Pulsars as probes of Black Hole physics

Claus Lämmerzahl
with E. Hackmann
April 26, 2017

Do Black Holes really exist? - The physics and philosophy of Black Holes
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Universität Bremen*
*EXZELLENT.
Gewinnerin in der Exzellenzinitiative

CENTER OF
APPLIED SPACE TECHNOLOGY
AND MICROGRAVITY
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Outline

Introduction

- Motivation
- Some mathematics

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Relativistic orbits in axially symmetric space–times

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Orbits in "non–standard" space–times

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Motivation II

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

Jürgen Ehlers 2006

For classifying solutions in general it is usual to focus primarily on properties of the metric and not on the matter variables (which may even be absent). But sometimes it is of interest, not least since it is matter (including radiation) that is observed, to characterize solutions in terms of the properties of matter.

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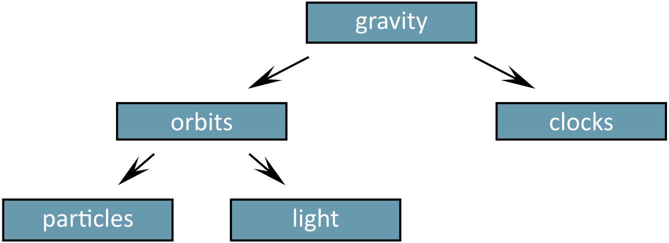

Motivation

Main questions

- How to proof the existence of Black Holes?
- What is a Black Hole?

Characteristics of a Black Hole

- Event horizon
- Singularity
- No hair / uniqueness

Black Hole foils

- Boson stars
- Planck stars
- gravastars
- Wormholes

- E. Berti et al.: Testing general relativity with present and future astrophysical observations (Topical Review), *Class. Quantum Grav.* **32**, 243001 (2015)
- L. Shao et al.: Advancing Astrophysics with the Square Kilometre Array, *Proceedings of Science*, PoS(AASKA14)042 (2015)

How to explore a Black Hole?

- particle orbits around Black Holes (stars, dust, accretion disk, ...)
  - point particles
  - particles with clocks
  - particles with structure (spin, mass multipoles)
  - continua (gas, fluid, plasma - viscosity)
- light effects (light defection, lensing, shadows, ...)
  - light rays
  - polarized light
  - waves
- merger, gravitational waves
  - structure of merger
  - structure of ring down

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Hyperelliptic problems

The mathematical problem

$$\left(\frac{du}{d\varphi}\right)^2 = P_n(u) \qquad \Leftrightarrow \qquad d\varphi = \frac{du}{\sqrt{P_n(u)}}$$

- $n = 1$: elementary: sin, cos
- $n = 3$: elliptic: $\wp$
- $n = 5$: hyperelliptic

$$u = -\frac{\sigma_1(\varphi, \varphi_1)}{\sigma_2(\varphi, \varphi_1)} \qquad \text{with} \qquad \sigma(\varphi, \varphi_1) = 0$$

- $n = 7$: hyperelliptic

$$u = -\frac{\sigma_{13}(\varphi_1, \varphi_2, \varphi_3)}{\sigma_{23}(\varphi_1, \varphi_2, \varphi_3)} \qquad \text{with} \quad \sigma(\varphi_1, \varphi_2, \varphi_3) = 0\,, \quad \sigma_3(\varphi_1, \varphi_2, \varphi_3) = 0.$$

Hyperelliptic problems

The mathematical problem

$$\left(\frac{du}{d\varphi}\right)^2 = P_n(u) \qquad \Leftrightarrow \qquad d\varphi = \frac{du}{\sqrt{P_n(u)}}$$

- arbitrary n: general hyperelliptic (Enolskii et al, JGP 2011)

$$u = -\frac{\dfrac{\partial^{M+1}}{\partial\varphi_1 \partial\varphi_g^M}\sigma}{\dfrac{\partial^{M+1}}{\partial\varphi_2 \partial\varphi_g^M}\sigma}$$

with

$$\varphi \in \Theta_1 := \left\{ u \in Jac(X_g) \mid \sigma(\varphi) = 0, \frac{\partial^j}{\partial\varphi_g^j}\sigma = 0, \forall j = 1, \dots, g-2 \right\}$$

Hyperelliptic problems

The general problem

$$\int R(x,y)dx = t \qquad \text{with} \qquad X_g : y^2 = 4x^{2g+1} + \lambda_{2g}x^{2g} + ... + \lambda_0$$

partial fraction decomposition

$$E(x) + \sum_{k=1}^{g} a_k \int du_k + \sum_{k=1}^{g} b_k \int dr_k + \sum_{k=1}^{g} c_k \int d\Omega_{\alpha_k,\beta_k} = t$$

(Enolskii et al, JMP 2012)

Why analytic methods?

- arbitrary accuracy
- complete set of solutions
- systematic study of the manifold of solutions
- test cases for numerical codes
- better start solution for new analytic approximation methods (post-Schwarzschild, post-Kerr, ...)
- clear definition of observables
- better understanding of effects
- better discussion of stability of solutions
- is a scientific value by itself

Analytic solutions of equations of motion

- Analytic solutions for geodesic equations in electrovac space–times
  - Schwarzschild (Hagihara, JJGA 1931)

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  - Kerr–Newman (Hackmann, Xu, PRD 2013)
  - Taub–NUT (Kagramanova, Kunz, Hackmann, C.L., PRD 2010)
  - Einstein-Maxwell-Dilaton-axion (Flathmann, Grunau, PRD 2015)
  - $f(R)$ Black Holes (Soroushfar, Saffani, Kunz, C.L., PRD 2015)
  - cylindrically symmetric conformal spacetime (Hoseini et al, PRD 2016)
  - Kerr-Newman-(A)dS (Sorousfar et al, PRD 2016)
  - $U(1)^2$ dyonic rotating black holes (Flathmann, Grunau, PRD 2016)

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  - Schwarzschild–de Sitter (Hackmann & C.L. PRL 2008, PRD 2008)

