Regular solutions of the Einstein equations with parametric transition to black holes

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1. Introduction

• In the case of spherical symmetry, an "unstoppable gravitational collapse" (resulting in a spacetime singularity – within classical general relativity) necessarily leads to a Schwarzschild black hole (cf. Oppenheimer & Snyder 1939) or, if there is some electric charge, to a Reissner–Nordstrøm–Weyl black hole.

• Without spherical symmetry, for example in the case of a collapsing *rotating* star, the result of a complete collapse is much more difficult to predict. The (weak) cosmic censorship conjecture (Penrose 1969), combined with (i) the assumption that the exterior gravitational field settles down to a stationary state and (ii) the black hole uniqueness ("no-hair theorem": Israel, Carter, Hawking, Robinson, Mazur, . . .), predicts the formation of a Kerr (or Kerr–Newman) black hole.

• With rotation (and/or electric charge), *quasi-stationary* collapse scenarios, described by sequences of equilibrium configurations becoming more and more compact, are possible. Do they lead to black holes or to naked singularities?

• A continuous sequence of stationary and axisymmetric, uniformly rotating perfect fluid bodies reaches a black hole limit if and only if

$$M - 2\Omega J \to 0 \qquad\qquad (G = c = 1)$$

in the limit (Meinel 2006). The limit leads to an *extreme* Kerr black hole.

• The existence of such a limit was first demonstrated for rotating discs of dust, numerically by Bardeen & Wagoner (1971) and analytically by Neugebauer & M. (1995).

• The fact that the quasi-stationary collapse of a rotating disc of dust leads to the formation of a black hole and *not to a naked singularity* is in remarkable agreement with the cosmic censorship conjecture.

• Further numerical examples, for genuine fluid bodies, were provided by the "relativistic Dyson rings" (Ansorg et al. 2003) and their generalizations.

• Examples for parametric transitions to black holes have also been found in the context of the static Einstein-Yang-Mills-Higgs equations (Breitenlohner et al. 1995, Lue & Weinberg 2000).

• Simplest possibility: The parametric compression of static configurations of "electrically counterpoised dust", also called "Bonnor stars", leads to an extreme Reissner–Nordstrøm–Weyl black hole (Bonnor & Wickramasuriya 1975, Lemos & Weinberg 2004, Bonnor 2010, M. & Hütten 2011). [Corresponding solution class: Papapetrou-Majumdar solutions]

• The quasi-stationary collapse of a rotating disc of electrically charged dust leads to the formation of an extreme Kerr-Newman black hole (Breithaupt et al. 2015).

2. Black hole limit of relativistic figures of equilibrium
(a) Necessary and sufficient conditions

Four-velocity of the fluid:

$$u^i = e^{-V}(\xi^i + \Omega\,\eta^i), \quad \Omega = \text{constant}$$

with Killing vectors: $\xi = \partial/\partial t,\ \eta = \partial/\partial\varphi$

[$\xi^i\xi_i \rightarrow -1$ at spatial infinity. We assume asymptotic flatness; the spacetime signature is chosen to be $(+\ +\ +\ -)$. The orbits of the spacelike Killing vector η are closed and η is zero on the axis of symmetry.]

$$\Omega = u^\varphi/u^t, \quad e^{-V} = u^t$$

$$u^i u_i = -1 \quad \Rightarrow \quad (\xi^i + \Omega\,\eta^i)(\xi_i + \Omega\,\eta_i) = -e^{2V}$$

Energy-momentum tensor: $T_{ik} = (\mu + p)\,u_i u_k + p\,g_{ik}$

“Cold” equation of state, $\mu = \mu(p)$, following from

$$p = p(\mu_\mathrm{b}, T), \quad \mu = \mu(\mu_\mathrm{b}, T)$$

for $T = 0$, where μ_b is the “baryonic mass-density” [with $(\mu_\mathrm{b} u^i)_{;i} = 0$] and T the temperature. The specific enthalpy

$$h = \frac{\mu + p}{\mu_\mathrm{b}}$$

can be calculated from $\mu(p)$ via the thermodynamic relation

$$dh = \frac{1}{\mu_\mathrm{b}}\, dp \qquad (T = 0)$$

leading to

$$\frac{dh}{h} = \frac{dp}{\mu + p} \quad \Rightarrow \quad h(p) = h(0)\, \exp\left[\int_0^p \frac{dp'}{\mu(p') + p'}\right].$$

[$h(0) = 1$ in most cases.]

$$T^{ik}{}_{;k} = 0 \quad \Rightarrow \quad h(p)\, e^{V} = h(0)\, e^{V_0} = \mathrm{constant}$$

Relative redshift z of zero angular momentum photons emitted from the surface of the fluid and received at infinity:

$$z = e^{-V_0} - 1$$

Equilibrium models, for a given equation of state, are fixed by two parameters, for example Ω and V_0. (When we discuss a "sequence" of solutions, what is meant is a curve in the two-dimensional parameter space.)

