Classical aspects of Black Holes I:

Static and Stationary black holes

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Do Black Holes exist? - The physics and philosophy of Black Holes.
Physics Center, Bad Honnef, Germany

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2. Spherically symmetric vacuum case: Schwarzschild Solution
3. Reissner Nordstrom Solution
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The basic idea

- When the gravitational field of a central body is large enough, gravitational collapse will take place and spacetime singularities will form. This is inevitable at the future endpoint of evolution of massive enough stars (with mass greater than the Chandrasekhar mass).
- In the spherically symmetric case, an event horizon will form; no information can escape from the interior to the exterior.
- This also occurs in the case of rotating black holes (provided cosmic censorship is true).
- Static (Schwarzschild) and stationary (Kerr) black holes are in principle possible, that have existed forever. Because of their symmetries, they enable a clear study of key black hole properties (this lecture).
- In reality, black holes form dynamically and there are major differences to the properties of stationary or static black holes (next lecture).
- They have been observed to occur at stellar, intermediate, and large masses. There might possibly also be small priordial black holes.

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Spherical Vacuum Solution

The unique spherically symmetric vacuum solution of the Einstein Field Equations is the Schwarzschild solution. It turns out to be a black hole.

In standard coordinates, the exterior Schwarzschild metric for a body of mass $m := GM/c^2$ may be written in the form:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

for $r > 2m$, where $d\Omega^2 := (d\theta^2 + \sin^2\theta d\phi^2)$. The Schwarzschild radius r_{Sch} is defined by $r_{Sch} := 2m$.

Schwarzschild Solution

The exterior solution

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- 3: Thus is automatically static (or stationary): this does not have to be added as an extra condition. This is Birkhoff's theorem.
- 4: It can be joined smoothly onto a spherically symmetric interior star metric provided the surface is at $R_S > 2M$. This is true whether the interior stellar solution is static or dynamic: possibly $R_S = R_S(t)$. The exterior remains static (no spherical gravitational waves).

Schwarzschild Solution

Matter motion

- 1: Matter paths are timelike geodesics. The effective radial potential is
$$V^2(h,r) \equiv \left(1 - \frac{2m}{r}\right)(1 + \frac{h^2}{r^2}) \qquad (2)$$
where h is angular monentum, so $V \to 1$ as $r \to \infty$
- 2. Particle orbits reproduce Newtonian results for planetary motion but with the correct perihelion precession, confirming the EFE and identifying m as the mass.
- 3. The lower bound for any circular particle orbit, is $r = \frac{3GM}{c^2} = \frac{3r_s}{2}$. These are unstable orbits

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- 4. Innermost stable circular matter orbit (ISCO) $r_{isco} = 3\,r_s = \frac{6\,GM}{c^2}$,
- 5. Matter: infalling matter will generically circle many times before falling in, forming accretion discs that will heat up and emit x-rays. These are power sources for astrophysical objects, of crucial importance for QSOs

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- 2. Light rays are bent by the gravitational field of the central star, leading to gravitational lensing and time delays in multiple images of time-varying sources. If there is a central mass with $R_S >> r_s$, this will give weak lensing. This further confirms the EFE.
- 3. Photon sphere: Unstable circular orbits are possible for light rays travelling at $r = \frac{3GM}{c^2} = \frac{3r_s}{2}$ (and only at that radius) – they are held at this distance by gravitational attraction. These are unstable orbits. The photon sphere casts a shadow.

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- 4. If one has a black hole with no central star, strong lensing will occur: with photons circling the horizon numerous times to create many multiple images. But dust may obscure them.

- 1: The metric has an apparent singularity at

$$\left(1 - \frac{2m}{r}\right) = 0 \Leftrightarrow r = r_{Schw} = 2m \tag{4}$$

where the metric tensor breaks down: as $r \to r_{Schw}$, $g_{00} \to 0$, $g_{11} \to \infty$. However no scalar invariants diverge there.

