Self-force in Bopp-Podolsky theory

Volker Perlick

ZARM, Univ. Bremen, Germany

Self-force of point particles in

(a) standard Maxwell theory

(b) Born-Infeld theory

(c) Bopp-Podolsky theory

Collaboration with Robin Tucker and Jonathan Gratus (Lancaster U)

Maxwell equations: $dF = 0\,, \qquad dH = 0$ outside of sources.

Constitutive equation in vacuo:

(a) Standard Maxwell: $H = {}^*F\,.$

(b) Born-Infeld theory:
$$H = \frac{{}^*F - \dfrac{{}^*(F\wedge F)}{8b^2}F}{\sqrt{1 + \dfrac{{}^*(F\wedge {}^*F)}{8b^2} + \dfrac{{}^*(F\wedge F)}{8b^4}}}\,.$$

M. Born, L. Infeld: "Foundations of the new field theory" Proc. Roy. Soc. London A 144, 425–451 (1934)

(c) Bopp-Podolsky theory: $H = \left(1 - \ell^2\Box\right){}^*F\,.$

F. Bopp: "Eine lineare Theorie des Elektrons" Annalen der Physik 430, 345–384 (1940)

B. Podolsky: "A generalized electrodynamics. Part I: Non-quantum" Phys. Rev. 62, 68–71 (1942)

Field of a static point charge:

(a) Standard Maxwell:

$$\nabla \times \vec{E} = \vec{0}\,, \qquad \nabla \cdot \vec{D} = 0 \qquad \text{for } r \neq 0\,.$$

Constitutive equation:

$$\vec{D} = \vec{E}\,.$$

Solution is the standard Coulomb field:

$$\vec{E} = \frac{q}{4\pi r^2}\,\vec{e}_r\,, \qquad \vec{D} = \frac{q}{4\,\pi\,r^2}\,\vec{e}_r\,.$$

The field energy in a ball K_R of radius R around the origin

$$W = \int_{K_R} \frac{1}{2}\,\vec{D}\cdot\vec{E}\,r^2 \sin\vartheta\,dr\,d\vartheta\,d\varphi = \frac{q^2}{8\,\pi}\int_0^R \frac{\cancel{r^2}\,dr}{\cancel{r^2}\,r^2}$$

is infinite.

(b) Born-Infeld theory:

$$\boldsymbol{\nabla} \times \vec{E} = \vec{0} \,, \qquad \boldsymbol{\nabla} \cdot \vec{D} = 0 \qquad \text{for } r \neq 0 \,.$$

Constitutive equation:

$$\vec{D} = \frac{\vec{E}}{\sqrt{1 - \frac{1}{b^2} |\vec{E}|^2}} \,.$$

Solution is (Born and Infeld, 1934):

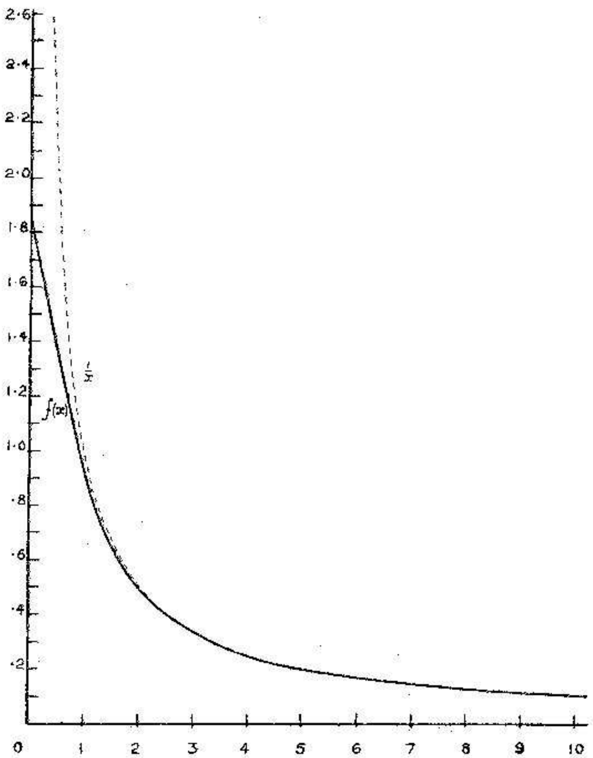
$$\vec{E} = \frac{q}{4\pi \sqrt{{r_0}^4 + r^4}} \vec{e}_r \,, \qquad \vec{D} = \frac{q}{4\pi r^2} \vec{e}_r \,, \qquad {r_0}^2 = \frac{q}{4\pi b}$$

Hence $|\vec{E}| \to b$ for $r \to 0$. The field energy in a ball K_R of radius R around the origin

$$W = \int_{K_R} \frac{1}{2} \vec{D} \cdot \vec{E}\, r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi = \frac{q^2}{8\pi} \int_0^R \frac{\cancel{r^2}\, dr}{\cancel{r^2} \sqrt{{r_0}^4 + r^4}}$$

is finite.