Analytic solutions of equations of motion

- Analytic solutions for geodesic equations in electrovac space–times
  - Spherically symmetric space–times in higher dimensions (Hackmann, Kagramanova, Kunz, C.L., PRD 2008, Enolskii et al, JGP 1011)
  - Plebański–Demiański (Hackmann, Kagramanova, Kunz, C.L., EPL 2009)
  - Kerr–de Sitter (Hackmann, Kagramanova, Kunz, C.L., PRD 2010)
  - Myers–Perry (Kagramanova, Reimers PRD 2012, PRD 2012)
  - higher dimensional black string space-time (Grunau, Karamanova, Kunz, C.L., PRD 2012, Grunau, Kagramanovs, Kunz, PRD 2012)
  - Hořava–Lifshitz (Enolskii et al, JMP 2012)
  - Ayon-Beato–Garcia regular black hole (Garcia, Hackmann, Kunz, C.L., Macias, 2015)

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- Analytic solutions for geodesic equations in nonvacuum space–times
  - Schwarzschild–string (Hackmann, Hartmann, C.L., Sirimachan, PRD 2010)
  - Kerr–string (Hackmann, Hartmann, C.L., Sirimachan, PRD 2010)
  - rotating black string (Grunau, Khamesra, PRD 2013)

Analytic solutions of equations of motion

- Further developments
  - analysis of observables (Hackmann, C.L., PRD 2013)
  - orbits of particles with spin (Hackmann, C.L., Obukhov, Pützfeld, Schaffer, PRD 2014)
  - analytic timing (Hackmann, C.L., Philipp, in prep.)
  - gravitomagnetic clock effect (Hackmann, C.L., Merkle, 2013)
  - quartic problems (Garcia, Hackmann, Kunz, C.L., Macias, 2015)

Outline

Plebański–Demiański space-time

stationary axially symmetric metric

$$ds^2 = \frac{\Delta_r}{p^2}(dt - A_\vartheta \, d\varphi)^2 - \frac{p^2}{\Delta_r} dr^2 - \frac{\Delta_\vartheta}{p^2} \sin^2\vartheta (a dt - A_r \, d\varphi)^2 - \frac{p^2}{\Delta_\vartheta} d\vartheta^2$$

where

$$\begin{aligned}
p^2 &= r^2 + (n - a\cos\vartheta)^2 \\
\Delta_\vartheta &= 1 + \tfrac{1}{3} a^2 \Lambda \cos^2\vartheta - \tfrac{4}{3} \Lambda a n \cos\vartheta \\
\Delta_r &= \left(1 - \tfrac{1}{3}\Lambda r^2\right)\left(r^2 + a^2\right) - 2Mr - n^2 + Q_e^2 + Q_m^2 - \Lambda n^2 \left(2r^2 + a^2 - n^2\right) \\
A_\vartheta &= a \sin^2\vartheta + 2n\cos\vartheta \\
A_r &= r^2 + a^2 + n^2
\end{aligned}$$

- $M =$ mass, $a =$ Kerr parameter, $\Lambda =$ cosmological constant, $n =$ NUT parameter, $Q_e =$ electric charge, $Q_m =$ magnetic charge,
- this metric contains all standard black hole space–times, Petrov Type D
- Plebański & Demiański, AP 1976; Griffiths & Podolski, IJMP 2006

Plebański–Demiański space-time

stationary axially symmetric metric

$$ds^2 = \frac{\Delta_r}{p^2}\left(dt - A_\vartheta \beta d\varphi\right)^2 - \frac{p^2}{\Delta_r} dr^2 - \frac{\Delta_\vartheta}{p^2} \sin^2\vartheta (a dt - A_r \beta d\varphi)^2 - \frac{p^2}{\Delta_\vartheta} d\vartheta^2$$

where

$$p^2 = r^2 + (n - a\cos\vartheta)^2$$
$$\Delta_\vartheta = 1 + \tfrac{1}{3}a^2\Lambda\cos^2\vartheta - \tfrac{4}{3}\Lambda a n\cos\vartheta$$
$$\Delta_r = \left(1 - \tfrac{1}{3}\Lambda r^2\right)\left(r^2 + a^2\right) - 2Mr - n^2 + Q_e^2 + Q_m^2 - \Lambda n^2\left(2r^2 + a^2 - n^2\right)$$
$$A_\vartheta = a\sin^2\vartheta + 2n\cos\vartheta$$
$$A_r = r^2 + a^2 + n^2$$

- $M =$ mass, $a =$ Kerr parameter, $\Lambda =$ cosmological constant, $n =$ NUT parameter, $Q_e =$ electric charge, $Q_m =$ magnetic charge, β deficit angle
- this metric contains all standard black hole space–times, Petrov Type D
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Conservation laws

There are two Killing vectors ∂_t and ∂_φ
$\Rightarrow$ two conservation laws

$$E := g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}$$
$$-L := g_{\varphi t}\dot{t} + g_{\varphi\varphi}\dot{\varphi}$$

or

$$E = \frac{\Delta_r}{p^2}(\dot{t} - A_\vartheta\dot{\varphi}) - a\frac{\Delta_\vartheta}{p^2}\sin^2\vartheta(a\dot{t} - A_r\dot{\varphi})$$
$$L = A_\vartheta\frac{\Delta_r}{p^2}(\dot{t} - A_\vartheta\dot{\varphi}) - A_r\frac{\Delta_\vartheta}{p^2}\sin^2\vartheta(a\dot{t} - A_r\dot{\varphi})\,,$$

this corresponds to

- energy
- angular momentum in z–direction

Solution of geodesic equation

geodesic equation

$$0 = \frac{d^2x^\mu}{ds^2} + \left\{ {\mu \atop \rho\sigma} \right\} \frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} \qquad\qquad g_{\mu\nu}u^\mu u^\nu = \epsilon$$

is equivalent to the Hamilton–Jacobi equation

$$2\frac{\partial S}{\partial s} = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\mu}$$

separation ansatz

$$S = \tfrac{1}{2}\epsilon s - Et + L\varphi + S_r(r) + S_\vartheta(\vartheta)$$

- insertion into Hamilton–Jacobi
- separation of r and ϑ equations
- separation constant $= k =$ Carter constant (Carter, PR 1968)
- introduction of Mino time τ through $d\tau = \rho^2 ds$ (Mino, PRD 2003)
- substitution $\xi = \cos\vartheta$
- renormalization: all quantities in units of r_S

