x}(s)\big)}{\partial x^a(s)}$$

$$\mathcal{L}(x, v) = g_{ab}(x, v)v^a v^b$$

Light signals (geodesics with $\mathcal{L} = 0$) and freely falling particles (geodesics with $\mathcal{L} < 0$) are well defined

Proper time is well defined

$$\tau = \int_{t_0}^{t} \sqrt{-\mathcal{L}\big(\gamma(t), \dot{\gamma}(t)\big)}\, dt$$

Multiple light cones possible; under certain additional conditions there is a unique light cone

E. Minguzzi: "Light cones in Finsler spacetime" Commun. Math. Phys. 334, 1529 (2015)

Radar method works, but synchronous surfaces are not in general smooth

C. Pfeifer: "Radar orthogonality and radar length in Finsler and metric spacetime geometry" Phys. Rev. D 90, 064052 (2014)

Characterising standard clocks with light signals and freely falling particles .... (to be worked out)

Clock transport

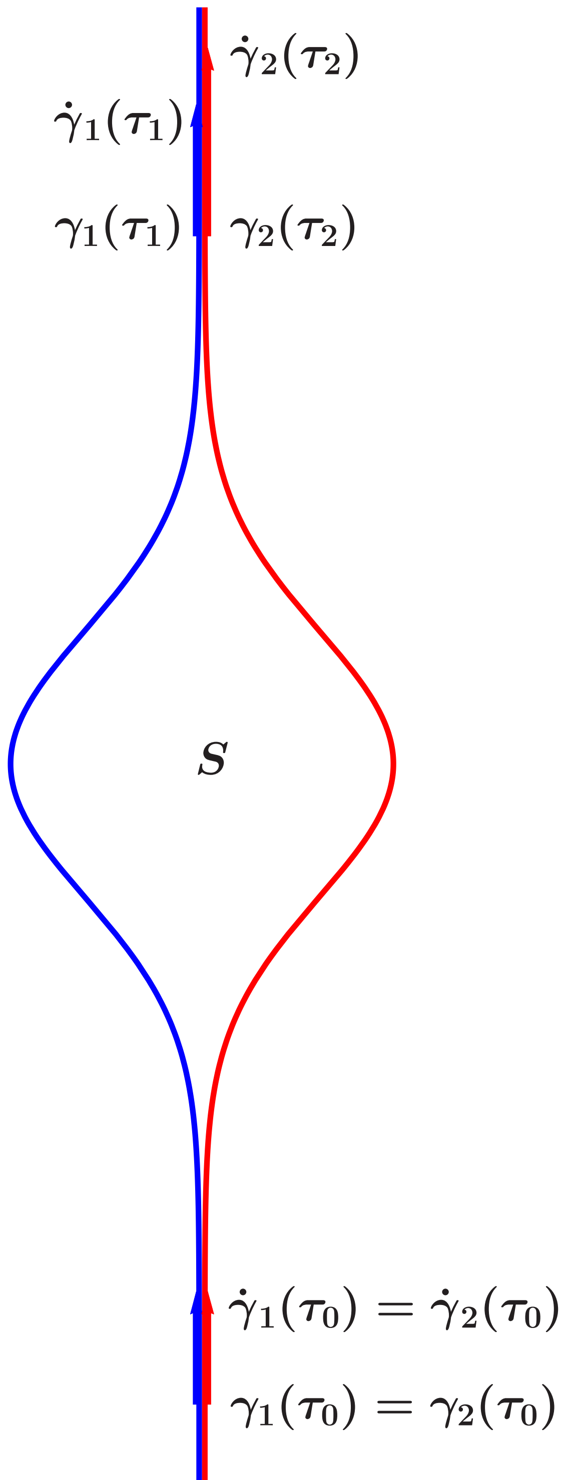
First clock effect: $\tau_1 \neq \tau_2$