Plot of $|\phi(r)|$, where $\vec{E} = \phi'(r)\vec{e}_r$:



From

M. Born, L. Infeld: "Foundations of the new field theory" Proc. Roy. Soc. London A 144, 425–451 (1934)

(c) Bopp-Podolsky theory:

$$\nabla \times \vec{E} = \vec{0}\,, \qquad \nabla \cdot \vec{D} = 0 \qquad \text{for } r \neq 0\,.$$

Constitutive equation:

$$\vec{D} = \left(1 - \ell^2 \Delta\right) \vec{E}\,.$$

Solution is (Bopp, 1940, Podolsky, 1942):

$$\vec{E} = \frac{q}{4\,\pi\,r^2}\Big\{1 - A\,\ell\,(r+\ell)\,e^{-r/\ell} + B\,\ell\,(r-\ell)\,e^{r/\ell}\Big\}\,\vec{e}_r\,, \qquad \vec{D} = \frac{q}{4\,\pi\,r^2}\,\vec{e}_r\,.$$

$\vec{E}$ vanishing at infinity: $B = 0$.

$\vec{E}$ finite everywhere: $A = \ell^{-2}$, hence $|\vec{E}| \to q/(4\pi\ell^2)$ for $r \to 0$. Then the field energy in a ball K_R of radius R around the origin

$$W = \int_{K_R} \frac{1}{2}\,\vec{D}\cdot\vec{E}\,r^2 \sin\vartheta\,dr\,d\vartheta\,d\varphi =$$

$$\frac{q^2}{8\,\pi}\int_0^R \Big\{\frac{1}{r^2} - \frac{e^{-r/\ell}}{r^2} - \frac{e^{-r/\ell}}{r\ell}\Big\}dr = \frac{q^2}{8\pi}\Big\{\frac{1}{\ell} + \frac{e^{-R/\ell} - 1}{R}\Big\}$$ is finite.

Field of an accelerating point charge:

