The following text is a preprint and extended version (including mathematical derivations) of the paper:

N. Fries, M. Dreyer; An Analytic Solution of Capillary Rise Restrained by Gravity; Journal of Colloid and Interface Science 320: 259–263, 2008.

www.elsevier.com/locate/jcis or

http://dx.doi.org/10.1016/j.jcis.2008.01.009

This text is intended as online supplementary material.

An Analytic Solution of Capillary Rise Restrained by Gravity

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Abstract

We derive an analytic solution for the capillary rise of liquids in a cylindrical tube or a porous medium in terms of height h as a function of time t. The implicit t(h) solution by Washburn is the basis for these calculations and the Lambert W function is used for its mathematical rearrangement. The original equation is derived out of the 1D momentum conservation equation and features viscous and gravity terms. Thus our h(t) solution, as it includes the gravity term (hydrostatic pressure), enables the calculation of the liquid rise behavior for longer times than the classical Lucas-Washburn equation. Based on the new equation several parameters like the steady state time and the validity of the Lucas-Washburn equation are examined. The results are also discussed in dimensionless form.

Key words: Capillary rise, Capillary tube, Analytic solution, Liquid penetration, Porous medium, Imbibition, Lucas-Washburn equation, Washburn equation, Lambert W function

Introduction

Capillary driven flow is an important field of research as many applications in science, industry and daily life rely on capillary transport. For example in hydrology the movement of groundwater is influenced by capillary transport as well as in heat pipes, spacecraft propellant management devices (PMDs), marker pens, candle wicks and sponges. Mostly this transport occurs in complex formed structures, however many flow or layout calculations adopt models for cylindric tubes to match the flow in arbitrary shaped capillaries. Often a porous medium, no matter how its pores are formed microscopically, can be described with sufficient precision by the "bundle of capillary tubes" model or the Darcy law. When regarding the behavior of a liquid brought into contact with a vertical, small tube (as shown in Fig. 1) it can be seen that at first a fairly fast flow into it will develop. Later the rising of the liquid will continuously slow down until finally a steady state is reached. The description of the liquid rise over time by mathematical methods and its prediction are of great interest as can be seen from the following brief literature review. In 1918 Lucas [1] and 1921 Washburn [2] are the first to give an analytic explanation of the rate of liquid rise in a capillary tube. They consider a flow regime where the influence of inertia and the influence of gravity can be neglected. In 1922 Rideal [3] and 1923 Bosanquet [4] try to expand the Lucas-Washburn solution to cases including inertia and gravity by means of series expansion. 1976 Levine et al. [5] and [6] develop theories for the capillary rise in tubes as well as in parallel plate channels. Marmur and Cohen [7], [8] characterize porous media by analyzing the kinetics of capillary penetration. Ichikawa and Satoda [9] describe the interface dynamics of capillary flow and derive dimensionless variables. In 1997 Quere [10] investigates the capillary rise dominated by inertial forces and finds oscillations to occur if the fluid viscosity is low enough. Delker et al. [11] and Lago and Araujo [12] write about the rise of liquids in columns of glass beads and find Lucas-Washburn behavior for small times, however deviations for later times. In 2000 Zhmud et al. [13]

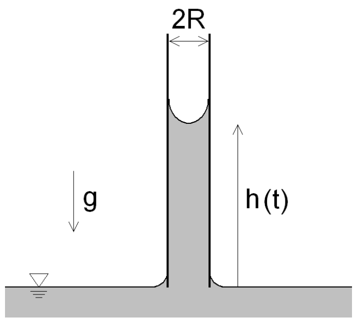

Fig. 1. Setup for capillary driven flow showing the liquid reservoir and a tube.

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give a good overview over the solutions for the different time regimes and derive short- and long time asymptotic solutions. Siebold et al. [14] carry out capillary rise experiments in glass capillaries and packed powder to investigate the effect of the dynamic contact angle. Hamraoui and Nylander [15] provide an analytical approach for setups with a highly dynamic contact angle. 2004 Chan et al. [16] give factors affecting the significance of gravity on infiltration of a liquid into a porous medium. Lockington and Parlange [17] find an equation for the capillary rise in porous media. Xue et al. [18] write about dynamic capillary rise with hydrostatic effects. In a recent paper Chebbi [19] investigates the dynamics of liquid penetration and compares numerical results with asymptotic solutions.
To look at the problem in more detail the momentum balance of a liquid inside a tube shall be presented. The following assumptions hold: i) the flow is one dimensional, ii) no friction or inertia effects by displaced air occur, iii) no inertia or entry effects in the liquid reservoir, iv) the viscous pressure loss inside the tube is given by the Hagen-Poiseuille respectively the Darcy law both valid for laminar flow, and v) the constant capillary pressure can be calculated with the static contact angle θ and the tube (or pore) radius R (see Appendix 1). With these assumptions the momentum balance of a liquid inside a capillary tube gives:

$$\frac{2\sigma\cos\theta}{R} = \rho g h \sin\psi + \frac{8\mu h}{R^2}\dot{h} + \rho\frac{d(h\dot{h})}{dt}. \qquad (1)$$

Here σ refers to the surface tension, R to the inner tube radius, ρ to the fluid density, g to gravity and μ to the dynamic viscosity. In Eq. (1) the individual terms refer to (left to right):
- The capillary pressure
- The gravity term (hydrostatic pressure)
- The viscous pressure loss (Hagen-Poiseuille)
- The inertia term

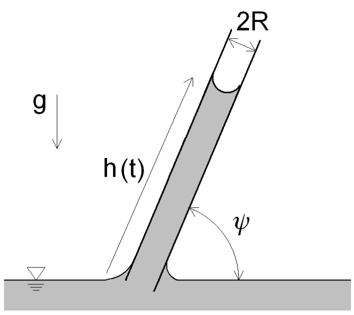

Fig. 2. Setup with an inclined tube.