Second clock effect: $\dot{\gamma}_1(\tau_1) \neq \dot{\gamma}_2(\tau_2)$

First clock effect occurs already in Specal Relativity

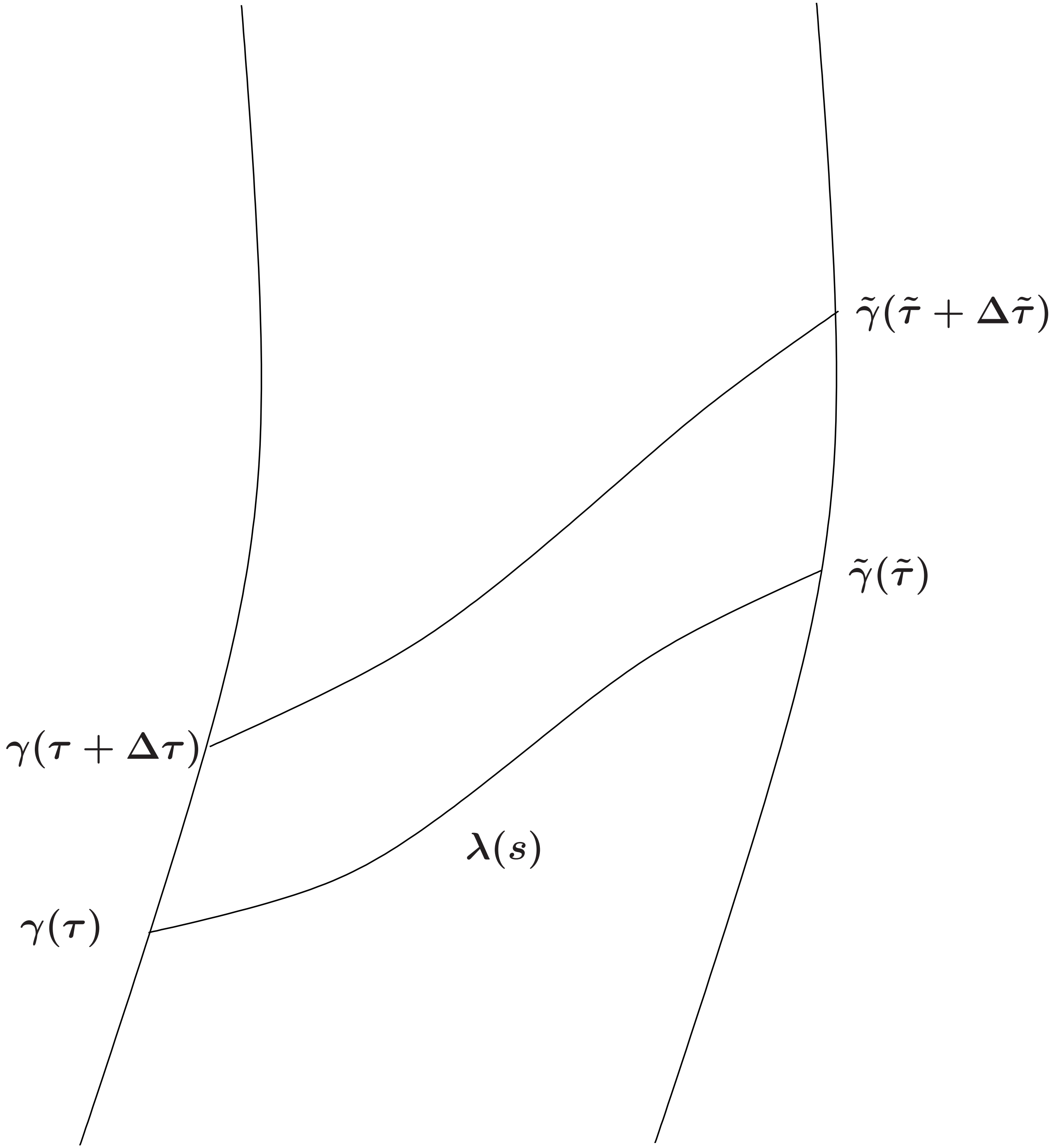
Second clock effect occurs only in non-reducible Weyl geometry and is proportional to

$$\int_S F = \oint \varphi$$



2. Redshift

For comparing the ticking of two standard clocks γ and $\tilde{\gamma}$, we send light rays from one to the other



Introduce the frequency ratio

$$\frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau\to 0}\frac{\Delta\tilde{\tau}}{\Delta\tau} =$$