Consider Minkowski space,

$$g = \eta_{ab} dx^a dx^b.$$

Fix a timelike curve $\xi^a(\tau)$ with

$$\eta_{ab} \dot{\xi}^a \dot{\xi}^b = -1$$

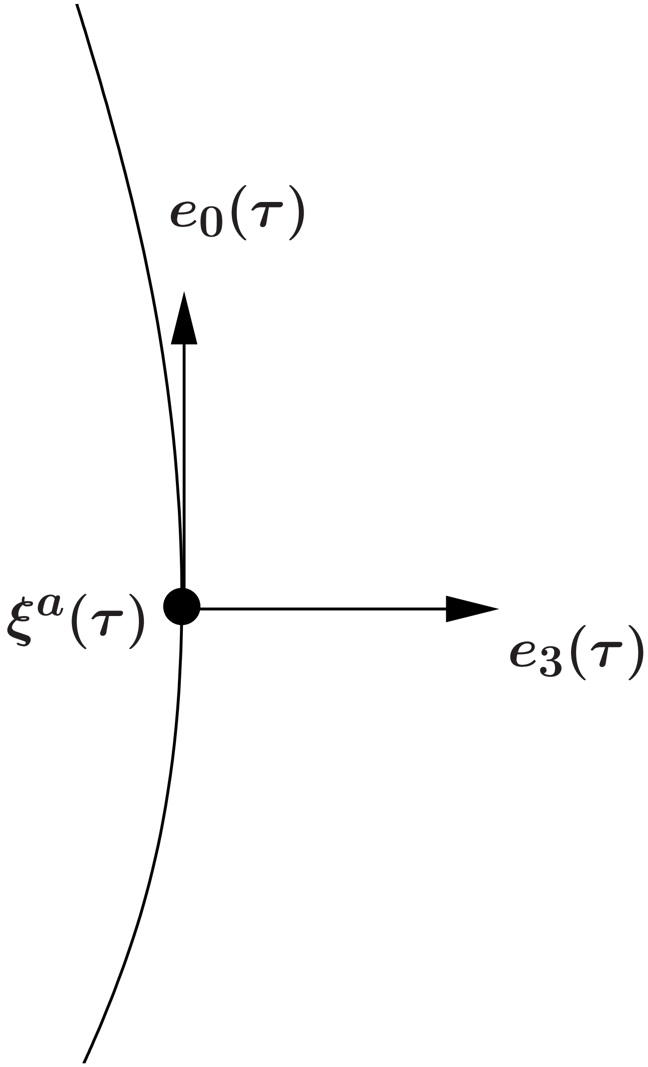
Choose a tetrad

$$\big(e_0(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\big)$$

along the worldline of the charged particle such that

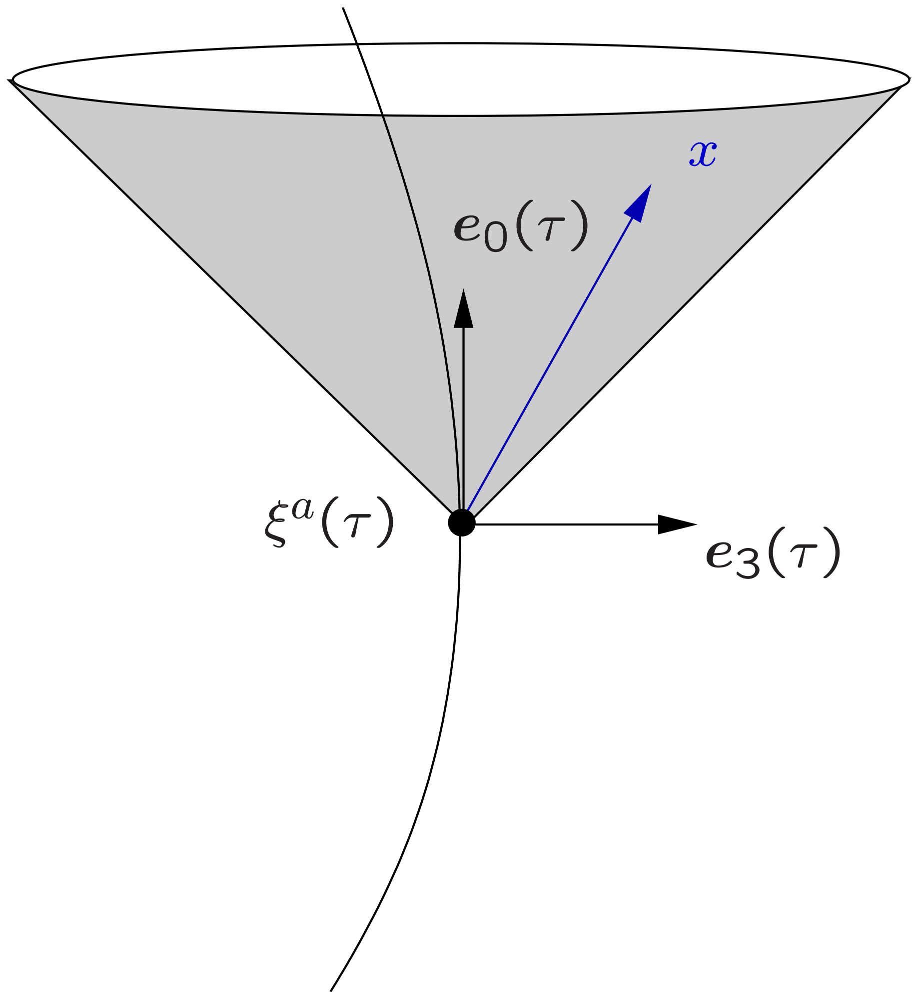
$e_0^a(\tau) = \dot{\xi}^a(\tau)$,

$a(\tau) e_3^b(\tau) = \ddot{\xi}^b(\tau)$.



Retarded light-cone coordinates $(\tau, r, \vartheta, \varphi)$:

E. T. Newman and T. W. J. Unti: "A class of null flat-space coordinate systems" J. Math. Phys. 4, 1467 (1963).



$$\boldsymbol{x^a = \xi^a(\tau) + r\Big(\dot{\xi}^a(\tau) + n^a(\tau, \vartheta, \varphi)\Big)}$$