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Solution of geodesic equation

$$\left(\frac{dr}{d\tau}\right)^2 = \left((r^2+a^2+n^2)E - aL\right)^2 - \Delta_r(\epsilon r^2 + k) \qquad =: R(r)$$

$$\left(\frac{d\xi}{d\tau}\right)^2 = \Delta_\xi(1-\xi^2)\left(k - \epsilon(n-a\xi)^2\right) - (L - A_\xi E)^2 \qquad =: \Theta(\xi)$$

$$\frac{d\varphi}{d\tau} = a\frac{(r^2+a^2+n^2)E - aL}{\Delta_r} + \frac{L - A_\xi E}{\Delta_\xi(1-\xi^2)} \qquad =: f(r) + g(\xi)$$

$$\frac{dt}{d\tau} = A_r\frac{(r^2+a^2+n^2)E - aL}{\Delta_r} + \frac{A_\xi\left(L - A_\xi E\right)}{\Delta_\xi(1-\xi^2)} \qquad =: h(r) + j(\xi)$$

analytic solution given by hyperelliptic functions

$$r(\tau) = \mp\frac{\sigma_2^{(r)}(\vec{x})}{\sigma_1^{(r)}(\vec{x})} + r_0 \qquad \text{with} \qquad \sigma^{(r)}(\vec{x}) = 0\,, \quad \vec{x} = \begin{pmatrix}\tau_1\\ \tau\end{pmatrix}$$

$$\xi(\tau) = \mp\frac{\sigma_2^{(\xi)}(\vec{y})}{\sigma_1^{(\xi)}(\vec{y})} + \xi_0 \qquad \text{with} \qquad \sigma^{(\xi)}(\vec{y}) = 0\,, \quad \vec{x} = \begin{pmatrix}\tau_1\\ \tau\end{pmatrix}$$

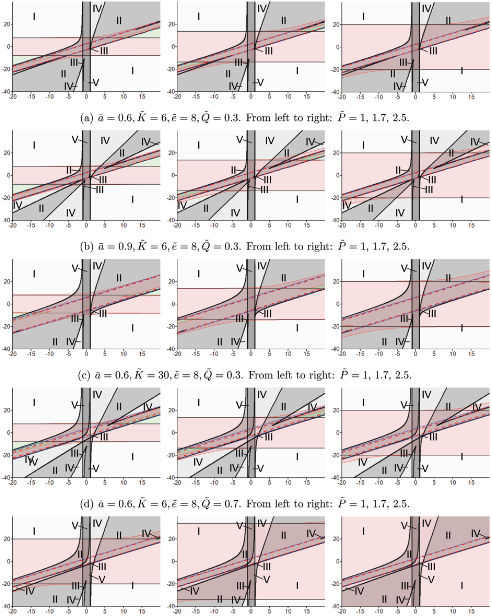
Solution of geodesic equation

integration of φ and t motion

$$\varphi - \varphi_0 = \int_{r_0}^{r(\tau)} f(r) \frac{dr}{\sqrt{R}} + \int_{\xi_0}^{\xi(\tau)} g(\xi) \frac{d\xi}{\sqrt{\Theta(\xi)}}$$
$$t - t_0 = \int_{r_0}^{r(\tau)} h(r) \frac{dr}{\sqrt{R}} + \int_{\xi_0}^{\xi(\tau)} j(\xi) \frac{d\xi}{\sqrt{\Theta(\xi)}}$$

f, g, h, and j are rational functions
(Hackmann, Kagramanova, Kunz, C.L., EPL 2009)

Kerr–Newman space–time



(a) $\bar{a} = 0.6, \tilde{K} = 6, \tilde{e} = 8, \tilde{Q} = 0.3$. From left to right: $\tilde{P} = 1, 1.7, 2.5$.

(b) $\bar{a} = 0.9, \tilde{K} = 6, \tilde{e} = 8, \tilde{Q} = 0.3$. From left to right: $\tilde{P} = 1, 1.7, 2.5$.

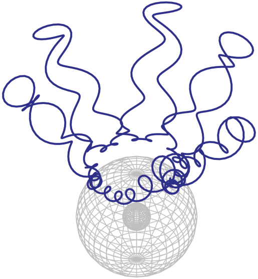
(c) $\bar{a} = 0.6, \tilde{K} = 30, \tilde{e} = 8, \tilde{Q} = 0.3$. From left to right: $\tilde{P} = 1, 1.7, 2.5$.

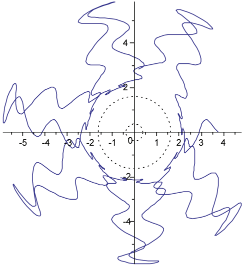
(d) $\bar{a} = 0.6, \tilde{K} = 6, \tilde{e} = 8, \tilde{Q} = 0.7$. From left to right: $\tilde{P} = 1, 1.7, 2.5$.

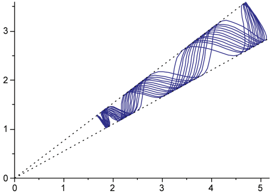
combined r and ϑ parameter plots for charged particle motion in Kerr–Newman space–time
Hackmann, Xu, PRD 2012

for Kerr: Hackmann, thesis 2009

Kerr–Newman with charged particle



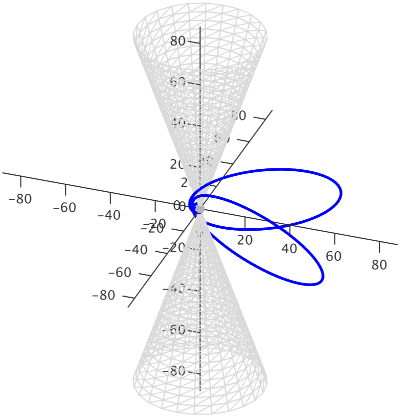




bound orbit

Hackmann & Xu, PRD 2013
Britzen, et al, Astron. Nachr. 2015

Kerr–de Sitter



bound orbit

escape orbit

Hackmann, Kagramanova, Kunz, C.L., PRD 2010

Taub–NUT space–time

metric

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - 2n\cos\vartheta d\varphi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