ψ (see Fig. 2) is the angle formed between the inclined tube and the free liquid surface. It shall be mentioned that for an inclined setup the height h is not the absolute height in relation to the liquid reservoir level but the distance covered within the tube.

When it comes to the momentum equation of a liquid in a generic porous medium (see Fig. 3), the Darcy law can be used. It gives the viscous pressure loss as

$$\nabla p = -\frac{\mu}{K}v_s, \qquad (2)$$

where v_s is the volume averaged velocity (superficial velocity) and K the permeability of the porous medium. Comparing the Hagen-Poiseuille law and Eq. (2) shows that

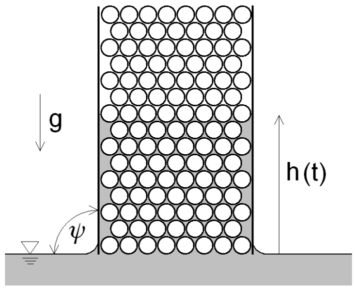

Fig. 3. Setup using a porous medium.

both laws are interchangeable with each other giving

$$R^2 = \frac{8K}{\phi}, \qquad (3)$$

with ϕ being the porosity of the material. The porosity is included as both laws are defined for the intersticial (Hagen-Poiseuille) and the superficial velocity (Darcy) respectively. Thus the momentum equation in a porous medium using the Darcy law reads

$$\frac{2\sigma\cos\theta}{R} = \rho g h \sin\psi + \frac{\phi\mu h}{K}\dot{h} + \rho\frac{d(h\dot{h})}{dt}. \qquad (4)$$

For porous media one possible approach to experimentally determine the two parameters R and K is to do a first evaluation of the maximum reachable height (static case) to obtain the radius R for the capillary pressure. Later the permeability K can be obtained by fitting the calculated liquid rise curve to experimental values.
The differential equations (1) and (4) cannot be easily solved analytically, but numerical methods may be used. However, as an analytical solution is favorable, solutions can be found for certain flow regimes where individual terms of Eqs. (1) and (4) can be neglected. Stange [20] claims that there are four time regimes. For small times the inertia term dominates, later the convective losses in the entry region (not modeled here), then the viscous term and finally the hydrostatic term. For infinite times a steady state is reached where the hydrostatic pressure balances the capillary pressure. As applications of capillary flow or experiments are often bound to certain time regimes, it is feasible to neglect the corresponding terms in Eq. (1) to obtain analytic solutions. In the following two of these will be presented.

Viscous dominated flow

Lucas [1] 1918 and Washburn [2] 1921 consider a flow regime where the influence of inertia and the influence of gravity can be neglected, thus simplifying Eq. (1) to

$$\frac{2\sigma\cos\theta}{R} = \frac{8\mu h\dot{h}}{R^2}. \quad (5)$$

Rearranging gives

$$h\frac{dh}{dt} = \frac{\sigma R\cos\theta}{4\mu}. \quad (6)$$

Solving this ordinary differential equation with the initial condition $h(0) = 0$ by means of separation of variables leads to the well-established Lucas-Washburn equation:

$$h^2 = \frac{\sigma R\cos\theta}{2\mu}t. \quad (7)$$

The Lucas-Washburn solution is probably the most used equation when it comes to the characterization of capillary transport or "wicking" in capillary tubes, porous media or the capillarities in a pack of powder. Unfortunately the fairly simple - and thus nice to handle - Eq. (7) has some limitations. For small times near zero the fluid velocity is approaching infinity, which is not feasible. This discrepancy can be explained with the neglect of the inertia term. Also when flow is occurring in a vertical capillary under gravity there is no limit to the maximum reachable height, which originates from the neglect of the gravity term. In the following sections the gravity term shall not be neglected which still allows to give an analytic solution as already shown by Washburn in 1921, however in terms of $t(h)$ and not $h(t)$ as we seek it.

Viscous and hydrostatic dominated flow regime

To extend the just derived Lucas-Washburn equation to flows where gravity and thus hydrostatic pressure have to be taken into account one is only allowed to neglect the inertia term of Eqs. (1) or (4) giving (here shown for Eq. (4)):

$$\frac{2\sigma\cos\theta}{R} = \rho g h\sin\psi + \frac{\phi\mu h}{K}\dot{h}. \quad (8)$$

Rearranging gives

$$\dot{h} = \frac{2\sigma\cos\theta}{\phi\mu}\frac{K}{R}\frac{1}{h} - \frac{\rho K g\sin\psi}{\phi\mu}, \quad (9)$$

valid for $h \neq 0$ as there is a singularity.

