How a black hole forms – gravitational collapse

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Where do black holes come from?

- At the end of a star's lifetime, its internal pressure from nucleosynthesis is insufficient to prevent the star from collapsing under its own gravity
- The collapse may be stopped by the degeneracy pressure of electrons ($\rightarrow$ white dwarf) or neutrons ($\rightarrow$ neutron star)
- If the mass of the remnant star exceeds the Tolman-Oppenheimer-Volkoff limit ($\sim 3M_\odot$), no known mechanism is able to halt the collapse and a black hole is expected to form

- Primordial black holes may have formed shortly after the big bang from gravitational collapse of density fluctuations

Gravitational collapse

- This talk is concerned with fundamental aspects of gravitational collapse in classical general relativity
- A proper mathematical understanding requires the notion of an initial value problem
- Einstein's equations are known to form singularities under quite general circumstances (Hawking-Penrose singularity theorems)
- In the standard picture of gravitational collapse, a black hole with event horizon forms that makes the singularity invisible to distant observers
- The weak cosmic censorship conjecture asserts that this happens for generic initial data, i.e. no naked singularities exist
- A related questions is how small a black hole one can make by tuning the initial data, leading to Choptuik's discovery of critical phenomena via numerical simulations
- While much is understood in spherical symmetry for simple matter models, little is known in axisymmetry or beyond

Outline of the talk

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Causal structure of spacetime

- Spacetime is a four-dimensional, smooth, connected Lorentzian manifold (M, g)
- Consider a spacelike hypersurface $\Sigma \subset M$
- Future domain of dependence $D^+(\Sigma) := \{p \in M :$ Every past inextendible causal (timelike or null) curve through p intersects $\Sigma\}$
- Past domain of dependence $D^-(\Sigma) := \{p \in M :$ Every future inextendible causal curve through p intersects $\Sigma\}$
- If $D(\Sigma) := D^+(\Sigma) \cup D^-(\Sigma) = M$ then M is globally hyperbolic and Σ is a Cauchy surface

Initial value problem

- Given a spacetime (M, g_{ab}) solving Einstein's equations

  $$G_{ab}[g] = \kappa T_{ab},$$

  the induced data on a given spacelike hypersurface Σ (the intrinsic metric h_{ab} and extrinsic curvature K_{ab} of Σ in M) must satisfy the constraint equations

  $$R[h] + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$
  $$D_b(K^{ab} - h^{ab}K) = \kappa J^a$$

- Conversely, given (Σ, h_{ab}, K_{ab}) satifying the constraints, is there a unique spacetime (M, g_{ab}) that induces these data?

Initial value problem

Theorem (Fourès-Bruhat 1952, Choquet-Bruhat & Geroch 1969)

Let Σ be a smooth 3D manifold with smooth Riemannian metric h_{ab}, and let K_{ab} be a smooth symmetric tensor field on Σ, such that (h_{ab}, K_{ab}) satisfy the vacuum constraint equations.
Then there exists a unique spacetime (M, g_{ab})—the maximal Cauchy development of (Σ, h_{ab}, K_{ab})—such that

- (M, g_{ab}) is a solution to the vacuum Einstein equations
- (M, g_{ab}) is globally hyperbolic with Cauchy surface Σ
- h_{ab} is the induced metric and K_{ab} the extrinsic curvature of Σ
- Every other spacetime satisfying the above can be mapped isometrically into a subset of (M, g_{ab})
- Diffeomorphic initial data give rise to isometric maximal Cauchy developments
- The solution g_{ab} depends continuously on the initial data (h_{ab}, K_{ab})

Singularities

- Singularities are by definition *not* part of the manifold M that is *determined* by Einstein's equations
- Unlike any other field theory!
- Coordinate descriptions are problematic because a mere breakdown of the coordinate chart might be mistaken for a singularity
- Often some curvature invariant (e.g. R, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$) blows up as a singularity is approached but not all singularities are of this type
- E.g. remove a wedge from Minkowski spacetime $\Rightarrow$ conical singularity
- Most robust definition of a singularity: existence of incomplete geodesics, i.e. inextendible in at least one direction but only finite range of affine parameter