with

$$\rho^2 = r^2 + n^2\,, \qquad \Delta = r^2 - 2Mr - n^2$$

$n =$ magnetic mass

horizons

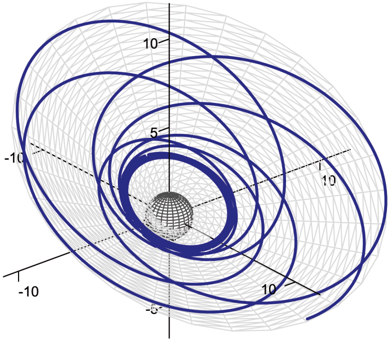
$$r_\pm = M \pm \sqrt{M^2 + n^2}$$

circle in equatorial plane

$$ds^2 = -\left(r^2 + n^2\right)d\varphi^2 \qquad \Rightarrow \qquad \text{circumference} = 2\pi\sqrt{r^2 + n^2}$$

Taub–NUT space–time

- orbits always lie on a cone
- orbits may proceed to negative r



bound orbit

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Taub–NUT space–time

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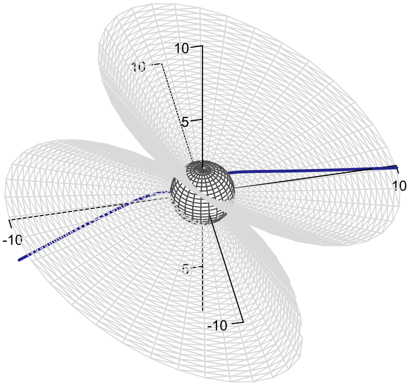
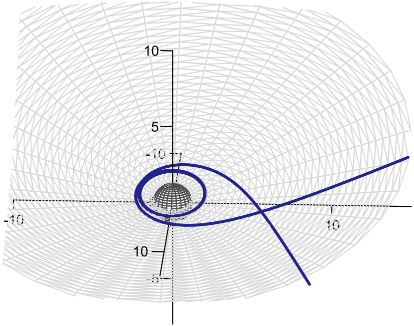


crossover transit orbit

Taub–NUT space–time

- orbits always lie on a cone
- orbits may proceed to negative r

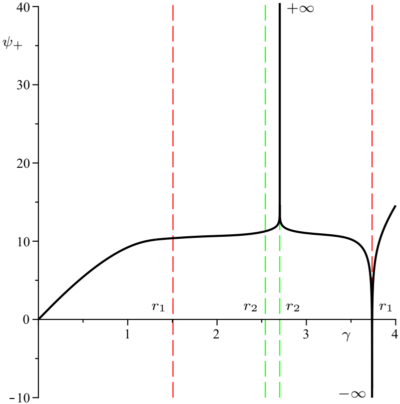


escape orbit

Taub–NUT space–time: incompleteness

- Taub–NUT space–times possess no curvature singularity
- but is geodesically incomplete ... during second transition through a horizon proper time terminates

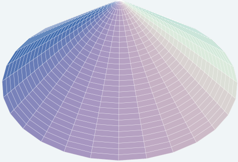
Hackmann, Kagramanova, Kunz, C.L. 2010



Schwarzschild pierced by string

Metric

conical Minkowski space–time



$$ds^2 = dt^2 - dr^2 - r^2\left(d\vartheta^2 + \beta^2 \sin^2\vartheta d\varphi^2\right)$$

conical Schwarzschild space–time

$$\begin{aligned} ds^2 &= g_{00}dt^2 - g_{rr}dr^2 - r^2 d\vartheta^2 - r^2\beta^2 \sin^2\vartheta d\varphi^2 \\ &= \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2\left(d\vartheta^2 + \beta^2 \sin^2\vartheta d\varphi^2\right) \end{aligned}$$

additional string of matter along z-axis

- geodesic equation looks similar to Schwarzschild
- φ–motion modified by β
- leads to additional perihelion shift, light deflection (implications for possible observations)

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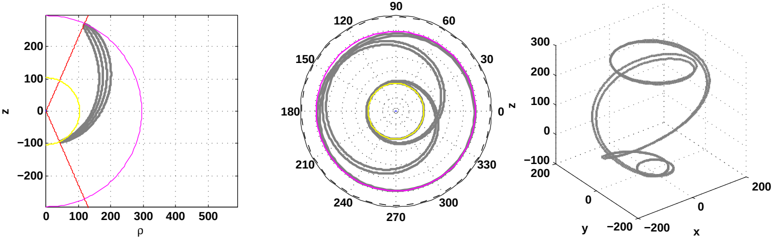
Schwarzschild pierced by string

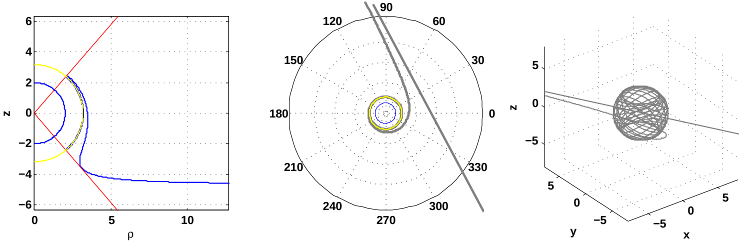
- motion in general does not remain in equatorial plane



bound orbit – Poincaré’s double circle limit

Schwarzschild pierced by string

- motion in general does not remain in equatorial plane



spiral escape orbit —tends to cover whole sphere

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Schwarzschild pierced by string

