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Local and Nonlocal Measurements of the Riemann Tensor

JÜRGEN AUDRETSCH and CLAUS LÄMMERZAHL

Fakultät für Physik der Universität Konstanz, Postfach 5560, D-775 Konstanz, Federal Republic of Germany

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Abstract

Quantum mechanically described test particles enable a local measurement of the Riemann tensor via the interaction with the elementary particle spin. The corresponding procedure is discussed in detail. It is compared with three nonlocal methods, which are based on the behavior of classical macroscopic test particles. A central question thereby is if the complete set of components of the Riemann tensor can be determined.

§(1): *Introduction*

The inclusion of quantum mechanically described matter into the classical theory of general relativity offers, already on the level of first quantization, several new answers to old questions. One of these fundamental questions is: how can the basic quantities of a metric theory of gravitation be measured at least in principle?

To measure the torsion of space-time, it seems to be inevitable to base the respective experiment on the elementary particle spin (Audretsch [1]). The other fundamental quantity of space geometry is the *curvature* as described by the Riemann tensor. And it is again the inclusion of quantum mechanically described particles with spin which enables a new type of measurement of the Riemann tensor with the important property of being a *local* one.

In the following we describe this method and compare it with three nonlocal methods, which are based on the behavior of classical test particles. To do so, all four methods will be presented in detail. In each case the central question will be, can the complete set of components of the Riemann tensor be determined?

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§(2): *Four Methods*

In a space-time theory of gravitation formulated in a Riemann space,[1] tidal gravitational fields are represented by the curvature of space-time. Mathematically, curvature is described by the Riemann tensor[2] $R^{\alpha}{}_{\beta\gamma\delta}$. It manifests itself in properties of the parallel transport which differ from those in flat space-time. Among others these are the following: (i) The failure of parallel geodesics to remain equidistant: the relative acceleration b^{α} of two neighbor timelike geodesics with tangent vector v^{α} and orthogonal connecting vector[3] r^{α} is (geodesic deviation)

$$b^{\alpha} = -R^{\alpha}{}_{\beta\gamma\delta} v^{\beta} r^{\gamma} v^{\delta} \tag{1}$$

(ii) the failure of a vector, when parallely transported around a closed loop, to return to its original direction (nonintegrability of the affine connection); (iii) the failure of the covariant derivative to commute (Ricci identity)

$$A_{\beta[\|\gamma\|\delta]} = \tfrac{1}{2} R^{\alpha}{}_{\beta\gamma\delta} A_{\alpha} \tag{2}$$

Each of these properties can be taken as basis for a measurement of the Riemann tensor.

The mathematical concepts of (i) and (ii) can be most directly interpreted in terms of classical test particle paths with tangent vector v^{α} and in the particles' classical internal angular momentum per proper mass S^{α} with $S^{\alpha} v_{\alpha} = 0$ and $S^{\alpha} S_{\alpha} = \text{const}$ so that the angular momentum is Fermi propagated:

$$\nabla_v S^{\alpha} = -(a_{\mu} S^{\mu})\, v^{\alpha} \tag{3}$$

The 4-acceleration a^{α} of such a gyroscope is given by

$$a^{\alpha} = \nabla_v v^{\alpha} = -R^{*\alpha}{}_{\beta\gamma\delta} v^{\beta} S^{\gamma} v^{\delta} \tag{4}$$

where

$$R^{*\alpha\beta\gamma\delta} = \tfrac{1}{2} R^{\alpha\beta\mu\nu} \eta_{\mu\nu}{}^{\gamma\delta} \tag{5}$$

[1] We assume vanishing torsion.

[2] We use the following conventions: Signature of the metric tensor $g_{\alpha\beta}$: $(+,---)$. $\alpha, \beta, \ldots =$ 0, 1, 2, 3 are tensor indices raised and lowered with $g_{\alpha\beta}$. $a, b \ldots = 0, 1, 2, 3$ and $\hat{a}, \hat{b}, \ldots =$ 1, 2, 3 are tetrad indices raised and lowered with $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. The corresponding object is a Riemann scalar with regard to $a, b, \ldots$. Particular values of $a, b, \ldots$ are denoted by brackets: $A^{(0)} = A^{a=0}$, $\eta^{(0)(1)(2)(3)} = -1$. Velocity of light: $c = 1$.

[3] See the theory of timelike congruences for details of the definition.

is the right dual of the Riemann tensor and ∇_v denotes the directional derivative along v^α (Papapetrou [2]).[4]

Classical test particles with vanishing internal angular momentum ($S^\alpha = 0$) move on geodesics ($a^\alpha = 0$). It has been suggested by Pirani [5] to use the observation of the relative acceleration of two such particles to measure components of the Riemann tensor on the basis of Eq. (1). A convenient device to do so is the gravity gradiometer [6] and the gravitational compass [7]. We will indicate below how the complete set of components of the Riemann tensor can be measured this way. Because two distinct particles at a distance r^α are used, the method is essentially nonlocal.

