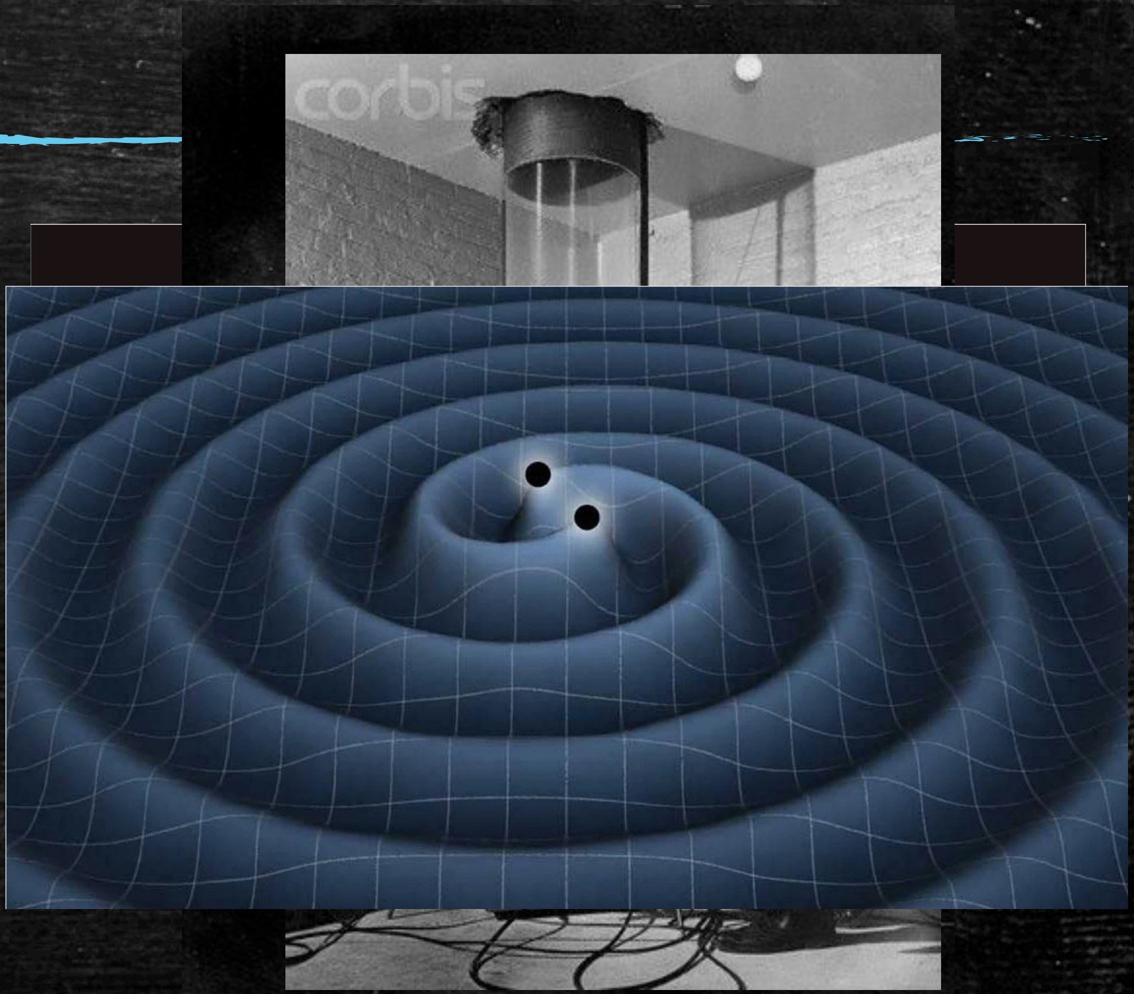
The interpretation of vacuum solutions to the Einstein field equations

Dennis Lehmkuhl, Einstein Papers Project, Caltech

Email: lehmkuhl@caltech.edu

The four classical tests

1. Deriving the perhihelion of Mercury
2. Light bending by the Sun
3. Gravitational redshift
4. Gravitational waves



- Note: All four classical tests only need solutions to the *vacuum* Einstein equations. (3 not even that.)

Question of the talk

- Given that all four classical tests involve material systems, it is prima facie surprising that we need only solutions to the *vacuum* field equations to predict what is confirmed in these tests.

- Question: how do we need to *use* and *interpret* these vacuum solutions to make this possible?

Standard Interpreter vs. Practice Interpreter

- Standard Interpreter: Interpret by asking yourself what the world would be like if a given theory were *exactly* true.
- Here: Ask yourself what the universe *would be like* if a given solution to the vacuum field equations were to describe it exactly.
- Practice Interpreter: Interpret by asking yourself how a given theory is used to describe (parts of) the *actual* world.
- Here: Ask yourself *in which different ways* a given solution to the vacuum field equations can be used to describe (parts of) the *actual world.*

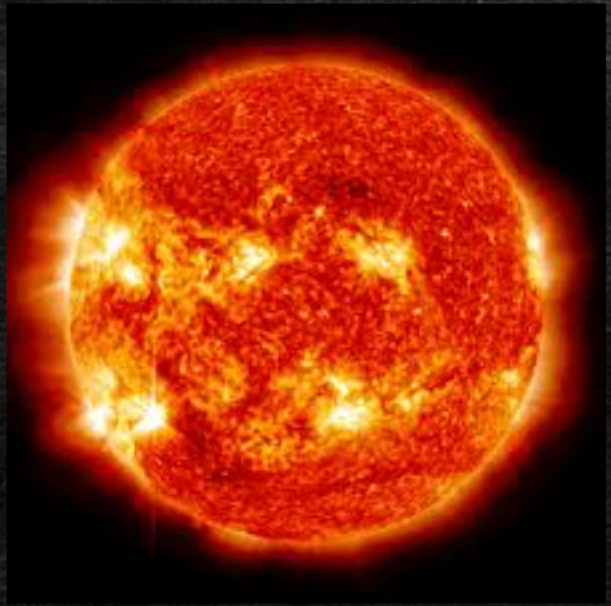
Outline

1. Introduction: Different ways of interpreting solutions
2. The Schwarzschild solution: spherically symmetric, static
3. The Weyl class of solutions: axially symmetric, static
4. Einstein's reinterpretation of Weyl's results
5. Conclusion

Outline

1. Introduction: Different ways of interpreting solutions
2. The Schwarzschild solution: spherically symmetric, static
3. The Weyl class of solutions: axially symmetric, static
4. Einstein's reinterpretation of Weyl's results
5. Conclusion

