Black Holes in Quantum Gravity

Claus Kiefer

Institut für Theoretische Physik
Universität zu Köln

Contents

Black holes and thermodynamics

Table: Analogies between the laws of thermodynamics and the laws of black-hole mechanics

Law	Thermodynamics	Stationary black holes
Zeroth	T constant on a body in thermal equilibrium	κ constant on the horizon of a black hole
First	$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$	$\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \Omega_{\mathrm{H}}\mathrm{d}J + \Phi\mathrm{d}q$
Second	$\mathrm{d}S \geq 0$	$\mathrm{d}A \geq 0$
Third	$T = 0$ cannot be reached	$\kappa = 0$ cannot be reached

κ is the surface gravity of the black hole.

Compare $\mathrm{d}E = T\mathrm{d}S$ with $\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A$. Write

$$T = \frac{\kappa}{G\zeta}, \quad S = \frac{\zeta A}{8\pi}.$$

For dimensional reasons, k_{B}/ζ must have the dimension of a length squared. A universal length is not available in the classical theory, but if $\hbar$ is taken into account, one can use the Planck length

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33}\ \mathrm{cm}.$$

Hawking radiation and black-hole entropy

- Black holes radiate with a temperature proportional to $\hbar$, the "Hawking temperature" (Stephen Hawking 1974)

$$\boxed{T_{\mathrm{BH}} = \frac{\hbar c^3}{8\pi G k_{\mathrm{B}} M} \approx 6.2 \times 10^{-8} \frac{M_\odot}{M}\ \mathrm{K}}$$

- They therefore have a finite lifetime: the black hole Cygnus X-1, for example, evaporates after 10^{68} years, which is about 10^{58} times the age of the Universe!

Entropy of black holes

With the above result for the Hawking temperature, one finds the following expression for the black-hole entropy:

$$S_{\mathrm{BH}} = k_{\mathrm{B}} \frac{Ac^3}{4\hbar G} \overset{\text{Schwarzschild}}{\approx} 1.07 \times 10^{77} k_{\mathrm{B}} \left(\frac{M}{M_\odot}\right)^2$$

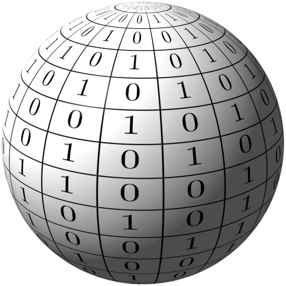
“Bekenstein–Hawking entropy”

For the collapse of a solar-mass star, this corresponds to an increase in entropy of about 20 orders of magnitude!

Main open problems

- Final evaporation phase
- Fate of black-hole singularity in quantum gravity
- Microscopic derivation of black-hole entropy
- Astrophysical relevance (primordial black holes)

Microscopic explanation of S_{BH}?



Cf. John Wheeler's “It from Bit”

$$S_{\mathrm{BH}} = -k_{\mathrm{B}} \mathrm{tr}\,(\rho \ln \rho)$$

Quantum gravity?

Black-hole spectroscopy

Bekenstein and Mukhanov (1995) assume a quantization condition for the area:

$$A_N = \alpha l_{\mathrm{P}}^2 N$$

with some undetermined constant α. The energy level N will be degenerate with multiplicity $g(N)$, so one would expect

$$S = \frac{A}{4l_{\mathrm{P}}^2} + \text{ constant } = \ln g(n).$$

With $g(1) = 1$ one gets

$$g(n) = \mathrm{e}^{\alpha(n-1)/4} \ .$$

Since this must be an integer, one has the options

$$\alpha = 4 \ln k \ , \quad k = 2, 3, \ldots$$

For information-theoretic reasons ('it from bit'), $k = 2$ seems to be preferred.