$$= \frac{\omega_{\text{emitter}}}{\omega_{\text{receiver}}} = 1 + z$$

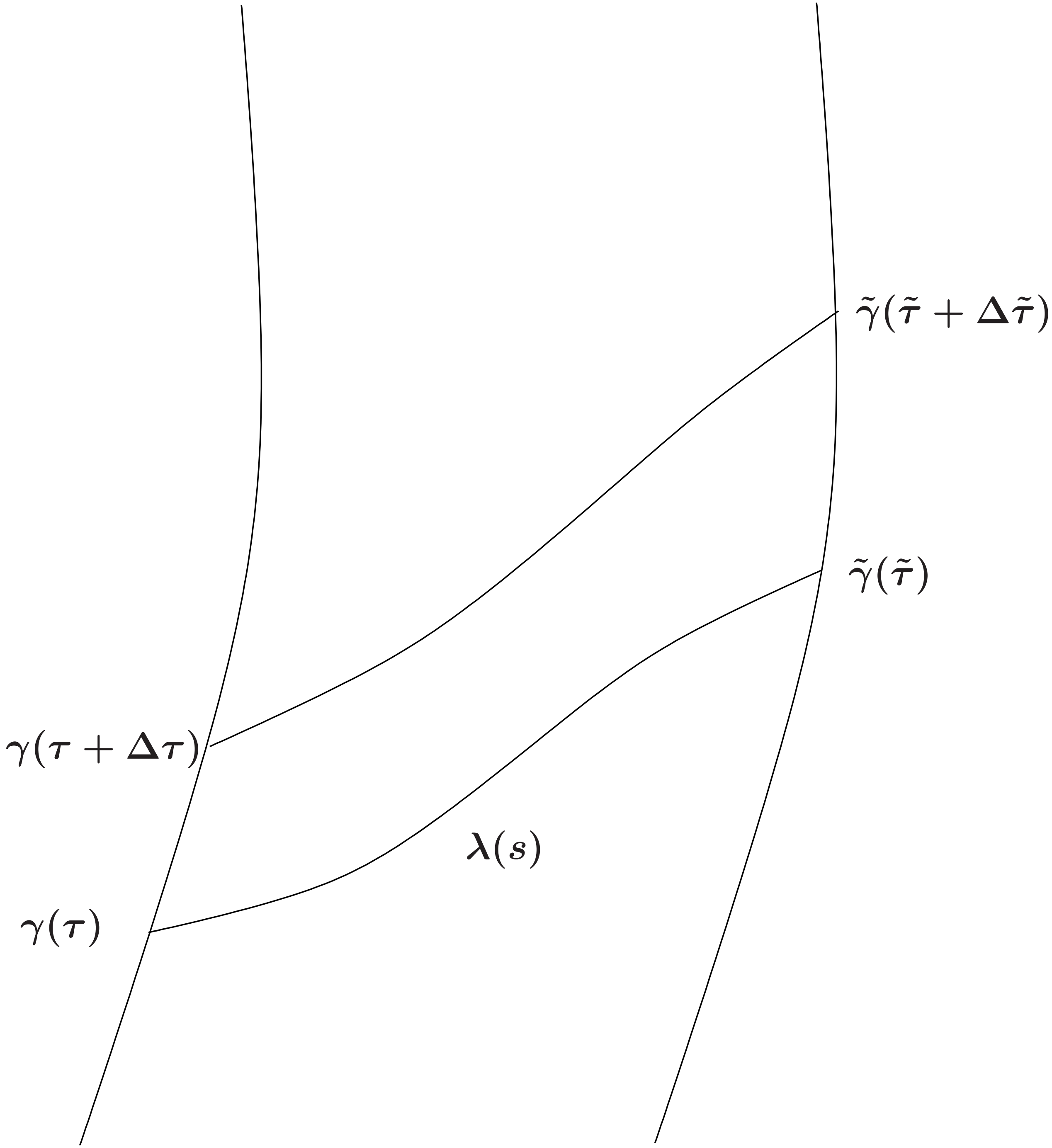
This defines the redshift

$$z = \frac{\omega_{\text{emitter}} - \omega_{\text{receiver}}}{\omega_{\text{receiver}}}$$

Universal redshift formula for standard clocks in general relativity:

$$1 + z = \frac{g_{ab}(\lambda(s_1))\,\frac{d\lambda^a}{ds}\Big|_{s=s_1}\frac{d\gamma^b}{d\tau}}{g_{cd}(\lambda(s_2))\,\frac{d\lambda^c}{ds}\Big|_{s=s_2}\frac{d\tilde{\gamma}^d}{d\tilde{\tau}}}$$

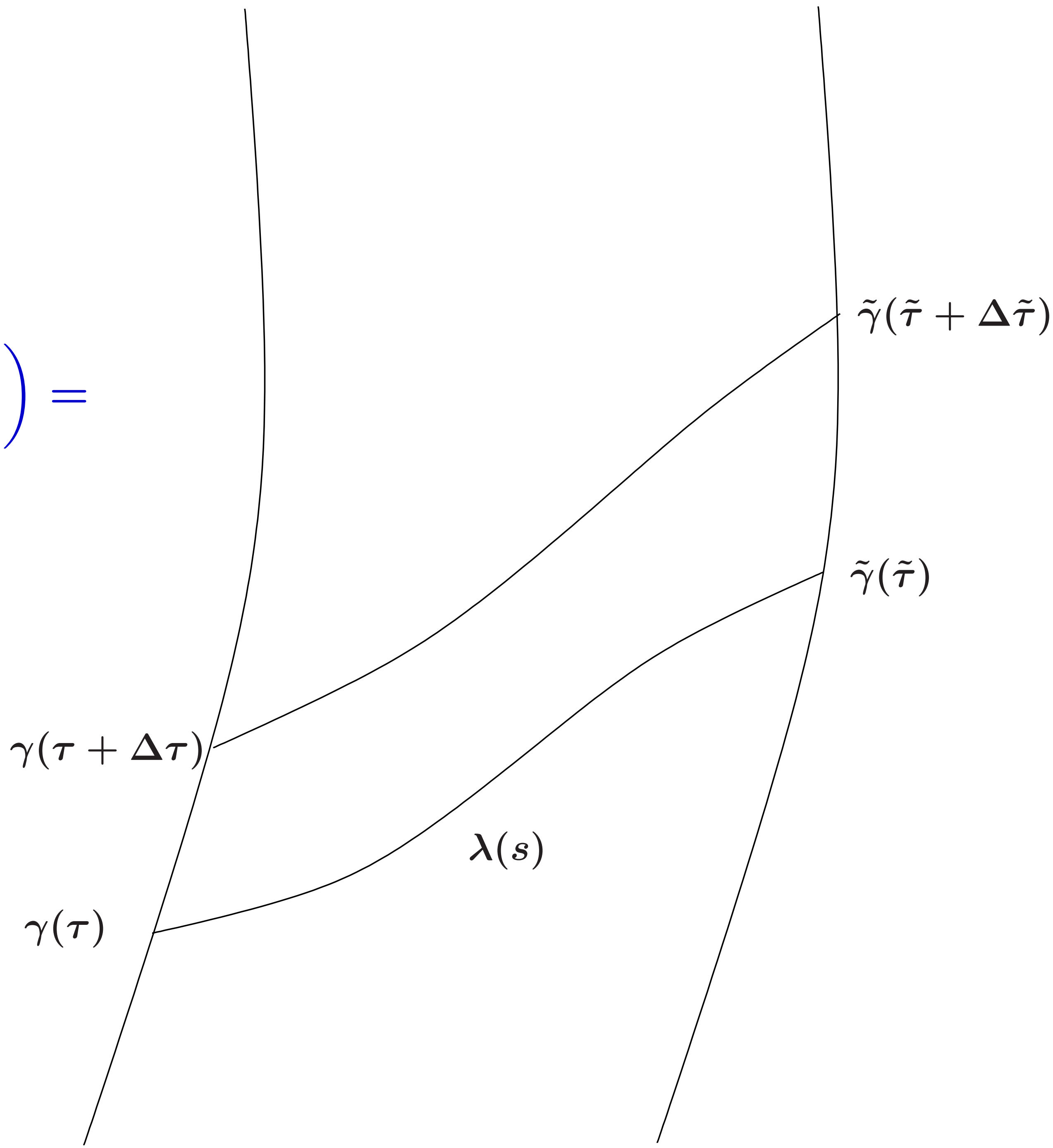
W. O. Kermack, W. H. McCrea and E. T. Whittacker: "On properties of null geodesics and their application to the theory of radiation", Proc. Roy. Soc. Edinburgh 53, 31 (1932)



