A tale of light and darkness: Black holes in Maxwell-Einstein Theory

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Plan of the talk

1. Main result
2. Brief introduction to Lorentzian geometry
3. First toolbox: Conformal extensions
4. Second toolbox: Singularity theorems, and proof sketch
5. By-product: Decision on an old bet

1.1 A long-standing conjecture

Thorne's *Hoop Conjecture* (1972): If sufficient mass/energy is concentrated in a region U of a Cauchy surface S then a black hole forms.

O'Murchadha et al. 2010: Hoop conjecture holds in spherically symmetric and time-symmetric case for $k = 2\pi$ and Brown-York mass.
Extended by Malec-Xie (2015) to non-spherically symmetric but still time-symmetric case, with stronger geometric conditions.
But: **General case open for almost 50 years!**

1.2 Our choice of matter model

Usually, Hoop Conjecture independent of specific matter model. Here: focus on **Einstein-Maxwell**, mass=rest mass of Maxwell field.

Einstein theory (gravity) and Maxwell theory (electromagnetism): the only long-range fundamental interactions well-tested as classical field theories.

Lagrangian density: $L(g, A) := \mathrm{scal}^g + F(A) \wedge F(A)$ for $F(A) = dA$.
Euler-Lagrange equations: $d^* dA = 0, \mathrm{Ein}^g = T(g, A)$
where $T(g, A)(e_i, e_j) = \frac{1}{4\pi}\big(\sum_a (F_{ai} F_j^a) - \frac{1}{4}\sum_{ab}(F^{ab} F_{ab}) g_{ij}\big)$.

1.3 Schoen-Yau radius

For an open set U of a Riemannian manifold S, denote by K_U the space of simple closed curves in U that are contractible in U, and for $k \in K_U$ define

$$\mathrm{Rad}(U,k) := \sup\{r > 0 | d(k, \partial U) > r, k \notin K_{B_r(k)}\},$$

$$\mathrm{Rad}(U) := \sup\{\mathrm{Rad}(U,k) | k \in K_U\}.$$

Examples:

- $\mathrm{Rad}(B_r^{\mathbb{R}^3}(p)) = r/2$
- $\mathrm{Rad}(\mathbb{S}^2_{\mathrm{round}}(r) \times (-L, L)) = \min\{\frac{\pi r}{2}, L\}$

1.4 Schoen-Yau concentration result

Let an initial hypersurface (S, g, h) be given and let $U \subset S$ be open in S. Define $\mu \in S$ resp. $J \in \Omega^1(S)$ by

$$\mu := \frac{1}{2}(\mathrm{scal} - \sum_{i,j} h^{ij} h_{ij} + (\sum_i h^i_i)^2), \;\; J^i = \sum_j \nabla_j (h^{ij} - (\sum_k h^k_k) g^{ij}).$$

We say that the *Schoen-Yau concentration condition* is satisfied iff

$$\mathrm{SY}(g, h, U) := \mathrm{Rad}(U) \cdot \left(\min\{\underbrace{\mu(q) - ||J(q)||}_{\geq 0 \forall q (\text{D.E.C.})} : q \in U\}\right)^{1/2} \geq \sqrt{\frac{3}{2}}\pi.$$

Schoen-Yau (1981) show that the concentration condition implies the existence of an 'marginally outer trapped surface'.

1.5 The main result

We call a point in (M, g) **black** iff there is no future timelike curve of infinite length starting at p. The **black hole** $\mathrm{BH}(M, g)$ **of** (M, g) is the subset of all black points (can be empty!).

Theorem (M. 2016)

Let $(S, g_0, K_0, A_0, \dot{A}_0)$ be Zipser-asymptotically flat initial values for four-dimensional Einstein-Maxwell theory. Let $U \subset S$ be an open precompact subset of S and V an open neighborhood of ∂U satisfying the Schoen-Yau concentration condition, then U is black.

Einstein-Maxwell initial values $(S, g_0, K_0, A_0, \dot{A}_0)$ are called **Zipser-asymptotically flat (of mass m)** iff $\mathrm{tr}^{g_0}(K_0) = 0$ and

- $(g_0)_{ij} \in (1 + 2m/r)\partial_{ij} + o_4(r^{-3/2})$,
- $(k_0)_{ij} \in o_3(r^{-5/2})$,
- $(F(A_0, \dot{A}_0))_{ij} \in o_3(r^{-5/2})$.