Baryonic mass M_{b}, gravitational mass M and angular momentum J:

$$M_{\mathrm{b}} = -\int_{\Sigma} \mu_{\mathrm{b}}\, u_i\, n^i d\mathcal{V}, \quad M = 2\int_{\Sigma} (T_{ik} - \frac{1}{2} T^j_j\, g_{ik}) n^i \xi^k d\mathcal{V}, \quad J = -\int_{\Sigma} T_{ik}\, n^i \eta^k d\mathcal{V},$$

where Σ is a spacelike hypersurface ($t = \mathrm{constant}$) with the volume element $d\mathcal{V} = \sqrt{{}^{(3)}g}\, d^3x$ and the future pointing unit normal n^i.

A combination of the previous relations leads to the formula

$$M = 2\Omega J + h(0)\, e^{V_0} \int \frac{\mu + 3p}{\mu + p}\, dM_\mathrm{b}\,.$$

We assume μ and p to be non-negative and $0 < M_\mathrm{b} < \infty,\ 0 < h(0) < \infty$.

$\Rightarrow \quad 1 \le (\mu + 3p)/(\mu + p) \le 3 \quad \Rightarrow$

$$M = 2\Omega J \quad \Leftrightarrow \quad V_0 \to -\infty \quad (z \to \infty)$$

This condition is necessary and sufficient for approaching a black hole limit (Meinel 2006).

Surface of the fluid: $\quad (\xi^i + \Omega\, \eta^i)(\xi_i + \Omega\, \eta_i) = -e^{2V_0}$

Black hole horizon: $\quad (\xi^i + \Omega_\mathrm{h}\, \eta^i)(\xi_i + \Omega_\mathrm{h}\, \eta_i) = 0$

Ω_h : “angular velocity of the horizon” ; $\quad V_0 \to -\infty$: $\ \Omega \to \Omega_\mathrm{h}$

$M = 2\Omega J \quad \Rightarrow$ Impossibility of black hole limits of non-rotating (uncharged) equilibrium configurations, cf. “Buchdahl's inequality”.

Together with

$$\Omega = \Omega_\mathrm{h} = \frac{J}{2M^2\left[M + \sqrt{M^2 - (J/M)^2}\,\right]}$$

$\Rightarrow \quad J = M^2 \qquad$ (*extreme* Kerr black hole).

Note: The last conclusion makes use of the Kerr black hole uniqueness *including the extreme case.*

(b) Extreme Kerr uniqueness

In Weyl's canonical coordinates, the stationary and axisymmetric *vacuum* line element takes the form

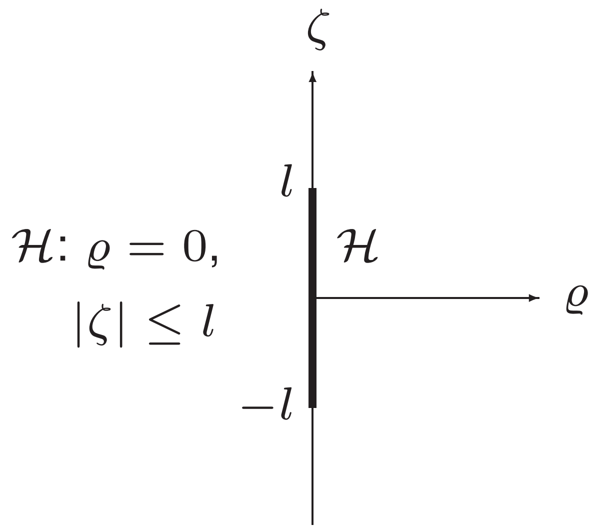
$$ds^2 = e^{2\alpha}(d\varrho^2 + d\zeta^2) + \varrho^2 e^{-2\nu}(d\varphi - \omega\, dt)^2 - e^{2\nu} dt^2,$$