Universal redshift formula for standard clocks in Weyl spacetime:

$$\left(1+z\right)\exp\left(-\int_{s_1}^{s_2}\varphi_a\frac{d\lambda^a}{ds}\,ds\right)=\frac{g_{\mu\nu}\big(\lambda(s_1)\big)\,\frac{d\lambda^\mu}{ds}\Big|_{s=s_1}\frac{d\gamma^\nu}{d\tau}}{g_{\rho\sigma}\big(\lambda(s_2)\big)\,\frac{d\lambda^\rho}{ds}\Big|_{s=s_2}\frac{d\tilde\gamma^\sigma}{d\tilde\tau}}$$

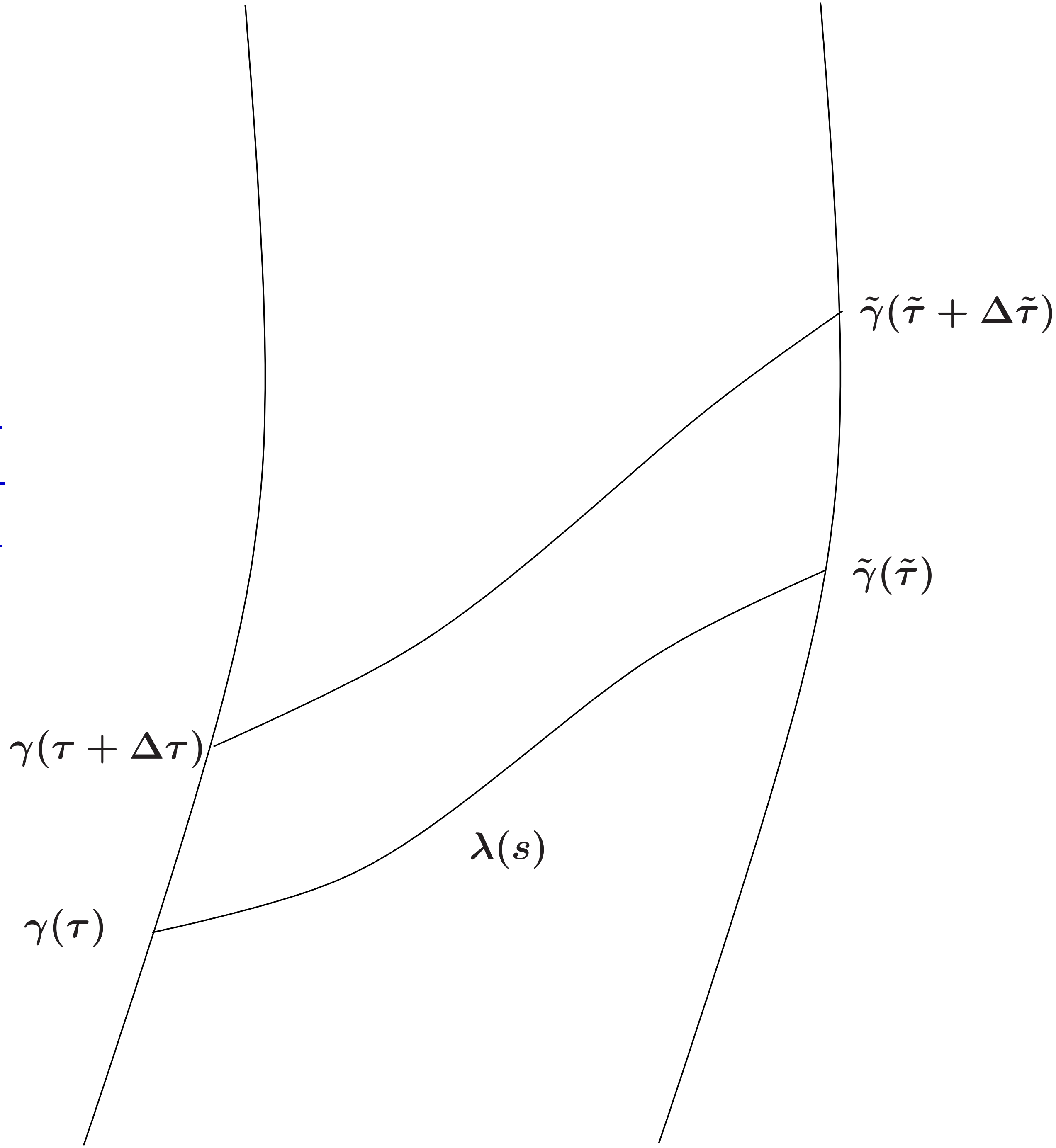
VP: PhD Thesis (1989)



Universal redshift formula for standard clocks in Finsler spacetime:

$$1 + z = \frac{g_{\mu\nu}\big(\lambda(s_1), d\lambda/ds\big)\, \frac{d\lambda^\mu}{ds}\Big|_{s=s_1} \frac{d\gamma^\nu}{d\tau}}{g_{\rho\sigma}\big(\lambda(s_2), d\lambda/ds\big)\, \frac{d\lambda^\rho}{ds}\Big|_{s=s_2} \frac{d\tilde{\gamma}^\sigma}{d\tilde{\tau}}}$$

W. Hasse and VP (in preparation)



Existence of a redshift potential for standard observer field V

$$\ln(1+z) = f\big(\tilde{\gamma}(\tau)\big) - f\big(\gamma(\tau)\big)$$

in general relativity:

f is a redshift potential if and only if $e^{f}V$ is a conformal Killing vector field.

