Gravitating non-Abelian solitons and hairy black holes in higher dimensions

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Contents

- Gravitating Yang-Mills fields in string theory.
Einstein and Yang-Mills in D=4

Pure gravity /attraction/

\[ \mathcal{L}_E = -\frac{R}{16\pi G} \]

has no solitons /Lichenrowitz/, there are black holes. Pure Yang-Mills /repulsion/

\[ \mathcal{L}_{YM} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \]

is scale invariant \(\Rightarrow\) no solitons /Deser, Coleman/. Gravity + Yang-Mills = attraction+repulsion

\[ \mathcal{L}_{EYM} = -\frac{R}{16\pi G} - \frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \]
EYM solutions in D=4

- Solitons of Bartnik-McKinnon ⇒ the first example of globally regular gravitational solitons.

- EYM black holes ⇒ the first example of hairy black holes, \( B_k^a \sim 1/r^3 \) /Galtsov+M.S.V./

- Generalizations: non-spherically symmetric, non-static solitons/black holes, coupling to other fields . . . /Kleihaus-Kunz+ . . . /

- Gravitating t’Hooft-Polyakov /Breitenlohner, Forgacs, Maison/

Manifest counter-examples to a number of electroweak theorems /uniqueness, staticity, circularity, Israel’s theorem, no-hair theorems . . . /

D=5, pure gravity

• if \( g^{E}_{\mu\nu} \) is a regular D=4 gravitational instanton \( \Rightarrow \)

\[
g_{MN}dx^M dx^N = dt^2 - g^{E}_{\mu\nu}dx^\mu dx^\nu
\]

is 5D Ricci flat. No AE instantons \( \Rightarrow \) no solitons.

• Localized but not regular: black holes

\[
g_{MN}dx^M dx^N = N dt^2 - \frac{dr^2}{N} - r^2 d\Omega_3^2, \quad N = 1 - \left( \frac{rg}{r} \right)^2
\]

• black strings: if \( g_{\mu\nu} \) is 4D Ricci flat (black hole) \( \Rightarrow \)

\[
g_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu - (dx^4)^2
\]
pure YMA in D=5, particles

Pure Yang-Mills is not scale invariant in $D \neq 4$ (the coupling $g$ is dimensionful) $\Rightarrow \exists$ solitons. If $A^a_\mu(x^\nu)$ is a solution of 4D Euclidean YM equations, then

$$A^a_M = (0, A^a_\mu(x^\nu)) \quad / \mu = 1, 2, 3, 4/$$

will be a soliton in 5D with the energy

$$E = \frac{1}{4g^2} \int (F^a_{\mu\nu})^2 d^4x \geq \frac{8\pi^2|n|}{g^2}$$

Self-dual 4D YM instantons $\Rightarrow$ ‘YM particles’ in D=5.
pure YM in D=5, vortices

If $\partial/\partial x^4$ is a symmetry $\Rightarrow$

$$A^a_M = (0, A^a_i(x^k), H^a(x^k)), \quad /i, k = 1, 2, 3/$$

the energy per unit $x^4$,

$$E = \frac{1}{2g^2} \int ((\partial_i H^a + \varepsilon_{abc} A^b_i H^c)^2 + \frac{1}{2} (F^a_{ik})^2) d^3 x \geq \frac{4\pi|n|}{g^2}$$

coincides with the energy of the D=3 YM-Higgs system $\Rightarrow$

$\Rightarrow$ monopoles, when lifted back to D=5 they become

‘YM vortices’.
Pure Yang-Mills in D=5 admits ‘particle’ and ‘vortex’ solutions.

What happens to them if they are coupled to gravity?
Gravitating YM particles

\[ S = \int \left( -\frac{1}{16\pi G} R - \frac{1}{4g^2} F^a_{MN} F^{aMN} \right) \sqrt{g} \, d^5 x. \]

\[ F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \varepsilon_{abc} A^b_M A^c_N \quad (a = 1, 2, 3), \]

\[ [G^{1/3}] = [g^2] = [\text{length}] \Rightarrow \text{dimensionless coupling} \]

\[ \kappa = \frac{8\pi G}{g^6} \quad / \kappa = 2\alpha^2 / \]