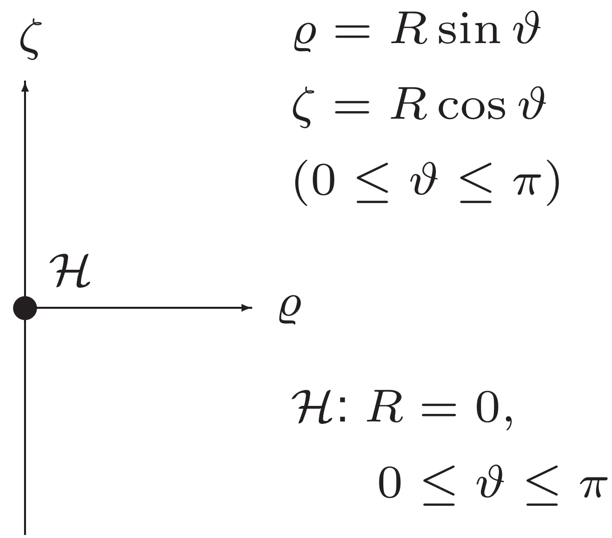
where

$$\varrho^2 = (\xi^i \eta_i)^2 - \xi^i \xi_i \eta^k \eta_k = (\chi^i \eta_i)^2 - \chi^i \chi_i \eta^k \eta_k.$$

$\Rightarrow \quad \varrho = 0$ on the horizon $\quad$ ($\mathcal{H}$: $\chi^i \chi_i = 0$, $\chi^i \eta_i = 0$ $\quad$ with $\quad \chi^i \equiv \xi^i + \Omega_{\mathrm{h}} \eta^i$)

Therefore, the $t = \mathrm{constant}$, $\varphi = \mathrm{constant}$ slice of the horizon of a single stationary and axisymmetric black hole surrounded by a vacuum can only be a finite intervall or a single point on the ζ-axis. In *both* cases, the corresponding boundary value problem can uniquely be solved by means of the “inverse scattering method” .





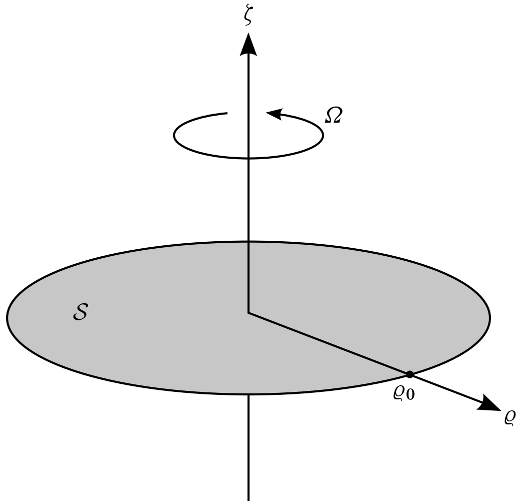
Result: Kerr with $J < M^2$ $\quad$ Kerr with $J = M^2$

$[\, l = \sqrt{M^2 - (J/M)^2}\,]$

- The Kerr (-Newman) black holes – including the extreme case – are the only stationary and axisymmetric black holes (with a single connected horizon) surrounded by an asymptotically flat (electro-) vacuum (Meinel et al. 2008, Meinel 2012).

Other proofs of the extreme Kerr (-Newman) uniqueness have been published by Amsel et al. (2010), Figueras & Lucietti (2010) and Chrusciel & Nguyen (2010).

3. Rigorous results for discs of dust

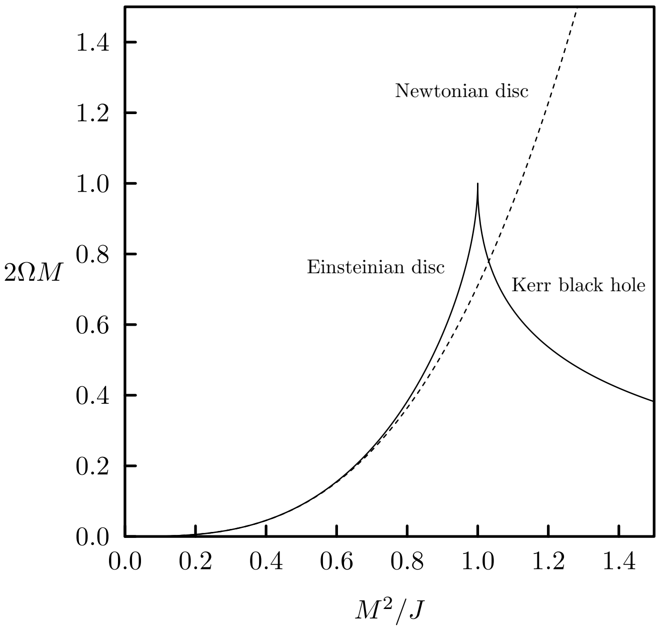


Two parameters: ϱ_o, Ω

The exact solution to this problem has been found in terms of hyperelliptic theta functions by solving the corresponding boundary value problem via the "inverse scattering method" (Neugebauer & M. 1995). It depends on the normalized coordinates $\varrho/\varrho_\mathrm{o}$, ζ/ϱ_o or ϱ/M, ζ/M and the previously introduced parameter e^{V_0}, which is given here by

$$e^{2V_0} = -(\xi^i + \Omega\,\eta^i)(\xi_i + \Omega\,\eta_i)\Big|_{\mathcal{S}} = \mathrm{constant}.$$