Trapped surfaces

- Consider a 2D compact, smooth, spacelike submanifold $S \subset M$, e.g. a closed 2-surface in a spacelike 3-slice Σ of M
- Consider the outgoing (tangent k^a) and ingoing (tangent l^a) null geodesics emanating from S
- Define the expansions of the null geodesic congruences

$$\Theta_+ := m^{ab}\nabla_a k_b, \qquad \Theta_- := m^{ab}\nabla_a l_b,$$

where $m_{ab} = g_{ab} + k_a l_b + l_a k_b$ is the induced 2-metric on S
- S is a trapped surface if $\Theta_\pm < 0$ everywhere on S
- Marginally trapped if $\Theta_\pm \leqslant 0$
- The outermost marginally trapped surface is the apparent horizon

Singularity theorems

Theorem (Penrose 1965)

Let (M, g_{ab}) be a connected, *globally hyperbolic* spacetime with a non-compact Cauchy surface Σ. Suppose that the null energy condition holds:

$$T_{ab}k^a k^b \geqslant 0$$

for all null vector fields k^a. Suppose further that M contains a trapped surface S, and let $\Theta_0 < 0$ be the maximum of $\Theta_\pm$ on S.
Then at least one inextendible future directed orthogonal null geodesic from S has affine length no greater than $2/|\Theta_0|$.

Remarks:

- The null energy condition follows from the weak energy condition: $\rho = T_{ab}u^b u^b \geqslant 0$ for all timelike u^a, where ρ is the energy density measured by an observer with 4-velocity u^a
- Similar theorems in a cosmological context (Hawking & Penrose 1965–70)

Asymptotic flatness

- We need to clarify what we mean by saying “a singularity is (in)visible to distant observers”
- Restrict to asymptotically flat spacetimes

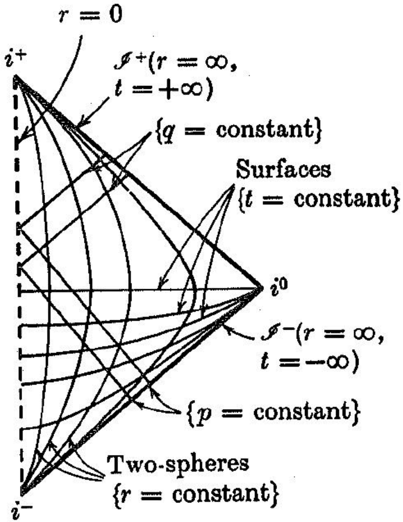
Definition (Penrose 1963–5)

A vacuum spacetime (M, g_{ab}) is asymptotically simple if there is a conformally isometric spacetime $(\tilde{M}, \tilde{g}_{ab})$ with $\tilde{g}_{ab} = \Omega^2 g_{ab}$ such that the conformal factor Ω is smooth up to $\partial\tilde{M}$, with $\Omega = 0$ and $\tilde{\nabla}_a\Omega \neq 0$ on $\partial\tilde{M}$, and such that every null geodesic in M has future and past endpoints on $\partial\tilde{M}$.

- This is too strong really as it excludes black holes
- (M, g_{ab}) is weakly asymptotically simple if it has an open set $U \subset M$ isometric to the neighbourhood of $\partial\tilde{M}$ of some asymptotically simple spacetime

