Time in the vicinity of black holes

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geo

Research Training Group
Graduiertenkolleg
Models of Gravity

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Gewinnerin in der Exzellenzinitiative

CENTER OF
APPLIED SPACE TECHNOLOGY
AND MICROGRAVITY

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Introduction

Notion of time

- In Newtonian theory time is absolute; all clocks tick at the same rate
- In Special Relativity we have to distinguish between coordinate time and proper time; different standard clocks tick with different rates if they are in relative motion
- In General Relativity proper time does in addition depend on the gravitational field

Clock effects in General Relativity (not complete)

- Gravitational redshift
- Shapiro delay
- Gravitomagnetic clock effect

Gravitational redshift

Schwarzschild solution ($G = 1$ and $c = 1$)

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Difference between proper time and coordinate time

- Proper time of an observer at rest at radius r_1 with proper time τ_1, from $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$

$$\frac{dt}{d\tau_1} = \frac{1}{\sqrt{1 - \frac{2M}{r_1}}}$$

Gravitational redshift

Consider another observer at rest at radius r_2 with proper time τ_2,

$$\frac{d\tau_2}{d\tau_1} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}} = \frac{\nu_1}{\nu_2}$$

- The above formula is the redshift between two clocks at rest
- For $r_1 \to 2M$ the redshift becomes infinitely large
- An object falling onto $r = 2M$ slows down ('freezes') and redshifts out of detectable frequency range

Crossing the horizon

However, an observer takes only finite proper time to reach $r = 2M$

- Killing vector ∂_t gives $g_{tt}\frac{dt}{d\tau} = E$
- Radial free fall: $\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)$
- Starting from r_0: $E^2 = \left(1 - \frac{2M}{r_0}\right)$
- Leads to $\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r} - \frac{2M}{r_0}$
- Integration gives a finite result

See Kassner 2016 for a discussion of two infalling observers

Shapiro delay

Consider a photon moving radially in a Schwarzschild spacetime

$$0 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$
$$\Rightarrow \quad \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)$$

- This can be easily integrated to

$$t = r - r_0 + 2M \ln \frac{r - 2M}{r_0 - 2M}$$

- Newtonian travel time: $ct = r - r_0$
- In addition there appears a logarithmic term
- This is the Shapiro delay

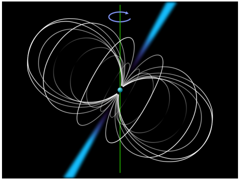
Observation of clock effects

These effects can and have been measured in the weak field regime

What about the strong field regime?

- We can not place a man-made clock near a back hole to observe these effects
- But: there are 'astronomical clocks': Some pulsars rotate so regularly that they rival the accuracy of the best man-made clocks!
- There is an ongoing search for pulsars orbiting a black hole
- Pulsars closely orbiting Sgr A* are the ideal laboratory to explore the supermassive black hole

Pulsar timing - introduction



- rapidly rotating neutron stars
- radio emission
- stable rotation
- ca. 10% with a companion

- emission encodes information about the gravitational field near the pulsar
- reconstruction of the emission time from the arrival time
- phase–connected solutions: up to 100 nanoseconds post-fit accuracy
- pulsar around a black hole: the holy grail

Image: commons.wikimedia.org/wiki/File:Pulsar_schematic.svg by Use

Pulsar timing - clock effects

Regarding clock effects there are two delays which are relevant

- Einstein delay
  - difference between coordinate time of the binary barycenter and the proper time of the pulsar
  - changing gravitational redshift and Doppler effect along pulsar orbit
  - gravitomagnetic clock effect!?
- Shapiro delay
  - propagation delay of photons due to the gravitational field
  - with respect to a reference orbit and an observer at infinity

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Standard Shapiro delay

In pulsar timing the Shapiro delay is modelled in the post-Newtonian framework

- Is this approximation still valid in the strong gravitational field near a supermassive black hole?
- In the case of a pulsar orbiting Sgr A* (extreme mass ratio) we can find an exact expression
- Test the accuracy of the pN approximation!