SO(4)-symmetry: if \( \theta^a \) are the invariant forms on \( S^3 \),

\[ ds^2 = \sigma(r)^2 N(r) dt^2 - \frac{dr^2}{N(r)} - r^2 \, d\Omega_3^2, \quad A^a = (1 + w(r)) \, \theta^a, \]

the length scale is \( g^2 \).
Field equations

\[ r^2 N w'' + r w' + \kappa (m - (w^2 - 1)^2) \frac{w'}{r} = 2 (w^2 - 1)w , \]

\[ rm' = r^2 N w'^2 + (w^2 - 1)^2 , \]

\[ N \equiv 1 - \kappa m(r)/r^2 \]

\[ \sigma' = \kappa \frac{w'^2}{r} \sigma \]

If \( \kappa = 0 \implies \sigma = N = 1 \), the YM particle with \( M_{\text{ADM}} = \frac{8}{3} \)

\[ w = \frac{1 - b r^2}{1 + b r^2} , \]

\( b \in \mathbb{R} \) is a scale parameter. What happens if \( \kappa \neq 0 \)?
No finite mass solutions with $\kappa \neq 0$

$M_{\text{ADM}} < \infty \Rightarrow w(\infty) = \pm 1$. Either

- $w = 1 - 2br^2 + O(r^4)$, $m = O(r^3)$ as $r \to 0$ /solitons/ or
- $\exists r_h: N(r_h) = 0$, $N'(r_h) > 0$, $w(r_h) < \infty$ /black holes/

$$M_{\text{ADM}}[w(r)] = m(\infty) =$$

$$= \frac{r_h^2}{\kappa} + \int_{r_h}^{\infty} \frac{dr}{r} \left( r^2 w'^2 + (w^2 - 1)^2 \right) \exp \left( -\kappa \int_{r_h}^{\infty} \frac{w'^2}{r} \, dr \right)$$

One should have

$$\frac{d}{d\lambda} M[w(\lambda r)] = 0 \text{ for } \lambda = 1 \text{ but } \frac{d}{d\lambda} M[w(\lambda r)] < 0 \text{ } \forall \lambda$$
The SO(4) YM particles get completely destroyed by gravity. They resemble a dust, since

\[ T_{\mu\nu} = \epsilon(r) \delta^0_\mu \delta^0_\nu \]

and they can be scaled to an arbitrary size ⇒ repulsion and attraction are not balanced ⇒ equilibrium states are not possible.

The globally regular solutions with infinite mass are quasi-periodic /infinite sequence of static spherical shells/ with the metric approaching the flat metric at large \( r \) (but not fast enough).
Quasi-periodic solutions

\[ m(r) \sim \ln r. \text{ Remedy: } F^4, R^2 \text{ etc terms} \]
Gravitating Yang monopole in D=6

YM field is again = 4D instanton, but this time on $S^4$

$$ds^2 = \sigma(r)^2 N(r) dt^2 - \frac{dr^2}{N(r)} - r^2 (d\xi^2 + \sin^2 \chi d\Omega_3^2), \quad A^a = (1+w(\chi)) \theta^a,$$

YM equations decouple $\Rightarrow w(\chi) = \cos \chi$, Einstein eq-s $\Rightarrow$

$\sigma = 1,$

$$N = 1 - \frac{2Gm(r)}{r^3}, \quad m' = 8\pi, \quad m(r) = 8\pi r + m_0,$$

$\Rightarrow$ mass issue linearly divergent

/Gibbons, Townsend ’06/

$D = 2k + 2$, $\text{SO}(2k)$, $m \sim r^{2k-3}$. 
Gravitating YM vortices

If $\partial/\partial x^4$ is hypersurface orthogonal Killing vector, then

$$g_{MN} dx^M dx^N = e^{-\zeta} g_{\mu\nu} dx^\mu dx^\nu - e^{2\zeta} (dx^4)^2$$

$$A^a_M dx^M = A^a_\mu dx^\mu + H^a dx^4$$

reducing the 5D EYM to the 4D EYM-Higgs-dilaton theory

$$\sqrt{5} g \mathcal{L}_{EYM} = 
\left(-\frac{(4) R}{2\kappa g^6} + \frac{3}{\kappa g^6} (\partial_\mu \zeta)^2 \right) + \frac{1}{2g^2} e^{-2\zeta} (D_\mu H^a)^2 - \frac{1}{4g^2} e^{\zeta} (F^a_{\mu\nu})^2 \right) \sqrt{-\left(4\right) g}$$