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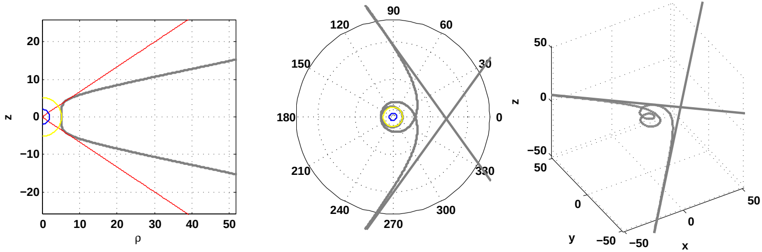

escape orbit

Schwarzschild with cosmic string: Hackmann, Hartmann, Sirimachan, C.L., PRD 2010

Kerr with cosmic string: Hackmann, Hartmann, Sirimachan, C.L., PRD 2010

Outline

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Reissner–Nordström

Standard spherically symmetric metric

$$ds^2 = g_{tt}dt^2 - g_{rr}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \qquad \text{with} \qquad g_{tt} = \frac{1}{g_{rr}}$$

Reissner–Nordström

$$g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

charge acts antigravitating

- 2, 1 degenerate, or no horizon
- singularity at $r = 0$

complete set of orbits for Reissner–Nordström: Grunau & Kagramanova, PRD 2011

question: can astrophysical Black Holes be charged? therare mechanisms which describe charging of Black Holes (embedded in plasma, with magnetic fieldd, ...)

Ayon-Beato–Garcia space–time

Standard spherically symmetric metric

$$ds^2 = g_{tt}dt^2 - g_{rr}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \qquad \text{with} \qquad g_{tt} = \frac{1}{g_{rr}}$$

Ayon-Beato–Garcia

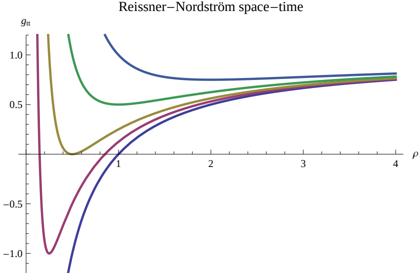
non–linear Maxwell can avoid singularity: regular black hole (Ayon-Beato & Garcia, PRL 1998)

$$g_{tt} = 1 - \frac{2Mr^2}{(r^2+Q^2)^{\frac{3}{2}}} + \frac{Q^2r^2}{(r^2+Q^2)^2}$$

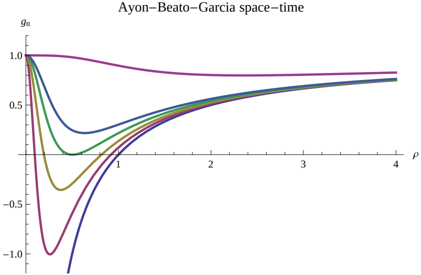
regular, Kretschmann scalar everywhere finite

Metrics

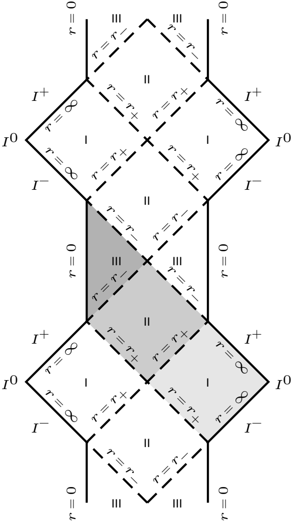
Reissner–Nordström



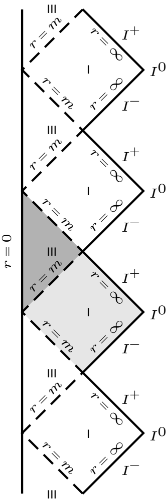
Ayon-Beato–Garcia



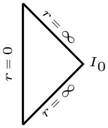
Carter–Penrose–diagram



Ayón-Beat–Garcia black hole space–time



Extremal Ayón-Beato–Garcia



No black hole

Geodesic equation

with

$$\rho = \frac{r}{2M}\,, \qquad \lambda = \frac{(2M)^2}{L^2} > 0\,, \qquad \kappa^2 = \frac{Q^2}{(2M)^2} > 0$$

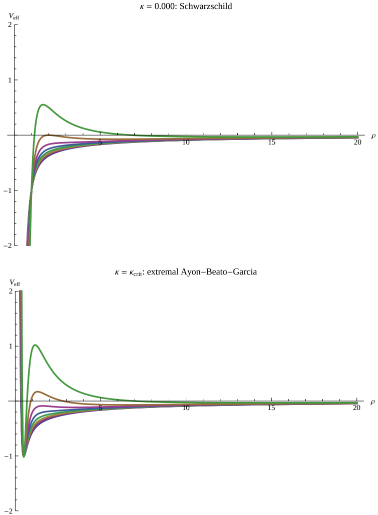
we have

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \lambda\rho^4 \left(E^2 - \left(1 - \frac{\rho^2}{(\rho^2+\kappa^2)^{\frac{3}{2}}} + \frac{\kappa^2\rho^2}{(\rho^2+\kappa^2)^2}\right)\left(\epsilon + \frac{1}{\lambda\rho^2}\right)\right)$$

effective potential

$$V_{\text{eff}} = g_{tt}\left(\epsilon + \frac{L^2}{r^2}\right) - 1 = \left(1 - \frac{\rho^2}{(\rho^2+\kappa^2)^{\frac{3}{2}}} + \frac{\kappa^2\rho^2}{(\rho^2+\kappa^2)^2}\right)\left(\epsilon + \frac{1}{\lambda\rho^2}\right) - 1$$

Effective potentials



Geodesic equation

introduce new variable

$$u = \frac{1}{\sqrt{\rho^2 + \kappa^2}} \qquad \rho^2 = \frac{1}{u^2} - \kappa^2$$

removes the square root

$$\left(\frac{1}{1 - \kappa^2 u^2} \frac{du}{d\varphi} \right)^2 = P_6(u)$$

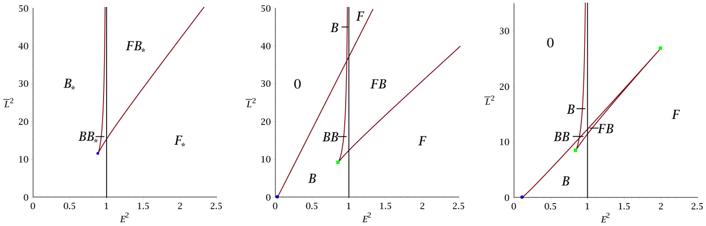
with the 6th order polynomial

$$\begin{aligned} P_6(u) \quad = \quad & \kappa^4 \left(1 - \epsilon\kappa^2\lambda\right) u^6 - \kappa^2 \left(1 - \epsilon\kappa^2\lambda\right) u^5 - \kappa^2 \left(1 - 2\epsilon\kappa^2\lambda\right) u^4 \\ & + \left(1 - 2\epsilon\kappa^2\lambda\right) u^3 - \left(1 + \kappa^2\lambda\mu\right) u^2 + \epsilon\lambda u + \lambda\left(\mu - \epsilon\right) \end{aligned}$$