Conformal infinity

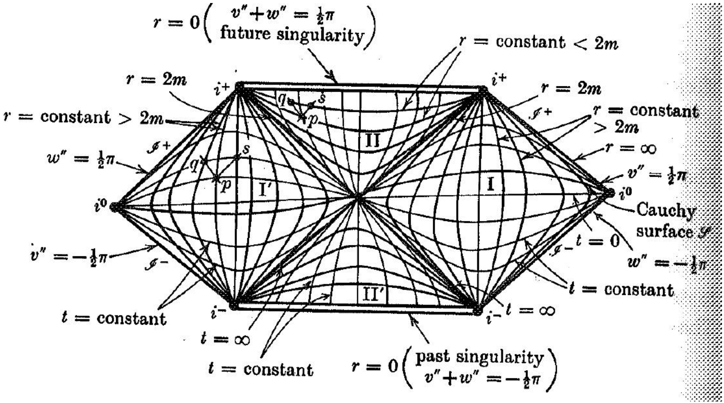
- Identify different parts of the conformal boundary:
  - Future (past) directed timelike geodesics approach future (past) timelike infinity i^+ (i^-)
  - Future (past) directed null geodesics approach future (past) null infinity $\mathscr{I}^+$ ($\mathscr{I}^-$)
  - Spacelike geodesics approach spatial infinity i^0
- Example: Penrose diagram of flat (Minkowski) spacetime



From Hawking & Ellis (1973)

Event horizons

- An event horizon is the boundary of the causal past of future null infinity: $H = \partial J^-(\mathscr{I}^+)$
- The black hole region is $B = M \setminus J^-(\mathscr{I}^+)$
- Example: Penrose diagram of Schwarzschild spacetime



From Hawking & Ellis (1973)

Trapped surfaces lie inside event horizons

- Assume spacetime M is strongly asymptotically predictable from a Cauchy surface Σ, i.e. $\overline{J^-(\mathscr{I}^+)}$ is contained in $D^+(\Sigma)$.

Theorem (cf. Hawking and Ellis 1973)

Let S be a marginally trapped surface in a strongly asymptotically predictable spacetime satisfying the null energy condition.
Then S lies inside the black hole region, $S \subset B$.

- In numerical simulations, apparent horizons are much easier to detect than event horizons because they only depend on the geometry on the slice Σ at a given instant of time
- The absence of an apparent horizon does not imply the absence of an event horizon: time slices Σ of Schwarzschild spacetime exist that come arbitrarily close to the singularity but do not contain any trapped surfaces (Iyer & Wald 1991)

Weak cosmic censorship

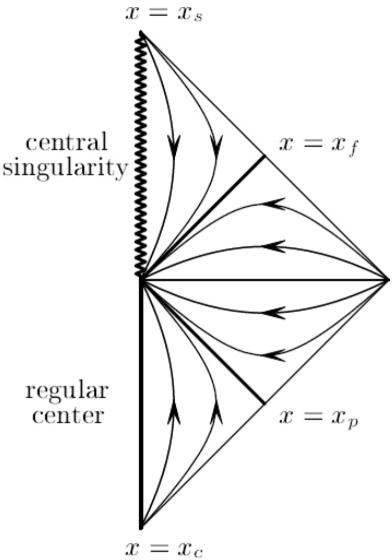
Conjecture (weak cosmic censorship, Penrose 1969)

Let Σ be a 3-manifold which is topologically $\mathbb{R}^3$ outside a compact submanifold. Specify non-singular, *asymptotically flat* (h_{ab}, K_{ab}) as well as initial data for *suitable matter* on Σ.
Then, *generically*, the maximal Cauchy development of this data is a spacetime (M, g_{ab}) which is asymptotically flat with geodesically complete $\mathscr{I}^+$.

Remarks:

- Essentially says there are no naked singularities visible from $\mathscr{I}^+$
- The emphasised properties are somewhat vaguely defined
- Easy to construct counterexamples but these usually have "unphysical" matter

Example of a naked singularity



from Gundlach & Martín-García, *Living Rev. Relativity* **10**, 5 (2007)

The bet

John Preskill, Kip Thorne, Stephen Hawking (1991)

The bet

Whereas Stephen W. Hawking firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,
And whereas John Preskill and Kip Thorne regard naked singularities as quantum gravitational objects that might exist unclothed by horizons, for all the Universe to see,
Therefore Hawking offers and Preskill/Thorne accept, a wager with odds of 100 pounds sterling to 50 pounds sterling, that

When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, the result can never be a naked singularity.