The Schwarzschild solution: spherically symmetric, static, asymptotically flat



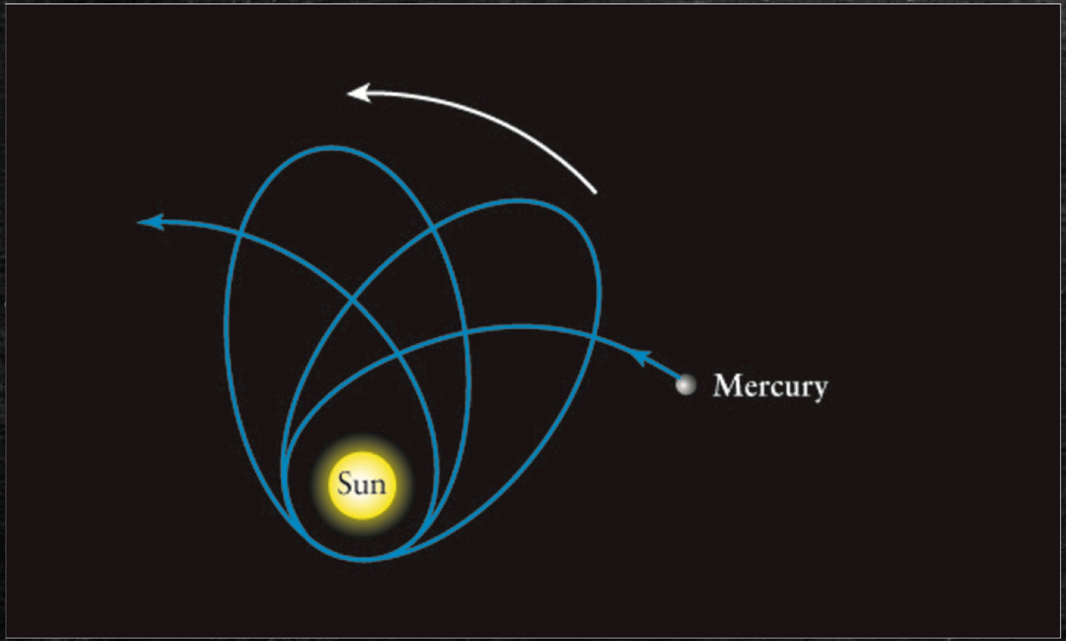
$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

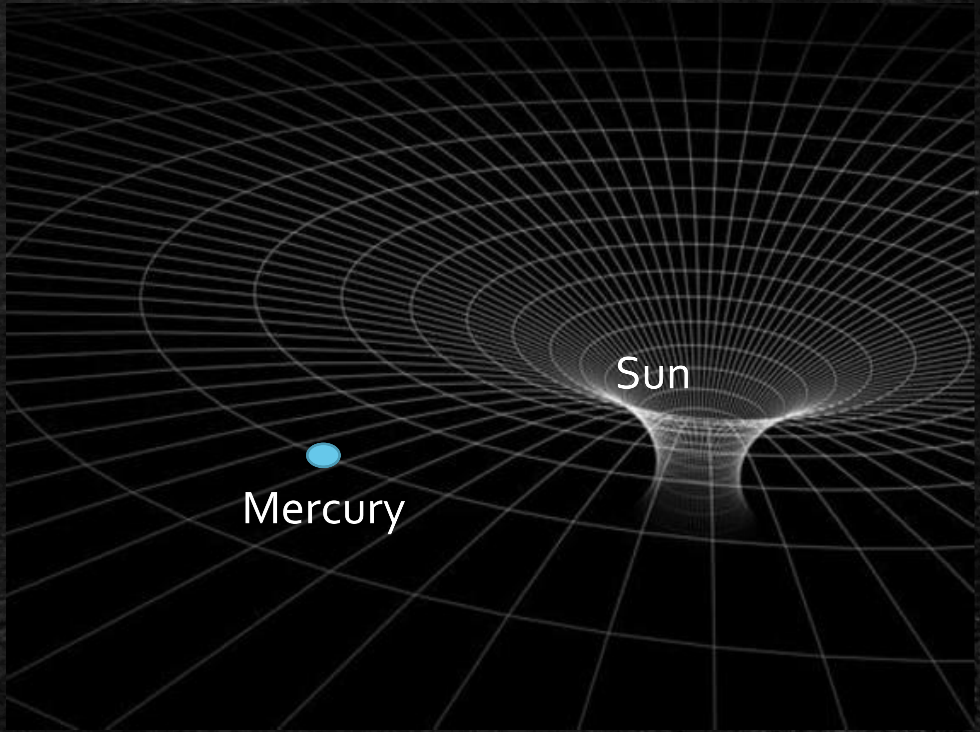
The Schwarzschild solution: spherically symmetric symmetry, static, asymptotically flat

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

- Nothing in the solution itself tells you what the parameter m means, or even how the coordinates should be interpreted (or constrained).
- Different applications to represent different systems will demand different interpretations and constraints.
- One question: there is a coordinate singularity at r=2m,and a genuine singularity at r=0. How to interpret them in different applications?