- 2. This is a coordinate singularity, and by using a null coordinate

$$v = t + r^*, \;\; r^* := r + 2m \ln\left(\frac{r}{2m} - 1\right) \Leftrightarrow dr^* = \frac{dr}{1 - \frac{2m}{r}} \tag{5}$$

the solution can be extended beyond the apparent singularity at $r = 2m$ to a time dependent domain ending at a physical singularity at $r = 0$.

The extended metric is

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Coordinate singularities

Metric singularities may be due to a breakdown of coordinates rather than a physical problem. Null coordinates can extend across such surfaces.

Schwarzschild Solution

Eddington-Lemaitre extension

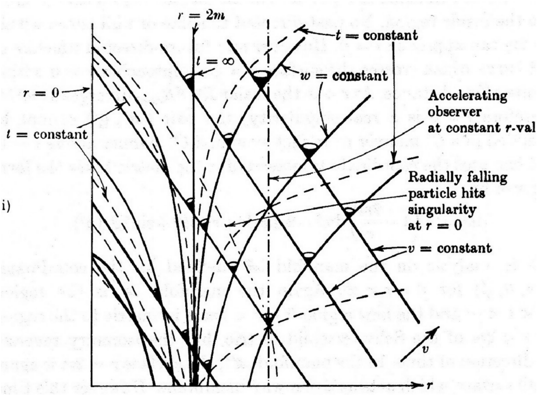

Figure: Diagram of event horizon and central singularity. Note that solution is time asymmetric. Radially infalling matter is seen crossing the horizon with infinite redshift: "frozen stars"

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- 3. It has *closed trapped surfaces* inside the event horizon: a highly counterintuitive aspect. Their existence is key to the Penrose singularity theorem and its extensions.
- 4. This solution is time asymmetric, and null geodesically incomplete to the past. There is a time-reverse time asymmetric extension with null coordinate $w = t - r^*$.

It can be further extended to a maximal time symmetric solution that cannot be extended any further. All geodesics then either extend to infinity, or end at a singularity. This maximal (Kruskal) metric is

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dX^2) + r^2 d\Omega^2 \tag{8}$$

where $r(T, X)$ is determined by

$$\left(1 - \frac{r}{2GM}\right) e^{r/2GM} = T^2 - X^2. \tag{9}$$

- 1. Unexpected unavoidable global structure: Its maximal extension has two scalar singularities , where the Weyl tensor diverges, at $r = 0$: one in the future, one in the past; plus two asymptotically flat domains, linked by a minimal throat. This forms a wormhole that in some coordinates opens and closes. However the wormhole is spacelike and hence is not traversable.

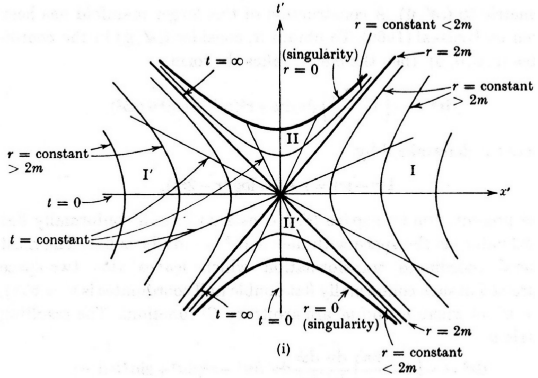

Figure: Maximally extended solution, which is time symmetric. It has two singularities bounding two spatially homogeneous evolving regions, and two asymptotically flat static regions.

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- 6. Closed trapped surfaces are obvious from the Kruskal diagram
- 4. The interior regions $r < 2m$ are not static: they are spatially homogeneous and evolving in time. There is no central world line at $r = 0$.

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Microphysics:

This solution shows that like in quantum physics, the idea of point particle is not viable in general relativity. You cannot model a gravitating particle by a timelike worldline with an series of moments defined on the world line, as you can in special relativity. The minimum radius for a particle is r_s.