Standard propagation delay

$$
\begin{aligned}
t_{\mathrm{arr}} - t_{\mathrm{em}} &\approx |\vec{r}_{\mathrm{E}}(t_{\mathrm{arr}}) - \vec{r}(t_{\mathrm{em}})| + 2M \ln\left(\frac{2r_{\mathrm{E}}}{r + \vec{r}\cdot\vec{n}}\right) \\
&= (\Delta t)_{\mathrm{Roemer}} + \mathrm{const} \cdot \underbrace{2M \ln\left(\frac{1 + e\cos\phi}{1 - \sin i \sin(\omega + \phi)}\right)}_{=:(\Delta t)_{\mathrm{Shap}}}
\end{aligned}
$$

Standard Shapiro delay

Standard Shapiro delay

$$(\Delta t)_{\text{Shap}} = 2M \ln\left(\frac{1 + e\cos\phi}{1 - \sin i \sin(\omega + \phi)}\right) = 2M \ln\left(\frac{a(1 - e^2)}{r - r_{||}}\right)$$

where $r_{||} = r \sin i \sin(\omega + \phi)$

- does not take into account the bending of the path
- edge-on orbits ($i = \pi/2$), superior conjunction ($\omega + \phi = \pi/2$): expression diverges
- is independent of the semi major axis of the pulsar

Bent path

Shapiro delay taking lensing into account Lai & Rafikov 2005

$$(\Delta t)_{\rm lens} = 2M \ln\left(\frac{a(1-e^2)}{\sqrt{r_{||}^2 + R_\pm^2} - r_{||}}\right)$$

where $R_\pm = \frac{1}{2}\left(R_s \pm \sqrt{R_s^2 + 4R_E^2}\right)$, $R_s^2 = r^2 - r_{||}^2$,
$R_E^2 = 4Ma_{||}$ the Einstein radius.

Geometric delay

$$(\Delta t)_{\rm geom} = 2M\left(\frac{|R_\pm - R_s|}{R_E}\right)^2$$

Schwarzschild spacetime

- Schwarzschild black hole spacetime

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Constants of motion: specific energy E, specific angular momentum L, $\theta \equiv \pi/2$, normalisation $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$
- Equations of motion for photons

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2} - r^2 + 2Mr =: b^{-2} R(r)$$

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 \frac{R}{r^4}$$

Here $b = L/E$ is the impact parameter

Exact Shapiro delay

From radial geodesics ($b = 0$)

- expect a linear and a logarithmic divergence for an observer at infinity
- linear corresponds to Roemer delay

In general Dhani, Master thesis 2017

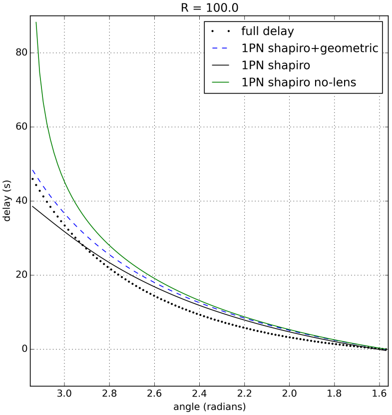
- Emitter-observer problem: for given emitter at (r_e, ϕ_e) find b from

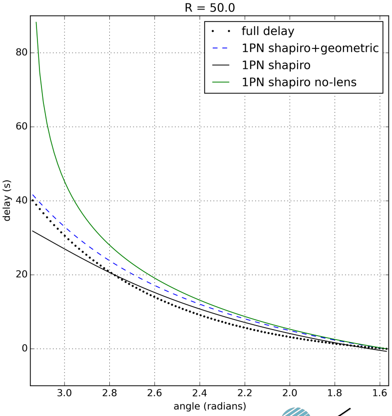
$$\phi_e = \int_{r_e}^{\infty} \frac{dr}{\sqrt{\frac{r^4}{b^2} - r^2 + 2Mr}} = b \int_{r_e}^{\infty} \frac{dr}{\sqrt{R}}$$

- solve for $t = \int_{r_e}^{\infty} \frac{r^3}{r-2M} \frac{dr}{\sqrt{R}}$
- exact expression in terms of elliptic integrals
- identify logarithmic divergence for observer at infinity
- find Δt with respect to a reference orbit $\rightarrow$ get rid of infinities

Comparison

- Propagation delay (in s) as function of the angle along a circular edge-on orbit of radius R (in units of M)