Newtonian limit: $|V_0| \ll 1$,   Black hole limit: $V_0 \to -\infty$

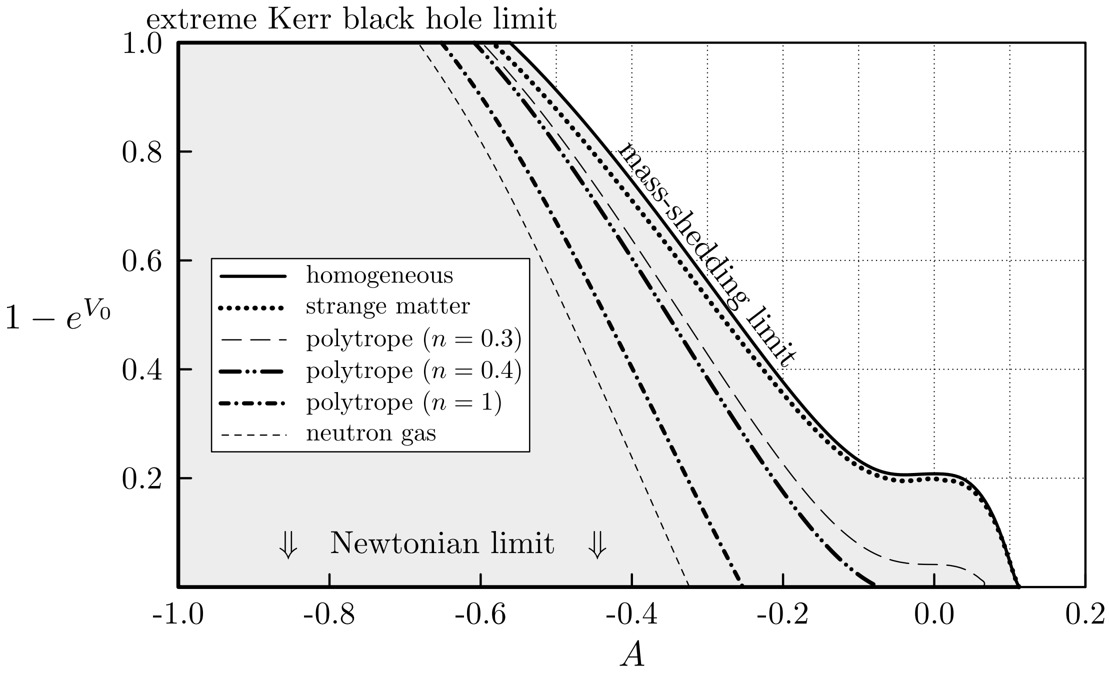


In the black hole limit, the disc shrinks to the origin of the ϱ/M, ζ/M coordinate system, since $\varrho_o/M \to 0$; and the solution becomes precisely the extreme Kerr solution (outside the horizon).
(Note that the limit in the ϱ/ϱ_o, ζ/ϱ_o coordinates is different: It gives a non-asymptotically flat solution with the extreme Kerr "throat geometry" at spatial infinity!)

4. Numerical results for fluid rings with various equations of state

$$ds^2 = e^{2\alpha}(d\varrho^2 + d\zeta^2) + W^2 e^{-2\nu}(d\varphi - \omega\, dt)^2 - e^{2\nu} dt^2$$