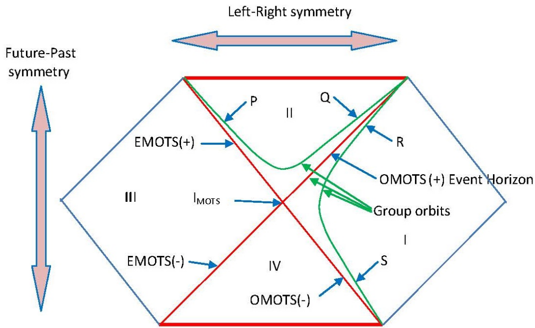
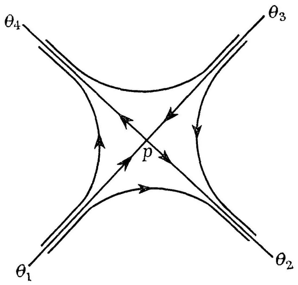


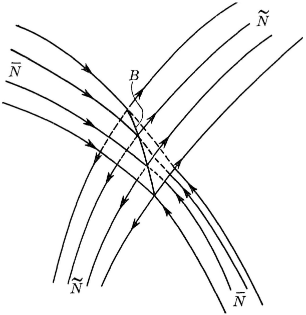
Figure: Penrose diagram of Symmetries of Maximally extended solution. Each point represents a 2-sphere. One can create different topologies by identifications under these groups (the field equations do not determine this topology) but this may create extendible singularities and closed timelike lines.

Schwarzschild Solution

Bifurcating horizons

R H Boyer and J Ehlers: *Proc Roy Soc* A311:245-252 (1969)





Where a Killing vector field ξ changes from timelike to spacelike, (1) θ_i are null geodesic Killing orbits with fixed points p; (2) Bifurcating Killing Horizons with branches $\tilde{N}, \overline{N}$ and fixed points B occur. The Killing vector parameter t relates to the affine parameter v by $u = \exp^{ct}$ and $k = \exp^{-ct} \xi$ is constant on θ so $t \to \pm\infty$ as $u \to \pm 0$.

Schwarschild solution

Infalling particles and redshift

- Radially infalling particles are represented by timelike geodesics with zero angular momentum
- From the outside they are seen to fade out with infinite time dilation and infinite redshift
- However they experience nothing special as they cross the horizon
- They can send no signals to the exterior after horizon crossing, due to the event horizon. No light or other radiation emitted by the inside region II can reach the outside region I.
- Infalling matter is doomed to fall into the singularity at the centre and be crushed by diverging tidal forces

Black holes

Existence of the event horizon is the key feature of black holes.

Uniqueness Theorem

Werner Israel: first uniqueness theorem, for Schwarzschild solution (*Phys Rev* 164 (1967): 1776)

A space-time manifold is static if it admits a hypersurface-orthogonal Killing vector field ξ which is timelike over some domain. In a simply connected region which has $\xi.\xi < 0$ throughout, there will exist a scalar field t and coordinates x^a such that the line element reduces locally to

$$ds^2 = g_{\alpha\beta}(x^1, x^2, x^3)dx^\alpha dx^\beta - V^2 dt^2, \qquad (10)$$

where $V = (|\xi.\xi|)^2 = V(x^1, x^2, x^3)$, $(\xi.\nabla)x^\alpha = 0$.

In a static space-time, let Σ be any spatial hypersurface $t = const$, maximally extended consistent with $\xi.\xi < 0$. We consider the class of static fields such that the following conditions are satisfied on Σ:

1. Σ is regular, empty, noncompact, and “asymptotically Euclidean.”
2. The equipotential surfaces $(V = const > 0,\ t = const)$ are regular, simply connected closed 2-spaces.
3. The invariant $R^{ABCD}R_{ABCD}$ formed from the four-dimensional Riemann tensor is bounded on Σ.
4. If V has a vanishing lower bound on Σ, the intrinsic geometry (characterized by ${}^{(2)}R$) of the 2-spaces $V = c$ approaches a limit as $c \to 0+$, corresponding to a closed regular 2-space of finite area.