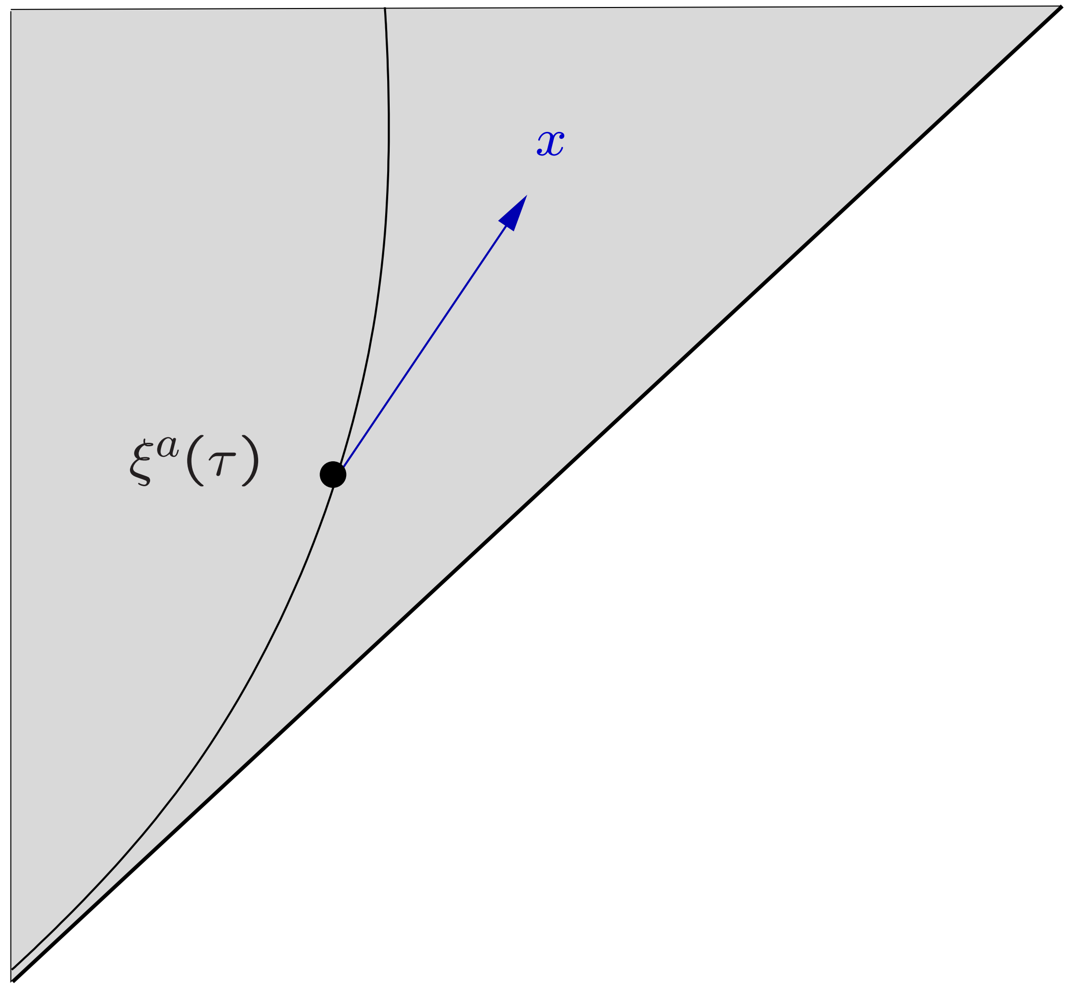
$$\boldsymbol{n^a(\tau, \vartheta, \varphi) = \cos\varphi \sin\vartheta\, e_1^a(\tau)}$$

$$\boldsymbol{+ \sin\varphi \sin\vartheta\, e_2^a(\tau) + \cos\vartheta\, e_3^a(\tau)}$$