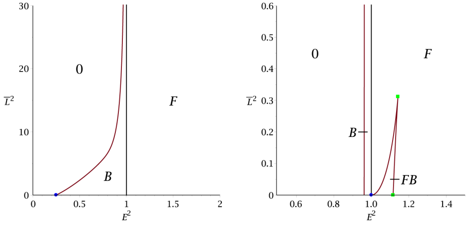
types of orbits $\leftrightarrow$ zeros of $P_6(u)$ $\leftrightarrow$ values of parameters E, L, Q
boundary of regions with different numbers of zeros

$$P_6(u) = 0 \qquad \text{and} \qquad \frac{dP_6(u)}{du} = 0$$

Types of orbits



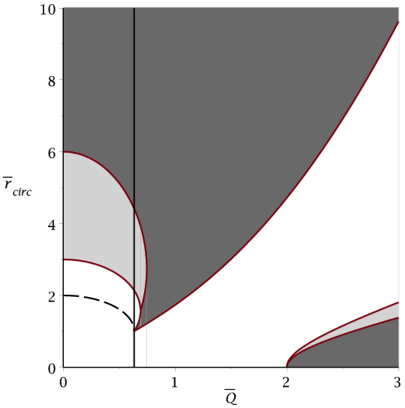
(a) $\bar{Q} = 0.3$ ($\bar{Q} < \bar{Q}_{\rm crit}$)
(b) $\bar{Q} = 0.65$ ($\bar{Q}_{\rm crit} < \bar{Q} < \bar{Q}_c$)
(c) $\bar{Q} = 0.7$ ($\bar{Q}_c < \bar{Q} < \bar{Q}_t$)



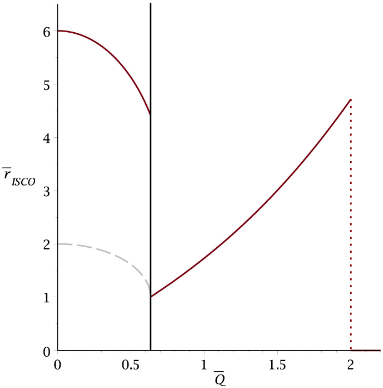
(d) $\bar{Q} = 0.8$ ($\bar{Q}_t < \bar{Q} < 2$)
(e) $\bar{Q} = 5$ ($2 < \bar{Q}$)

- characteristic charges $Q_{\rm crit}$, $Q_{\rm c}$, and $Q_{\rm t}$
- F flyby orbits
- B bound orbits
- * crossing both horizons
- blue dots: ISCO
- green squares: boundaries of regions of unstable circular orbits

Circular orbits



dark gray: stable circular orbits
light gray: unstable circular orbits
white: no circular orbits possible

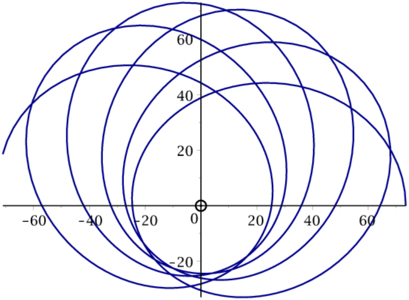


radial coordinate of innermost circular orbit

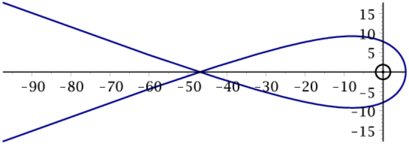
Orbits

$$\left(\frac{1}{1-\kappa^2u^2}\frac{du}{d\varphi}\right)^2 = P_6(u)$$ hyperelliptic integral of third kind

$\rightarrow$ can be solved analytically (Garcia, Hackmann, Kunz, C.L., Macias, 2013)



bound orbit with perihelion shift



flyby orbit

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Observable: Perihelion shift

analytic expression for Perihelion shift

$$\Omega_r = 2\int_{r_p}^{r_a} \frac{dr}{\sqrt{P_6(r)}} - 2\pi$$

expansion yields linearized Schwarzschild term + term proportional to Q^2

$$\Omega_{r,Q^2} = \frac{1}{p^{\frac{3}{2}}\sqrt{p+2e-6}}\Big(\left(3p^2-8p+e^2+3\right)K(k) + \frac{3p^3-24p^2+75p-7pe^2-12(1-e^2)}{2e-p+6}E(k)\Big)$$

K, E complete elliptic integrals of first and second kind
p semilatus rectum, e eccentricity
Reissner–Nordström

$$\Lambda_{r,Q^2} = \frac{1}{p^{\frac{1}{2}}\sqrt{p+2e-6}}\left((p-2)K(k) - \frac{p^2-6p-2e^2+18}{p-2e-6}E(k)\right)$$

(Garcia, Hackmann, Kunz, C.L., Macias, JMP 2015)