Theorem

The only static space-time satisfying (1), (2), (3), and (4) is Schwarzschild's spherically symmetric vacuum solution.

Essentially: this requires

- a static spacetime
- asymptotic flatness
- simply connected equipotential surfaces
- a regular Killing horizon

It does not requires spherical symmetry, it deduces it

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Reissner Nordstrom Solution

Charged version of Schwarzschild:

Reissner Nordstrom Solution

The Reissner Nordstrom Solution is the unique electrovac solution of the Einstein Field Equations

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 - \left(1 - \frac{r_{2m}}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2\, d\Omega^2, \quad (11)$$

where e is the electrical charge at the centre in suitable units.

- For large enough e (e.g. electron) this has a naked singularity
- This solution has no astrophysical or particle physics applications.
- It has two apparent singularities and associated horizons where $\left(1 - \frac{r_S}{r} + \frac{e^2}{r^2}\right) = 0$, and a consequent complex global structure.

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Kerr stationary axisymmetric vacuum solution. The metric is

$$d\tau^2 = \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \left(r^2 + a^2 \right) \sin^2 \theta d\phi^2 - dt^2 + \frac{2mr}{\rho^2} \left(a \sin^2 \theta d\phi - dt \right)^2 \quad (12)$$

where (r, θ, ϕ) are standard spherical coordinates, ma is the angular momentum measured at infinity, and

$$\rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta,$$

$$\Delta(r) = r^2 - mr + a^2.$$

This gives the Schwarzschild solution when $a = 0$.
No exact interior solution.

The Kerr solution

Rotating black hole

- More complex metric than Schwarschild when $a \neq 0$ (not spherically symmetric), with various coordinate systems
- Two radii r_+ and r_- where $\Delta(r)$ vanishes, and an ergosphere between the stationary limit surface (a bifurcating Killing horizon) and the event horizon $r = r_+$.
- There is a ring singularity at $r = 0$; one can continue through the ring to negative values of r
- Maximal extension with infinite number of Killing horizons and asymptotically flat regions; : but this is unstable because of infinite blueshifts at $r = r_-$
- Closed timelike lines inside inner horizon because light cones tilt over
- Killing tensors that can be used to determine geodesics and lensing. Much more complex orbits and images than Schwarzschild.

The Kerr Solution

Ergosphere

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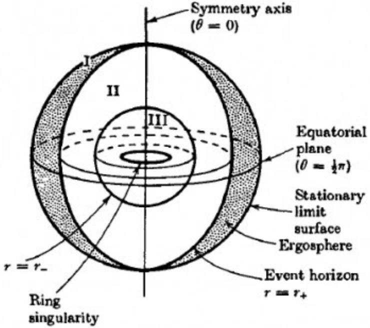

FIGURE 29. In the Kerr solution with $0 < a^2 < m^2$, the ergosphere lies between the stationary limit surface and the horizon at $r = r_+$. Particles can escape to infinity from region I (outside the event horizon $r = r_+$) but not from region II (between $r = r_+$ and $r = r_-$) and region III ($r < r_-$; this region contains the ring singularity).

Figure: The Kerr solution, rotating about the symmetry axis

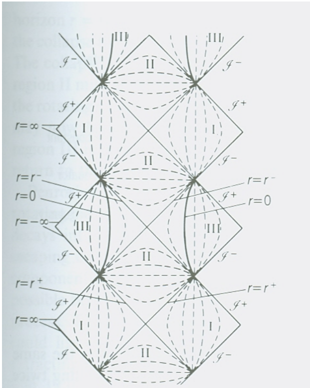

Fig. 19.6 Penrose diagram of maximal extension of Kerr solution ($a^2 < m^2$) along symmetry axis.