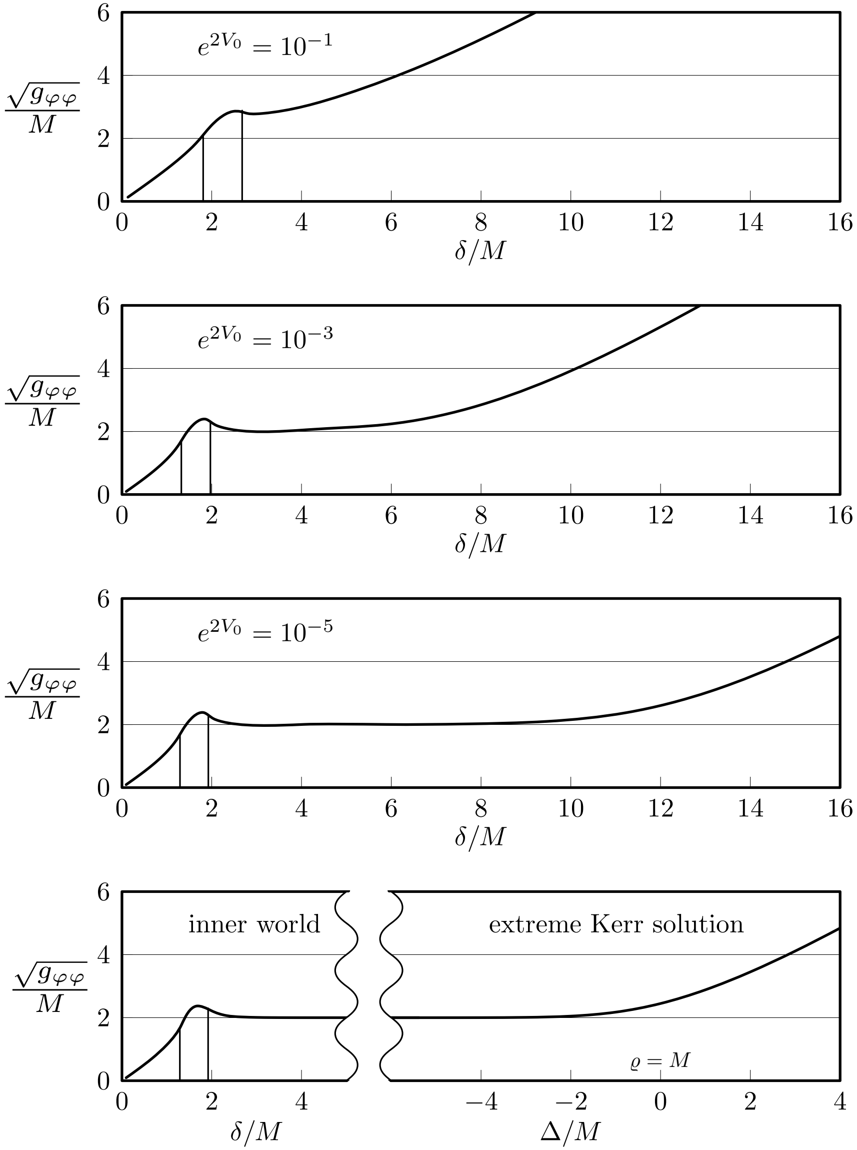
$$\alpha = \alpha(\varrho, \zeta)$$
$$\nu = \nu(\varrho, \zeta)$$
$$\omega = \omega(\varrho, \zeta)$$
$$W = W(\varrho, \zeta)$$



toroidal bodies: $A := -\varrho_\mathrm{i}/\varrho_\mathrm{o} < 0$

sheroidal bodies: $A := \zeta_\mathrm{p}/\varrho_\mathrm{o} > 0$

Ansorg et al. (2003), Fischer et al. (2005), Labranche et al. (2007)



$$2\pi\sqrt{g_{\varphi\varphi}(\varrho,0)} = 2\pi W(\varrho,0)e^{-\nu(\varrho,0)}:$$

proper circumference of a circle $\varrho = \text{constant}$ in the "equatorial plane" ($\zeta = 0$, $t = \text{constant}$)

$$\delta = \int\limits_0^{\varrho} e^{\alpha(\varrho',0)}d\varrho'$$: proper radius of that circle

$$\Delta = \int\limits_M^{\varrho} e^{\alpha(\varrho',0)}d\varrho'$$

5. Electrically counterpoised dust (ECD) configurations

$$ds^2 = S^2(dx^2 + dy^2 + dz^2) - S^{-2}dt^2$$

Energy-momentum tensor:

$$T_{ik} = \rho u_i u_k + T_{ik}^{(\mathrm{em})}, \quad u^i = \delta_4^i\, S, \quad \rho \geq 0, \quad S > 0$$

with $\quad T_{ik}^{(\mathrm{em})} = \frac{1}{4\pi}(F_{ij}F_k{}^j - \frac{1}{4}F^{mn}F_{mn}g_{ik}), \quad F_{ik} = A_{k,i} - A_{i,k}$

$$A_i = -\delta_i^4\,\phi, \quad \phi = -\epsilon(S^{-1} - 1), \quad \epsilon = \pm 1$$

Einstein-Maxwell equations for the ECD case ($J^i = \sigma u^i$, $\sigma = \epsilon\rho$):

$$R_{ik} - \frac{1}{2}R\,g_{ik} = 8\pi T_{ik},\ F^{ik}{}_{;k} = 4\pi J^i \Longleftrightarrow \Delta S \equiv \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} = -4\pi S^3\rho$$

$\Longleftrightarrow \Delta V = 4\pi\mu \quad$ with $\quad S = 1 - V, \quad \rho = \frac{\mu}{S^3}$

General solution with a localized ECD distribution:

$$V = -\int \frac{\mu(\mathbf{r}')d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \quad \mathbf{r} = (x, y, z).$$