To simplify the equation one may introduce the constants (capillary tube and Darcy version)

$$a = \frac{\sigma R\cos\theta}{4\mu} \widehat{=} \frac{2\sigma\cos\theta}{\phi\mu}\frac{K}{R} \quad (10)$$

and

$$b = \frac{\rho g R^2\sin\psi}{8\mu} \widehat{=} \frac{\rho K g\sin\psi}{\phi\mu}. \quad (11)$$

Thus Eq. (9) reduces to

$$\dot{h} = \frac{a}{h} - b. \quad (12)$$

As mentioned above, an analytic solution to this differential equation is given by Washburn [2] or Lukas and Soukupova [21]. It is calculated as follows: Eq. (12) can be rewritten as

$$dt = \frac{hdh}{a - bh}. \quad (13)$$

After integration as shown in Appendix 2 one obtains

$$t = -\frac{h}{b} - \frac{a}{b^2}\ln(a - bh) + C. \quad (14)$$

To find the unknown constant C the initial condition

$$h(t \to 0) = 0 \quad (15)$$

can be used to give

$$C = \frac{a}{b^2}\ln(a). \quad (16)$$

This leads to following implicit analytic form

$$t = -\frac{h}{b} - \frac{a}{b^2}\ln\left(1 - \frac{bh}{a}\right), \quad (17)$$

which is the result of Washburn or Lukas and Soukupova in terms of $t = t(h)$. Hamraoui and Nylander [15] find this solution to diverge as the liquid approaches the equilibrium height. In 2000 Zhmud et al. [13] evolve a long term asymptotic solution in terms of $h(t)$, shown here rearranged as

$$h(t) = \frac{a}{b}(1 - e^{-\frac{b^2 t}{a}}). \quad (18)$$

To obtain a more accurate solution for $h(t)$ we follow a new approach. Eq. (17) can be multiplied by $-b^2/a$ and by subtracting 1 on both sides one obtains

$$-1 - \frac{b^2 t}{a} = \frac{hb}{a} - 1 + \ln\left(1 - \frac{bh}{a}\right), \quad (19)$$

which by taking the power of e gives after rearrangement

$$-e^{-1-\frac{b^2 t}{a}} = \left(\frac{hb}{a} - 1\right)e^{\frac{hb}{a}-1}. \quad (20)$$

At this point the Lambert W function $W(x)$ named after Johann Heinrich Lambert, and defined by an inverse exponential function

$$x = W(x)e^{W(x)} \quad (21)$$

can be used to solve for h. It can be seen that Eq. (20) follows the form

$$y(t) = x(h)e^{x(h)}. \quad (22)$$

By definition the W function can be written as

$$y(t) = W(y(t))e^{W(y(t))}. \quad (23)$$

Relating Eq. (22) and Eq. (23) gives

$$x(h)e^{x(h)} = W(y(t))e^{W(y(t))}. \quad (24)$$

From this it can be seen that

$$x(h) = W(y(t)). \quad (25)$$

Coming back to Eq. (20) the inverse properties of the Lambert W function can be used to give

$$\frac{bh}{a} - 1 = W\left(-e^{-1-\frac{b^2 t}{a}}\right). \quad (26)$$

After rearranging one obtains

$$h(t) = \frac{a}{b}\left[1 + W(-e^{-1-\frac{b^2 t}{a}})\right], \quad (27)$$

which is a full analytic solution in terms of $h = h(t)$, and can be verified as shown in Appendix 3. From now on we will refer to it as the extended solution. In Fig. 4 this extended solution is calculated and plotted for a setup using silicon fluid (0.93 cSt) in a 0.1 mm radius borosilicate glass capillary. The material properties are found to be $\theta_s = 16.3°$, $\mu = 7.6 \times 10^{-4}$ Pa s and $\sigma = 15.9 \times 10^{-3}$ N/m. The other lines in Fig. 4 refer to the Lucas-Washburn equation (Eq. (7)) and the long time asymptotic solution by Zhmud et al. [13] (Eq. (18)). The two numerical simulations are calculated with constant contact angle and dynamic contact angle as done by Chebbi [19], respectively. For the numeric simulations inertia is neglected and for the dynamic contact angle the equation given by Jiang et al. [22] (see Appendix 1) is used.

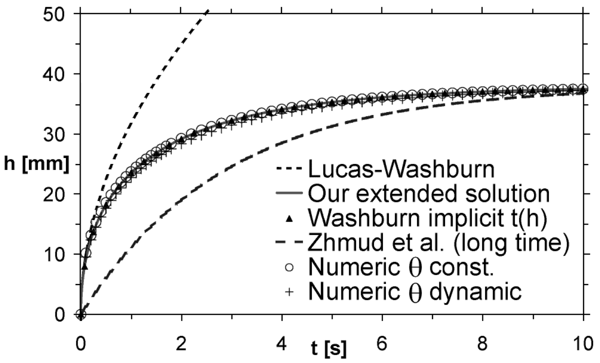

Fig. 4. Diagram showing the different solutions for silicon fluid 0.93 cSt in a 0.1 mm radius borosilicate glass capillary. Height h is plotted versus time t.

For certain cases it might be of interest to consider a more general definition of the initial condition (Eq. (15)) like

$$h(t_0) = h_0. \tag{28}$$

This leads to

$$h(t) = \frac{a}{b}\left[1 + W\left(\frac{(-a + bh_0)e^{-1+\frac{b(bt_0+h_0-bt)}{a}}}{a}\right)\right], \tag{29}$$

as is shown in Appendix 4. Regarding Eq. (29) it can be seen that for $t_0 = 0$ and $h_0 = 0$ it is equal to Eq. (27), the extended solution derived before.