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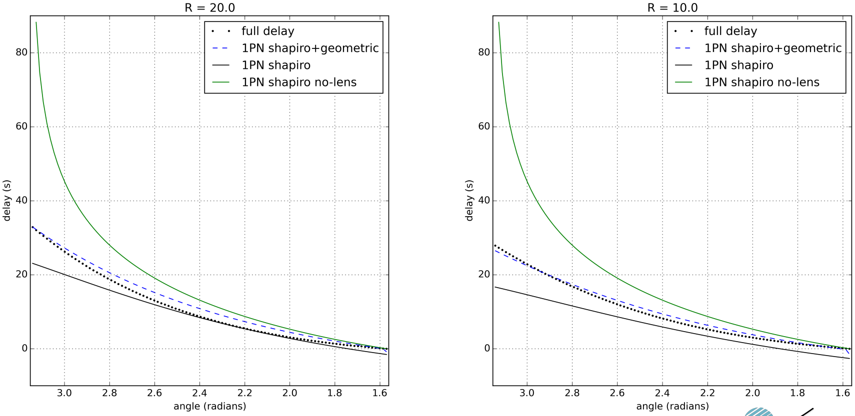

Conclusions

In the considered setting

- the lensed Shapiro delay + the geometric delay fits the exact delay best
- the usual first order Shapiro delay quickly deviates from the exact expression by several seconds

$\rightarrow$ also applies to less inclined orbits

- the geometric delay should only be used together with the lensed Shapiro delay
- for 'good' pulsars closely orbiting Sgr A* at least the second order post-Newtonian approximation should be used

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The gravitomagnetic clock effect

The setup

- Two clocks on circular orbits in the equatorial plane of a rotating astronomical object
- One clock on prograde orbit, one on retrograde orbit
- Compare the measured time after a full revolution of 2π

Also called observer-dependent two-clock clock effect

Cohen and Mashhoon (Phys. Lett. A, 181:353, 1993)

- $\tau_+ - \tau_- \approx \frac{4\pi J}{mc^2}$
- For the Earth: time difference of about $10^{-7}\mathrm{sec}$ per revolution

$\rightarrow$ Large effect!?

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Problems and Goals

Problems

- Identical initial conditions required
- Identical orbits required
- Idealized circular orbits required

→ Generalisations to eccentric and inclined orbits exist

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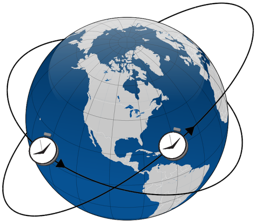

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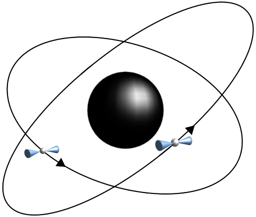

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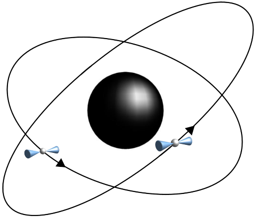


Generalisation: Fully general relativistic definition

→ Consider bound geodesic orbits in Kerr spacetime

→ Derive an expression for $\tau(\pm 2\pi)$, τ proper time

→ Use fundamental frequencies

Kerr spacetime

in Boyer-Lindquist (BL) coordinates

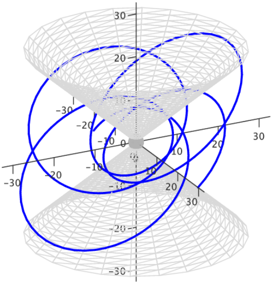
$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}(adt - (r^2+a^2)d\varphi)^2 + \rho^2 d\theta^2$$

where $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$,
$M = \frac{Gm}{c^2}$ the mass, $a = J/(mc)$ the spin.

Equations of motion (using $d\tau = \rho^2 d\lambda$)

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r)\,, \quad \frac{d\varphi}{d\lambda} = \Phi_r(r) + \Phi_\theta(\theta)\,,$$
$$\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta)\,, \quad \frac{dt}{d\lambda} = T_r(r) + T_\theta(\theta)$$

Periodic motion



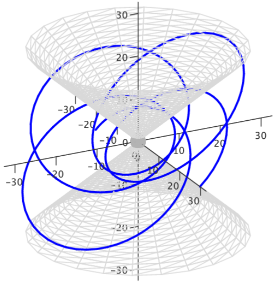
For bound orbits outside the horizons:

- The radial motion is periodic, $r \in [r_\mathrm{p}, r_\mathrm{a}]$
- The polar motion is periodic, $\theta \in [\theta_\mathrm{min}, \theta_\mathrm{max}]$