Application 1: Einstein's 1915 derivation of Mercury's perihelion

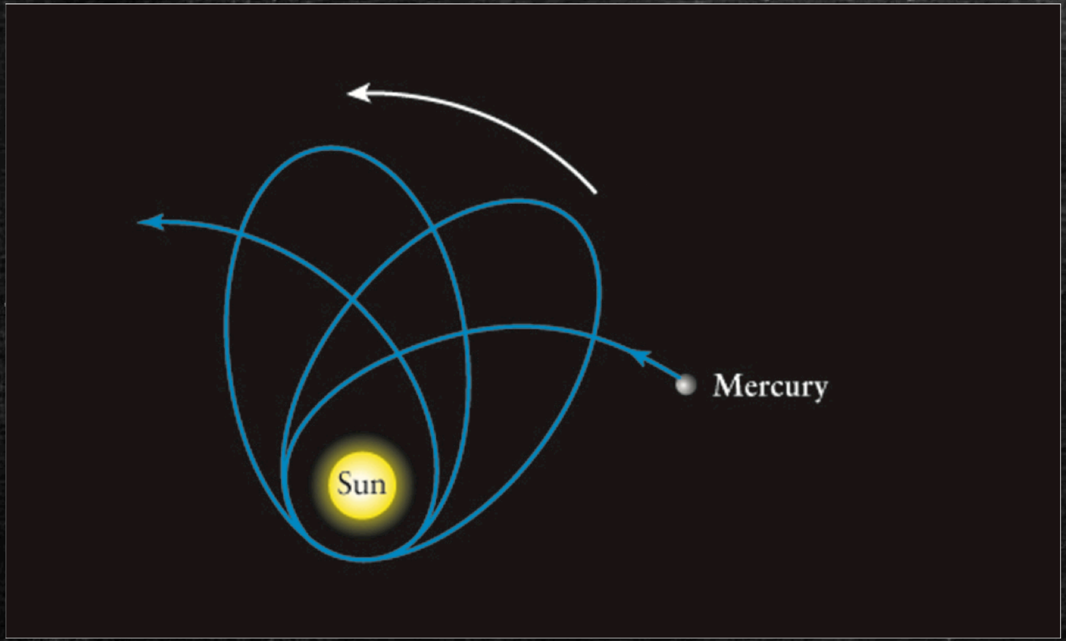




$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

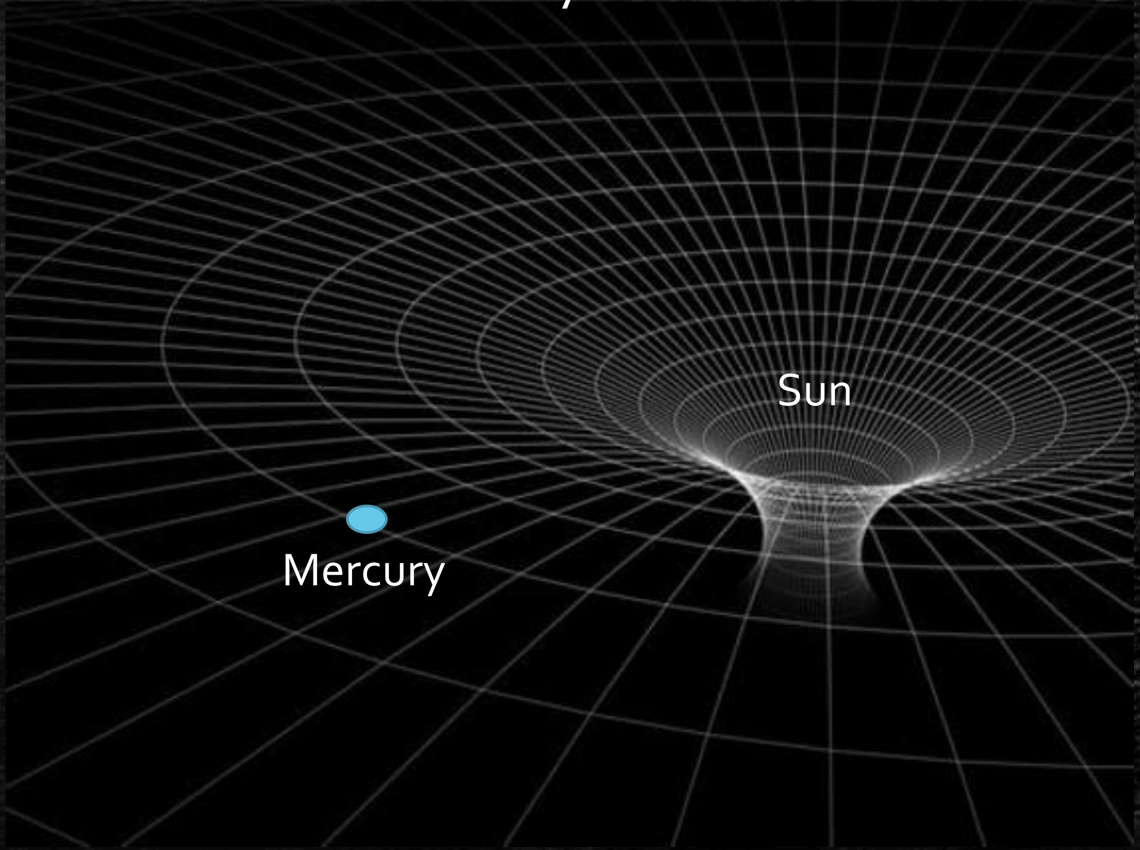
How do we interpret the singularity at the center in this context?

Application 1: Einstein's 1915 derivation of Mercury's perihelion



The Schwarzschild metric should not be interpreted as representing the Sun, but as representing its exterior gravitational field.

The singularity at the center of the Schwarzschild solution can be interpreted as a *place-holder* for a theory of matter.



Exterior solutions as guidelines for finding interior solutions

- The standard interpreter would focus on a Schwarzschild universe and ask how we should interpret the Schwarzschild metric for the entire range of coordinates.
- In applying the Schwarzschild metric to calculate the perihelion of Mercury, one can restrict the r coordinate to $r > 2m$ and thus avoid thinking about singularities: we obtain the exterior Schwarzschild metric, and more is not needed for deriving the orbit of Mercury.
- Birkhoff's Theorem: The exterior Schwarzschild solution is the unique spherically symmetric solution to the vacuum Einstein equations.
- Still, an exterior solution can be used as a guideline to find a more adequate interior solution that actually describes the Sun: Schwarzschild's second paper of 1916. Or: Embrace the idea by Rainich to represent bodies *only* by their exterior gravitational fields.

Application 2: A Schwarzschild Black Hole

- Even though the first application of the Schwarzschild metric was to represent the exterior field of an active star, we now know that the solution can also be used to describe the final state of a collapsed star: a black hole.
- In this context, it is more conceivable that we should take the entire coordinate range of the r coordinate to represent the astrophysical object and take the singularity at the center seriously.
- But still note: because of the event horizon at r=2m, the interior of the black hole is causally isolated from the rest of the universe. Thus, for astrophysical purposes, we can easily get along with characterizing the black hole by the exterior Schwarzschild solution.
- Indeed, the presence of an event horizon gives real juice to the Rainich approach of representing bodies only by their exterior fields.

Interpreting solutions by looking at their Newtonian counterparts. Case 1: Schwarzschild

The geodesics of the Schwarschild metric:

$$\dot{r}^2 + (1 - \frac{2m}{r})(\epsilon_0 + \frac{L^2}{r^2}) = E^2$$

Equations of motion around a central source in Newtonian theory:

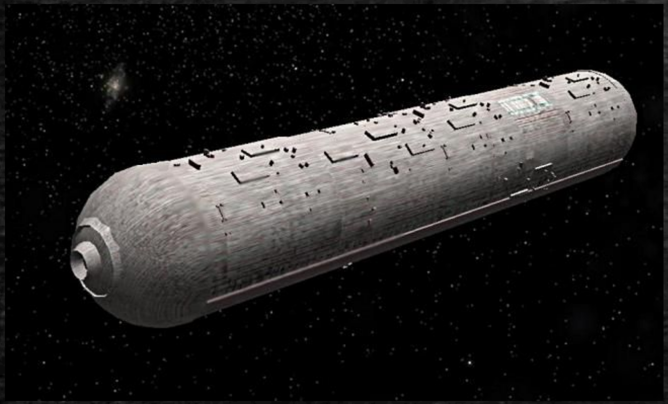
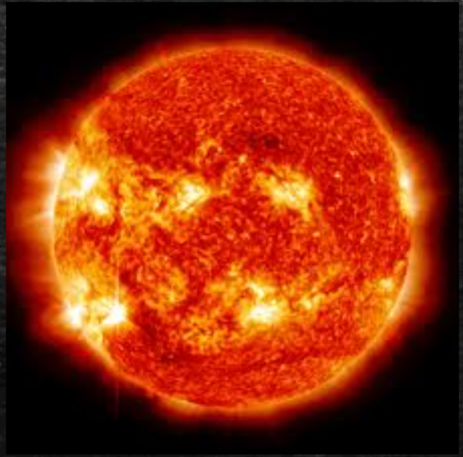
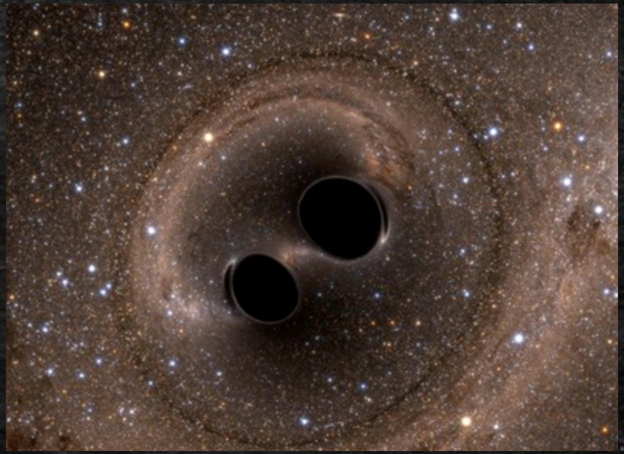
$$\dot{r}^2 + (1 - \frac{2m}{r} + \frac{L^2}{r^2}) = E^2$$