Logarithmic corrections

N spin-1/2 particles out of which n point up and $N - n$ point down:

$$\downarrow \uparrow \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \uparrow$$

$$S = \ln \begin{pmatrix} N \\ N - n \end{pmatrix} = \ln \begin{pmatrix} N \\ n \end{pmatrix}$$

For the 'equilibrium case' $n = N/2$, using Stirling's formula, one gets, neglecting terms of order $1/N$,

$$S = N \ln 2 - \frac{1}{2} \ln N + \frac{1}{2} \ln \frac{2}{\pi}$$

With $S_0 := N \ln 2$, one can write

$$S \approx S_0 - \frac{1}{2} \ln S_0$$

In the Bekenstein–Mukhanov model, we have

$$A_N = (4 \ln k) l_{\mathrm{P}}^2 N$$

For $k = 2$ ('it from bit') and using the spin model from above, one gets

$$S = \frac{A_N}{4l_{\mathrm{P}}^2} - \frac{1}{2} \ln \frac{A_N}{4l_{\mathrm{P}}^2} + \frac{1}{2} \ln \frac{2}{\pi} + \frac{1}{2} \ln(\ln 2)$$

(Loop quantum gravity predicts the same logarithmic correction term.)

Except for very small black holes, this yields almost the same result as the exact expression

$$S = \ln \frac{\left(\frac{A_N}{4l_{\mathrm{P}}^2 \ln 2}\right)!}{\left[\left(\frac{A_N}{8l_{\mathrm{P}}^2 \ln 2}\right)!\right]^2} .$$

C. K. and G. Kolland (2008)

Information-loss problem

- Black holes have a finite lifetime:

$$\tau_{\mathrm{BH}} \approx 8895 \left(\frac{M_0}{m_{\mathrm{P}}}\right)^3 t_{\mathrm{P}} \approx 1.159 \times 10^{67} \left(\frac{M_0}{M_\odot}\right)^3 \mathrm{yr}$$

  from the emission of gravitons and photons (D. Page 1976)
- The semiclassical approximation breaks down if the black hole approaches the Planck mass m_{P}.
- If the black hole left only thermal radiation behind, a pure state for a closed system would evolve into a mixed system (information-loss problem)
- This would be in contradiction to ordinary quantum theory where the entropy

$$S = -k_{\mathrm{B}} \mathrm{Tr}(\rho \ln \rho)$$

  is conserved for a closed system (unitary evolution); the problem would more properly be called the “unitarity problem”.

Options

- Information is lost during the evaporation,

$$\rho \to \$\rho \neq S\rho S^\dagger$$

  (Hawking's original opinion (1976))
- The full evolution is unitary, but this cannot be seen in the semiclassical approximation (now the most popular option)
- The black hole leaves a remnant carrying all the information

Final answer only within quantum gravity!

- At no point in the calculation by Hawking (and others) is an exact mixed (canonical) state used in the formalism.
- The coherent superposition used by Hawking is indistinguishable from a local thermal mixture (L. Parker 1975).
- The reduced state of each mode in a two-mode squeezed state is a thermal state (canonical ensemble); in the special case of a black hole, the temperature is independent of k (universality).

- Squeezed states are very sensitive to decoherence.
- This sensitivity is responsible, for example, for the quantum-to-classical transition for the primordial fluctuations in the early Universe. (During inflation, the Wigner ellipse becomes frozen.)
- The degree of decoherence can conveniently be studied with the Wigner function (C.K. 2001)

Main Approaches to Quantum Gravity

No question about quantum gravity is more difficult than the question, "What is the question?"
(John Wheeler 1984)

- Quantum general relativity
  - Covariant approaches (perturbation theory, path integrals, spin foam, ...)
  - Canonical approaches (geometrodynamics, connection dynamics, loop dynamics, ...)
- String theory
- Other approaches
  (Causal sets, group field theory, ...)