A second method is based on the comparison of two gyroscopes. According to (3)-(5) the angular momentum vector S^α of a test gyroscope with 4-velocity v^α is parallely propagated along its world line up to terms of second order in S^α. This behavior can be used to give a physical demonstration of the mathematical property (ii) above:

To introduce the closed loop, we take a pair of neighboring gyroscopes, displace at two different times the angular momentum of the second parallely along the orthogonal connecting vector r^α with $r^\alpha v_\alpha = 0$ to the world line of the first one, and compare it with the angular momentum there. Projection of the resulting difference vector $-\nabla_r S^\alpha$ into the local rest space of the observer accompanying the first gyroscope by means of the projection tensor

$$\underset{v}{P}^{\alpha\beta} = g^{\alpha\beta} - v^\alpha v^\beta \tag{6}$$

gives the difference of the directions of the two gyroscopes

$$\Delta S^\alpha = \underset{v}{P}^\alpha{}_\beta \nabla_r S^\beta \tag{7}$$

as seen by this observer. The change in time of this difference, again as registered by this observer, is then finally the measured quantity. It is given by

$$(\Delta S^\alpha)^{\cdot} = \underset{v}{P}^\alpha{}_\beta \nabla_v (\Delta S^\beta) \tag{8}$$

To work out the quantities above, it is easier to use instead of r^α the connecting vector n^α which is Lie transferred along the v^α world line

$$\underset{v}{\mathcal{L}} n^\alpha = 0 \tag{9}$$

Accordingly we have

$$(\nabla_n \nabla_v - \nabla_v \nabla_n) S^\alpha = R^\alpha{}_{\beta\gamma\delta} S^\beta n^\gamma v^\delta \tag{10}$$

[4] This paper is more of a formal than experimental character. We are not discussing the details of the experimental realizations of the procedures in Sections 4-6. The equations (3)-(4) have up to now no experimental justification. For the current status of gyro-satellite experiments see [3] and [4].

which implies with (3) and (4)

$$\nabla_v \nabla_n S^\alpha = -R^\alpha{}_{\beta\gamma\delta} S^\beta n^\gamma v^\delta + O(S^2) \tag{11}$$

We introduce this into (8) and obtain with (7) and (4)

$$(\Delta S^\alpha)^\cdot = -\underset{v}{P}{}^\alpha{}_\epsilon R^\epsilon{}_{\beta\gamma\delta} S^\beta r^\gamma v^\delta + O(S^2) \tag{12}$$

The replacement of n^α by $r^\alpha = \underset{v}{P}{}^\alpha{}_\beta n^\beta$ amounts thereby to corrections of the order S^2.

Using the left dual of the Riemann tensor

$${}^*R^{\alpha\beta\gamma\delta} = \tfrac{1}{2}\eta^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu\gamma\delta} \tag{13}$$

and introducing

$$H_\gamma = {}^*R_{\gamma\chi\lambda\mu} v^\chi r^\lambda v^\mu \tag{14}$$

Eq. (12) can be given the form

$$(\Delta S^\alpha)^\cdot = \eta^{\alpha\beta\gamma\delta} v_\beta H_\gamma S_\delta + O(S^2) \tag{15}$$

This represents the 3-equation

$$(\Delta \mathbf{S})^\cdot = \mathbf{H} \times \mathbf{S} \tag{16}$$

which refers to the rest space of the observer. Equation (16) can be found with a misprint in the Appendix of a paper of Sachs [8]. This second method, which is based on the comparison of two gyroscopes, is again explicitly a nonlocal one.

A third approach to measure the Riemann tensor can be based on the influence of the quantum mechanical spin on the trajectory v^α of a Dirac particle in curved space-time. It has been worked out by Audretsch [9], that in a WKB approximation after a Gordon decomposition the generalized "force equation" for the v^α congruence is given by

$$a^\alpha = \nabla_v v^\alpha = \lambda_c \tfrac{1}{4} R^\alpha{}_{\beta\gamma\delta} v^\beta S^{\gamma\delta} + O(\hbar^2) \tag{17}$$

where $\lambda_c = \hbar/m$ is the Compton wavelength of the particle and

$$S^{\gamma\delta} = \frac{\bar{\Psi}\sigma^{\gamma\delta}\Psi}{\bar{\Psi}\Psi} \tag{18}$$

is the tensor of the spin density.

The Riemann tensor causes via the quantum mechanical spin a nongeodesic behavior of the Dirac particle streamlines. This is essentially the case, because the convection current of the Gordon decomposition (as the Schrödinger current in the nonrelativistic limit) contains already a first derivative of the Dirac field ψ. Accordingly, a force equation (nongeodesic behavior) contains for a single world line second derivatives, and a Ricci identity for spinor fields can be applied. Because of this structure it is possible to measure components of the Riemann tensor in observing the influence of the spin on the bending of one world line only. We call this a *local measurement*. Substituting into (17) the normalized spin S^α by means of

$$S^{\alpha\beta} = \eta^{\alpha\beta\gamma\delta} v_\gamma S_\delta \tag{19}$$

the acceleration which is to be measured may be rewritten as

$$a^\alpha = -\frac{\lambda_c}{2} R^{*\alpha}{}_{\beta\gamma\delta} v^\beta S^\gamma v^\delta \tag{20}$$

Finally a fourth method can be based on the interaction of curvature with the classical internal angular momentum of a gyroscope as described by (4). Although a corresponding measurement uses one test object only, the classical angular momentum makes this method a nonlocal macroscopic one. Because Eqs. (4) and (20) essentially agree, the respective results can simply be transcribed.