Further information on the Lambert W function as defined in Eq. (21), and its applications shall be given [23], [24] and [25]. Its solutions are partly in the complex plane, but switch to real values for $-1/e \leq x$ as shown in Fig. 5. Also, the Lambert W function has been used to solve differential equations before. For example in 1993 Barry et al. [26] use the Lambert W function to give an analytical solution to a transcendental equation related to the problem the present work deals with. To calculate water movement in unsaturated soil they use a differential equation which is a simplified form of Eq. (12) with $a = b$ (here shown rearranged and in our variables)

$$\dot{h} = \frac{a}{h} - a. \tag{30}$$

Solving for the initial condition $h(t \to 0) = 0$ gives an equation featuring the Lambert W function

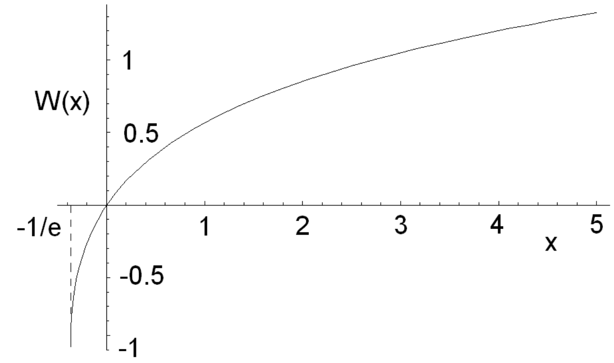

Fig. 5. Upper branch of the Lambert W function for $-1/e \leq x \leq 5$.

$$h(t) = 1 + W(-e^{-1-at}). \tag{31}$$

Barry et al. mention that it can be used to calculate infiltration as well as capillary rise of moisture in soils.

Practical evaluation of the Lambert W function

When applying Eq. (27) to practical problems it is important to be able to calculate the numeric value of the Lambert W function. In many commercial mathematical programs the Lambert W function is already included with its call being W[x] or ProductLog[x]. It is also possible to use spreadsheet calculation programs that don't feature Lambert W. In this case it is required to use an approximation expression for $W(x)$ that covers the relevant range. For our case such an approximate function is given by Barry et al. [26] (slightly rearranged) as

$$W(x) \approx -1 + \frac{\sqrt{2 + 2ex}}{1 + \frac{4.13501\sqrt{2+2ex}}{12.7036+\sqrt{2+2ex}}}, \tag{32}$$

with e being Euler's number. This equation accounts for the relevant range of $-e^{-1} \leq x \leq 0$ (upper branch) with a maximum relative error of 0.1%.

Steady state time

Regarding Eq. (27) one may notice that for infinite time the height gain converges into a maximum value h_{max}. This is the point where the hydrostatic pressure balances the capillary pressure, or mathematically $W(0) = 0$. This gives

$$h_{max} = h(t \to \infty) = \frac{a}{b} = \frac{2\sigma\cos\theta}{R\rho g \sin\psi}. \tag{33}$$

Here a and b are defined by Eqs. (10) and (11). As h approaches h_{max} at some point the height increase is so small that one may speak of a steady state. To find the time needed to reach this point, we define the steady state time t_s to be exactly that time where h has reached 99% of h_{max}. Thus one may write

$$h(t_s) = 0.99h_{max}. \tag{34}$$

Using Eq. (33) and Eq. (27) gives

$$0.99\frac{a}{b} = \frac{a}{b}(1 + W(-e^{-1-\frac{b^2 t_s}{a}})). \tag{35}$$

After rearranging one obtains

$$-0.01 = W(-e^{-1-\frac{b^2 t_s}{a}}), \tag{36}$$

and with Eq. (21)

$$-0.01e^{-0.01} = -e^{-1-\frac{b^2 t_s}{a}}. \tag{37}$$

To obtain t_s

$$-1 - \frac{b^2 t_s}{a} = \ln(0.01) - 0.01, \tag{38}$$

and finally

$$t_s = \frac{a}{b^2}\left(-\ln(0.01) + 0.01 - 1\right) \approx \frac{3.62a}{b^2}. \tag{39}$$

This may also be written with all variables to give

$$t_s = 3.62\frac{16\sigma\mu\cos\theta}{R^3\rho^2 g^2 \sin^2\psi} \,\widehat{=}\, 3.62\frac{2\phi\sigma\mu\cos\theta}{R\rho^2 g^2 K \sin^2\psi}. \tag{40}$$

Flow velocity

To obtain the flow velocity $\dot{h}(t)$ it is necessary to differentiate the height $h(t)$. For the Lucas-Washburn equation one obtains

$$\dot{h}(t) = \sqrt{\frac{a}{2t}}, \tag{41}$$

while using the extended solution including gravity yields

$$\dot{h}(t) = \frac{-b\,W(-e^{-1-\frac{b^2 t}{a}})}{1 + W(-e^{-1-\frac{b^2 t}{a}})}. \tag{42}$$

Both velocity functions are only defined for $t > 0$. Further details on differentiating the Lambert W function can be found in Appendix 3.

Dimensional analysis

To generalize the obtained solutions the introduction of dimensionless numbers is always of interest. Zhmud et al. [13] use

$$h^* = \frac{h(t)}{h_{max}}, \tag{43}$$

and

$$t^* = \frac{\rho g r^2 t}{8\mu h_{max}}. \tag{44}$$

Ichikawa and Satoda [9] give dimensionless numbers that also allow to account for the dynamic contact angle. For the time they find

$$t^* = \frac{t}{T_0}, \tag{45}$$

where T_0 is defined as the characteristic time of viscous effects

$$T_0 = \frac{\rho D^2}{32\mu}. \tag{46}$$

To scale the height they use following expression

$$h^* = \sqrt{\frac{128}{\mathrm{Me}}}\frac{h}{D}, \tag{47}$$

where Me is defined as Reynolds number divided by Capillary number

$$\mathrm{Me} = \frac{\sigma_0 \rho D}{\mu^2}. \tag{48}$$

Here σ_0 refers to the static surface tension. Ichikawa and Satoda scale the driving force by

$$D_f = \frac{\sigma\cos\theta}{\sigma_0} + \frac{\mathrm{Bo}}{4}, \tag{49}$$

where Bo is the Bond number as given in Eq. (52). In our work, regarding Eqs. (39) and (40) we derive a dimensionless number for the time, the "capillary time number" TN

$$\mathrm{TN} = \mathrm{t}\frac{\mathrm{b}^2}{\mathrm{a}} = \frac{\mathrm{t}\rho^2\mathrm{g}^2\mathrm{R}^3\sin^2\psi}{16\sigma\mu\cos\theta} \,\widehat{=}\, \frac{\mathrm{tR}\rho^2\mathrm{g}^2\mathrm{K}\sin^2\psi}{2\phi\sigma\mu\cos\theta}. \tag{50}$$