Approach used here: Canonical quantum geometrodynamics

(For more details on all approaches, see e.g. C.K., *Quantum Gravity*, 3rd ed., Oxford 2012)

Black-hole entropy and quantum gravity

Microscopic explanation of entropy?

$$\boxed{S_{\mathrm{BH}} = k_{\mathrm{B}}\frac{A}{4l_{\mathrm{P}}^2}}$$

- *Loop quantum gravity*: microscopic degrees of freedom are the spin networks; S_{BH} only follows under certain assumptions
- *String theory*: microscopic degrees of freedom are the "D-branes"; S_{BH} only follows for special (extremal or near-extremal) black holes
- *Quantum geometrodynamics*: one can find $S \propto A$ in particular models

Singularity avoidance: an exact model

- Spherically-symmetric thin shell consisting of particles with zero rest mass (“null dust shell”);
- Classical theory: collapse to a black hole, or expansion from a white hole (usually excluded for thermodynamical reasons)
- Our quantization will lead to a singularity-free quantum state (“superposition of black and white hole”)

(P. Hájíček and C.K. 2001)

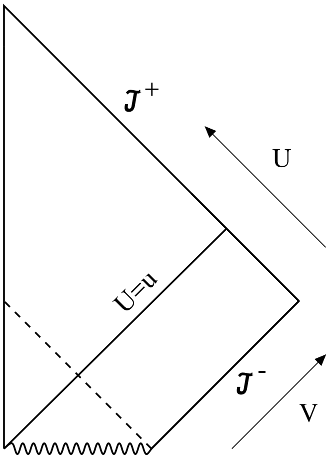


Figure: Penrose diagram for the outgoing shell in the classical theory. The shell is at $U = u$.

Wave packets

Represent the shell by a narrow wave packet; start at $t = 0$ with

$$\psi_{\kappa\lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} p^{\kappa+1/2} \mathrm{e}^{-\lambda p}$$

Expectation value for the energy and variance:

$$\langle E \rangle_{\kappa\lambda} := \int_0^\infty \frac{\mathrm{d}p}{p} \, p \psi^2_{\kappa\lambda}(p) = \frac{\kappa + 1/2}{\lambda},$$

$$\Delta E_{\kappa\lambda} = \frac{\sqrt{2\kappa + 1}}{2\lambda}$$

Since the time evolution of the packet is generated by $-\hat{p}_t$, one has

$$\psi_{\kappa\lambda}(t, p) = \psi_{\kappa\lambda}(p) \mathrm{e}^{-\mathrm{i}pt}$$

Exact time evolution in the r-representation:

$$\Psi_{\kappa\lambda}(t,r) = \frac{1}{\sqrt{2\pi}}\frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}}\left[\frac{\mathrm{i}}{(\lambda+\mathrm{i}t+\mathrm{i}r)^{\kappa+1}} - \frac{\mathrm{i}}{(\lambda+\mathrm{i}t-\mathrm{i}r)^{\kappa+1}}\right]$$

Important consequence:

$$\lim_{r\to 0}\Psi_{\kappa\lambda}(t,r) = 0$$

This means that the probability of finding the shell at vanishing radius is zero! In this sense, the singularity is avoided in the quantum theory. The quantum shell bounces and re-expands, and no event horizon forms.

Expectation value and variance of the shell radius:

$$\langle R_0 \rangle_{\kappa\lambda} := 2G\langle E \rangle_{\kappa\lambda} = (2\kappa + 1)\frac{l_{\mathrm{P}}^2}{\lambda},$$

$$\Delta(R_0)_{\kappa\lambda} = 2G\Delta E_{\kappa\lambda} = \sqrt{2\kappa + 1}\frac{l_{\mathrm{P}}^2}{\lambda}$$

It turns out that the wave packet can be squeezed below its Schwarzschild radius if its energy is greater than the Planck energy—a genuine quantum effect!