Asymptotic behaviour:

$$r \equiv |\mathbf{r}| \to \infty : \quad V \to -\frac{M}{r}, \quad g_{44} = -S^{-2} \to -\left(1 - \frac{2M}{r}\right)$$

with $M = \int \mu(\mathbf{r})d^3\mathbf{r}$ Note: $Q \equiv \int \sigma S^3 d^3\mathbf{r} = \epsilon M$

Any ECD distribution of finite extent: $\mu(\mathbf{r}) = f(\mathbf{r})$ with $f(\mathbf{r}) \equiv 0$ for $r > R$

Corresponding one-parameter family of solutions: $\mu(\mathbf{r}) = \alpha^3 f(\alpha\mathbf{r})$ $(\alpha > 0)$

i.e. $\mu(\mathbf{r}) \equiv 0$ for $r > \dfrac{R}{\alpha}$ Note: M independent of α

Sufficiently small α : Newtonian limit $(|V| \ll 1)$ $\alpha \to \infty$: black hole limit

(a) The exterior point of view

$$\alpha \to \infty \quad \Rightarrow \quad \mu(\mathbf{r}) = M\delta(\mathbf{r}) \quad \Rightarrow \quad \underline{r > 0:} \;\; V = -\frac{M}{r}, \quad S = 1 + \frac{M}{r}$$

$$\Rightarrow \quad ds^2 = \left(1 + \frac{M}{r}\right)^2 (dx^2 + dy^2 + dz^2) - \left(1 + \frac{M}{r}\right)^{-2} dt^2$$

• metric of an extremal Reissner-Nordström black hole outside the event horizon

Note: horizon at $r = 0$ in the isotropic coordinates used here
(relation to radial Schwarzschild coordinate r_S: $r_S = r + M$)

(b) The interior point of view

Limit $\alpha \to \infty$ *after* the coordinate transformation

$$\tilde{x} = \alpha x, \; \tilde{y} = \alpha y, \; \tilde{z} = \alpha z, \; \tilde{t} = \alpha^{-1} t$$

$$\Rightarrow \quad ds^2 = \tilde{S}^2 (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) - \tilde{S}^{-2} d\tilde{t}^2 \qquad (\tilde{S} = \alpha^{-1} S)$$

$$\alpha \to \infty \quad \Rightarrow \quad \tilde{S} = \alpha^{-1} S\Big|_{\alpha\to\infty} = \alpha^{-1}(1 - V)\Big|_{\alpha\to\infty} = -\alpha^{-1} V\Big|_{\alpha\to\infty}$$

$$\Rightarrow \quad \tilde{S} = \int \frac{f(\tilde{\mathbf{r}}')d^3\tilde{\mathbf{r}}'}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}'|}, \quad \tilde{\mathbf{r}} = (\tilde{x}, \tilde{y}, \tilde{z})$$

Note: Finite values of $\tilde{r} \equiv |\tilde{\mathbf{r}}|$ correspond to $r = 0$ in the limit!

<u>Asymptotic behaviour:</u>

$$\tilde{r} \to \infty : \quad \tilde{S} \to \frac{M}{\tilde{r}} \quad \Rightarrow \quad ds^2 = \frac{M^2}{\tilde{r}^2}(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) - \frac{\tilde{r}^2}{M^2} d\tilde{t}^2$$

• extreme Reissner-Nordström "near-horizon geometry" (also known as the Bertotti-Robinson metric or $AdS_2 \times S^2$ spacetime, in fact going back to Levi-Civita 1917)

• *Spherically symmetric case:* $AdS_2 \times S^2$ not only asymptotically, but for all $\tilde{r} > R$

(c) A generic example

$$f(\mathbf{r}) = \begin{cases} \mu_0 & \text{for} \quad x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