This TN can be transferred to t^* (Eq. (44)) used by Zhmud et al.. The TN can be interpreted as a specialized form of the following term:

$$\frac{(\text{Gravity force})^2}{\text{Viscous force} \times \text{ Surface tension force}} = \frac{\mathrm{Bo}^2}{\mathrm{Ca}}, \tag{51}$$

with the Bond number relating gravity forces to surface tension forces

$$\mathrm{Bo} = \frac{\rho\mathrm{g}\sin\psi\mathrm{R}^2}{\sigma}, \tag{52}$$

and the Capillary number relating viscous forces to surface tension forces being

$$\mathrm{Ca} = \frac{\mu\mathrm{v}}{\sigma} \sim \frac{\mu\mathrm{R}}{\sigma\mathrm{t}}. \tag{53}$$

From Eq. (39) it can be derived that if TN is larger than 3.62 a steady state has been reached as shown in Fig. 6. To give a dimensionless number including the height, the "capillary height number" HN, one may relate the height h to the maximum obtainable height h_{max} as done by Zhmud et al.

$$\mathrm{HN} = \mathrm{h}\frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{hR}\rho\mathrm{g}\sin\psi}{2\sigma\cos\theta}. \tag{54}$$

The HN can also be regarded as a "capillary Bond number" since it relates gravity forces to surface tension forces.
Using these dimensionless numbers the Lucas-Washburn equation (Eq. (7)) as well as the extended solution (Eq. (27)) can be made dimensionless giving

$$\mathrm{HN} = \sqrt{2\mathrm{TN}}, \tag{55}$$

for the Lucas-Washburn equation and

$$\mathrm{HN} = 1 + \mathrm{W}(-\mathrm{e}^{-1-\mathrm{TN}}) \tag{56}$$

for the extended solution including the gravity term. Fig. 6 shows that in the beginning the Lucas-Washburn solution fits very good to the extended solution (Eq. (27)), however tends to deviate to higher values for longer times due to the neglect of gravity. For TN > 3.62 the extended solution reaches a steady state. Regarding the velocity of the ex-

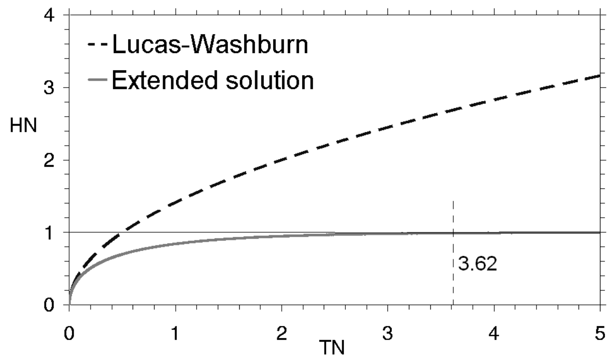

Fig. 6. Dimensionless representation of the Lucas-Washburn equation and Eq. (27). Steady state reached for TN $\geq$ 3.62.

tended solution (Eq. (42)) one may derive a dimensionless number for the flow velocity, the "capillary velocity number" VN

$$\mathrm{VN} = \frac{\dot{\mathrm{h}}}{\mathrm{b}} = \frac{8\dot{\mathrm{h}}\mu}{\rho \mathrm{g} \mathrm{R}^2 \sin\psi} \mathrel{\widehat{=}} \frac{\phi\dot{\mathrm{h}}\mu}{\rho \mathrm{g} \mathrm{K} \sin\psi}. \tag{57}$$

The VN can be interpreted as viscous forces in relation to gravity forces

$$\mathrm{VN} = \frac{\mathrm{Ca}}{\mathrm{Bo}}. \tag{58}$$

Using TN and VN the flow velocity can be rewritten as

$$\mathrm{VN} = \sqrt{\frac{1}{2\mathrm{TN}}} \tag{59}$$

for the Lucas-Washburn equation and

$$\mathrm{VN} = \frac{-\mathrm{W}(-\mathrm{e}^{-1-\mathrm{TN}})}{1 + \mathrm{W}(-\mathrm{e}^{-1-\mathrm{TN}})} \tag{60}$$

for the extended solution including the gravity term. These results are plotted in Fig. 7.

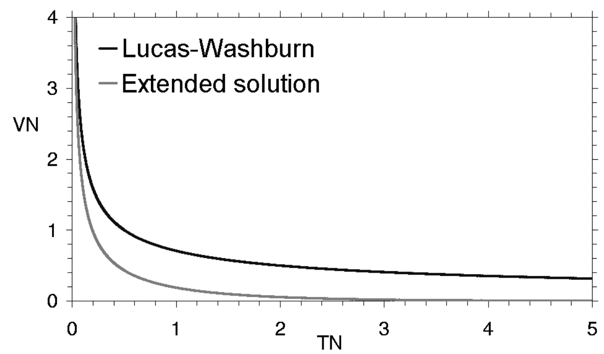

Fig. 7. Dimensionless representation of flow velocity calculated by differentiating the Lucas-Washburn equation and our extended solution Eq. (27).