The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable concessionary message.

Stephen W. Hawking, John P. Preskill, Kip S. Thorne
Pasadena, California, 24 September 1991

Outline of the talk

Gravitational collapse in spherical symmetry

- Assumption of spherical symmetry simplifies analysis ($1+1$ effective dimensions)
- Birkhoff's theorem $\Rightarrow$ gravitational field has no dynamical degrees of freedom in spherical symmetry, need matter
- Dust (perfect fluid with zero pressure, $T^{ab} = \rho u^a u^b$) has been widely studied (Eardley & Smarr 1979, ...)
- Naked singularities can occur but these typically arise from shell crossing singularities already present in flat spacetime
- Better matter model: massless scalar field

$$\nabla^a \nabla_a \phi = 0$$

$$G_{ab} = \kappa \left(\nabla_a \phi \nabla_b \phi - \tfrac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right)$$

- Rare example for which we have an essentially complete understanding of weak cosmic censorship, restricted to spherical symmetry (Christodoulou 1987–97)

Outline of the talk

Characteristic initial value problem

- Specify initial data on a future null cone C_0^+ with vertex at $r = 0$
- Initial data characterised by function

$$\alpha := \frac{d}{dr}(r\phi),$$

  where r is areal radius
- If α has bounded variation on C_0^+ then a unique solution to the Einstein-Klein-Gordon equations with bounded variation exists (Christodoulou 1993)

Naked singularities

- Consider solutions with the additional symmetry

$$g_{ab} \to \lambda^2 g_{ab}, \quad r \to \lambda r, \quad \phi \to \phi - k \ln \lambda \tag{1}$$

- These can be truncated to yield asymptotically flat data
- Choices of initial data exist which evolve to naked singularities (Christodoulou 1994)
- Singularity propagates out along null cone, reaching $\mathscr{I}^+$ at finite retarded time ($\mathscr{I}^+$ incomplete)
- Curvature remains bounded on the singular null cone
- The corresponding initial data are not smooth

Non-genericity of naked singularities

Three classes of evolutions:

1. No singularities at all occur, $\mathscr{I}^+$ is complete (small initial data)
2. A singularity forms within the black hole region (bounded by a non-singular event horizon), $\mathscr{I}^+$ is complete
3. Neither of the above two cases holds

Theorem (Christodoulou 1997)

Consider any initial data characterised by α_0 that evolve to a spacetime in class 3 above.
Then there exists a smooth function f such that for any $c \in \mathbb{R}$, the initial data characterised by $\alpha = \alpha_0 + c\,f$ evolve to a spacetime in class 2.

- Naked singularities are non-generic in this precise sense

Outline of the talk

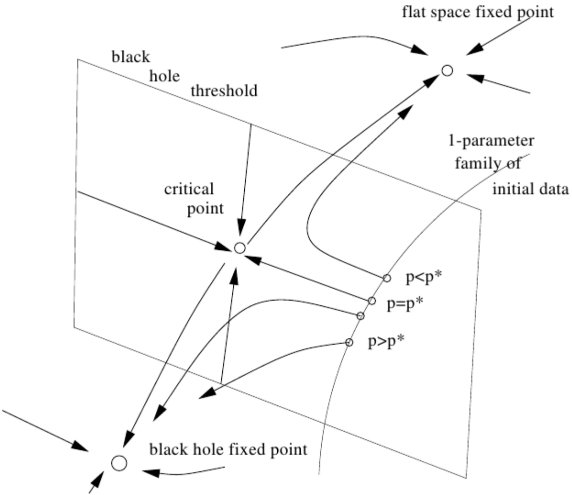
Critical collapse

- Consider one-parameter (p) family of *smooth* Cauchy initial data and evolve them numerically (Choptuik 1993)
- Dispersal for $p < p_*$, black hole formation fo $p > p_*$
- The black hole mass scales as

$$M \sim (p - p_*)^\gamma$$

  with a *universal* critical exponent γ
- In particular, one can make arbitrarily small black holes
- The critical solution as $p \to p_*$ is discretely self-similar (cf. (1) for a fixed λ)
- It has a naked singularity (Hamadé & Stewart 1996)