with $\mu_0 = \mathrm{constant}$

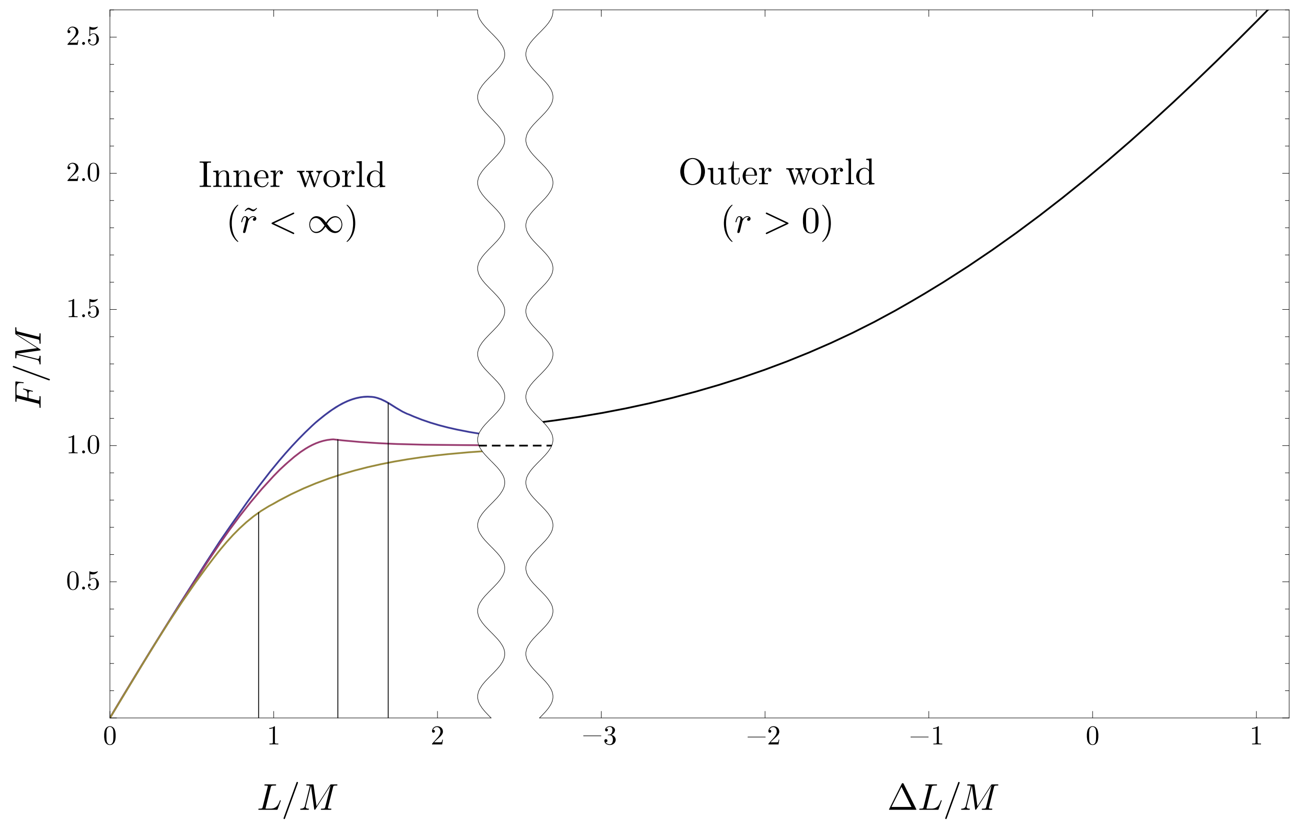
$$\Rightarrow \quad M = \frac{4\pi}{3} abc\,\mu_0$$

Visualization of the limit $\alpha \to \infty$:

$$F \equiv \begin{cases} \tilde{r}\tilde{S} & \text{for} \quad \tilde{r} < \infty \quad \text{("inner world")} \\ rS & \text{for} \quad r > 0 \quad \text{("outer world")} \end{cases}$$

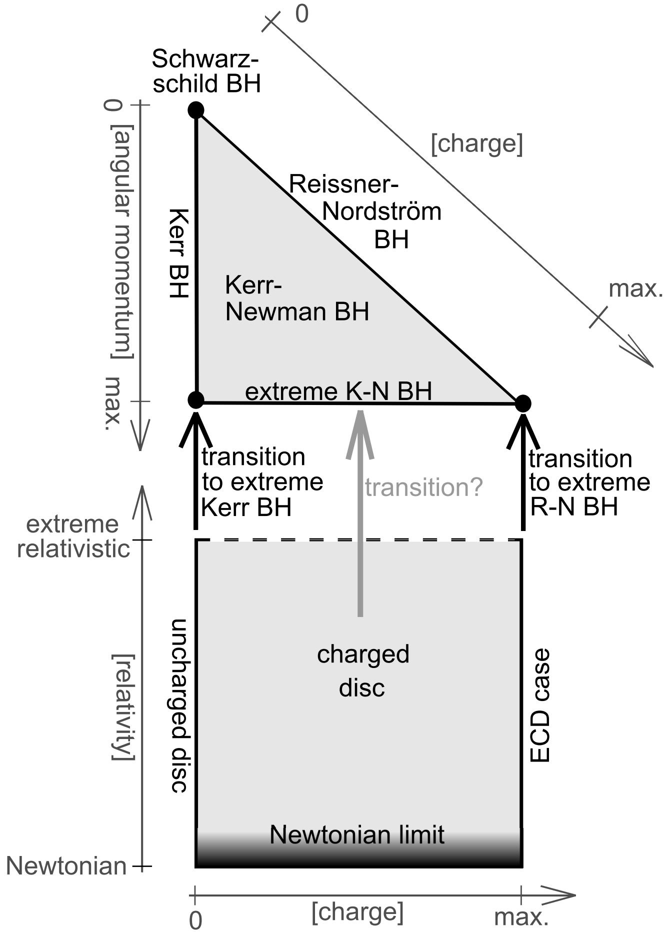
Note: $\tilde{r}\tilde{S} = rS$ for finite α