(Also: both the Komar mass and the ADM mass of a Schwarzschild spacetime turn out to be m)

Outline

1. Introduction: Different ways of interpreting solutions
2. The Schwarzschild solution: spherically symmetric, static
3. The Weyl class of solutions: axially symmetric, static
4. Einstein's reinterpretation of Weyl's results
5. Conclusion

1917: The Weyl class of solutions: axially symmetric, static

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi}[r^2 d\theta^2 + e^{2\gamma}(dr^2 + dz^2)]$$

where

$$\Delta\psi = \frac{1}{r}[\frac{\partial}{\partial z}(r\frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})] = 0$$

$$d\gamma = 2r\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}dz + r(\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial z^2})dr$$

Peculiarities of the Weyl class of solutions

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}[r^2 d\theta^2 + e^{2\gamma}(dr^2 + dz^2)]$$

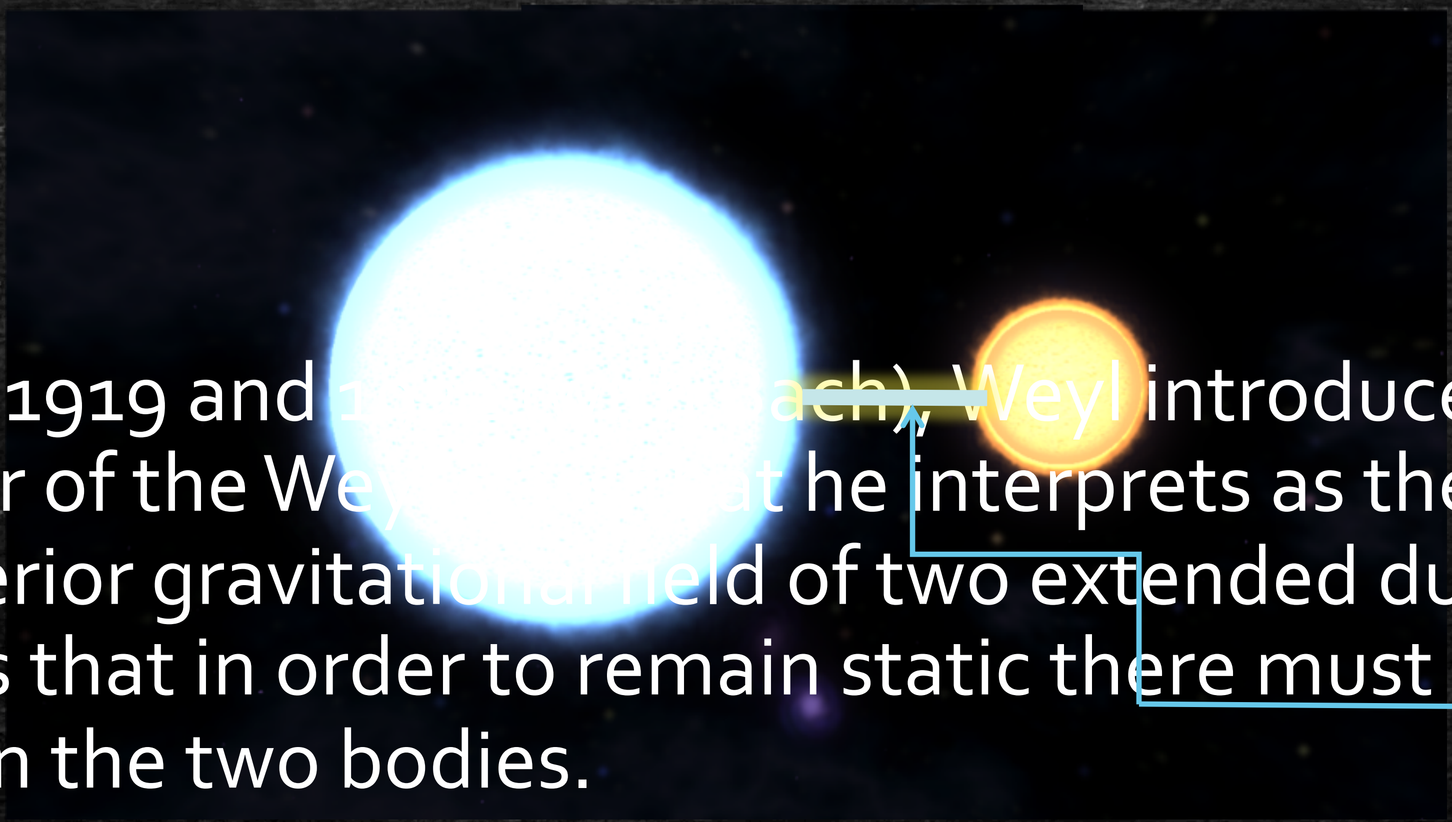
where

$$\Delta\psi = \frac{1}{r}[\frac{\partial}{\partial z}(r\frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})] = 0$$

$$d\gamma = 2r\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}dz + r(\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial z^2})dr$$

- The latter two equations are equivalent to the Einstein equations for axially symmetric fields.
- The second equation is Laplace’s equation, i.e. Poisson’s equation for vanishing mass density. It’s a linear equation. Link to Newtonian gravity?
- The third equation embodies the non-linearity.

Special Case: Weyl's static two-body solution



- In 1917, 1919 and 1 [illegible] ach), Weyl introduces a special member of the We [illegible] t he interprets as the static, exterior and interior gravitatio [illegible] eld of two extended dust bodies.
- He finds that in order to remain static there must be stresses between the two bodies.

"Weyl strut"

$$T^1_1 + T^2_2 = 0$$

Weyl makes clear that the introduction of a "Weyl strut" avoids a singularity along the rotation axis, for it ensures that $\gamma = 0$ along the axis.