From $\left(\frac{dr}{d\lambda}\right)^2 = R$, $\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta$:

- Radial period $\varLambda_r$: $r(\lambda + \varLambda_r) = r(\lambda)\,, \quad \varLambda_r = 2\int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{dr}{\sqrt{R}}$, $\Upsilon_r = \frac{2\pi}{\Lambda_r}$
- Polar period $\varLambda_\theta$: $\theta(\lambda + \varLambda_\theta) = \theta(\lambda)\,, \quad \varLambda_\theta = 2\int_{\theta_\mathrm{min}}^{\theta_\mathrm{max}} \frac{d\theta}{\sqrt{\Theta}}$, $\Upsilon_\theta = \frac{2\pi}{\Lambda_\theta}$

Fundamental Frequencies



- φ, t, and τ are not periodic
- can be expressed as a linear function in λ + periodic oscillations
- Ansatz: $\varphi(\lambda) = \Upsilon_\varphi \lambda + \Phi^r_{osc} + \Phi^\theta_{osc}$
  Υ_φ infinite $\lambda-$average
- Analogously: $\tau(\lambda) = \Upsilon_\tau \lambda +$ osc.;
  $t(\lambda) = \Upsilon_t \lambda +$ osc.

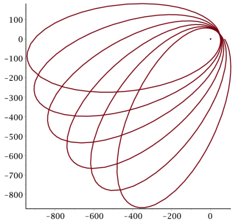
Proper time as function of φ:

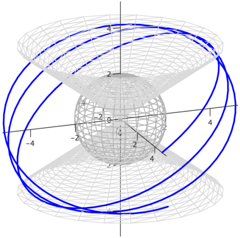
- Use averaged $\tau = \Upsilon_\tau \lambda$ and $\varphi = \Upsilon_\varphi \lambda$

$\rightarrow$ $\tau : \varphi \mapsto \tau(\lambda(\varphi)) = \Upsilon_\tau \Upsilon_\varphi^{-1} \varphi$

- In the Newtonian limit we obtain from this the Keplerian time of revolution

Periapsis precession and Lense-Thirring effect





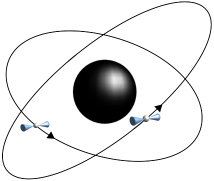
Periapsis precession

- mismatch of radial and angular frequency wrt coordinate time
- $\dot{\omega} = \Omega_r - \Omega_\varphi = \frac{\Upsilon_r}{\Upsilon_t} - \frac{\Upsilon_\varphi}{\Upsilon_t}$
  $= (2\pi - \Lambda_r \Upsilon_\varphi)/P_r$
- $P_r = \Lambda_r \Upsilon_t$ anomalistic period

Lense-Thirring effect

- mismatch of polar and angular frequency wrt coordinate time
- $\dot{\Omega} = \Omega_\theta - \Omega_\varphi = (2\pi - \Lambda_\theta \Upsilon_\varphi)/P_\theta$
- $P_\theta = \Lambda_\theta \Upsilon_t$ draconitic period

The gravitomagnetic clock effect



Consider two clocks on arbitrary geodesics

- Orbital parameters $r_{\mathrm{p},n}$, $r_{\mathrm{a},n}$, $\theta_{\max,n}$, $n = 1, 2$
- Proper time of a full revolution: $\tau_n(\pm 2\pi, J)$

Generalised definition

- *Gravitomagnetic* clock effect:

$$\Delta\tau_{\mathrm{gm}} = \tau_1(\pm 2\pi, J) + \alpha\tau_2(\pm 2\pi, J)$$

- with α such that *gravitoelectric* effects cancel: $\Delta\tau_{\mathrm{gm}} = 0$ for $J = 0$, i.e. $\alpha = -\frac{\tau_1(\pm 2\pi, 0)}{\tau_2(\pm 2\pi, 0)}$

Post-Newtonian expansion

For a one-year orbit around Sgr A*: $a/r \leq M/r \lesssim 5 \times 10^{-4}$

- Expansion for small $\frac{a}{r} = \frac{J}{mcr}$ and small $\frac{M}{r} = \frac{Gm}{c^2 r}$