Plot of F/M for $b = 0.8\,a, \;\; c = 0.5\,a$:



$$L = \int_0^{\tilde{r}} \tilde{S}(\tilde{r}')d\tilde{r}' \qquad \Delta L = \int_M^{r} S(r')dr'$$

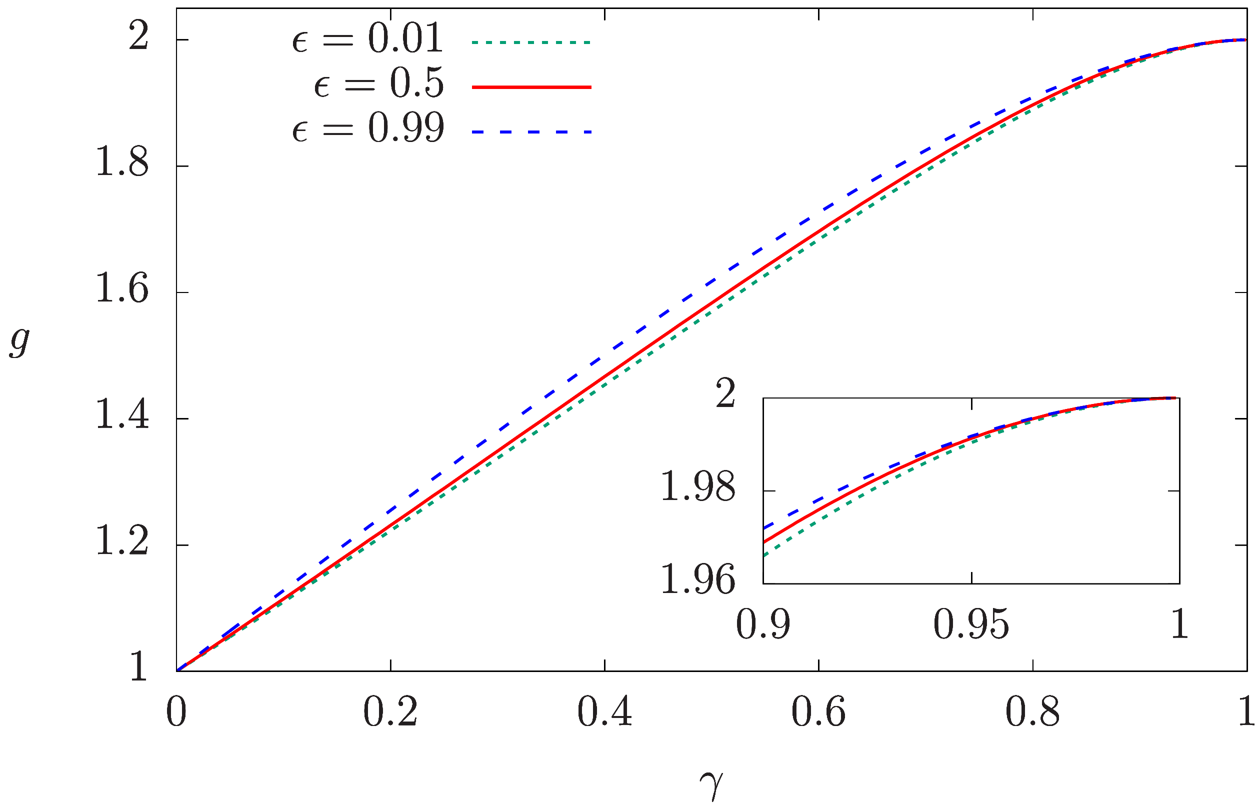
6. Rotating discs of electrically charged dust



Palenta & M. (2013)

Breithaupt et al. (2015)

Liu Pynn et al. (2016)



g : gyromagnetic factor ($= \frac{\mu_{\rm m}}{QJ/2M}$), ϵ : specific electric charge ($0 < \epsilon < 1$)

$\gamma = 1 - e^{V_0} = z/(1+z)$

Conclusion

All considered quasi-stationary collapse scenarios lead to black holes, *and not to naked singularities*.

- These results, which are highly non-trivial because of the absence of spherical symmetry, are in remarkable agreement with the cosmic censorship conjecture.

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