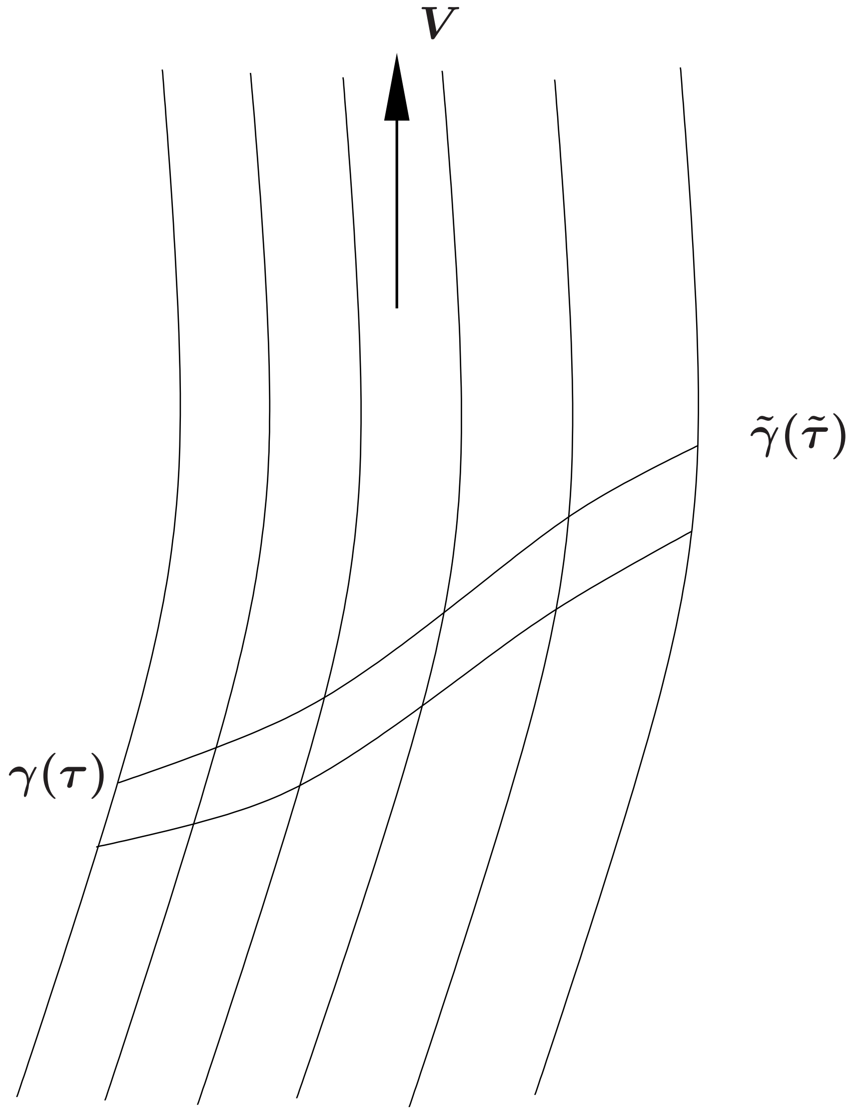
In coordinates $(x^0 = t, x^1, x^2, x^3)$ with $\partial_t = e^{f}V$ the metric reads

$$g_{ab}dx^a dx^b =$$

$$e^{2f}\Big(-\big(dt + \psi_\mu dx^\mu\big)^2 + h_{\mu\nu}dx^\mu dx^\nu \Big)$$

with $\partial_t \psi_\mu = \partial_t h_{\mu\nu} = 0$

W. Hasse and VP: "Geometrical and kinematical characterization of parallax-free world models", J. Math. Phys. 29, 2064 (1988)



Existence of a time-independent redshift potential for standard observer field V

$$\ln(1+z) = f\big(\tilde{\gamma}(\tau)\big) - f\big(\gamma(\tau)\big)$$

$$df(V) = 0$$

in general relativity:

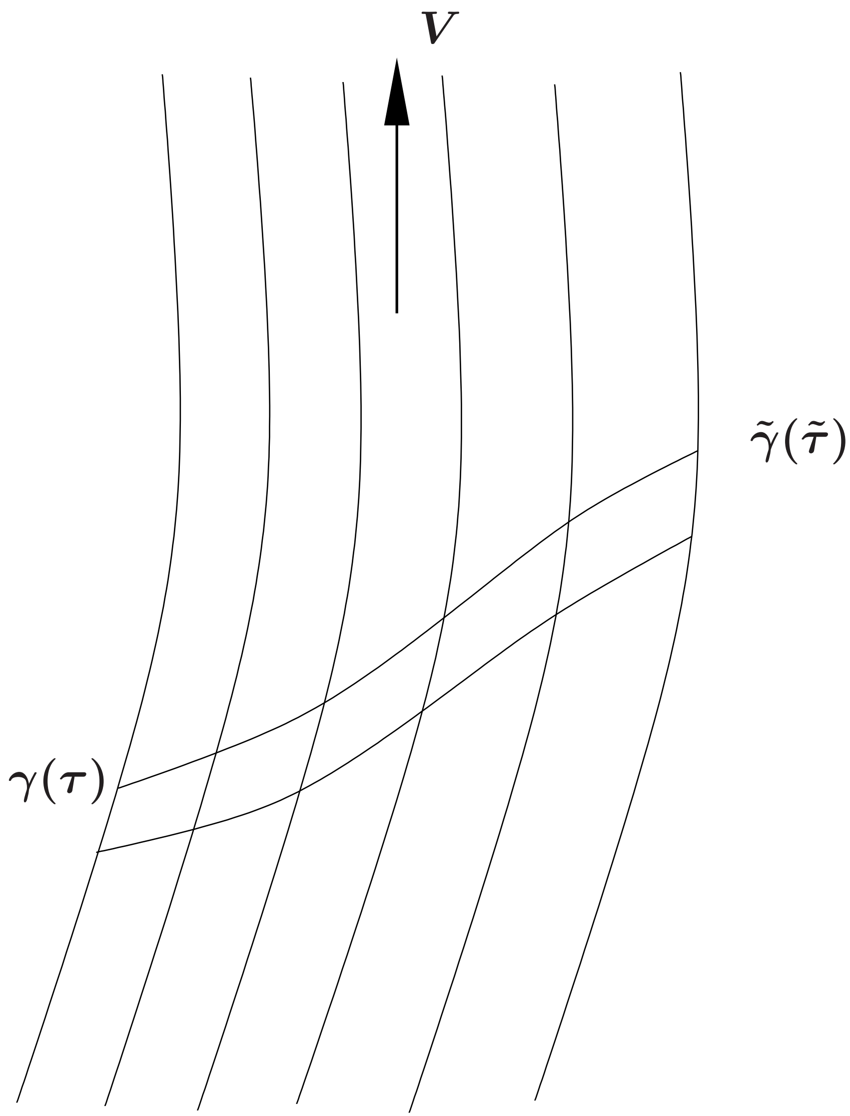
f is a time-independent redshift potential if and only if $e^f V$ is a Killing vector field.

In coordinates $(x^0 = t, x^1, x^2, x^3)$ with $\partial_t = e^f V$ the metric reads

$$g_{ab} dx^a dx^b =$$

$$e^{2f}\Big(-\left(dt + \psi_\mu dx^\mu\right)^2 + h_{\mu\nu} dx^\mu dx^\nu \Big)$$

with $\partial_t \psi_\mu = \partial_t h_{\mu\nu} = \partial_t f = 0$



In stationary spacetime redshift can be split into

- gravitational (seen by stationary observers)

- Doppler (redshift from motion relative to stationary observer)

Experimental verifications:

- Pound and Rebka (1959), Pound and Snider (1965): In a Laboratory on Earth

- Brault (1962): In the gravitational field of the Sun

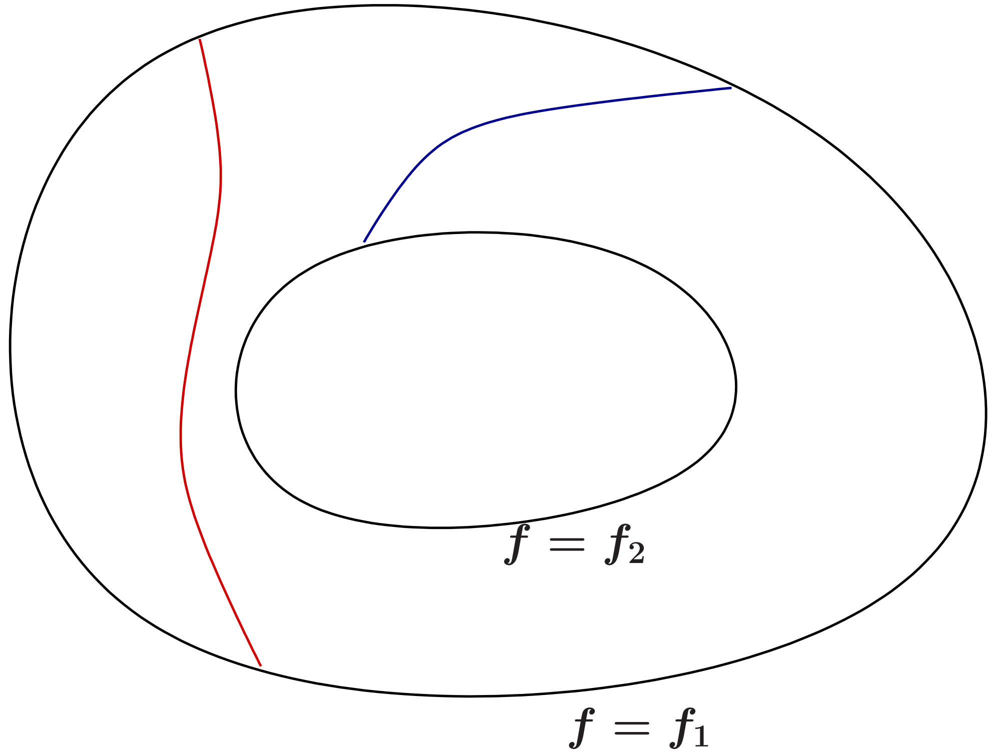
- Gravity Probe A (1976): With a sounding rocket in the gravitational field of the Earth