§(3): *Six-Dimensional Notation*

To visualize the 20 independent components of the full *Riemann tensor*, we introduce a six-dimensional notation which directly represents the respective symmetry properties. With regard to a timelike normalized vector u^α we define the tensors

$$X_{\alpha\beta} = {}^*R^*{}_{\alpha\chi\beta\lambda} u^\chi u^\lambda \tag{21a}$$

$$Y_{\alpha\beta} = R_{\alpha\chi\beta\lambda} u^\chi u^\lambda \tag{21b}$$

$$Z_{\alpha\beta} = -R^*{}_{\alpha\chi\beta\lambda} u^\chi u^\lambda \tag{21c}$$

$$Z^t{}_{\alpha\beta} = -{}^*R_{\alpha\chi\beta\lambda} u^\chi u^\lambda \tag{21d}$$

with the properties

$$X_{\alpha\beta} = X_{\beta\alpha} \tag{22a}$$

$$Y_{\alpha\beta} = Y_{\beta\alpha} \tag{22b}$$

$$Z^t{}_{\alpha\beta} = Z_{\beta\alpha} \tag{22c}$$

$$Z^\alpha{}_\alpha = 0 \tag{22d}$$

We complete u^α to an orthonormal tetrad

$$h^\alpha_a h^\beta_b g_{\alpha\beta} = \eta_{ab}, \qquad h^\alpha_{(0)} = u^\alpha \tag{23}$$

and introduce the following notation for the components R_{abcd} of the Riemann tensor with regard to this tetrad: each index pair ab is replaced by a single index Ω running from 1 to 6, according to the scheme,

$$\begin{array}{c|cccccc} ab & 10 & 20 & 30 & 23 & 31 & 12 \\ \hline \Omega & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{24}$$

The full Riemann tensor and its right and left dual can then be represented by the matrices

$$(R_{\Omega\Sigma}) = \begin{bmatrix} Y_{\hat{a}\hat{b}} & Z_{\hat{a}\hat{b}} \\ Z^t{}_{\hat{a}\hat{b}} & X_{\hat{a}\hat{b}} \end{bmatrix} \tag{25a}$$

$$(R^*{}_{\Omega\Sigma}) = \begin{bmatrix} Z_{\hat{a}\hat{b}} & Y_{\hat{a}\hat{b}} \\ X_{\hat{a}\hat{b}} & Z^t{}_{\hat{a}\hat{b}} \end{bmatrix} \tag{25b}$$

$$({}^*R_{\Omega\Sigma}) = \begin{bmatrix} Z^t{}_{\hat{a}\hat{b}} & X_{\hat{a}\hat{b}} \\ Y_{\hat{a}\hat{b}} & Z_{\hat{a}\hat{b}} \end{bmatrix} \tag{25c}$$

where the first index denotes the row and the second the column. For example, $R_{1032} = -R_{\Omega=1,\Sigma=4} = -Z_{11}$. Indices $\hat{a}, \hat{b}, \ldots = 1, 2, 3$ refer to the three-dimensional spatial tetrad indices, i.e., to the contractions with the tetrad vector $h^\alpha_{\hat{a}}$, $h^\alpha_{\hat{b}}, \ldots$.

We note that in vacuum, assuming Einstein's field equations, the Riemann tensor reduces to the *Weyl tensor*. In this special case we have additionally

$$X_{\hat{a}\hat{b}} = -Y_{\hat{a}\hat{b}} \tag{26a}$$

$$Y^{\hat{a}}{}_{\hat{a}} = 0 \tag{26b}$$

$$Z_{\hat{b}\hat{a}} = Z_{\hat{a}\hat{b}} \tag{26c}$$

and therefore

$$(C_{\Omega\Sigma}) = \begin{bmatrix} Y_{\hat{a}\hat{b}} & Z_{\hat{a}\hat{b}} \\ Z_{\hat{a}\hat{b}} & -Y_{\hat{a}\hat{b}} \end{bmatrix} \tag{27}$$

Below we will make use of the measurements of different observers in one space-time point. The respective tetrads (frames of reference) are related by *Lorentz transformations*. We restrict to the boosts in the three direction $h^\alpha_{\hat{a}}$ of the original tetrad with $h^\alpha_{(0)} = u^\alpha$. Let i be the number of the respective boosted tetrad ($i = 1, 2, 3$), then the nonvanishing components of the transformations

$$\underset{i}{h}{}^\alpha_a = \underset{i}{L}{}_a{}^b h^\alpha_b \tag{28}$$

are given by

$$\begin{aligned} \underset{i}{L}_0{}^0 &= \underset{i}{L}_i{}^i = (1 - v_i{}^2)^{-1/2} \\ \underset{i}{L}_0{}^i &= \underset{i}{L}_i{}^0 = v_i(1 - v_i{}^2)^{-1/2} \end{aligned} \tag{29}$$

(no summation convention, tetrad indices), where v_i are the respective relative velocities measured by the original observer.