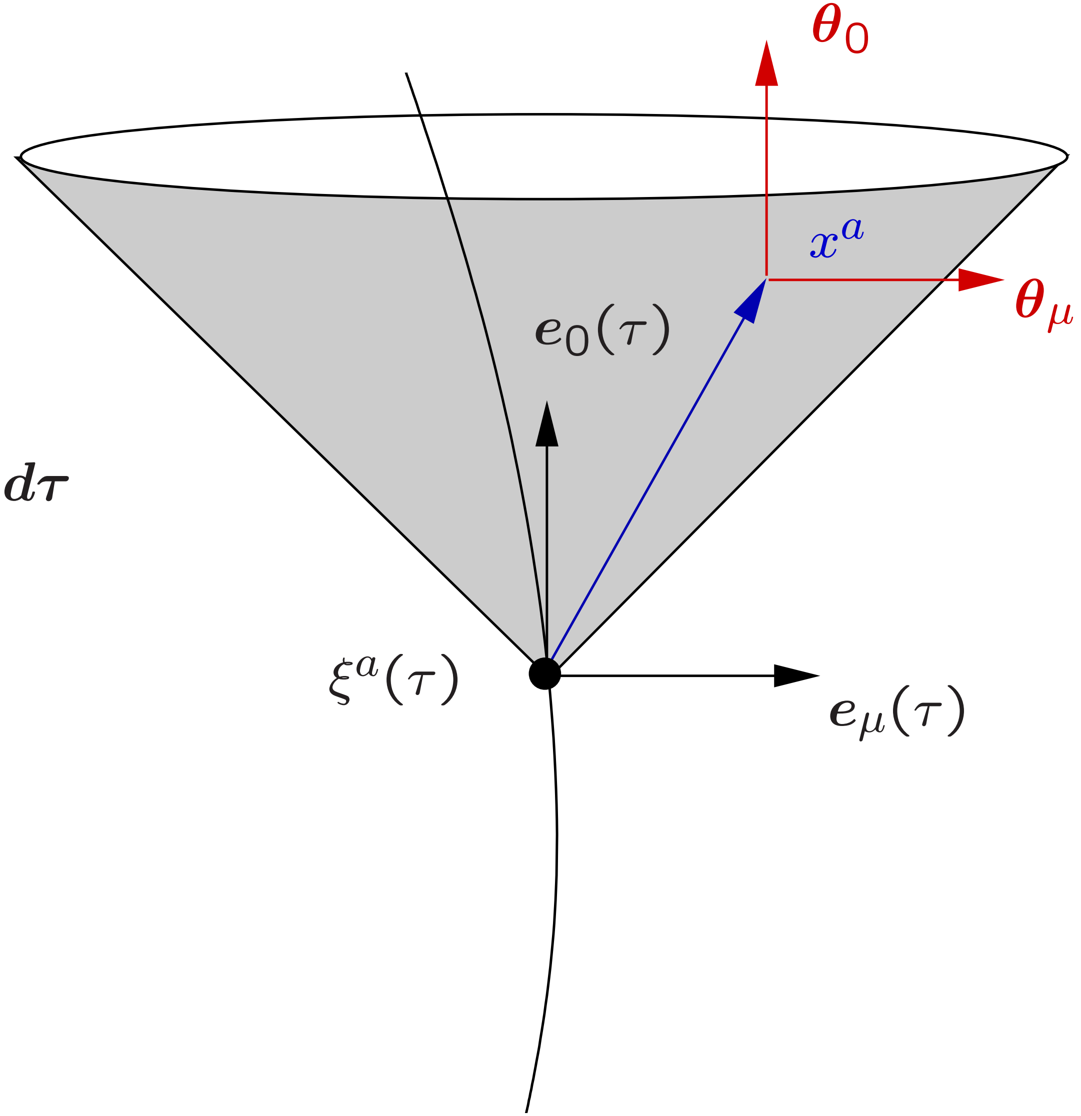
(Advanced) light cone coordinates where first introduced in General Relativity by G. Temple: "New system of normal co-ordinates for relativistic optics" Proc. R. Soc. London, Ser. A 168, 122–148 (1938)