"Superposition of black and white hole"

More complicated models

Lemaître–Tolman–Bondi (LTB) model:
self-gravitating dust cloud with $T_{\mu\nu} = \epsilon(\tau, \rho) u_\mu u_\nu$

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \frac{(\partial_\rho R)^2}{1 + 2E(\rho)} \mathrm{d}\rho^2 + R^2(\rho)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

- exact quantum states of a particular type (cloud consists of decoupled shells)
- Hawking radiation and greybody factors
- BTZ black hole: Hawking radiation as well as microscopic derivation of black-hole entropy (next slide)

(S. Gutti, C. K., J. Müller-Hill, T. P. Singh, C. Vaz, L. C. R. Wijewardhana, L. Witten in various combinations 2003–2008)

Entropy of the BTZ black hole

Jacob Bekenstein 1973:

It is then natural to introduce the concept of black-hole entropy as the measure of the *inaccessibility* of information (to an exterior observer) as to which particular internal configuration of the black hole is actually realized in a given case.

- Discrete mass spectrum for the shells collapsing to the black hole;
- black-hole entropy is number of possible distributions of $\mathcal{N}$ identical shells between these levels;

$$S \approx 2\pi \sqrt{\left(1 - \frac{48 l M_0}{\hbar}\right) \frac{lM}{6\hbar}}$$

  with $l = |\Lambda|^{-1/2}$;

- this is equal to the Bekenstein–Hawking entropy for

$$M_0 = -\frac{1}{16G} + \frac{\hbar}{48l}$$

Vaz *et al.* (2008)

A simple model of black-hole evaporation

Quantum black hole (Wheeler–DeWitt Hamiltonian) embedded into a semiclassical Universe with WKB time t:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x,y,z,t) = \left(\frac{\hbar^2}{2m_{\mathrm{P}}}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m_y}\frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m_z}\frac{\partial^2}{\partial z^2}\right.$$
$$\left. + \frac{m_{\mathrm{P}}\omega_x^2}{2}x^2 + \frac{m_y\omega_y^2}{2}y^2 + \frac{m_z\omega_z^2}{2}z^2\right)\Psi(x,y,z,t)$$

- x: mimics Schwarzschild radius of the black hole
- y: mimics Hawking radiation
- z: mimics further quantum degrees of freedom (will be neglected in the following)

(C.K., Marto, Moniz 2009)

Solving the Schrödinger equation

Separation ansatz:

$$\Psi(x,y,t) = \psi_x(x,t)\psi_y(y,t)$$

$$\begin{aligned}
i\hbar\dot{\psi}_x(x,t) &= \left(\frac{\hbar^2}{2m_{\mathrm{P}}}\frac{\partial^2}{\partial x^2} + \frac{m_{\mathrm{P}}\omega_x^2}{2}x^2\right)\psi_x(x,t)\ , \\
i\hbar\dot{\psi}_y(y,t) &= \left(-\frac{\hbar^2}{2m_y}\frac{\partial^2}{\partial y^2} + \frac{m_y\omega_y^2}{2}y^2\right)\psi_y(y,t)\ .
\end{aligned}$$

We see that ψ_x^* obeys a Schrödinger equation with standard kinetic term, but with the sign of the potential being *reversed* ("upside-down oscillator").

Squeezed ground state

For an initial ground state

$$\psi^g_{x0}(x',0) = \left(\frac{m_\mathrm{P}\omega_x}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m_\mathrm{P}\omega_x}{2\hbar}x^2\right)$$

we find

$$\psi^g_x(x,t) = \left(\frac{m_\mathrm{P}\omega_x}{\pi\hbar(1-\mathrm{i}\sinh 2\omega_x t)}\right)^{1/4} \exp\left(-\frac{m_\mathrm{P}\omega_x}{2\hbar\cosh 2\omega_x t}(1+\mathrm{i}\sinh 2\omega_x t)x^2\right)$$

This is a squeezed ground state with $\phi = \pi/4$ and $r = \omega_x t$.