The components of the Riemann tensor in the different tetrads are then related according to

$$\underset{i}{R}_{abcd} = \underset{i}{L}_a{}^k \underset{i}{L}_b{}^l \underset{i}{L}_c{}^m \underset{i}{L}_d{}^n R_{klmn} \tag{30}$$

or

$$\underset{i}{R}_{\Omega\Sigma} = \underset{i}{\Lambda}_\Omega{}^\Pi \underset{i}{\Lambda}_\Sigma{}^\Phi R_{\Pi\Phi} \tag{31}$$

in the six-dimensional notation, respectively. For the boosts, the nonvanishing components of the symmetric 6 X 6 matrices $\underset{i}{\Lambda}_\Omega{}^\Pi$ are explicitly given by (no summation convention)

$$\begin{aligned} \underset{i}{\Lambda}_{\Omega=i}{}^{\Sigma=i} &= \underset{i}{\Lambda}_{\Omega=i+3}{}^{\Sigma=i+3} = 1 \\ \underset{i}{\Lambda}_\Omega{}^\Omega &= \underset{i}{L}_0{}^0 \qquad \text{for} \quad \Omega \neq i, \quad \Omega \neq i+3 \\ \underset{1}{\Lambda}_3{}^5 &= -\underset{1}{\Lambda}_2{}^6 = \underset{1}{L}_0{}^1 \\ \underset{2}{\Lambda}_1{}^6 &= -\underset{2}{\Lambda}_3{}^4 = \underset{2}{L}_0{}^2 \\ -\underset{3}{\Lambda}_1{}^5 &= \underset{3}{\Lambda}_2{}^4 = \underset{3}{L}_0{}^3 \end{aligned} \tag{32}$$

§(4): *Local Experiments Using the Elementary Particle Spin*

4.1. One Observer. The local experiment is based on the curvature-induced acceleration (20), as measured in a coinciding reference system h_a^α of an observer with 4-velocity $u^\alpha = h_{(0)}^\alpha$. The measured components of a quantity are obtained by projection onto one of the spacelike vectors. The three components of the quantum mechanical spin are, for example,

$$S^{\hat{s}} = S^\alpha h_\alpha^{\hat{s}} \tag{33}$$

In the various experiments the orientations of the spin and the particle velocity parallel to the different directions $h^{\alpha}_{\hat{a}}$ are fixed by the preparation, and particular components of the resulting acceleration of the particle are measured. To simplify the description of the experiments we introduce the set of numbers (V, S, A) with $V, S, \ldots = 0, 1, 2, 3$ and the notation

$$v^{\hat{v}} = v_V \delta^{\hat{v}}_V \tag{34a}$$

$$S^{\hat{s}} = S_S \delta^{\hat{s}}_S \tag{34b}$$

$$a^{\hat{a}} = a_A \delta^{\hat{a}}_A \tag{34c}$$

(no summation convention for the capital letters). The combination $(V = 1, S = 2, A = 3)$, for example, then means: Prepare an experiment with a particle flying parallel to $h^{\alpha}_{\hat{a}=1}$ with velocity v_1 (i.e., $v^{\hat{v}} = v_1 \delta^{\hat{v}}_1$) and having a spin parallel to $h^{\alpha}_{\hat{a}=2}$ (i.e., $S^{\hat{s}} = S_2 \delta^{\hat{s}}_2$). Measure under these conditions the component a_3 of the acceleration in the direction $h^{\alpha}_{\hat{a}=3}$. This example is a particular measurement of type 5 of Table I.

Writing (20) with respect to components, we obtain as a relation which is to be evaluated

$$a^{\hat{a}} = -\frac{\lambda_c}{2} [R^{*\hat{a}}{}_{\hat{v}\hat{s}\hat{d}} v^{\hat{v}} S^{\hat{s}} v^{\hat{d}} + (R^{*\hat{a}}{}_{\hat{v}\hat{c}(0)} v^{\hat{v}} + R^{*\hat{a}}{}_{(0)\hat{c}(0)})(S^{\hat{c}} v^{(0)} - v^{\hat{c}} S^{(0)}) + R^{*\hat{a}}{}_{(0)\hat{s}\hat{v}} S^{\hat{s}} v^{\hat{v}} v^{(0)}] \tag{35}$$

By always using the results of the previous measurements, the components of the Riemann tensor can successfully be obtained by performing the experiments in the following order (cf. Table I):

1. Type of measurements: For particles at rest we have $u^{\alpha} = v^{\alpha}$ and accordingly $v^{\hat{v}} = 0$ and $V = 0$. This implies with (35)

$$a^{\hat{a}} = -\frac{\lambda_c}{2} R^{*\hat{a}}{}_{(0)\hat{s}(0)} S^{\hat{s}} \tag{36}$$

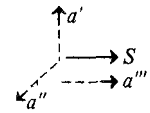
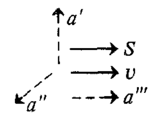
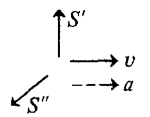
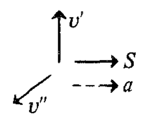
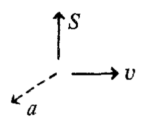
The measured components of the Riemann tensor for all orientations of the spin $(S = 1, 2, 3)$ are therefore the complete matrix $Z_{\hat{a}\hat{b}}$.