2.1 Lorentzian geometry: Basics

Lorentzian metric: nondegenerate symmetric bilinear form of signature $(1, n)$ on tangential bundle $\tau M : TM \to M$
(Obstructed ionly on compact M, by Euler characteristic

- $I_p := \{v \in T_pM | g(v, v) < 0\} = I_p^+ \dot\cup I_p^-$: **timelike** vectors
- $J_p := \{... \leq 0\} = \bar{I}_p$: **causal**, $J_p \setminus I_p$: **null** vectors

(M, g) **time-oriented** $:\Leftrightarrow I^g := \bigcup_{p \in M} I_p$ has 2 components.
A C^1-curve $c : I \to M$ is called
future/past causal $\Leftrightarrow c'(t) \in J^{\pm}_{c(t)} \setminus \{0\} \ \forall t \in I$,
future/past timelike $\Leftrightarrow c'(t) \in I^{\pm}_{c(t)} \ \forall t \in I$.
$J^{\pm}(x) := \{y \in M | \ \exists \ \text{zukunfts/vergangenheitskausale} \ c : x \rightsquigarrow y\}$
- correspondingly $I^{\pm}(x)$ with timelike curves.

2.2 Causality conditions

A time-oriented Lorentzian manifold is called

- **causal** $\Leftrightarrow$ (M, g) has no closed causal curves. (Counterexample: $\mathbb{R}^{1,n}/\mathbb{Z}^n$)
- **diamond compact** $\Leftrightarrow$ $J^+(p) \cap J^-(q)$ compact $\forall p, q \in M$. (Counterexample: $\mathbb{R}^{1,n} \setminus \{0\}$, $\{x \in \mathbb{R}^{1,n} | x_n \in [-1, 1]\}$)
- **globally hyperbolic(g.h.)**$\Leftrightarrow$ M causal & diamond compact.

Examples of globally hyperbolic manifolds:

- $(\mathbb{R} \times N, g = -dt^2 + h)$ for (N, h) complete
- causally convex subsets of g.h. manifolds
- (M, g), if (M, k) g.h. and $I^g \subset I^k$
  ($\rightsquigarrow$ 'g.h.' is conformally invariant notion)

2.3 Geodesics and Cauchy sets

If (M, g) g.h., $p \in M$ and $q \in J^+(p)$, then there is a maximal (!) causal geodesic from p to q.

Cauchy set := Subset of (M, g) intersected by every causal C^0-inextendible curve in M exactly once. Each Cauchy set is a C^0 hypersurface ($\rightsquigarrow$ "Cauchy surface")

(M, g) g.h. $\Leftrightarrow$ M contains a Cauchy surface.

Theorem (Bernal-Sánchez 2005, M.-Sánchez 2009)

Let (M, g) globally hyperbolic. Then (M, g) is isometric to $(\mathbb{R} \times N, -f^2 dt^2 + g_t)$, where $f \in C^\infty(M)$ bounded and g_t a smooth one-parameter family of Riemannian metrics on N. Levelset $t^{-1}(x)$ is Cauchy surface for all $x \in \mathbb{R}$.

2.4 Analytic Relevance

(M, g) g.h. An $\mathrm{ord}(1)$-differential operator $P : C^\infty(\pi) \to C^\infty(\pi)$ on a vector bundle π over M with BLF h is called **symmetric-hyperbolic (s.h.)** $\Leftrightarrow$

1. The image of $\mathrm{symb}P : \tau^* M \to \mathrm{End}(\pi)$ consists of h-symmetric endomorphisms,
2. $\mathrm{symb}P(g(v, \cdot))$ is h-positive definite for all $v \in J^g$.

Examples of s.h. operators: Dirac operator, (canonical prolongation of) Lorentzian Laplace operator (wave operator), of the Yang-Mills operator...

Cauchy problem of *linear* s.h. PDE is **well-posed**, i.e., restriction of solutions to a Cauchy surface has a smooth inverse.

2.5 Cauchy development

Maxwell-Einstein Equations: Metric is now a dynamical variable!