Critical collapse: dynamical systems picture



from Gundlach & Martín-García, *Living Rev. Relativity* **10**, 5 (2007)

Critical collapse: two types

- Similar phenomena found in a variety of other matter models
- Two different types of critical behaviour:

Type II infinitesimal black hole mass at threshold
self-similar critical solution
mass scaling $M \sim (p - p_*)^\gamma$

Type I finite black hole mass at threshold
static (or time-periodic) critical solution
time spent near critical solution $T \sim -\lambda \ln |p - p_*|$

- Example of Type I: Einstein-Yang-Mills system, critical solution is Bartnik-McKinnon soliton (Movie)

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Stephen W. Hawking, John P. Preskill, Kip S. Thorne
Pasadena, California, 24 September 1991

Conceded on a technicality by Stephen W. Hawking, 5 February 1997
Message printed on T-shirts: Nature Abhors a Naked Singularity

The revised bet

Whereas Stephen W. Hawking (having lost a previous bet on this subject by not demanding genericity) still firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,
And whereas John Preskill and Kip Thorne (having won the previous bet) still regard naked singularities as quantum gravitational objects that might exist, unclothed by horizons, for all the Universe to see,
Therefore Hawking offers, and Preskill/Thorne accept, a wager that

When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, then
A dynamical evolution from **generic** initial conditions (i.e., from an open set of initial data) can never produce a naked singularity (a past-incomplete null geodesic from scri-plus).

The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable, **truly** concessionary message.

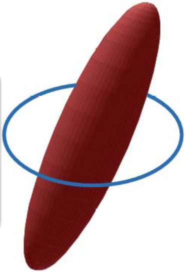
Stephen W. Hawking, John P. Preskill, Kip S. Thorne
Pasadena, California, 5 February 1997

Outline of the talk

The hoop conjecture

Conjecture (Thorne 1972)

Black holes with event horizons form when and only when a mass m gets compacted into a region whose circumference in *every* direction is $C \leqslant 4\pi m$.



- If gravitational collapse occurs in less than 3 spatial dimensions (to a 2D "pancake" or 1D "spindle"), the idea is that a naked singularity will form
- "Mass" or gravitational energy is ill defined (how about collapse of vacuum gravitational waves?)
- Formulation depends on spacetime slicing (hoop can always be distorted in null directions to decrease the circumference)

Outline of the talk

Brill waves

- Vacuum Einstein equations
- Axisymmetry: spacelike Killing vector field $\xi = \partial/\partial\phi$, assumed here to be hypersurface orthogonal
- Metric in cylindrical polar coordinates t, r, z, ϕ, quasi-isotropic gauge:

$$ds^2 = -\alpha^2 dt^2 + \psi^4 \{ e^{2rS} [(dr + \beta^r dt)^2 + (dz + \beta^z dt)^2] + r^2 d\phi^2 \}$$

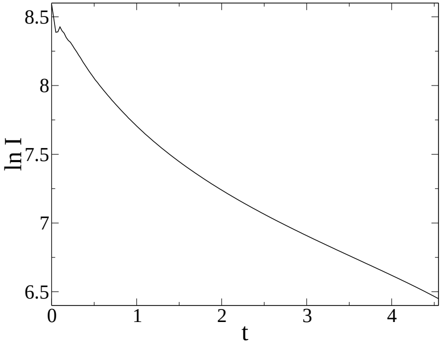
- Evolve on maximal slices ($\mathrm{tr}K = 0$)