In all other cases, i.e., $V \neq 0$, (35) reduces to

$$a^{\hat{a}} = -\frac{\lambda_c}{2} [R^{*\hat{a}}{}_{\hat{v}\hat{c}(0)} v^{\hat{v}} (S^{\hat{c}} v^{(0)} - v^{\hat{c}} S^{(0)}) + R^{*\hat{a}}{}_{(0)\hat{s}\hat{v}} S^{\hat{s}} v^{\hat{v}} v^{(0)}] + \text{known terms} \tag{37}$$

To obtain this, we have used the fact that the components $R^{*\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}$ agree according to (24) and (25b) with the components $Z^{t}{}_{\hat{a}\hat{b}}$, which are already determined in measurement 1. Note that because of $S^{\epsilon} S_{\epsilon} = -1$ the factor $S^{\epsilon} u_{\epsilon} = S^{(0)}$ is

Table I. Local Measurements of the Riemann Tensor Using the Elementary Particle Spin. (Components of the Particle Velocity: v^{v}, of the particle spin: $S^{\hat s}$, of the measured acceleration or force: $a^{\hat a}$)

Measurement type	Prepare successively the combinations	Measure for each combination the components	Symbolically	(Additionally) measured components
	$v^{\hat v} = v_V \delta^{\hat v}_V$ $S^{\hat s} = S_S \delta^{\hat s}_S$	$a^{\hat a} = a_A \delta^{\hat a}_A$	——, prepared – – –, measured	
1	$V = 0$ $S = 1, 2, 3$	$A = 1, 2, 3$		$R^{*\hat a}{}_{(0)\hat s(0)} \leftrightarrow Z_{\hat a \hat s}$
2	$V = 1, 2, 3$ $S = V$	$A = 1, 2, 3$		$R^{*\hat a}{}_{\hat v \hat v(0)} \leftrightarrow X_{\hat m \hat v}$, $\hat a \neq \hat v \neq \hat m \neq \hat a$
3	$V = 1, 2, 3$ $S \neq V$	$A = V$		$R^{*\hat v}{}_{(0)\hat s \hat v} \leftrightarrow Y_{\hat v \hat m}$, $\hat v \neq \hat s \neq \hat m \neq \hat v$
4	$S = 1, 2, 3$ $V \neq S$	$A = S$		Already obtained above
5	$V = 1, 2, 3$ $S \neq V$	$A \neq V \neq S$		Six linear dependent equations for the six components $X_{\hat n \hat n}$, $Y_{\hat n \hat n}$

known when the $S^{\hat a}$ are specified. Similarly, $v^{\epsilon} u_{\epsilon} = v^{(0)}$ is known when the $v^{\hat a}$ are fixed.

2. *Type of measurement:* The spin is directed parallely to the velocity ($S = V$). This implies with (34) and (37)

$$a^{\hat a} = -\frac{\lambda_c}{2} R^{*\hat a}{}_{VV(0)} v_V (S_V v^{(0)} - v_V S^{(0)}) + \text{known terms} \tag{38}$$

The acceleration is measured orthogonal to the direction of the velocity ($A \neq V$). All this can be done for three directions of the velocity ($V = 1, 2, 3$). The measured components of the Riemann tensor are $R^{*}{}_{AVV(0)}$, or according to our scheme (25) the terms X_{MV} with $V \neq A \neq M \neq V$. The case $A = V$ leads to terms which are already known.

3. Type of measurement: Now the spin is orthogonally oriented to the velocity ($S \neq V$), and for all orientations of the velocity ($V = 1, 2, 3$) the component of the acceleration parallel to the velocity ($A = V$) measured. In this case (37) reduces to

$$a^{\hat{a}} = -\frac{\lambda_c}{2} R^{*\hat{a}}{}_{(0)\hat{s}\hat{v}} S^{\hat{s}} v^{\hat{v}} v^{(0)} + \text{known terms} \tag{39}$$

It follows that the components $R^{*}{}_{V(0)SV}$ or correspondingly Y_{VM} with $S \neq V \neq M \neq S$ are measured. Note that because of $V \neq M$ the diagonal terms of Y_{VM} remain undetermined.

4. Type of measurement: Now conversely the velocity orthogonally oriented to the spin ($V \neq S$), and for orientations of the spin ($S = 1, 2, 3$) the component of the acceleration parallel to the spin ($A = S$) is observed. Equation (37) then reads

$$a^{\hat{a}} = -\frac{\lambda_c}{2} \{R^{*\hat{a}}{}_{\hat{v}\hat{s}(0)} v^{\hat{v}} S^{\hat{s}} v^{(0)} - R^{*\hat{a}}{}_{\hat{v}\hat{c}(0)} v^{\hat{v}} v^{\hat{c}} S^{(0)} + R^{*\hat{a}}{}_{(0)\hat{s}\hat{v}} S^{\hat{s}} v^{\hat{v}} v^{(0)}\}$$
$$+ \text{known terms} \tag{40}$$

Because of $A = S$ the first term has already been obtained in measurement 2, and the second term as well. The third term was the result of measurement 3. Therefore these types of measurements, although they have been tabulated to show the completeness of the whole scheme, do not lead to any new information.

5. Type of measurement: The only remaining case is that all three vectors are orthogonal with $V = 1, 2, 3$. Again going back to (37) and making use of the results of measurement 2 we find

$$a^{\hat{a}} = -\frac{\lambda_c}{2} (R^{*\hat{a}}{}_{\hat{v}\hat{s}(0)} + R^{*\hat{a}}{}_{(0)\hat{s}\hat{v}}) v^{\hat{v}} S^{\hat{s}} v^{(0)} + \text{known terms} \tag{41}$$