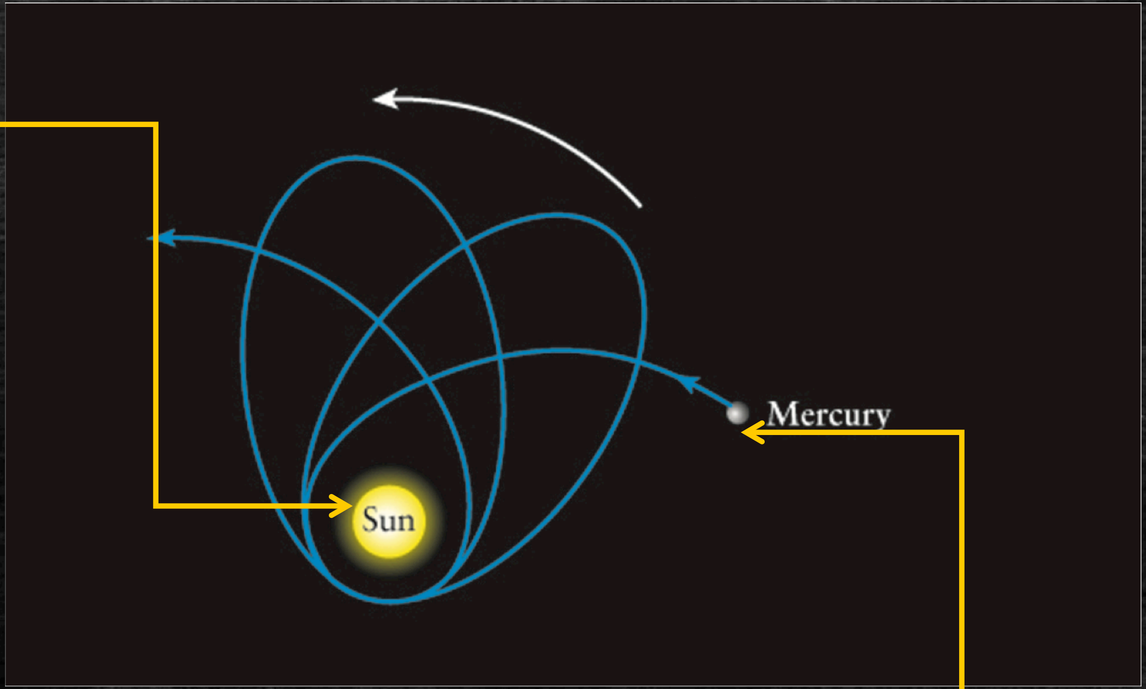
Outline

1. Introduction: Different ways of interpreting solutions
2. The Schwarzschild solution: spherically symmetric, static
3. The Weyl class of solutions: axially symmetric, static
4. Einstein’s reinterpretation of Weyl’s results
5. Conclusion

The context: Einstein's approach to the problem of motion

The Einstein field equations:

$$R_{\mu\nu} = 0$$

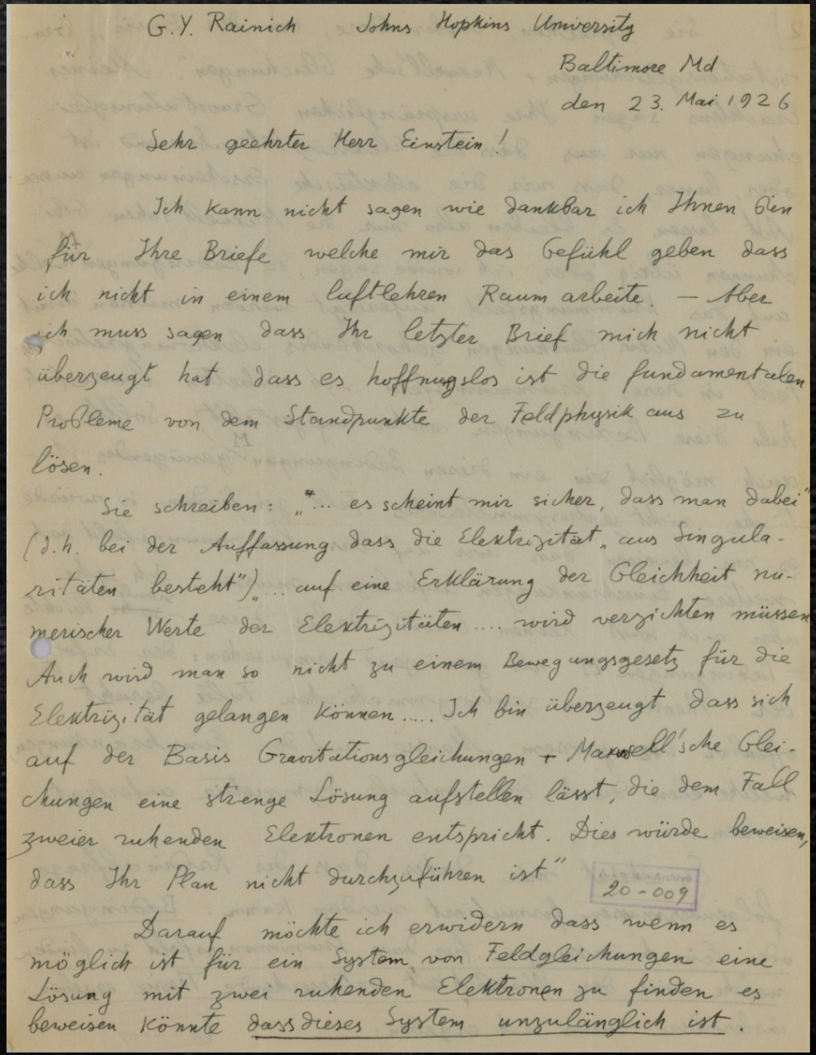


- BUT there is a price to pay: it seems matter is represented by singularities.

The geodesic equation:

$$\frac{d^2 x_\tau}{ds^2} + \Gamma^{\tau}{}_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0$$

Einstein’s reinterpretation of Weyl’s two-body solution during his Correspondence with Yuri Rainich, 1925-1926



G. Y. Rainich Johns Hopkins University

Baltimore Md

den 23. Mai 1926

Sehr geehrter Herr Einstein!

Ich kann nicht sagen wie dankbar ich Ihnen bin für Ihre Briefe welche mir das Gefühl geben dass ich nicht in einem luftleeren Raum arbeite. — Aber ich muss sagen dass Ihr letzter Brief mich nicht überzeugt hat dass es hoffnungslos ist die fundamentalen Probleme von dem Standpunkte der Feldphysik aus zu lösen.

Sie schreiben: „*... es scheint mir sicher, dass man dabei" (d.h. bei der Auffassung dass die Elektrizität „aus Singularitäten besteht") „... auf eine Erklärung der Gleichheit numerischer Werte der Elektrizitäten ... wird verzichten müssen. Auch wird man so nicht zu einem Bewegungsgesetz für die Elektrizität gelangen können..... Ich bin überzeugt dass sich auf der Basis Gravitationsgleichungen + Maxwell'sche Gleichungen eine strenge Lösung aufstellen lässt, die dem Fall zweier ruhenden Elektronen entspricht. Dies würde beweisen, dass Ihr Plan nicht durchzuführen ist"

A. Einstein Archive 20-009

Darauf möchte ich erwidern dass wenn es möglich ist für ein System von Feldgleichungen eine Lösung mit zwei ruhenden Elektronen zu finden es beweisen könnte dass dieses System unzulänglich ist.

Rainich on linear vs non-linear field equations

- Rainich pointed out to Einstein that in a *linear* theory the existence of a solution representing a static single-body solution would imply a static two-body solution.
- However, in a *non-linear* theory like GR, the existence of a two-body solution is not implied. In a letter to Einstein from 5 April 1926, Rainich adds that in contrast to a linear theory, in a non-linear theory the field of one particle may heavily constrain the properties the second particle can have.
- Rainich connects these remarks with his own research project: represent and investigate the behaviour of material bodies only in terms of their exterior gravitational fields.

Einstein on two-body solutions

``I am convinced that one could find an exact solution on the basis of the gravitational equations + Maxwell equations, which would represent the case of two electrons at rest (as singularities). For the case in which the particles in question have no electric charge this has already been shown by Weyl and Levi-Civita (special case of axial symmetry). This would show that your plan cannot be carried out.''

Einstein to Rainich, 18 April 1926.

Rainich insists

``I cannot tell you how grateful I am for your letters, which give me the feeling that I am not working in a vaccuum. - But I have to say that your last letter did not convince me… . […] " Rainich to Einstein, 23 May 1926.