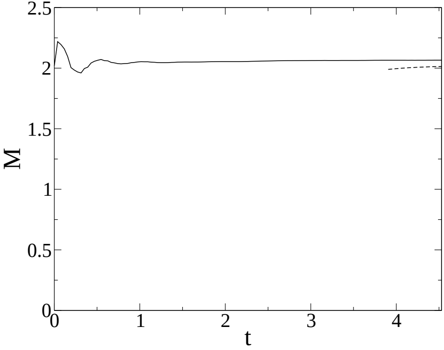
Naked spindle singularities?

- Abrahams *et al.* (1992) constructed families of prolate initial data with arbitrarily large Kretschmann scalar $I := R^{abcd} R_{abcd}$ but without apparent horizons
- Conjectured that these would collapse to naked singularities
- Garfinkle & Duncan (2001) evolved these data and found that I decreased and that an apparent horizon might form
- An apparent horizon was eventually found using an improved formulation and code (O.R. 2006)

Naked spindle singularities?

Left: Evolution of Kretschmann scalar
Right: ADM mass (solid) and apparent horizon mass (dashed)





Critical collapse

- Abrahams & Evans (1993) found evidence of Type II critical behaviour
- Despite several attempts (Alcubierre *et al.*, Choptuik *et al.*, O.R., Sorkin, Hilditch *et al.*, . . .), this important result has not been reproduced yet

Outline of the talk

Collisionless matter: the Einstein-Vlasov system

- Vlasov distribution function $f(x^a, p^a)$
- f is conserved along spacetime geodesics: Vlasov equation

$$p^a \frac{\partial f}{\partial x^a} - \Gamma^a{}_{bc} p^b p^c \frac{\partial f}{\partial p^a} = 0$$

- Energy-momentum tensor

$$T_{ab} = \int f \, \frac{p_a p_b}{m} \frac{1}{\sqrt{-g}} \mathrm{d}p_0 \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3$$

- Evolve using particle-in-cell method: ensemble of particles travelling along geodesics, each carrying its own conserved phase-space volume element

Naked spindle singularities? Collisionless gas

- Numerical simulations of highly prolate axisymmetric collisionless gas spheroids (Shapiro & Teukolsky 1991)
- Singularity formed just outside the ends of the spheroid but no trapped surfaces were found
- They conjectured that the singularity is naked
- Inconclusive because an event horizon might still exist
- Unclear if the matter model used is physically reasonable (no velocity dispersion / dust?)

- Simulations recently repeated by Yoo, Harada & Okawa (2016)
- Similar results, but singularities just within support of matter
- Apparent horizons formed for different initial data

Einstein-Vlasov revisited

- Ellery Ames, Håkan Andréasson & O.R. are developing a new axisymmetric code based on a symmetry reduction of the Einstein equations ((2+1)+1 formalism)
- Applications:
  - Revisit cosmic censorship using more reasonable initial data
  - Critical collapse
  - Stability of new stationary solutions
- Preliminary results: gravitational collapse simulations with apparent horizon formation
  - Movie: $\alpha, \psi, \tilde{\rho}_H, \tilde{J}^\phi$
  - Movie: apparent horizon

Outline of the talk

Conclusions

- In the standard picture of gravitational collapse, the formation of a spacetime singularity is accompanied by the formation of an event horizon
- The genericity of this picture is still an open problem (weak cosmic censorship conjecture)
- Weak cosmic censorship proven for the spherically symmetric Einstein-Klein-Gordon system (Christodoulou)
- Discovery of critical phenomena (Choptuik) showed that smooth but non-generic initial data can evolve to a naked singularity
- Cosmic censorship in axisymmetry (naked spindle singularities) still controversial, e.g.
  - prolate Brill waves
  - collisionless matter (Einstein-Vlasov)
- Numerical simulations can probe these systems