According to (24) and (25) these are six equations for the still unknown components $X_{11}, X_{22}, X_{33}, Y_{11}, Y_{22}, Y_{33}$ (tetrad indices):

$$\begin{aligned} X_{33} + Y_{11} &= A_1, & X_{22} + Y_{11} &= A_2 \\ X_{11} + Y_{22} &= A_3, & X_{33} + Y_{22} &= A_4 \\ X_{22} + Y_{33} &= A_5, & X_{11} + Y_{33} &= A_6 \end{aligned} \tag{42}$$

where the right-hand sides are fixed in the experiment. Because the determinant of this system of equations vanishes, the diagonal elements of the matrices X and Y remain undetermined. These are the following components of $R^*{}_{\Omega\Sigma}$ and $R_{\Omega\Sigma}$:

$$(R^*{}_{\Omega\Sigma}) = \left(\begin{array}{ccc|ccc} & & & ? & & \\ & & & & ? & \\ & & & & & ? \\ \hline ? & & & & & \\ & ? & & & & \\ & & ? & & & \end{array}\right), \quad (R_{\Omega\Sigma}) = \left(\begin{array}{ccc|ccc} ? & & & & & \\ & ? & & & & \\ & & ? & & & \\ \hline & & & ? & & \\ & & & & ? & \\ & & & & & ? \end{array}\right) \tag{43}$$

If we restrict ourselves to the vacuum space-times of the Einstein theory where the Riemann tensor agrees with the Weyl tensor, we have in addition to (42) the relations (26a, b). This system of equations can now be solved.[5]

For the vacuum space-times of Einstein theory the local method of the elementary particle spin enables a complete determination of the Riemann tensor.

4.2. Several Observers. The measurements of one single observer have turned out to be incomplete. It is therefore natural to ask if it is possible to close the gap in combining the information of several observers with different velocities.

We introduce three additional observers. As seen by the first observer, they move along the three directions $h^{\alpha}_{\hat{a}}$. All three observers measure within their comoving frame of reference $\underset{i}{h}{}^{\alpha}_{\hat{a}}$ of (28) all components of $\underset{i}{R}_{\Omega\Sigma}$ apart from the diagonal ones. In addition all three observers determine the right-hand side of six equations corresponding to (42):

$$\underset{i}{X}_{33} + \underset{i}{Y}_{11} = \underset{i}{A}_{1}, \qquad \underset{i}{X}_{22} + \underset{i}{Y}_{11} = \underset{i}{A}_{2} \quad \text{and so on} \tag{44}$$

Together with (42) these are 4×6 equations. Taking all this information as given, we have to check if Lorentz transformations to the original frame according to (28)-(32) fix the diagonal components of $(R_{\Omega\Sigma})$.

[5]The system (42) represents five independent equations for the six unknown diagonal components $X_{11}, \ldots Y_{11}, \ldots$. Therefore, assuming Einstein's field equations, the additional knowledge of the energy density $T_{\alpha\beta}u^{\alpha}u^{\beta}$ or one of the diagonal components $T_{\hat{a}\hat{a}}$ of the energy momentum tensor is already sufficient to make the system (42) solvable. Vacuum is a special case of this.

With regard to the first moving observer we obtain with (31), (32) and taking into account (25a)

$$\begin{aligned}
Y_{11} &= \underset{1}{Y}_{11} \\
Y_{22} &= (\underset{1}{L}_0{}^0)^2 \underset{1}{Y}_{22} + (\underset{1}{L}_0{}^1)^2 \underset{1}{X}_{33} + \text{known terms} \\
Y_{33} &= (\underset{1}{L}_0{}^0)^2 \underset{1}{Y}_{33} + (\underset{1}{L}_0{}^1)^2 \underset{1}{X}_{22} + \text{known terms} \\
X_{11} &= \underset{1}{X}_{11} \\
X_{22} &= (\underset{1}{L}_0{}^1)^2 \underset{1}{Y}_{33} + (\underset{1}{L}_0{}^0)^2 \underset{1}{X}_{22} + \text{known terms} \\
X_{33} &= (\underset{1}{L}_0{}^1)^2 \underset{1}{Y}_{22} + (\underset{1}{L}_0{}^0)^2 \underset{1}{X}_{33} + \text{known terms}
\end{aligned} \tag{45}$$

With regard to the second and third moving observer we find similar sets of six equations.

Taking all equations together, we have 42 (= 4 × 6 + 3 × 6) equations for the 24 (= 4 × 6) unknown terms $\mathrm{diag}(R_{\Omega\Sigma})$ and $\mathrm{diag}(\underset{i}{R}_{\Omega\Sigma})$, which must be solved with regard to $\mathrm{diag}(R_{\Omega\Sigma})$. An analysis (we omit the details) shows that such a solution of this system of equations is impossible.

Accordingly, apart from vacuum space-times[6] *of Einstein theory, the Riemann tensor cannot be determined completely by the local method of the elementary particle spin.*

§(5): *Nonlocal Experiments Based on the Geodesic Deviation*

In an appropriate device, for example, particles connected by springs, the relative acceleration b^β of (1) manifests itself as a force. We assume that the components of this force with regard to a reference system h_a^α of an observer with 4-velocity $u^\alpha = h_{(0)}^\alpha$ can be measured:

$$b^{\hat{b}} = b^\beta h_\beta^{\hat{b}} = b_B \delta_B^{\hat{b}} \tag{46}$$

The initial situations in the various measurements are characterized by the nonvanishing components of velocity and orthogonal connecting vector

$$v^{\hat{v}} = v_V \delta_V^{\hat{v}} \tag{47}$$

$$r^{\hat{r}} = r_R \delta_R^{\hat{r}} \tag{48}$$

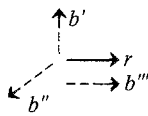
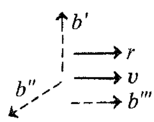
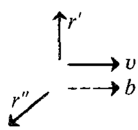
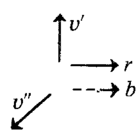
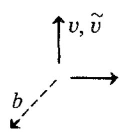
[6] Compare footnote 4.