A **Cauchy development** of an initial datum $I = (S, g_0, K, A_0, \dot{A}_0)$ is (M, g, A, J), where

- (M, g) g.h. manifold,
- (g, A) solves the Maxwell-Einstein equation,
- $J : S \to M$ embedding as Cauchy surface of M inducing the initial datum I: $J^* A = A_0, (\partial_t J)^* A = \dot{A}_0$,

  g_0 resp. K first resp. second fundamental form of (J, g).

[Choquet-Bruhat, Geroch]: For each initial datum I there is exactly one *maximal* Cauchy development $MCD(I)$.

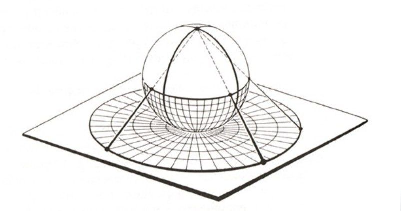
3.1 Conformal compactifications

Definition

Let (M, g) be a g.h. spacetime. A **conformal compactification of** (M, g) **(of regularity** C^k**)** is an open conformal embedding I of (M, g) into a g.h. spacetime (N, h) (with I and h of regularity C^k) such that $\overline{F(M)}$ is causally convex and compact.

Conformal compactifications can also be used to construct Hadamard states for Maxwell theory (Dappiaggi-Siemssen 2011)

3.2 The standard example



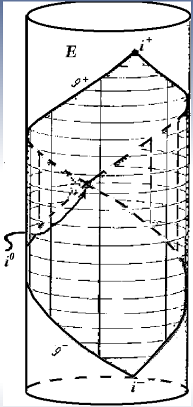


Abbildung: Left: stereographic projection $\mathbb{R}^n \to \mathbb{S}^n$, right: its unique Lor4entzian extension to the Penrose compactification $\mathbb{R}^{1,n} \to (\mathbb{R} \times \mathbb{S}^n, -dt^2 + g_{rund})$

3.3 Conformal extensions

Necessity of generalization of 'conformal compactification':

Theorem (M. 2016)

Let $(S, g_0, K_0, A_0, \dot{A}_0)$ be asymptotically flat initial values for matter-Einstein equations obeying the dominant energy condition, S spin or of dimension ≤ 7 and with not identically vanishing matter fields A_0, then for any open conformal embedding with precompact image, the metric h on N is not C^2 at spatial infinity.

3.4 Definition of 'conformal extension'

A subset of a spacetime is called **future compact** iff it is contained in the past of a compact subset.

Definition

A **conformal extension** of order k is an open conformal embedding I into another globally hyperbolic manifold (N, h) with I and h of regularity C^k such that the closure of $I(M)$ is causally convex and future compact. A **strong conformal extension** is a conformal extension with the property that the inverse ω of the conformal factor, as a function on $I(M)$, has a C^k extension.

- Introduced as central tool for small-initial value well-posedness of Dirac-Higgs-Yang-Mills theory in Ginoux - M. (2014)
- Conformal extensions can still be used to construct Hadamard states!

3.5 Abundance of strong conformal extensions

Zipser's stability theorem for Einstein-Maxwell theory (2009) $\rightsquigarrow$ maximal Cauchy developments of initial values contained in a neighborhood of trivial initial values are future causally complete.

Theorem (M. 2016)

If $I := (\psi_0, \Phi_0, A_0, A_1, g_0, h_0)$ are Zipser-asymptotically flat values of order k for Einstein-Maxwell theory then the maximal Cauchy development M of I

- *admits a strong conformal extension of order k,*
- *has 'standard' spatial ends: If for a future curve c in M we have $J^-(c(\mathbb{R}))$ spatially noncompact, then $J^-(c(\mathbb{R}))$ contains timelike curves of arbitrary length and all M-inextendible null geodesics in $J^-(c(\mathbb{R}))$ are of infinite affine length.*

4.1 Singularity theorems

A submanifold N is called **trapped** iff its mean curvature vector H_N is past and if there is $a > 0$ such that for all $p \in \mathbb{N}$: $\langle H_N, H_N \rangle < -a^2$.

Penrose's singularity theorem (1966): If (M, g)

1. satisfies the *null convergence condition (NCC)* $\mathrm{ric}(v, v) \geq 0$ for all $v \in TM$ null,
2. has a noncompact Cauchy surface and a trapped compact spacelike codimension-2 submanifold N.

then some C^0-inextendible future null geodesic from N is incomplete.