- In what follows, Rainich insists on the points of his previous letter: it is not clear that GR admits a solution that should be interpreted as representing two particles (represented as singularities) at rest with respect to one another.

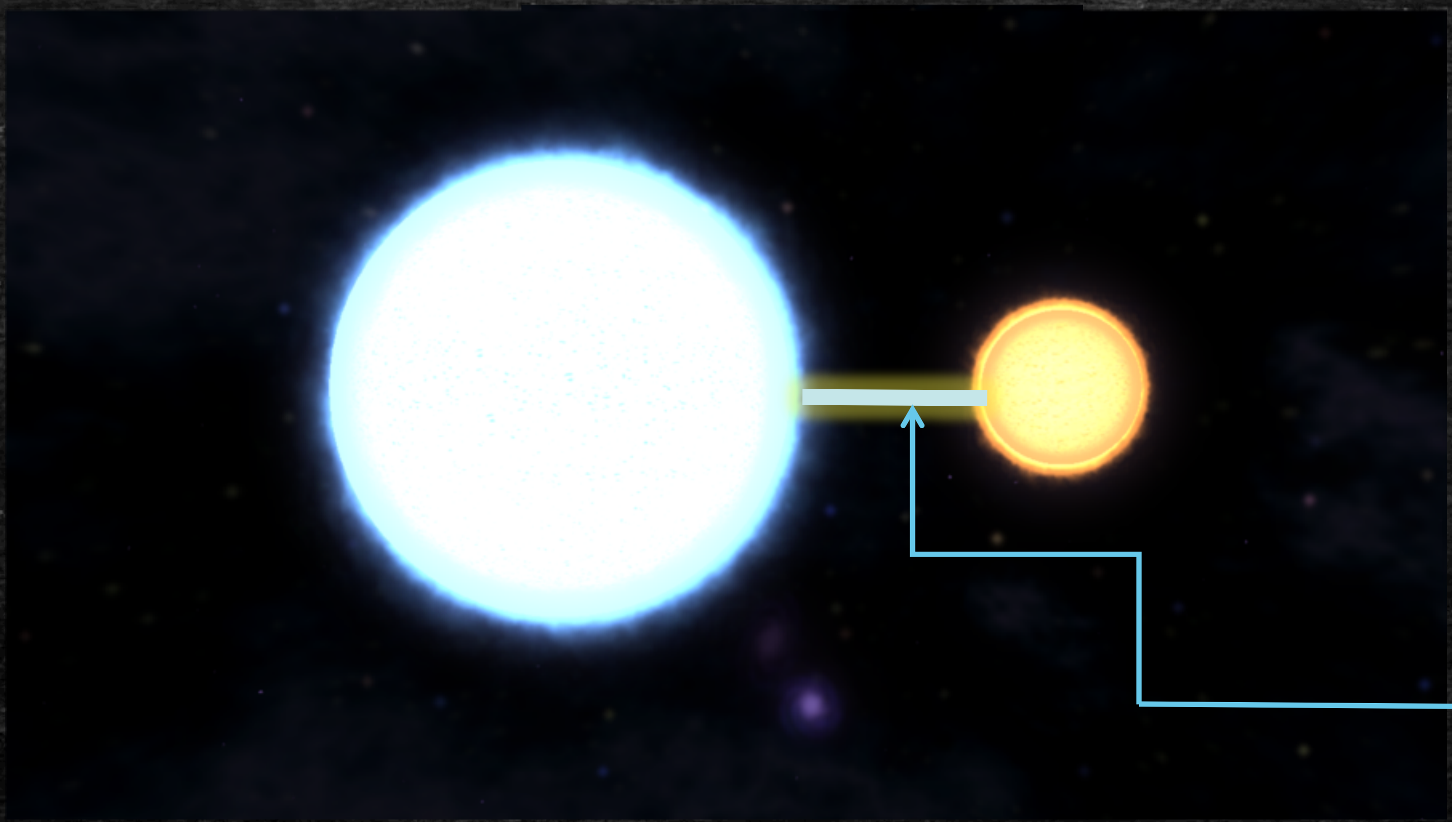
Einstein to Rainich: the U turn

``I completely agree with your main point. If a theory has a solution which represents two electrons *at rest*, then it is inadequate. This was indeed the reason why I thought that I had to reject a theory which regards electrons *as singularities*. For I had thought to have seen that any such theory would have solutions with electrons at rest. But it now seems that I was wrong about this." Einstein to Rainich, 6 June 1926 (emphasis in original).

Between 23 May and 6 June 1926

- On 18 April 1926, Einstein had pointed to Weyl and Levi-Civta's solutions as representing a static two-body solution. On 6 June 1926 he agrees with Rainich that a static two-body solution does not exist. What happened?
- I conjecture that between Rainich's letter of 23 May and Einstein's answer of 6 June, Einstein must have gone back to the papers by Levi-Civita and Weyl (and Bach) that he had referred to in his previous letter.
- He found reason to judge Wey's two-body solution as unsatisfactory, as a *non-physical* two-body solution.