Decomposition of (1) leads to

$$b^{\hat{b}} = -R^{\hat{b}}{}_{\hat{v}\hat{r}\hat{c}} v^{\hat{v}} r^{\hat{r}} v^{\hat{c}} - R^{\hat{b}}{}_{(0)\hat{r}\hat{v}} r^{\hat{r}} v^{\hat{v}} v^{(0)} - (R^{\hat{b}}{}_{\hat{v}\hat{c}(0)} v^{\hat{v}} + R^{\hat{b}}{}_{(0)\hat{c}(0)} v^{(0)})(r^{\hat{c}} v^{(0)} - v^{\hat{c}} r^{(0)}) \quad (49)$$

Performing successively the measurements of type 1 to type 5 as described in Table II (and always making use of the informations obtained in the previous measurements) one is able to determine directly all components of $R_{\Omega\Sigma}$ apart

Table II. Nonlocal Measurements of the Riemann Tensor Based on the Geodesic Deviation. (Components of the Particle Velocity: $v^{\hat{v}}$, of the Orthogoanl Connecting Vector: $r^{\hat{r}}$, of the Measured Relative Acceleration or Force: $b^{\hat{b}}$)

Measurement type	Prepare successively the combinations	Measure for each combination the components	Symbolically	(Additionally) measured components
	$v^{\hat{v}} = v_V \delta^{\hat{v}}_V$ $r^{\hat{r}} = r_R \delta^{\hat{r}}_R$	$b^{\hat{b}} = b_B \delta^{\hat{b}}_B$	——, prepared – – –, measured	
1	$V = 0$ $R = 1, 2, 3$	$B = 1, 2, 3$		$R^{\hat{b}}{}_{(0)\hat{r}(0)} \leftrightarrow Y_{\hat{b}\hat{r}}$
2	$V = 1, 2, 3$ $R = V$	$B = 1, 2, 3$		$R^{\hat{b}}{}_{\hat{v}\hat{v}(0)} \leftrightarrow Z^{t}_{\hat{m}\hat{v}}$, $\hat{b} \neq \hat{v} \neq \hat{m} \neq \hat{b}$
3	$V = 1, 2, 3$ $R \neq V$	$B = V$		Already obtained above
4	$R = 1, 2, 3$ $V \neq R$	$B = R$		$R^{\hat{r}}{}_{\hat{v}\hat{r}\hat{v}} \leftrightarrow X_{\hat{m}\hat{m}}$, $\hat{m} \neq \hat{r}, \hat{m} \neq \hat{v}$
5	$V = 1, 2, 3$ $R \neq V$ for V_r and $\tilde{V}_r$, $V_r \neq \tilde{V}_r$	$B \neq V$ $B \neq R$		Seven equations for the six components $Z_{\hat{m}\hat{m}}, X_{\hat{m}\hat{n}}$, $\hat{m} \neq \hat{n}$

from $Z_{\hat{a}\hat{a}}$ and $X_{\hat{b}\hat{c}}$ with $\hat{b} \neq \hat{c}$. The measurements of type 5 allow only to fix the right-hand sides B_1, B_2, B_3 of the equations (tetrad indices)

$$\begin{aligned} X_{32}v_1 + (Z_{33} - Z_{22})(1 - v_1{}^2)^{-1/2} &= B_1 \\ X_{31}v_2 + (Z_{11} - Z_{33})(1 - v_2{}^2)^{-1/2} &= B_2 \\ X_{12}v_3 + (Z_{22} - Z_{11})(1 - v_3{}^2)^{-1/2} &= B_3 \end{aligned} \tag{50}$$

Together with the cyclic identity (22d) we have four equations for the six unknown components. But, as compared with system (42), adjustable velocities v_v with $V = 1, 2, 3$ still appear. Repetition of the experiments with different values $\tilde{v}_v$ of the velocities and measurement of the quantities $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ adds to (50) three corresponding equations. The total system of equations now turns out to be solvable. This completes the proof that *measurements based on the geodesic deviation enable the determination of all components of the Riemann tensor*.

§(6): *Nonlocal Experiments Using Two Gyroscopes*

Equation (12) describes the change in time of the difference between the angular momenta of two neighboring gyroscopes as registered in the rest space of one of the gyroscopes. We assume that this quantity $(\Delta S^\alpha)^{\cdot}$, which is primarily defined in a kinematical way according to (8) and (7), can be measured *dynamically*. By this we mean that it is possible to construct in the rest space of the gyroscope a device which represents $(\Delta S^\alpha)^{\cdot}$ as a force, expanding for example a spring. We assume additionally that the experimental setup makes it possible that the corresponding quantity, e.g., the resulting length of the spring, can be measured by an observer with 4-velocity u^α which is not necessarily comoving. The observed quantity will then be

$$f^\alpha = P_u{}^\alpha{}_\beta (\Delta S^\beta)^{\cdot}, \qquad f^\alpha u_\alpha = 0 \tag{51}$$