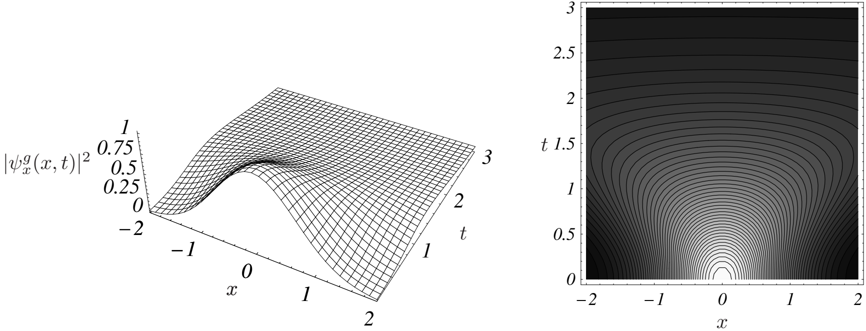


Figure: Evolution of a Gaussian state under the inverted oscillator propagator. We depict $|\psi_x^g(x,t)|^2$ with $m_{\mathrm{P}} = \hbar = \omega_x = 1$ for simplicity. In the contour plot the brighter areas correspond to higher values for $|\psi_x^g(x,t)|^2$.

Squeezed coherent state

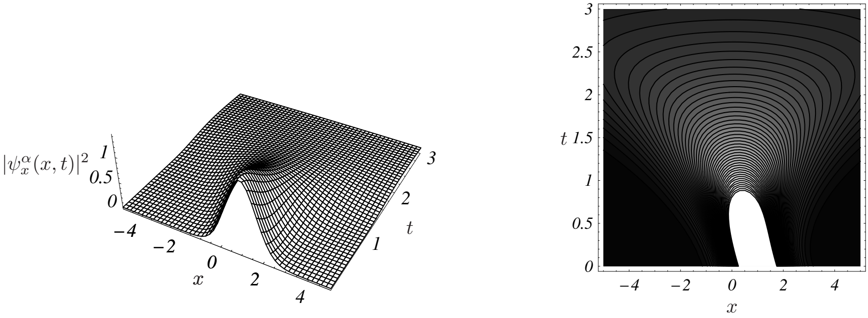
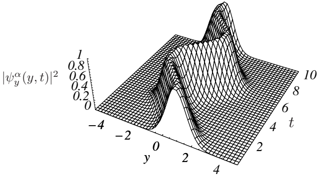


Figure: Evolution of $|\psi_x^\alpha(x,t)|^2$ under the inverted oscillator propagator, where $m_{\mathrm{P}} = \hbar = \omega_x = x_0 = 1$ for simplicity and with $p_0 = -1$.

Hawking radiation

For the y-part ("Hawking radiation"), one obtains the standard result for the time-dependent coherent state:



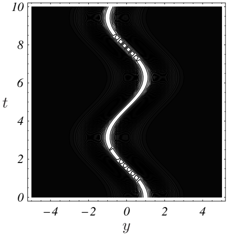


Figure: Evolution of $|\psi_y^\alpha(y,t)|^2$ under the ordinary oscillator propagator, with $m_y = \hbar = \omega_y = y_0 = p_{0y} = 1$ for simplicity.

In fact, a slight squeezing occurs, cf. Demers and C.K. (1996)

Inclusion of back reaction

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x,y,t) \quad &= \quad \left(\frac{\hbar^2}{2m_{\mathrm{P}}}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m_y}\frac{\partial^2}{\partial y^2}\right. \\
&\quad \left. + \frac{m_{\mathrm{P}}\omega_x^2}{2}x^2 + \frac{m_y\omega_y^2}{2}y^2 + \mu xy\right)\Psi(x,y,t)
\end{aligned}
$$

Initial state?

- x-part: again, a coherent state
- y-part:

$$
\psi_{y0}^H(y,t_0) \propto \exp\left(-\frac{m_y\omega_y}{2\hbar}\coth\left[\frac{2\pi\omega_y GM}{c^3} + \mathrm{i}\omega_y t_0\right]y^2\right) \ ,
$$

where M is the original mass of the (Schwarzschild) black hole, which corresponds to the initial value x_0 of the x-part of the quantum state.