Special Case: Weyl's static two-body solution

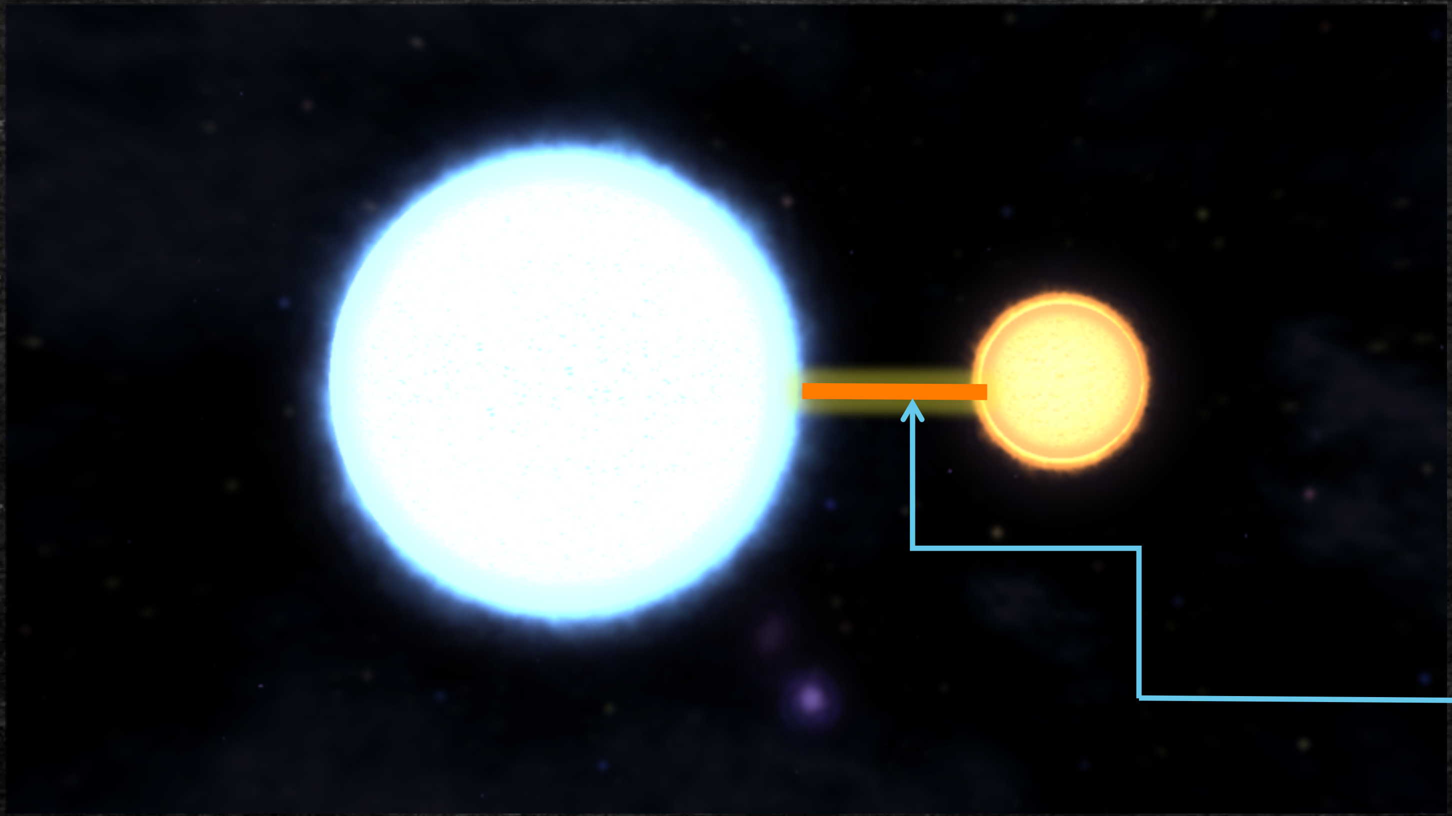


"Weyl strut"

$$T_1^1 + T_2^2 = 0$$

Weyl makes clear that the introduction of a "Weyl strut" is the only way to avoid a singularity along the rotation axis, for it ensures that $\gamma = 0$ along the axis.

Weyl’s static two-body solution without “Weyl strut”



Line singularity along the z-axis.

General Relativity as a hybrid theory → Good and bad singularities

- Einstein regarded general relativity as what I would call a hybrid theory:
  - fundamentally correct with regard to spacetime regions containing only gravitational fields, and
  - only phenomenologically correct with regard to spacetime regions in which matter is present. The energy-momentum tensor in GR was only a place-holder for an adequate (quantum) theory of matter not yet found.
- Thus, he was fine with introducing singularities to stand in for matter: it just meant switching one placeholder for another.
- But in spacetime regions free of matter no singularities where to be allowed.
- This implied a selection principle for physical vs. non-physical solutions.

In search for an acceptable solution

- Einstein now made two moves:
  1. He turned Weyl's two-body problem into the problem of finding an axially symmetric solution capable of representing one body subject to an external gravitational field.
  2. He chose a simpler ansatz: while Weyl aimed to find a solution capable of representing extended material bodies, Einstein wanted an axially symmetric solution capable of representing a point mass subject to an external field.

Interpreting solutions by looking at their Newtonian counterpart. Case 2: Weyl

- As we saw, the Weyl class of solution includes a Poisson-like equation:

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}[r^2 d\theta^2 + e^{2\gamma}(dr^2 + dz^2)]$$

where

$$\Delta\psi = \frac{1}{r}[\frac{\partial}{\partial z}(r\frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})] = 0$$

$$d\gamma = 2r\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}dz + r(\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial z^2})dr$$

- This suggest a solution-generating technique: start with the exact Newtonian potential ψ for some classical axially symmetric system in a flat space expressed in terms of standard cylindrical coordinates. Then...

Interpreting solutions by looking at their Newtonian counterpart. Case 2: Weyl

- This suggest a solution-generating technique: start with the exact Newtonian potential ψ for some classical axially symmetric system in a flat space expressed in terms of standard cylindrical coordinates.
- Plug ψ into the Laplace-like equation of the Weyl metric, and determine γ
- Together ψ and γ suffice to determine a particular axially symmetric solution, a specific member of the Weyl class of solutions.
- Interpret the solution as the gravitational field of the analogous Newtonian source.
- (This is what Einstein and Grommer did, as we will see in the following. Note, however, that the last step can be treacherous, as we will also see.)

From Newtonian point particle to the Curzon solution

Einstein and Grommer's Ansatz: $$\psi_1 = -\frac{m}{r^2 + z^2}$$

$$\Delta\psi = \frac{1}{r}[\frac{\partial}{\partial z}(r\frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})] = 0$$

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}[r^2 d\theta^2 + e^{2\gamma}(dr^2 + dz^2)]$$

From Curzon solution to a point particle subject to an external field

Einstein and Grommer's Ansatz:

$$\psi_{total} = \psi_1 + \hat{\psi}$$

with

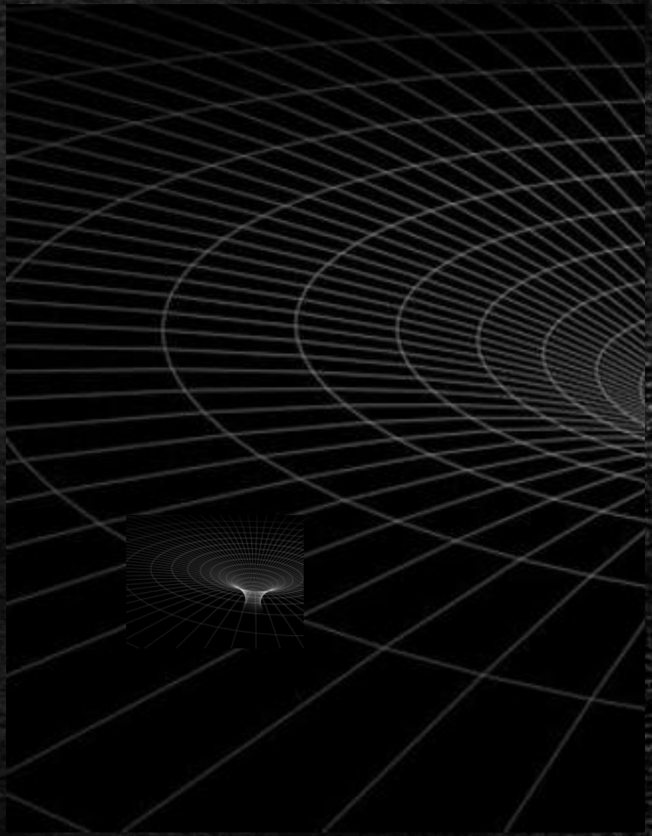
$$\psi_1 = -\frac{m}{r^2 + z^2}$$

- Like Weyl, Einstein and Grommer had argued that the only way to avoid a singularity along the rotation axis is to ensure that $\gamma = 0$ along the axis.
- They find that the only way to do this without introducing stresses is:

$$\text{No line singularity along z-axis} \iff \gamma = 0 \quad \text{when} \quad r \to 0 \iff \oint_{r \to 0} d\gamma = 0 \iff \hat{\psi} = 0$$

From two-body vaccuum solution to problem of motion

- Einstein and Grommer conclude that in the full, non-linear theory, there is no physical solution of a particle at rest but subject to an external gravitational field.
- Thus, they say, in GR it follows from the field equations that a particle cannot be at rest when subject to a gravitational field. (Big difference to Newtonian theory of gravity and Maxwellian theory of electrodynamics.)
- So the field equations predict whether a particle moves; they predict *that* it will move.
- From here it is only a small step to expect the field equations to determine *how* the particle will move.