When considering a setup using a horizontal capillary or an experiment under microgravity the Lucas-Washburn equation has to be applied. However it is not possible to use the "capillary time number" TN and the "capillary height number" HN derived before, since g or $\sin\psi$ would be zero. For these cases the following approach may be used: The Lucas-Washburn equation (Eq. (7)) can be rearranged to

$$\frac{h^2}{R^2} = \frac{\sigma\cos\theta}{2\mu R} t. \tag{61}$$

When using a dimensionless height

$$h^* = \frac{h}{R}, \tag{62}$$

and a modified capillary number

$$\mathrm{Ca}^* = \frac{2\mu \mathrm{R}}{\sigma \mathrm{t} \cos\theta} \mathrel{\widehat{=}} \frac{16\mu \mathrm{K}}{\phi \mathrm{R}\sigma \mathrm{t}\cos\theta}, \tag{63}$$

the Lucas-Washburn equation can be rewritten as

$$h^* = \sqrt{\frac{1}{\mathrm{Ca}^*}}. \tag{64}$$

Using these scalings, experimental values gathered with different liquids or different radii can all be expressed by a single curve as is shown later in the chapter on experimental evidence.

Validity of the Lucas-Washburn equation

When comparing the Lucas-Washburn equation Eq. (7) and the extended solution (Eq. (27)) one may notice that the Lucas-Washburn equation is preferable due to its simplicity, however it is not valid for increasing time. This rises the question to which time t_v it may be used when taking into account an acceptable error, where for later times the extended solution including gravity has to be considered. This chapter tries to give more insight to this question. If the acceptable discrepancy is - for example - 1% one may write:

$$(100\% - 1\%)\ h_{Lucas\ Washburn} = h_{extended\ solution} \tag{65}$$

equal to

$$(1 - 0.01)\sqrt{2at_v} = \frac{a}{b}(1 + W(-e^{-1-\frac{b^2 t_v}{a}})), \tag{66}$$

with

$$\sqrt{2at} = \sqrt{\frac{\sigma R\cos\theta}{2\mu} t} \tag{67}$$

being the Lucas-Washburn equation (Eq. (7)). To solve the transcendental equation Eq. (66) for t_v we use the following approach. Considering the dimensionless number TN derived in Eq. (50) one may assume that the solution (t_v) to Eq. (66) can be expressed by means of this number

$$t_v = \mathrm{TN_v}\frac{\mathrm{a}}{\mathrm{b}^2}, \tag{68}$$

with $\mathrm{TN_v}$ being the unknown value. Substituting t_v in Eq. (66) gives

$$0.99\sqrt{2\mathrm{TN_v}} = 1 + W(-e^{-1-\mathrm{TN_v}}). \tag{69}$$

This transcendental equation can now be solved by means of numerical methods giving $\mathrm{TN_v} = 0.0004523$. Thus the result is that the Lucas-Washburn equation can be used up to $t = 0.0004523a/b^2$ when accepting an error of 1%. The height reached at this point may again be expressed in terms of HN

$$\mathrm{HN_v} = \frac{\mathrm{h_v}}{\mathrm{h_{max}}} = 1 + \mathrm{W}(-\mathrm{e}^{-1-\mathrm{TN_v}}), \tag{70}$$

giving $\mathrm{HN_v} = \mathrm{h_v}/\mathrm{h_{max}} = 0.029775$ for an error of 1%. Values for further errors are given in Table 1.

Table 1
Further values for different errors

error	$TN_v = t_v\, b^2/a$	$HN_v = h_v/h_{max}$
1%	0.0004523	0.029775
5%	0.0115465	0.144366
10%	0.0475088	0.277424

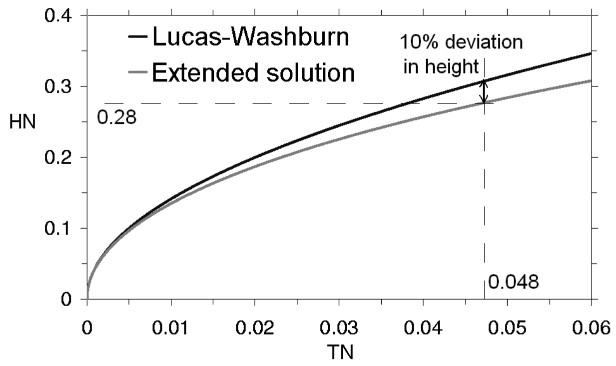

Fig. 8. Dimensionless representation of the Lucas-Washburn equation and the extended solution (Eq. (27)), mark at 10% deviation.

Generalizing it can be concluded that under gravity the Lucas-Washburn equation can be used up to about 10% of the maximum reachable height.

Experimental evidence

To verify the obtained results the investigation done by Stange [20] can be used as a benchmark. He examines the fluid rise in capillary tubes made of borosilicate glass with varying radius. Also the angle of inclination ψ is varied, while two different liquids, silicon fluid (1.0 cSt) and FC 77 are used. The height recordings are performed by optical means and are now plotted in dimensionless form in Fig. 9. The dimensionless capillary height number "HN" and the capillary time number "TN" introduced before are used.

Table 2
Experimental Data by Stange

Name	Inner radius	Fluid	Inclin. ψ	θ
Exp1	0.088 mm	SF1	32.3°	16.3°
Exp2	0.104 mm	FC77	32.3°	28.0°
Exp3	0.1405 mm	SF1	88.7°	16.3°

From the dimensionless plot it can be seen that the experimental data by Stange matches the values predicted by the extended solution Eq. (27). Especially for the steady state values a good consistency can be observed. For TN between 0.2 and 1.4 some deviation to lower results can be seen, this may be explained by a higher friction than expected originating from surface roughness, the neglect of entry effects or assuming a constant contact angle.