Equation of motion for charged particle

electrostatic potential

$$A_0 = -Q\frac{r^5}{(r^2+Q^2)^3} + \frac{3}{2}\frac{M}{Q}\frac{r^5}{(r^2+Q^2)^{\frac{5}{2}}}$$

with same substitution

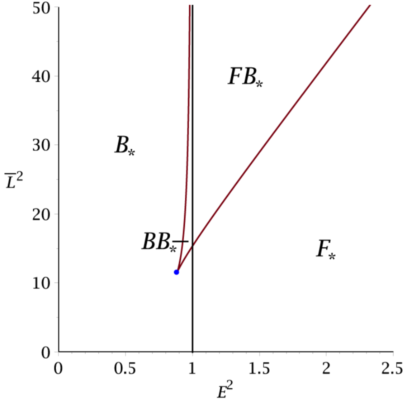
$$\left(\frac{1}{1-\kappa^2u^2}\frac{du}{d\varphi}\right)^2 = P_{14}(u) + 2E\lambda q\left(\kappa u - \frac{3}{4\kappa}\right)(1-\kappa^2u^2)^{\frac{7}{2}}$$

square root cannot be removed: squaring gives problem based on quartic algebraic curve

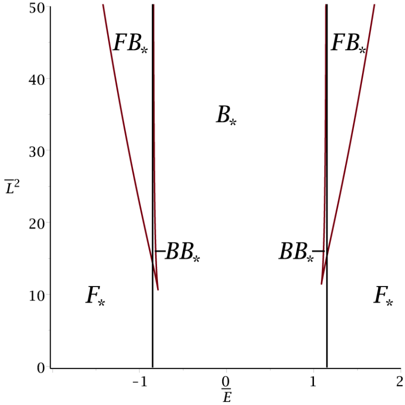
$$y^4 + P_n(x)y^2 + P_m(x) = 0 \qquad \text{with} \qquad y = \frac{1}{1-\kappa^2u^2}\frac{du}{d\varphi}$$

- parameter discussion possible (Garcia, Hackmann, Kunz, C.L., Macias, JMP 2015)
- analytic solution is under consideration ...

Types of orbits



neutral massive test body



small charge of massive test body

Hořava–Lifshitz space–time

metrics

$$ds^2 = N^2(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

HL black hole solution (Schwarzschild limit exists)

$$N^2 = f = 1 + c_1 r^2 - \sqrt{c_2 r^4 + c_3 r}$$

special case $c_2 = 0$:

$$N^2 = f = 1 + c_1 r^2 - \sqrt{c_3 r}$$

can make substitution $u = \sqrt{r}$ and again obtain hyperelliptic problem

$$\left(\frac{1}{u}\frac{du}{d\varphi}\right)^2 = P_n(u)$$

(hyperelliptic integral of second kind) with

- $n = 8$ for massive particles (Enolskii et al, JMP 2011)
- $n = 4$ for light (elliptic problem)

for $c_2 \neq 0$: quartic problem

Gauss–Bonnet gravity

Gauss–Bonnet gravity based on mdified Einstein–Hilbert action

$$\mathcal{L} = \sqrt{-g}\left(c_1 R + c_2\left(R^2 - 4R_{\mu\nu}r^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right)$$

spherically symmetric metric

$$g_{tt} = \frac{1}{g_{rr}} = 1 + \alpha r^2\left(1 \pm \sqrt{1 + \beta r^{1-d}}\right)$$

d = dimension of space–time

again gives a quartic problem

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Outline

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Observables: for bound orbits

For bound orbits

- two oscillatory coordinates: r and ϑ (generalized Lissajous figures)
- two (secularly) increasing coordinates: t and φ

Periods

- radial period

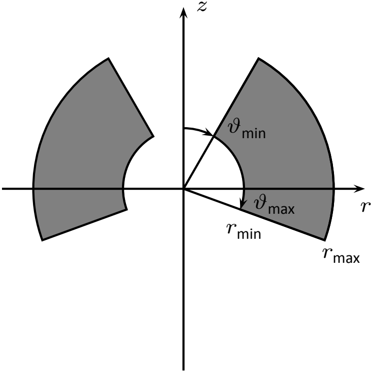
$$\omega_r = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{R}}$$

is time needed to go from r_{min} to r_{min}

- polar angle period

$$\omega_\vartheta = 2\int_{\vartheta_{min}}^{\vartheta_{\text{max}}} \frac{dr}{\sqrt{\Theta}}$$

is time needed to go from ϑ_{min} to ϑ_{max}



Observables: for bound orbits

Secular increases

- secular time increase

$$\Gamma = \left\langle \frac{dt}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\text{min}}}^{r_{\text{max}}} h(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\text{min}}}^{\vartheta_{\text{max}}} j(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

- secular azimuthal increase

$$Y = \left\langle \frac{d\varphi}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\text{min}}}^{r_{\text{max}}} f(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\text{min}}}^{\vartheta_{\text{max}}} g(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

orbital frequencies (Drasco & Hughes, PRD 2004; Schmidt, CQG 2004)

$$\Omega_r = \frac{2\pi}{\Gamma \omega_r}\,, \qquad \Omega_\vartheta = \frac{2\pi}{\Gamma \omega_\vartheta}\,, \qquad \Omega_\varphi = \frac{Y}{\Gamma}$$

- angular velocity of r–oscillations
- angular velocity of ϑ–oscillations
- secular angular velocity

Observables: for bound orbits

observables: self referential comparison, invariant

The observables

- **periastron shift**

$$\Omega_{\text{periastron}} := \Omega_\varphi - \Omega_r = \left(Y - \frac{2\pi}{\omega_r} \right) \frac{1}{\Gamma}$$

- **Lense–Thirring effect**

$$\Omega_{\text{Lense-Thirring}} := \Omega_\varphi - \Omega_\vartheta = \left(Y - \frac{2\pi}{\omega_\vartheta} \right) \frac{1}{\Gamma}$$

- **Conicity** of orbit: $\Delta_{\text{equator}} = \pi - (\vartheta_{\text{max}} + \vartheta_{\text{min}})$

- $\Omega_{\text{periastron}}$ compares the φ–advance for r_{min} with 2π
  $\rightarrow$ in weak field motion of r_{min} within orbital plane or orbital cone
- $\Omega_{\text{Lense-Thirring}}$ compares the φ–advance for ϑ_{min} with 2π
  $\rightarrow$ in weak field precession of orbital plane or orbital cone

explicit evaluation by hyperelliptic integrals (Hackmann & C.L. PRD 2012)