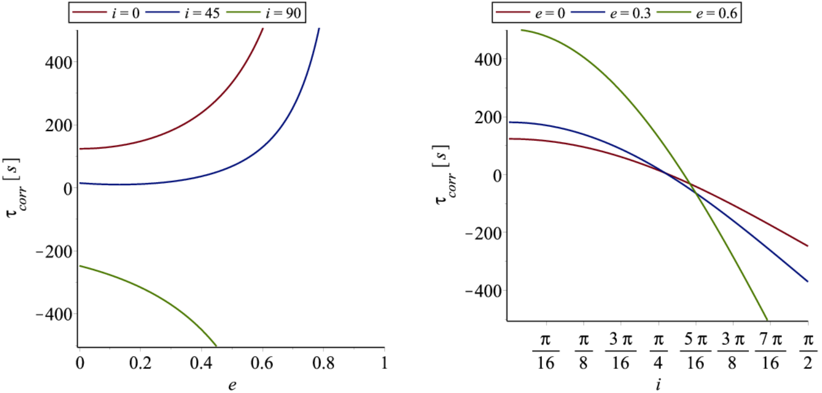
$$\tau(\pm 2\pi) \approx 2\pi\sqrt{\frac{\mathrm{a}^3}{Gm}}\left(1 - \frac{3(1+e^2)}{2(1-e^2)}\frac{M}{\mathrm{a}}\right) \pm \frac{2\pi(\cos i(3e^2+2e+3) - 2e - 2)}{(1-e^2)^{\frac{3}{2}}}\frac{J}{mc^2},$$

- a semimajor axis, e eccentricity, and i inclination
- $r_\mathrm{p} = \mathrm{a}(1-e)$, $r_\mathrm{a} = \mathrm{a}(1+e)$, and $\theta_\mathrm{max} = \pi/2 + i$

Astronomical object orbiting Sgr A*

Correction to proper orbital period due to frame dragging:

$$\tau_{\text{corr}} \approx \frac{2\pi J}{mc^2} \frac{\cos i(3e^2+2e+3)-2e-2}{(1-e^2)^{\frac{3}{2}}}$$



Clock effect for general orbits

For two clocks with arbitrary orbital parameters $\mathrm{a}_{1,2}$, $e_{1,2}$, $i_{1,2}$:

$$\Delta\tau_{\mathrm{gm}} \approx \frac{2\pi J}{mc^2}\left[s_1 \frac{\cos i_1(3e_1^2+2e_1+3)-2e_1-2}{(1-e_1^2)^{\frac{3}{2}}} - s_2\sqrt{\frac{\mathrm{a}_1^3}{\mathrm{a}_2^3}}\frac{\cos i_2(3e_2^2+2e_2+3)-2e_2-2}{(1-e_2^2)^{\frac{3}{2}}}\right]$$

- $s_{1,2}=+1$ for prograde motion, $s_{1,2}=-1$ for retrograde
- In particular: $s_1 = s_2$ possible!
- Identical orbital parameters: $\tau_+ - \tau_- \approx \frac{4\pi J}{mc^2}\frac{\cos i(3e^2+2e+3)-2e-2)}{(1-e^2)^{\frac{3}{2}}}$

Two examples

First example

- Sgr A* rotates with $J/(mc) = 0.9M$
- First pulsar: 0.5-year orbit, nearly equatorial and circular
- Second pulsar: 1-year orbit, quite eccentric and highly inclined
- Result: $\Delta\tau_{\rm gm} \approx 297{\rm s} \approx 2 \times 10^{-5}\,\tau(2\pi; {\rm J}=0)$

Second example

- Sgr A* rotates with $J/(mc) = 0.5M$
- First pulsar: 1-year orbit, nearly equatorial and circular
- Second pulsar: 2-year orbit, a bit eccentric and quite inclined
- Result: $\Delta\tau_{\rm gm} \approx 59{\rm s} \approx 2 \times 10^{-6}\,\tau(2\pi; {\rm J}=0)$

Summary

The gravitomagnetic clock effect for Earth satellites

- satellites orbiting the Earth: effect $\sim 10^{-8} - 10^{-7}\,\mathrm{s}$
- but ultra precise tracking necessary: semi major axis to at least mm accuracy!

The gravitomagnetic clock effect for general astronomical objects

- for arbitrary bound geodesic orbits in Kerr spacetime
- definition via fundamental frequencies
- objects orbiting Sgr A*: effect up to $\sim 10^2\mathrm{s}$
- detectable by pulsars?

Thank you for your attention!