Domain U of coordinate system $(\tau, r, \vartheta, \varphi)$:

$U =$ **causal future of the worldline, with the worldline itself omitted**



Associate orthonormal coframe $\boldsymbol{\theta^0, \theta^1, \theta^2, \theta^3}$ with the light-cone coordinates $(\tau, r, \vartheta, \varphi)$:



$$\theta^0 = d\tau + dr + r\,a(\tau)\cos\vartheta\,d\tau$$
$$\theta^1 = dr + r\,a(\tau)\cos\vartheta\,d\tau$$
$$\theta^2 = r\,d\vartheta - r\,a(\tau)\sin\vartheta\,d\tau$$
$$\theta^3 = r\sin\vartheta\,d\varphi$$

Want to solve Maxwell's equations $dF = 0, \quad dH = 0$ on U

and calculate the field energy in a ball around the charge.

(a) Standard Maxwell:

Solution is $F = dA$, $H = {}^*F$, where

$$A = -\frac{q\,\theta^0}{4\pi\,r} = -\frac{-q}{4\pi}\left(\frac{d\tau + dr}{r} + a(\tau)\cos\vartheta\,d\tau\right)$$

is the (retarded) Liénard-Wiechert potential.

$$A = -\frac{q\,\theta^0}{4\,\pi\,r}$$

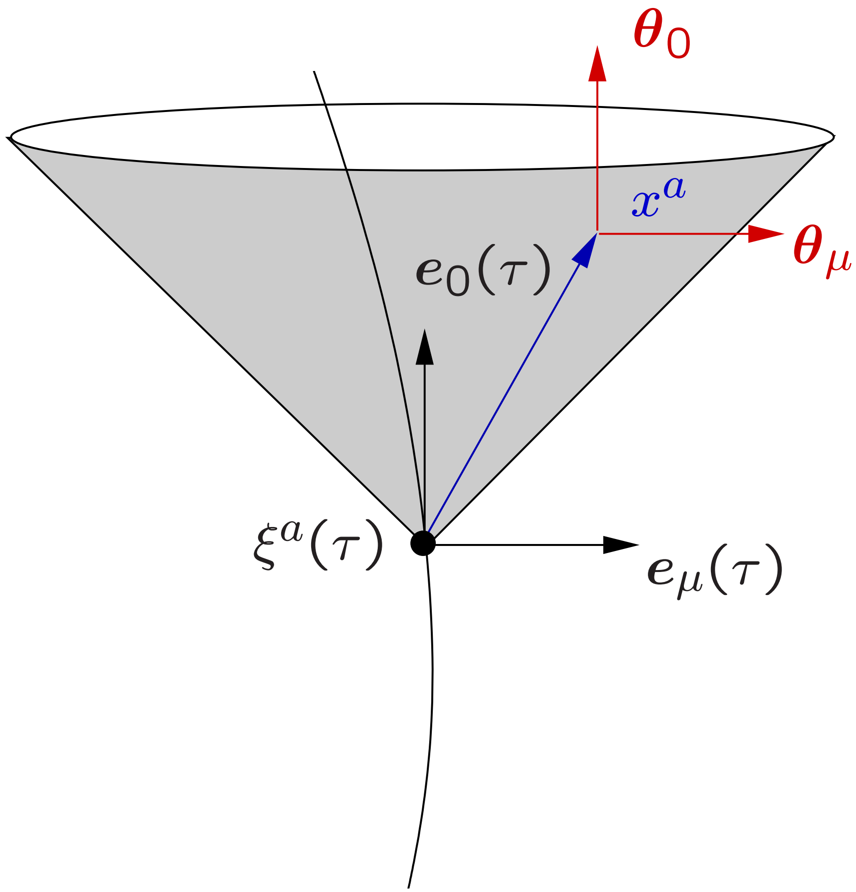
Write $F = dA$ as

$$F = E_\mu \theta^\mu \wedge \theta^0 + \frac{1}{2} B^\rho \varepsilon_{\rho\mu\nu} \theta^\mu \wedge \theta^\nu,$$

then

$$E_\mu \theta^\mu = \frac{q}{4\pi} \left\{ \frac{\theta^1}{r^2} + a(\tau) \sin\vartheta \, \frac{\theta^2}{r} \right\},$$

$$B_\mu \theta^\mu = \frac{q}{4\pi} a(\tau) \sin\vartheta \, \frac{\theta^3}{r}$$



hence the field energy in a ball $(r \leq R, \tau = \tau_0)$ is infinite.

(b) Born-Infeld theory:

Want to solve Maxwell's equations, $dF = 0, \quad dH = 0$ on U with the Born-Infeld constitutive law

$$H = \frac{{}^*F - \dfrac{{}^*(F \wedge F)}{8b^2} F}{\sqrt{1 + \dfrac{{}^*(F \wedge {}^*F)}{8b^2} + \dfrac{{}^*(F \wedge F)}{8b^4}}} \, .$$

Can the analogue of the Liénard-Wiechert potential be calculated?

Does the problem admit a solution that is regular everywhere away from the worldline of the charge?

Does it behave near the worldline in the same way as in the static case?

There are no regularity results on Born-Infeld field with time-dependent sources.

For time-independent regular sources ρ and $\vec{j}$, regularity of the solution has been proven only recently:

> M. Kiessling: "Convergent perturbative power series solution of the stationary Born-Infeld field equations with regular sources" J. Math. Phys. 52, 022902 (2011)

The proof uses series expansions in $1/b$.

The hard part is in the proof of convergence.

Behaviour of the fields near the worldline was discussed in

D. Chruściński: "Point charge in the Born-Infeld electrodynamics" Phys. Lett. A 240, 8–14 (1998)

based on ideas taken from

J. Kijowski: "Electrodynamics of moving particles" Gen. Rel. Grav. 26, 167–201 (1994)

But no proof of boundedness of the electric field strength is given.

Write $F = dA$ as a power series w.r.t. $1/b^2$:

$$F = \sum_{N=0}^{\infty} \frac{F_N}{b^{2N}} = F_0 + \frac{F_1}{b^2} + \ldots, \qquad F_N = dA_N$$

Insert into constitutive law:

$$H = \sum_{N=0}^{\infty} \frac{1}{b^{2N}} \left({}^*F_N + \mathcal{W}_N\big(F_0, \ldots, F_{N-1}\big) \right) =$$

The $F_N = dA_N$ are determined by $dH = 0$.

Solve this order by order:

Solution to zeroth order is known:

A_0 is the Liénard-Wiechert potential:

$$A_0 = -\frac{q\,\theta^0}{4\,\pi\,r} = \frac{-\,q}{4\,\pi}\Big(\frac{d\tau + dr}{r} + a(\tau)\cos\vartheta\,d\tau\Big)$$

Higher order solutions can be determined iteratively:

$$d\,\big({}^{*}dA_N + \mathcal{W}_N\big(dA_0, \ldots, dA_{N-1}\big)\,\big) = 0$$

In the Lorenz gauge: A_N can be written in terms of retarded potentials.

This gives the solution $F = dA$ as a formal power series.

Does this series converge? Don't know.

We do know that in the case of vanishing acceleration, $a(\tau) = 0$, it does converge.

One might conjecture that the same is true for small acceleration. However, it is not unlikely that for large acceleration singularities (e.g. "shock waves") may form, or the field may diverge too strongly towards the worldline.

(c) Bopp-Podolsky theory:

Want to solve Maxwell's equations, $dF = 0$, $dH = 0$ on U with the Bopp-Podolsky constitutive law

$$H = \left(1 - \ell^2 \Box\right) {}^*F \, .$$

Can the analogue of the Liénard-Wiechert potential be calculated?

Is it regular everywhere away from the worldline of the charge?