We want to show that by these measurements using two gyroscopes, the Riemann tensor cannot be determined completely. To do so we make first of all use of the fact that according to (15) measurements of $f^{\hat{a}}$ are essentially measurements of $H^{\hat{c}}$. Decomposing again with regard to the tetrad $h^\alpha_{\hat{a}}$ with $h^\alpha_{(0)} = u^\alpha$, we obtain from (51) and (15)

$$f^{\hat{f}} = \eta^{\hat{f}(0)\hat{h}\hat{s}} H_{\hat{h}} S_{\hat{s}} v^{(0)} + \eta^{\hat{f}\hat{v}(0)\hat{s}} v_{\hat{v}} S_{\hat{s}} H^{(0)} + \eta^{\hat{f}\hat{v}\hat{h}(0)} v_{\hat{v}} H_{\hat{h}} S^{(0)} \tag{52}$$

Because the Riemann tensor will afterwards be worked out using (14), we have to find all components $H_{\hat{h}}$ for all given combinations $v_{\hat{v}}$ and $r_{\hat{r}}$. Vectors will always be prepared parallel to one of the tetrad vectors. We introduce again the notation of (34) and

$$H^{\hat{h}} = H_H \delta_H^{\hat{h}} \tag{53a}$$

$$f^{\hat{f}} = f_F \delta_F^{\hat{f}} \tag{53b}$$

($H, F = 1, 2, 3$; no summation convention.)

For a comoving observer, $V = 0$, we get $H_{\hat{h}}$ directly from $f^{\hat{f}}$ according to (52). For the other cases, we begin with a measurement of the component of $f^{\hat{f}}$ parallel to $v^{\hat{v}}$ (i.e., $V = F$)

$$f^{\hat{v}} = \eta^{\hat{v}(0)\hat{h}\hat{s}} H_{\hat{h}} S_{\hat{s}} v^{(0)} \tag{54}$$

For given V we choose $S \neq V$ and obtain in this way the components $H_{\hat{h}}$ with $H \neq V$.

To determine $H_{\hat{h}}$ with $H = V$ we choose $S \neq V$ and measure the components of $f^{\hat{f}}$ with $F \neq S, F \neq V$. Because $S^{\alpha} V_{\alpha} = 0$ and $\hat{s} \neq \hat{v}$ we have $S^{(0)} = 0$ so that (52) reduces to

$$f^{\hat{f}} = \eta^{\hat{f}(0)\hat{v}\hat{s}} H_{\hat{v}} S_{\hat{s}} v^{(0)} + \eta^{\hat{f}\hat{v}(0)\hat{s}} v_{\hat{v}} S_{\hat{s}} H^{(0)} \tag{55}$$

Accordingly, this type of experiment allows us to fix

$$-v^{(0)} H_{\hat{v}} + v_{\hat{v}} H^{(0)} = X_{\hat{v}} \tag{56}$$

where $X_{\hat{v}}$ is a known quantity. Because of $H^{\alpha} v_{\alpha} = 0$, this can be completed by

$$H^{(0)} v^{(0)} - H_{\hat{v}} v_{\hat{v}} = 0 \tag{57}$$

(no summation convention). The equations (56) and (57) can then be solved with regard to $H_{\hat{v}}$. This completes the measurement of all components $H_{\hat{h}}$ for any given combination of components $v^{\hat{v}}$ and $r^{\hat{r}}$.

The final step is now based on Eq. (14). A comparison with (20) shows that all the remaining measurements and calculations are completely analogous to those treated in Section 4. The only difference is that now the components of ${}^*R_{\Omega\Sigma}$ instead of $R^*{}_{\Omega\Sigma}$ are determined. The respective values of Ω and Σ in the measurements of type 1-5 (cf. Table I) are thereby unchanged. This implies that we can easily transcribe the results of Section 4: For example, $R^*{}_{14} = Y_{11}$ remaining undetermined there means that ${}^*R_{14} = X_{11}$ remains undetermined here, and so on. Because of (43) and (25) these are the following elements of ${}^*R_{\Omega\Sigma}$ and $R_{\Omega\Sigma}$:

$$
{}^*R_{\Omega\Sigma} = \left(\begin{array}{ccc|ccc} & & & ? & & \\ & & & & ? & \\ & & & & & ? \\ \hline ? & & & & & \\ & ? & & & & \\ & & ? & & & \end{array}\right), \quad R_{\Omega\Sigma} = \left(\begin{array}{ccc|ccc} ? & & & & & \\ & ? & & & & \\ & & ? & & & \\ \hline & & & ? & & \\ & & & & ? & \\ & & & & & ? \end{array}\right) \tag{58}
$$

Accordingly, at least the diagonal elements of the matrices X and Y cannot be observed. *The method of the two gyroscopes is as incomplete as the one using the elementary particle spin.*

§(7): *Conclusion*

The inclusion of quantum mechanically described test particles into general relativity enables via the interaction with the elementary particles spin an extremely local measurement (using one particle only) of certain components of the Riemann tensor. The locality of this measurement is only limited by quantum mechanical restrictions: Particle production prevents extensions smaller than Compton wavelength. Related to this is that the WKB approximation used above would not be applicable furthermore.

With quantum mechanical measurements one can completely get the Riemann tensor of the Einstein vacuum which agrees with the Weyl tensor. But in the nonvacuum case the method remains incomplete. The same is the case for the nonlocal macroscopic method measuring the influence of the Riemann tensor on one or two gyroscopes. Only the nonlocal macroscopic measurement based on the geodesic deviation makes it possible to determine all components of the Riemann tensor.

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