- GRAVITY collaboration (2018): With the S2 star in the gravitational field of the supermassive object at the centre of our galaxy

- Delva et al. and Herrmann et al. (2018): With Galileo satellites in the gravitational field of the Earth

A time-independent redshift potential foliates the 3-space into surfaces $f = \mathrm{const.}$ (“isochronometric surfaces”)

$$g_{ab}dx^a dx^b =$$
$$e^{2f}\Big(- (dt + \psi_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu\Big)$$



Coordinate travel time of signal with speed of light along spatial path:

$$t_2 - t_1 = \int \sqrt{h_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}ds$$
$$- \int \psi_\mu \frac{dx^\mu}{ds}ds$$

is independent of the emission time

$\Longrightarrow$ redshift potential gives correct redshift also for signals sent through optical fibers

Define the geoid as an isochronometric surface:

D. Philipp, VP, D. Puetzfeld, E. Hackmann, C. Laemmerzahl: "Definition of the relativistic geoid in terms of isochronometric surfaces", Phys. Rev. D 95, 104037 (2017)

Basic idea:

A.Bjerhammer (1985): "The relativistic geoid is the surface where precise clocks run with the same speed and the surface is nearest to mean sea level."

In PN formalism:

M. H. Soffel, H. Herold, H. Ruder and M. Schneider: "Relativistic theory of gravimetric measurements and definition of the geoid" Manuscripta Geodaetica 13, 143 (1988)

S. M. Kopeikin, E. M. Mazurova and A. P. Karpik: "Towards an exact relativistic theory of Earth's geoid undulation" Phys. Lett. A 379, 1555 (2015)

Alternative definition of a fully relativistic geoid:

M. Oltean, R. J. Epp, P. L. McGrath and R. B. Mann: "Geoids in general relativity: geoid quasilocal frames" Class. Quantum Grav. 33, 105001 (2016)

Example:

Isochronometric surfaces in the Kerr spacetime:

$$g_{ab}dx^a dx^b = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 - \frac{4mra\sin^2\vartheta}{\rho^2} dt\, d\varphi$$

$$+\sin^2\vartheta \left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\rho^2}\right) d\varphi^2$$

$$\rho^2 = r^2 + a^2\cos^2\vartheta, \; \Delta = r^2 + a^2 - 2mr$$

Killing vector field ∂_t

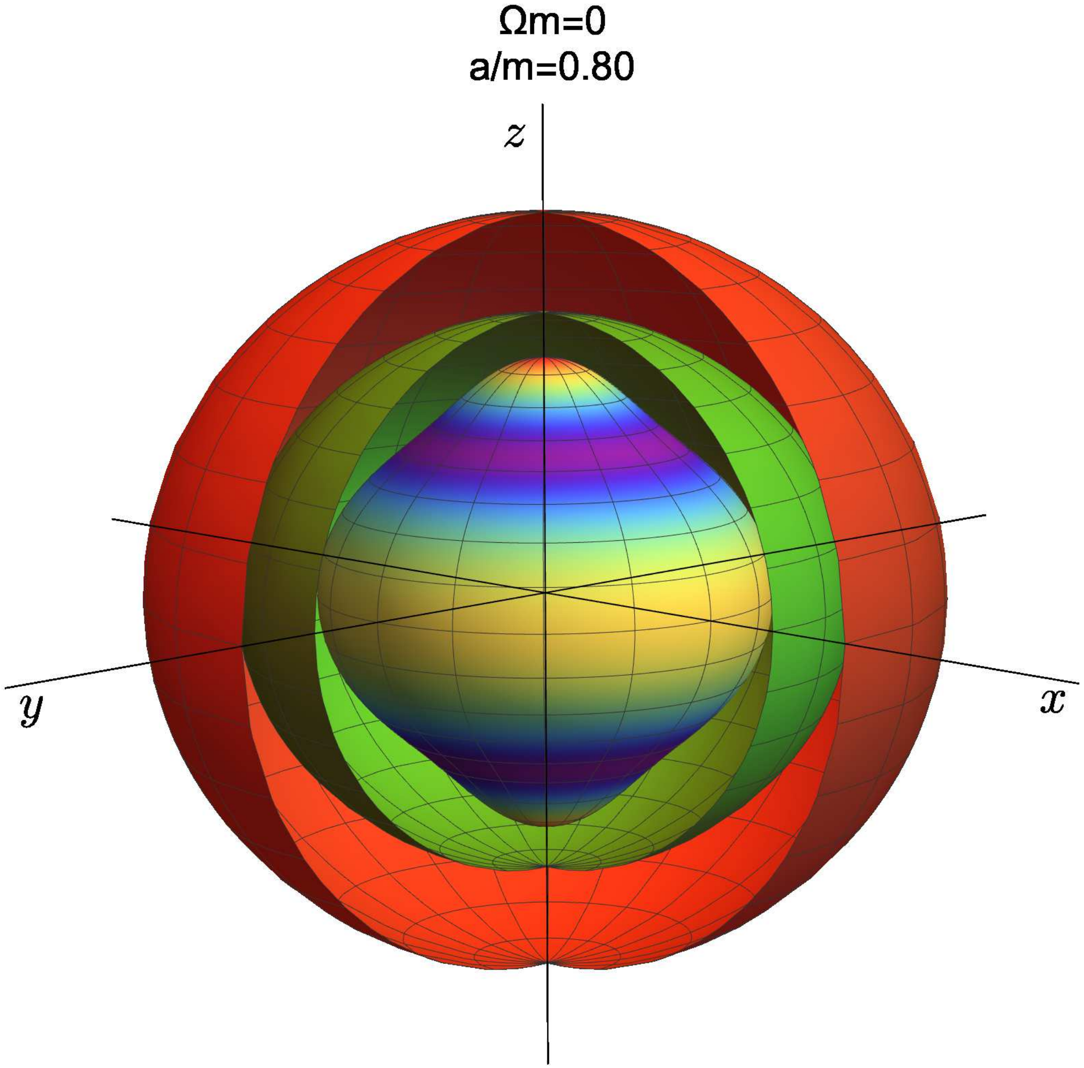
Redshift potential $e^{2f} = -g_{tt} = 1 - \dfrac{2mr}{\rho^2}$