Does it behave near the worldline in the same way as in the static case?

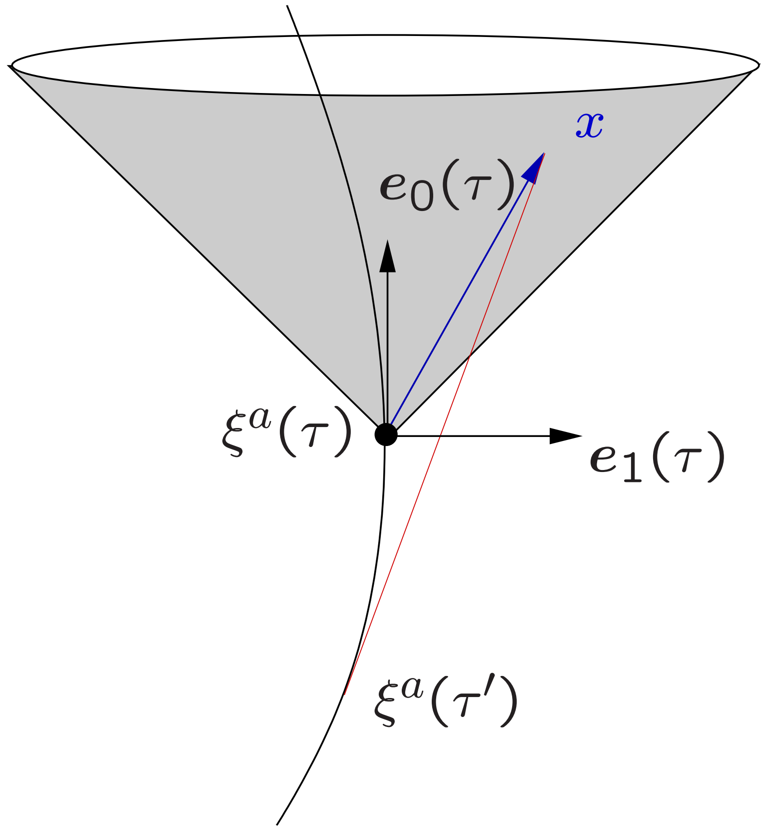
With $F = dA$ and choosing the Lorenz gauge, $d^*A = 0$, the field equation reads

$$\left(1 - \ell^2 \Box\right)\Box A = -4\pi j \,.$$

The Green function of $\left(1 - \ell^2 \Box\right)\Box$ is known, see

A.Landé, L.Thomas: "Finite self-energies in radiation theory. II" Phys. Rev. 60, 514–523 (1941)

Retarded solution for charge on world-line $\xi^a(\tau)$ is



$$A_a(x) = q \int_{-\infty}^{\tau} \frac{J_1(z/\ell)}{\ell\, z} \dot{\xi}_a(\tau')\, d\tau'$$

where

$$z^2 = -\left(x^a - \xi^a(\tau')\right)\left(x_a - \xi_a(\tau')\right) .$$

$$F_{ab}(x) = \frac{\partial A_b(x)}{\partial x_a} - \frac{\partial A_a(x)}{\partial x_b} =$$

$$\frac{q}{2\ell^2}\Big(\dot{\xi}_a(\tau) n_b(\tau,\vartheta,\varphi) - \dot{\xi}_b(\tau) n_a(\tau,\vartheta,\varphi)\Big)$$

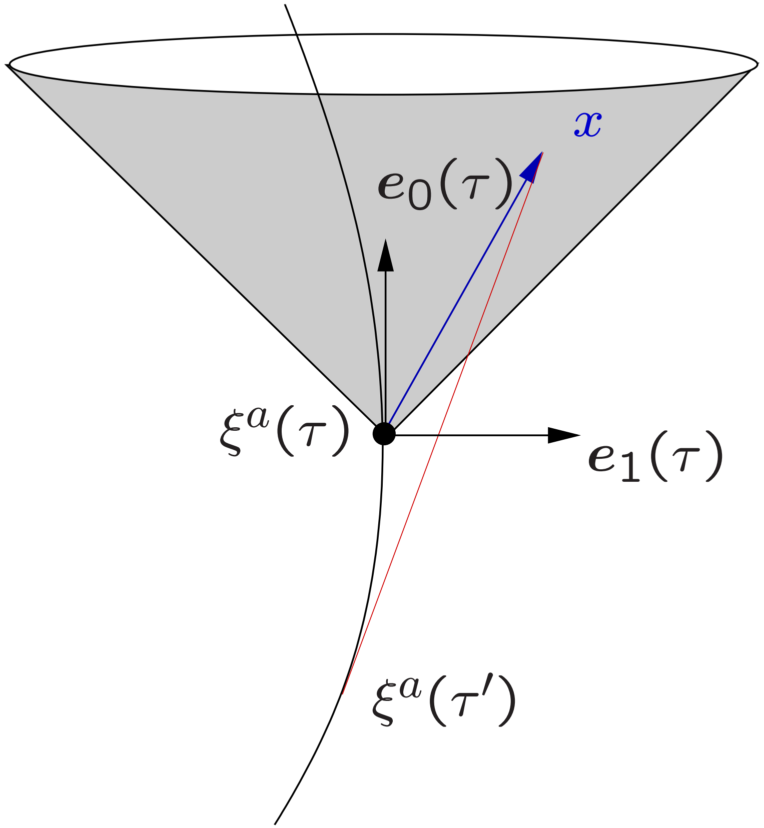
$$- q \int_{-\infty}^{\tau} \frac{J_2\big(\frac{z}{\ell}\big)}{z^2\ell^2} \Big(\big(x_b - \xi_b(\tau')\big)\dot{\xi}_a(\tau') - \big(x_a - \xi_a(\tau')\big)\dot{\xi}_b(\tau')\Big) d\tau' ,$$