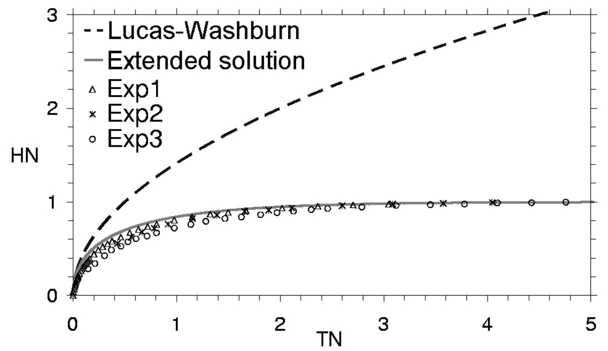

Fig. 9. Dimensionless experimental results ($\psi \neq 0$) by Stange [20] as described in Table 2.

As mentioned in the chapter "dimensional analysis" the scaling using TN and HN is not applicable for horizontal capillaries as $g \cdot \sin\psi$ is zero. In Fig. 10 the dimensionless height h^* and the modified capillary number Ca^* are used. It can be seen that both experimental rows for horizontal tubes by Stange collapse to the curve given by the Lucas-Washburn equation.

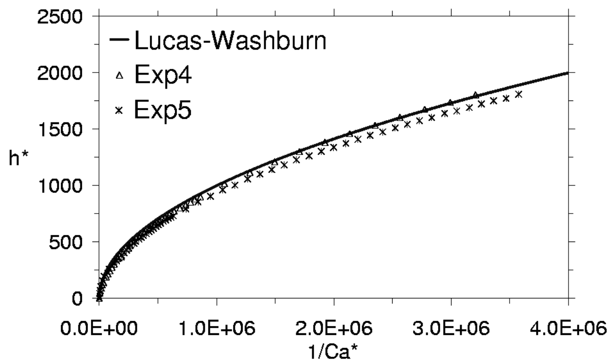

Fig. 10. Dimensionless experimental results ($\psi = 0$, horizontal capillary) by Stange as described in Table 3.

Table 3
Experimental Data by Stange

Name	Inner radius	Fluid	Inclin. ψ	θ
Exp4	0.109 mm	SF1	0°	16.3°
Exp5	0.104 mm	FC77	0°	28.0°

Conclusion

A method for deriving an analytic solution to the momentum balance of a liquid in a capillary tube is presented. The well-established Lucas-Washburn equation is shown as well as an extended solution introduced. The extended solution includes the gravity term (hydrostatic pressure) and enables the calculation of the liquid rise behavior for longer times. The time necessary to reach a steady state is examined and several relevant dimensionless numbers are found. By means of these numbers a dimensionless plot of the Lucas-Washburn equation and the extended solution including gravity can be plotted. The flow velocity is obtained by differentiating the height and a dimensionless number for its description is found. Also the error made when neglecting gravity and using the Lucas-Washburn equation is determined. As an outlook it can be stated that deriving further analytical solutions to the momentum equation is still of great interest, as it could lead to a solution that is also valid for shortest time regimes at the beginning of the capillary rise process.

Acknowledgement

The funding of the research project by the German Federal Ministry of Education and Research (BMBF) through the Research Training Group PoreNet is gratefully acknowledged.

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Appendix 1

When a liquid-gas interface is subject to motion the dynamic contact angle θ_d formed between the liquid and solid is different from the static contact angle θ_s. Jiang et al. [22] (based on data by Hoffman [27]) give the following correlation for the dynamic contact angle

$$\frac{\cos\theta_d - \cos\theta_s}{\cos\theta_s + 1} = -\tanh(4.94\mathrm{Ca}^{0.702}), \tag{71}$$

where the capillary number is defined as

$$\mathrm{Ca} = \frac{\mu\dot{\mathrm{h}}}{\sigma}. \tag{72}$$

Bracke et al. [28] find

$$\frac{\cos\theta_d - \cos\theta_s}{\cos\theta_s + 1} = -2\mathrm{Ca}^{0.5}. \tag{73}$$

When considering silicon fluid (0.93 cSt) in a borosilicate glass capillary (like in Fig. 4) and using some arbitrary rise rates one obtains the values shown in Table 4. Coming back to the numerical simulation presented in Fig. 4 (capillary radius is 0.1 mm) Figs. 11 and 12 show further results of the simulation featuring the dynamic contact angle. Due to

Table 4
Calculated values of the dynamic contact angle for different capillary rise rates. "eff. dev." refers to the effective deviation of the cosine values of the dynamic vs. static contact angle.

	Jiang et al.		Bracke et al.	
$\dot{h}$ [mm/s]	θ_d [°]	eff. dev. [%]	θ_d [°]	eff. dev. [%]
1	18.04	0.94	21.14	2.83
5	21.26	2.90	25.94	6.31
10	23.87	4.72	29.06	8.93
50	34.95	14.6	39.81	20.0

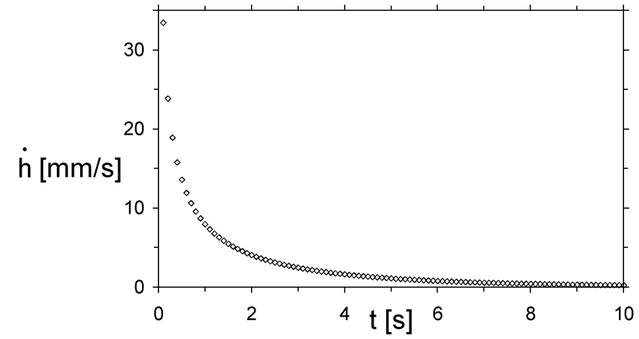

Fig. 11. Numerical results (interface velocity) for silicon fluid (0.93 cSt) in a 0.1 mm capillary.