- Regions I ($\infty > r > r_+$) are asymptoticaly flat with event horizons at $r = r_+$.
- Regions II ($r_+ > r > r_-$)are spatially homogeneous and contain closed trapped surfaces
- Regions III ($-\infty < r < r_-$) contain the ring singularity; $\exists$ closed timelike curves through every point in region III
- Bifurcating Killing horizons separate these regions

Maximal extension on symmetry axis (Carter) for $a^2 < m^2$ (naked singularity if $m^2 > a^2$)

Uniqueness Theorems

Kerr solution

Uniqueness theorems for rotating stationary vacuum space time:
Extending Israel's Theorem from static to stationary

- Hawking: All stationary but non-static black holes must be have spherical topology and be axisymmetric
- Carter: vacuum black holes that are stationary and axisymmetric form a 2-parameter family
- Robinson, Bunting, Mazur, Wald, Chruściel: Kerr back holes are the only possible stationary vacuum black holes

Thus an uncharged stationary black hole solution is completely described by the 2 parameters of the Kerr metric: mass and angular momentum.

Uniqueness (Carter 1999)

The Kerr family of metrics are believed to constitute, when $a^2 \leq m^2$, the unique family of asymptotically flat and stationary black hole solutions

This result includes Schwarzschild when $a = 0$.

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MacVittie solution

Black hole in an expanding universe

"The mass-particle in an expanding universe"
GC McVittie - *MNRAS* 93:325-339 (1933)

Spherical Mass in Expanding universe

The unique spherically symmetric solution of the Einstein Field Equations with a central mass in an expanding universe is the MacVittie solution

No longer asymptotically flat, but still simple
In "Schwarzschild" coordinates this is

$$ds^2 = -\left(1 - \frac{2M}{r\,a(t)}\right) dt^2 + \left(1 - \frac{2M}{r\,a(t)}\right)^{-1} a^2 dr^2 + a(t)^2 r^2 d\Omega^2 \quad (13)$$

where r is the radial coordinate $a = a(t)$ is the cosmological scale factor.

MacVittie solution

Black hole in an expanding universe

Roberto A. Sussman "Conformal structure of a Schwarzschild black hole immersed in a Friedman universe" GRG (1985) 17:251–291

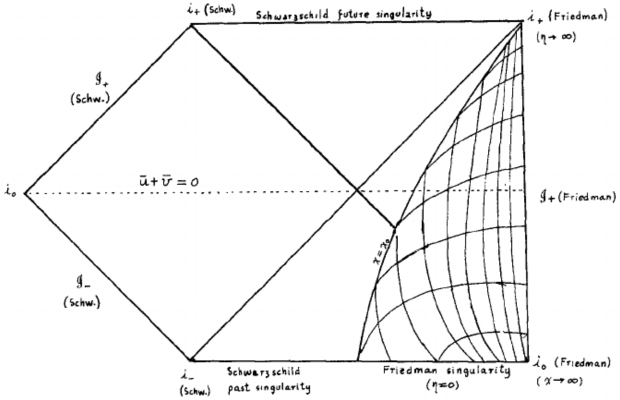

Figure:

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- Power source for QSO's: $10^8 M_\odot < M < 10^9 M_\odot$
- Primordial black holes, $M > 10^{-8}$ kg?? Dark matter???
- Black hole collisions producing gravitational waves

The significance of black holes

Black holes started off as a purely mathematical construct. They are now central to much of high energy astrophyiscs.

Dynamical black holes in a cosmological context are different than static or stationary (next lecture)

Cosmological contexts
Kerr-de Sitter Black Hole

Kerr-de Sitter Universe - Akcay and Matzner.
Class.Quant.Grav. 28 (2011) 085012 arXiv:1011.0479.

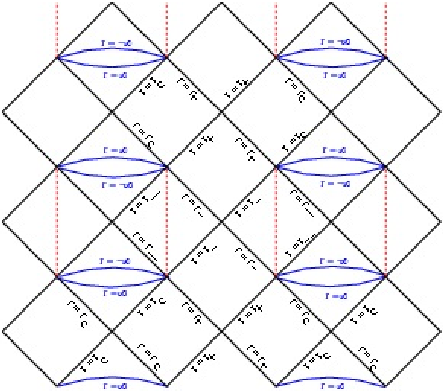

Figure: Kerr-de Sitter black hole