F stays finite for $r \to 0$, for a large class of worldlines.

The energy in a sphere around the charge is finite.

Lorentz force on charge $\tilde{q}$ at x with 4-velocity U:

$$f_a(x) \;=\; \tilde{q}\, F_{ab}(x)\, U^b$$



Self-force: $x \to \xi(\tau)$ **with** $q = \tilde{q}$ **and** $U_b = \dot{\xi}_b(\tau)$

$$f^s_a(\tau) \;=$$

$$-\,q^2 \int_{-\infty}^{\tau} \frac{J_2(\frac{\zeta}{\ell})}{\zeta^2\,\ell^2}\,\dot{\xi}^b(\tau)\Big(\big(\xi_b(\tau)-\xi_b(\tau')\big)\dot{\xi}_a(\tau') \;-\; \big(\xi_a(\tau)-\xi_a(\tau')\big)\dot{\xi}_b(\tau')\Big)d\tau'$$

where

$$\zeta^2 \;=\; -\,\big(\xi^a(\tau)-\xi^a(\tau')\big)\big(\xi_a(\tau)-\xi_a(\tau')\big)\,.$$

The self-force is finite for a large class of timelike curve ξ.

There is no need (and no justification) for mass renormalisation.

The equation of motion is

$$m_0 \, \ddot{\xi}_a(\tau) \, = \, f^s_a(\tau) \, + \, f^e_a(\tau)$$

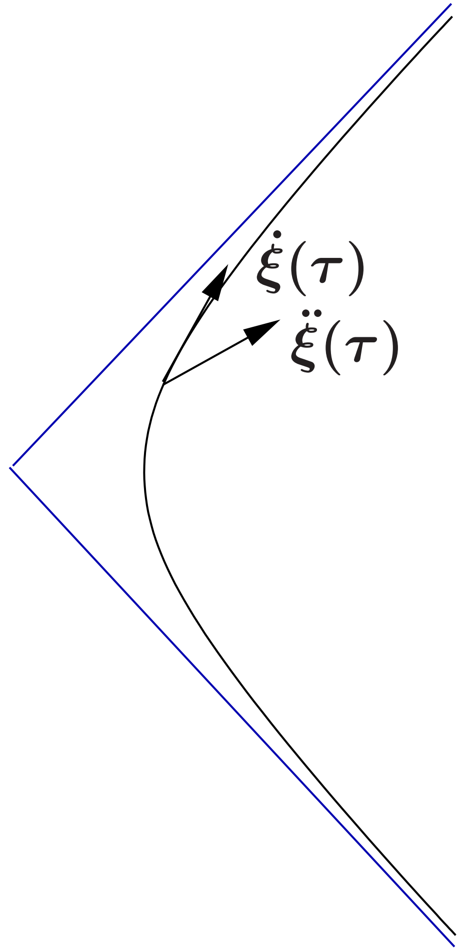
with a (bare) mass m_0.

This is an integro-differential equation for $\xi(\tau)$.

For $\ell \to 0$ and after mass renormalisation one gets the Lorentz-Dirac equation.

Example: Rindler motion (hyperbolic motion)

$$\xi(\tau) = \frac{1}{a} \begin{pmatrix} \sinh(a\tau) \\ \cosh(a\tau) \\ 0 \\ 0 \end{pmatrix}$$



$$f_a^s(\tau) = -\frac{q^2}{a\ell^2} I_1\big((a\ell)^{-1}\big) K_1\big((a\ell)^{-1}\big) \ddot{\xi}_a(\tau) \,.$$

$$m_0 \, \ddot{\xi}_a(\tau) = f_a^s(\tau) + f_a^e(\tau)$$

Cf. A. E. Zayats: "Self-interaction in the Bopp-Podolsky electrodynamics: Can the observable mass of a charged particle depend on its acceleration?" arXiv:1306.3966