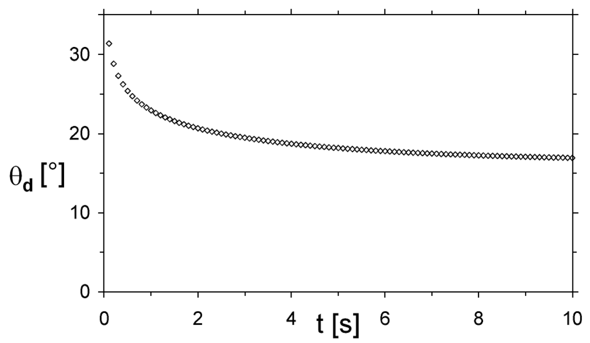

Fig. 12. Dynamic contact angle for same simulation.

the small deviation between the simulation with the static contact angle and the dynamic one it can be concluded that assuming a constant contact angle is feasible for capillary rise in the investigated flow regime as typical contact line rise rates (depending on radius, viscosity etc.) are mostly found to be in the range of some mm/s. However one must be aware that for very large capillaries and short times the contact angle can not be assumed as constant and inertia effects may as well be significant.

Appendix 2

To solve the integral of Eq. (13) the following approach may be used:

$$t = \int \frac{h}{a - bh} dh = \int \frac{(bh - a) + a}{b(a - bh)} dh. \qquad (74)$$

This may be rearranged to

$$t = \int -\frac{1}{b} dh + \int \frac{a}{b} \frac{1}{a - bh} dh. \qquad (75)$$

Preparing the substitution

$$y = a - bh \qquad (76)$$

by

$$dy = -bdh \qquad (77)$$

gives

$$t = \int -\frac{1}{b} dh - \int \frac{a}{b^2} \frac{1}{y} dy. \qquad (78)$$

Solving and reversing the substitute gives

$$t = -\frac{h}{b} - \frac{a}{b^2} \ln(y) = -\frac{h}{b} - \frac{a}{b^2} \ln(a - bh). \qquad (79)$$

Appendix 3

To show that the extended solution Eq. (27) fulfills the differential equation (Eq. (12)) one may use the derivative of the solution and put it back into the initial differential equation.

By differentiating the defining Eq. (21) for $W(x)$ [24] one obtains

$$\frac{dx}{dW(x)} = e^{W(x)} + W(x)e^{W(x)}. \qquad (80)$$

Rearranging gives

$$W'(x) = \frac{1}{e^{W(x)} + W(x)e^{W(x)}}, \qquad (81)$$

further

$$W'(x) = \frac{1}{e^{W(x)}(1 + W(x))}, \qquad (82)$$

and

$$W'(x) = \frac{W(x)}{W(x)e^{W(x)}(1 + W(x))}. \qquad (83)$$

Per definitionem:

$$x = W(x)e^{W(x)} \qquad (84)$$

and finally

$$W'(x) = \frac{W(x)}{x(1 + W(x))}. \qquad (85)$$

To ease the handling of Eq. (27) a coefficient z can be defined:

$$z = -e^{-1-\frac{b^2 t}{a}}, \qquad (86)$$

the derivative is

$$\frac{dz}{dt} = z\frac{-b^2}{a}. \qquad (87)$$

Putting Eq. (85), Eq. (87) (the inner derivative) and Eq. (27) into Eq. (12) gives

$$\frac{a}{b} \frac{W(z)}{z(1 + W(z))} z \frac{-b^2}{a} = \frac{ab}{a(1 + W(z))} - b. \qquad (88)$$

After some rearrangement one obtains

$$\frac{-b^2}{b} W(z) = \frac{-b^2}{b} W(z) \qquad (89)$$

which proves that Eq. (27) is a solution to Eq. (12).

Appendix 4

In following the derivation of the analytic solution for the general initial condition $h(c) = d$ shall be explained in more detail. Starting from Eq. (14)

$$t = -\frac{h}{b} - \frac{a}{b^2}\ln(a - bh) + C \tag{90}$$

one can use the initial condition to obtain

$$t = -\frac{h}{b} - \frac{a}{b^2}\ln(a - bh) + c + \frac{d}{b} + \frac{a}{b^2}\ln(a - bd). \tag{91}$$

Rearranging gives

$$\frac{b^2 c}{a} + \frac{bd}{a} - \frac{b^2 t}{a} - \frac{bh}{a} = \ln((a - bh)(bd - a)), \tag{92}$$

and

$$\frac{b^2 c}{a} + \frac{bd}{a} - \frac{b^2 t}{a} - \frac{bh}{a} = \ln((bh - a)(a - bd)), \tag{93}$$

which gives

$$\ln(-a+bd) - 1 + \frac{b}{a}(bc+d-bt) = \ln(bh-a) + \frac{bh}{a} - 1. \tag{94}$$

After some further rearrangement

$$\frac{(-a+bd)e^{-1+\frac{b(bc+d-bt)}{a}}}{a} = \left(\frac{bh}{a} - 1\right) e^{\left(\frac{bh}{a}-1\right)} \tag{95}$$

and applying the inverse properties of the Lambert W function as shown before one finally obtains

$$h(t) = \frac{a}{b}\left[1 + W\left(\frac{(-a+bd)e^{-1+\frac{b(bc+d-bt)}{a}}}{a}\right)\right]. \tag{96}$$