Linear effect due to a

Post-Schwarzschild

$$\Omega_{\mathsf{LT},a} = \frac{1}{Z}\left(2\frac{r_{04}}{r_{04}-2}\Pi(n_2,k) - 2K(k)\right)$$

Post-Newton

- $$\Omega_{\mathsf{P},a} \approx 2\frac{1-6\cos i}{d^3(1-\epsilon^2)^{\frac{3}{2}}}aM^2$$

  $i =$ inclination
- equatorial plane:
  perturbation against the direction of rotation
- test of

$$\chi = \frac{S}{M^2} \leq 1$$

Linear effects due to n: Conicity

Linear effects due to n

- Post-Schwarzschild: $\Delta_{\text{cone}} \approx \frac{4EL}{|L|\sqrt{C}} n$

$\rightarrow$ no motion in equatorial plane possible,
motion on cone with opening angle $\pi - \frac{4E}{\sqrt{C}n} + \mathcal{O}(n^2)$

- Post-Newton: $\Delta_{\text{cone}} \approx \frac{4L}{|L|\sqrt{d(1-\epsilon^2)}} nM^{\frac{1}{2}}$

Estimate of n from Solar system data

- for the Sun, using orbital data of Mercury: $|\Delta_{\text{cone}}| \leq 4.2$ arcsec
- Yields $|n| \leq 0.032$ (Hackmann & C.L., PRD 2013)

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Series expansion of observables

For bound orbits in Plebański–Demiański space–times

- Schwarzschild–de Sitter: fixed orbital plane

$$\Delta_{\text{perihelion}} \neq 0\,, \qquad \Delta_{\text{Lense-Thirring}} = 0\,, \qquad \Delta_{\text{equator}} = 0$$

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$$\Delta_{\text{perihelion}} \neq 0\,, \qquad \Delta_{\text{Lense-Thirring}} = 0\,, \qquad \Delta_{\text{equator}} \neq 0$$

Series expansion of observables

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- Kerr–de Sitter: precession of orbital plane, perihelion shift(!)

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- Kerr–NUT–de Sitter: precession of orbital cone

$$\Delta_{\text{perihelion}} \neq 0\,, \qquad \Delta_{\text{Lense-Thirring}} \neq 0\,, \qquad \Delta_{\text{equator}} \neq 0$$

- In general, for more complicated potentials there are several periods
  $\Rightarrow$ many perihelion shifts or Lense–Thirring effects

Outline

Dynamics of spinning particles

Mathisson-Papapetrou-equations

$$\begin{aligned} D_u p_\mu &= \frac{1}{2} R_{\mu\nu\rho\sigma} u^\nu S^{\rho\sigma}, \\ D_u S^{\mu\nu} &= 2p^{[\mu} u^{\nu]} \end{aligned}$$

then also

$$p^\mu = \bar{m} u^\mu + \frac{DS^{\mu\nu}}{ds} u_\nu \qquad \text{with} \qquad \bar{m} = p_\mu u^\mu$$

one needs a supplementary condition

- Tulczyjew

$$p_\mu S^{\mu\nu} = 0$$

- Frenkel

$$u_\mu S^{\mu\nu} = 0$$

in both cases

$$S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = const.$$

Spinning particles

for Tulczyjev

$$u^\mu = f^\mu(p, S, R)$$

for a Killing vector ξ

$$E_\xi = p_\mu \xi^\mu + \frac{1}{2} S^{\mu\nu} D_\mu \xi_\nu$$

in Kerr we have 2 such conserved quantities, together 4 constants of motion

$$S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} \qquad E = p_\mu \overset{t}{\xi}{}^\mu + \frac{1}{2} S^{\mu\nu} D_\mu \overset{t}{\xi}{}_\nu$$

$$m^2 = p_\mu p^\mu \qquad -J = p_\mu \overset{\varphi}{\xi}{}^\mu + \frac{1}{2} S^{\mu\nu} D_\mu \overset{\varphi}{\xi}{}_\nu$$

can eliminate spin components for equatorial orbits with polar spin

$$S^{rt} = -\frac{S p_\varphi}{mr}\,, \qquad S^{\varphi t} = \frac{S p_r}{mr}\,, \qquad S^{\varphi r} = -\frac{S p_t}{mr}$$

and

Spinning particles

$$p_t = \frac{E - \frac{MS}{mr^3}(J - aE)}{1 - \frac{MS^2}{m^2 r^3}}$$

$$p_\varphi = \frac{-J - \frac{aMS}{mr^3}\left[aE\left(1 - \frac{r^3}{a^2 M}\right) - J\right]}{1 - \frac{MS^2}{m^2 r^3}}$$

from that one can derive he 4 velocities

$$u^\mu = u^\mu(m, S, E, J, a, M)$$

and from that

$$\frac{dr}{d\varphi} = \frac{u^r}{u^\varphi} = \frac{\Delta(r^3 + S^2)}{rQ}\sqrt{P}$$

where P is a polynomial in r of order 8, and Q a polynomial of order 6
... analytic solution ...

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Outline

Summary

Summary: mathematics

- hyperelliptic problems completely solved
- more general problems (e.g. quartic) are under consideration

Summary 1: physics

- complete analytic solution of geodesic equation in Plebański–Demiański space–times = all electro–vac space-times for which Hamilton–Jacobi separates, and further space-times
- definition and calculation of observables
- tests of nonstandard GR

Further applications

Summary 2: physics

- point particles clock effects ($\rightarrow$ talk by Eva Hackmann)
- light rays ($\rightarrow$ talk by Volker Perlick, Thomas Müller)
- spinning particles
- effective one-body problem (Buonnano, Damour, Schäfer)
- bumpy black holes, chaotic motion (Lukes-Gerakopoulos)

Summary: philosophy

- eleatic principle ($\rightarrow$ talk by Andreas Eckart)
- ???

Thank you for your attention

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- DFG Collaborative Research Center ”Relativistic Geodesy” *geo*-Q
- DFG Collaborative Research Center ”Designed Quantum States of Matter” DQ-*mat*
- German Research Foundation DFG
- German Space Agency DLR
- Center of Excellence QUEST
- ERASMUS MUNDUS
- IRAP-PhD
- German Israeli Foundation