Problems:

- Are there also complete null geodesics from N?
- Massive observers follow *timelike*, not null curves.

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4.2 Singularity theorems, ctd.

Hawking's singularity theorem (1965): If (M, g)

1. satisfies the *timelike convergence condition (TCC)* $\mathrm{ric}(v, v) \geq 0$ for all $v \in TM$ timelike,
2. has a trapped, compact or noncompact Cauchy surface (which is a codimension-1 submanifold) N.

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Problem: In asymptotically flat case, we can't find Cauchy surface N with H_N uniformally past!

Question: Is there a synthesis between Penrose and Hawking?

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4.3 Variants of singularity theorems

The proof includes a new variant of Hawking's theorem...

Theorem (Compact Umbrella Theorem, M. 2016)

Let (M, g) be g.h. with timelike convergence condition. If $\exists S$ Cauchy surface: $J^-(J^+(p)) \cap S$ compact, then p black.

... and a slight generalization of Penrose's singularity theorem:

Theorem (M. 2016)

If (M, g) satisfies the null energy condition, has a noncompact Cauchy surface S and if there is a compact set U in (M, g) whose boundary is an outer trapped surface, then there is a C^0-inextendible incomplete null geodesic starting at ∂U.

Kodim.-2-Untermf. S **outer trapped** $:\Leftrightarrow g(H_S, \nu_+) < C < 0$ für *äußere* Licht-Vergangenheits-Normale ν_+ von S (trapped$\Leftrightarrow g(H_S, \nu_\pm) < C < 0$).

4.4 Synthesis, main proof

Theorem (Penrose-Hawking synthesis, M. 2016)

Let $\dim(S) = 3$, $I := (S, g_0, K_0, A_0, \dot{A}_0)$ *Zipser-asymptotic flat Maxwell-Einstein initial data,* U *open in* S *and precompact,* ∂U *an OTS. Then* U *is black in* $MCD(I)$*.*

Sketch of proof: OTS $\xRightarrow[\text{Penrose variant}]{}$ ring of incompl. null geos c $\xRightarrow[\text{Hawking variant}]{}$ BH or $i_0 \in J^-(c)$ for one of those c $\xRightarrow[\text{Analysis of symm.-hyperb. eq.}]{\text{conformal extension \&}}$ exclude option 2 $\square$

Main proof: SY Concentration $\xRightarrow[\text{Schoen-Yau}]{}$ MOTS $\xRightarrow[\text{Galloway rigidity}]{}$ OTS $\xRightarrow[\text{Penrose-Hawking Synthesis}]{}$ BH $\square$

4.5 Result, Perspectives

Theorem (Main result as above, M. 2016)

Let $(S, g_0, K_0, A_0, \dot{A}_0)$ be Zipser-asymptotically flat initial values for four-dimensional Einstein-Maxwell theory. Let $U \subset S$ be an open precompact subset of S and V an open neighborhood of ∂U satisfying the Schoen-Yau concentration condition, then U is black.

Ansatz for falsification of **Cosmic Censorship Conjecture** ("For opn and dense subset I of initial values, $MCD(I)$ is maximal as Lorentzian manifold")

NB: Completeness $\overset{\Rightarrow}{\not\Leftarrow}$ Maximality

Kruskal spacetime *is* maximal, Kerr spacetime is *not*.

5 By-product: Decision of a weak cosmic censorship bet

Whereas Stephen W. Hawking [...] believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,
And whereas John Preskill and Kip Thorne [...] regard naked singularities as quantum gravitational objects that might exist, unclothed by horizons, for all the Universe to see,
Therefore Hawking offers, and Preskill/Thorne accept, a wager that
When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, then
A dynamical evolution from generic initial conditions (i.e., from an open set of initial data) can never produce a naked singularity [...]
The loser will reward the winner with clothing to cover the winner's nakedness. [...]

Stephen W. Hawking, John P. Preskill, Kip S. Thorne Pasadena, CA, 5 February 1997

Conclusion from another step of the proof: Hawking wins the bet, even without genericity, for Einstein-Maxwell Theory and Zipser-asymptotically flat initial values!