➢ The problem of motion.

The perils of Newtonian starting points

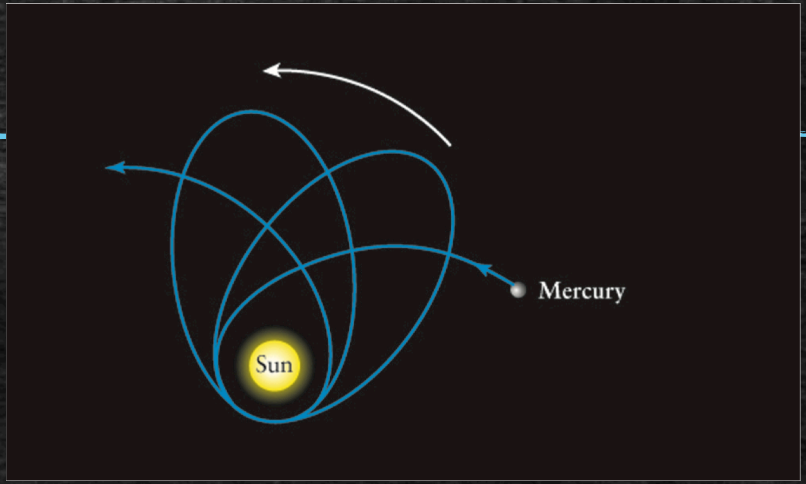
- Einstein and Grommer's ansatz was to start with the Newtonian potential of a point particle and plug it into the Weyl metric.
- They interpreted the resulting Curzon metric as representing the axisymmetric exterior gravitational field of a point particle according to GR.
- But the curvature singularity at the center has directional dependence and has the structure of a ring (Szekeres 1986).
- It is also a naked singularity; if we accept the cosmic censorship hypothesis as another selection principle, it should be dismissed as an unphysical solution on these grounds too.
- (Another point against trusting Newton: rewriting the Schwarzschild solution in Weyl coordinates gives us a ψ that is the Newtonian potential of a finite rod, rather than that of a spherical body.)

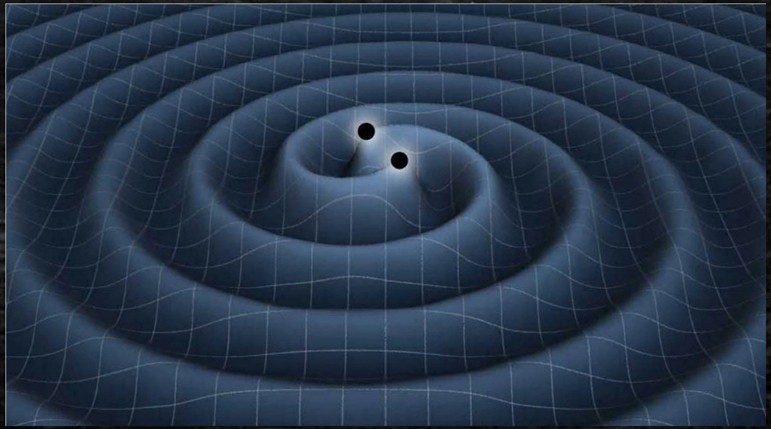
Summary

- I argued that in order to understand the different solutions to the Einstein equations, we should interpret how they can be used in practice to model actual systems in our universe.
- I described how historically both the Schwarzschild and the Weyl solutions were used for very different representational purposes, and have to be interpreted differently in different contexts.
- I showed that what Weyl took for an existence proof of static two-body solutions was repurposed as a non-existence proof of static bodies subject to exterior fields by Einstein.
- I showed that regularly solutions to GR are interpreted by appeal to their Newtonian counterparts, which *can* be misleading.

The four classical tests

1. Deriving the perhihelion of Mercury
2. Light bending by the Sun
3. Gravitational redshift
4. Gravitational waves



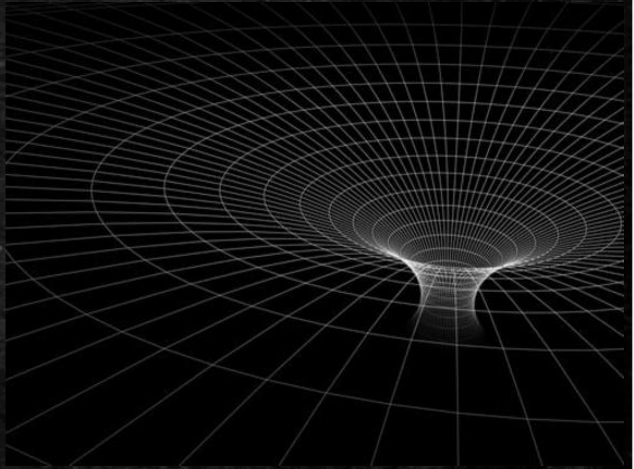


- Note: All four classical tests need only solutions to the vacuum Einstein equations.

Thank you!

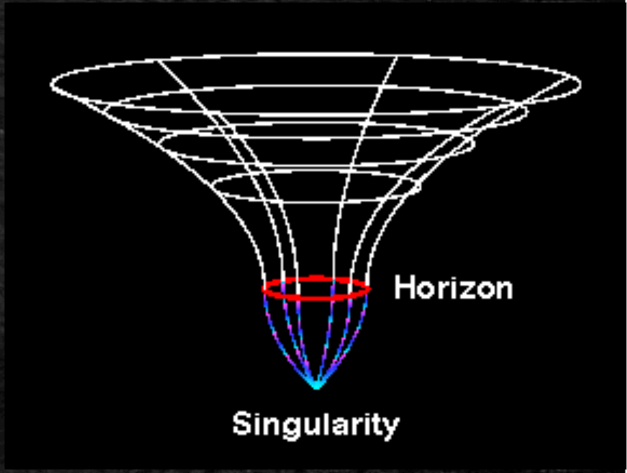
Thinking further along the Weyl-Rainich approach: Representing isolated bodies by vacuum spacetimes

- Ehlers (1979) suggested that for something to be “a model of an isolated system” in spacetime, the spacetime has to be asymptotically flat.
- This allows for vacuum spacetimes in which, as Thorne and Hartle (1985) put it, “one can separate spacetime into a part that represents the body and a part which represents the spacetime of the external universe”.
- Indeed, as we learned from Arnowitt-Deser-Misner (1960) and Bondi (1962), we can define mass, momentum and angular momentum for the isolated body represented by such a vacuum spacetime.
- But vacuum spacetimes need one more property in addition to be capable of representing material/astronomical bodies.



Note: Make sure that if your vacuum solution has a singularity, it's not a naked one

- Naked singularities threaten a breakdown of determinism but a non-naked singularity is "hidden" behind a black-hole event horizon: it is causally isolated from the exterior.
- If a singularity is non-naked, then for astrophysical purposes it does not really matter if it's there; a black hole is then just a very massive body.
- The Schwarzschild metric has a non-naked singularity at its center.



➢ In virtue of it being asymptotically flat and involving only non-naked singularities, we are able to represent an astronomical body like the Sun by the exterior Schwarzschild metric.