Solution of Schrödinger equation

With the above initial states, the solution reads as

$$\psi(x,y,t) = F(t)\exp\left(A(t)x^2 + B(t)x + C(t)y^2 + D(x,t)y\right) ,$$

with explicit (complicated) expressions for the time-dependent functions

Entangled state between the quantum black hole and its Hawking radiation

(recall that the black hole is an open quantum system)

Density matrix for the black hole

Restrict to diagonal elements ("probabilities")

$$\rho_{xx} = \mathrm{tr}_y \rho = \int |\langle x, y | x, y \rangle|^2 \mathrm{d}y$$

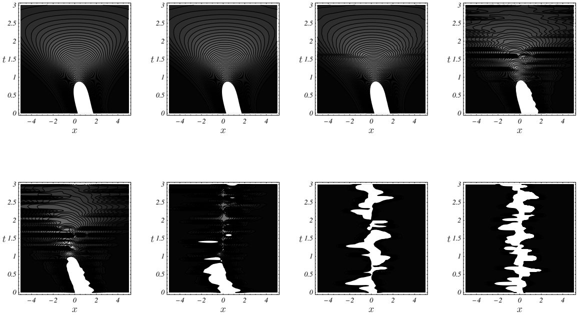


Figure: Time evolution of ρ_{xx}, with $m_{\mathrm{P}} = \hbar = \omega_x = x_0 = 1$; $t_0 = 0$ and $p_0 = -1$ for simplicity, and μ (graphics from left to right and top to bottom) assuming the values of the set $\{0, 0.5, 1, 5, 10, 20, 50, 100\}$, $\omega_y = \omega_x \times 10^{5/2}$, $m_y = m_{\mathrm{P}} \times 10^{-5}$.

Density matrix for the Hawking radiation

Restrict again to diagonal elements ("probabilities")

$$\rho_{yy} = \mathrm{tr}_x \rho = \int |\langle x, y | x, y \rangle|^2 \mathrm{d}x$$

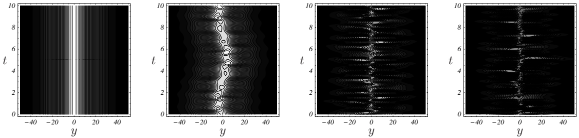
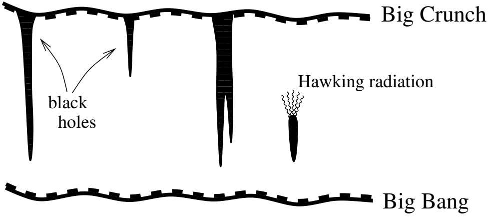
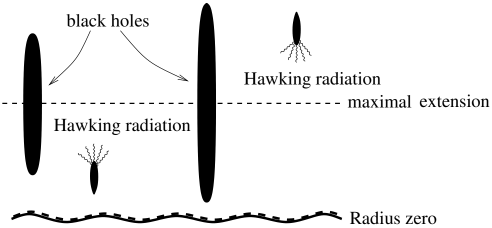


Figure: Time evolution of ρ_{yy}, with $m_{\mathrm{P}} = \hbar = \omega_x = x_0 = 1$ and $p_0 = -1$; $t_0 = 0$ for simplicity, and μ (graphics from left to right) assuming the values of the set $\{0, 1, 5, 10\}$, $\omega_y = \omega_x \times 10^{5/2}$, $m_y = m_{\mathrm{P}} \times 10^{-5}$. For large values of μ the results look qualitatively similar to the results for ρ_{xx}. If the back reaction is large, the difference between the black hole and the Hawking radiation begins to disappear.

Quantum black holes and cosmology





(C.K. and Zeh 1995)

Conclusion

- Strong indications that black holes are genuine quantum objects;
- black-hole horizon is a classical concept;
- interpretation of black-hole entropy and black-hole evaporation can be studied in approaches to quantum gravity.