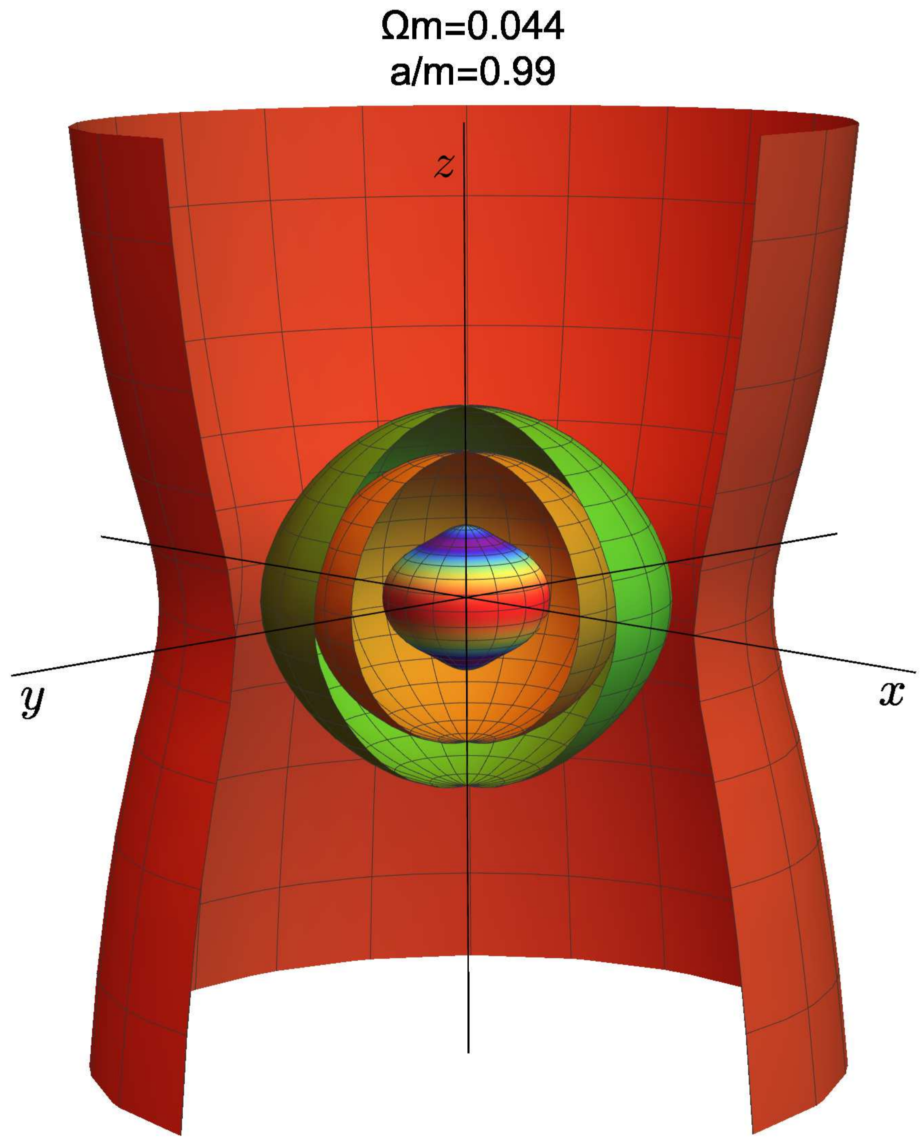
Coordinate transformation $\tilde{t} = t$, $\tilde{\varphi} = \varphi + \Omega t$, $\tilde{r} = r$, $\tilde{\vartheta} = \vartheta$

Killing vector field $\partial_{\tilde{t}} = \partial_t - \Omega \partial_\varphi$

Redshift potential $e^{2\tilde{f}} = -g_{\tilde{t}\tilde{t}} = -g_{tt} + 2\Omega g_{t\varphi} - \Omega^2 g_{\varphi\varphi}$

$$= 1 - \frac{2mr}{\rho^2} + 4\Omega \frac{mra\sin^2\vartheta}{\rho^2} - \Omega^2 \sin^2\vartheta \Big(r^2 + a^2 + \frac{2ma^2\sin^2\vartheta}{\rho^2} \Big)$$


Ωm=0
a/m=0.80
z
y
x


Ωm=0.044
a/m=0.99
z
y
x