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SO(3) symmetry

\[ e^{-\zeta(r)} ds^2 = e^{2\nu(r)} dt^2 - dr^2 - R^2(r) d\Omega_2^2, \]

\[ A^a_k dx^k = (w(r) - 1) \epsilon_{ai} n^i dn^k, \quad H^a = n^a e^{\zeta(r)} h(r) \]

the independent field equations can be represented as a seven-dimensional dynamical system

\[ \frac{d}{dr} y_k = F_k(y_s, \kappa) \]

with \( y_k = \{w, w', h, h', Z = \zeta', R, R'\} \).
Fixed points

I. The origin, \((w, h, Z, R) = (1, 0, 0, 0)\); as \(r \to 0\),
\[
\begin{align*}
w &= 1 - br^2 + O(r^2), \\
h &= ar + O(r^3), \\
Z &= O(r^2), \\
R &= r + O(r^3).
\end{align*}
\] (1)

II. Infinity, \((w, h, Z, 1/R) = (0, 1, 0, 0)\); as \(r \to \infty\),
\[
\begin{align*}
w &= Ar^C e^{-r} + o(e^{-r}), \\
Z &= \kappa Q r^{-2} + O(r^{-3} \ln r), \\
h &= 1 - Cr^{-1} + O(r^{-2} \ln r), \\
R &= r - m \ln r + m^2 r^{-1} \ln r - r_0 + \gamma r^{-1} + O(r^{-2} \ln r).
\end{align*}
\] (2)

The ADM mass
\[
M_{\text{ADM}} = 3(C + (2 + \kappa)Q)
\]
III. “Warped” $AdS_3 \times S^2$: If $4q^3 + 7q^2 + 11q = 1$, then $w^2 = q$,

$$R^2 = \frac{\kappa}{(4q^2 - 13q + 1)} \left(11q - 1\right) \left(1 - q\right), \quad h^2 = \frac{1 - q}{R^2}, \quad Z^2 = -\frac{4q^2 - 13q + 1}{(4q + 1)R^2}$$

Evaluating, $w = 0.29$, $h = \frac{1.27}{\sqrt{\kappa}}$, $Z = \pm \frac{0.31}{\sqrt{\kappa}}$, $R = 0.75\sqrt{\kappa}$ \Rightarrow essentially non-Abelian field, Kantowski-Sachs geometry

$$ds^2 = e^{2(1+\kappa h^2)} Zr \, dt^2 - dr^2 - e^{2Zr} (dx^4)^2 - R^2 \, d\Omega_2^2$$

The characteristic eigenvalues

$$\left( -\frac{2.77}{\sqrt{\kappa}}, -\frac{2.47}{\sqrt{\kappa}}, -\frac{2.12}{\sqrt{\kappa}}, -\frac{0.61}{\sqrt{\kappa}} \pm \frac{1.24}{\sqrt{\kappa}}, \frac{0.88}{\sqrt{\kappa}}, \frac{1.54}{\sqrt{\kappa}} \right)$$
Global solutions

- $\kappa=0$: BPS monopole
- $\kappa \ll 1$: weakly gravitating BPS monopole
- $\kappa \sim 1$: strongly gravitating BPS monopole

exterior solution
interior solution

strong gravity
For $\kappa_{max} = 3.22 > \kappa > \kappa_{min} = 0.11$ one has more than one solution (‘branches’).
Limiting solution

Strongly gravitating solutions have a regular core connected to the asymptotic region by a long throat. For the limiting solution the throat becomes infinite and the solution splits up into the union of two different solutions.

- **Interior solution** interpolates between the regular origin and the Kantowsksi-Sachs (KS).
- **Exterior solution** interpolates between the KS and infinity. Looks like an extreme non-Abelian black string: in the Schwarzschild gauge ($r = R$) one has

\[
\begin{align*}
g_{rr} &\sim (r - r_h)^{-2}, \quad g_{tt} \sim (r - r_h)^{2.02}, \quad g_{44} = e^{2\zeta} \sim (r - r_h)^{2.02} \\
\end{align*}
\]

with $r_h = 0.42$, $\kappa = 0.316$. 
Strong gravity limit

strong gravity  limiting solution

interior

exterior (extreme black string)

+
Limiting solution
Conclusions

Gravitating YM vortices form a one-parameter family (fundamental branch) of globally regular solutions that interpolates between the flat space BPS monopole and the extreme non-Abelian black string.

∃ also excited solutions for which the YM field amplitude \( w \) oscillates around zero value. These solutions do not have the flat space limit.

Generalizations of the fundamental YM vortices have been studied /Brihaye, Hartmann, Radu/. Excited solutions have never been considered.
From the 4D viewpoint the YM vortices are regular gravitating monopoles. Gravitating solitons can usually be generalized to include a small black hole with a non-degenerate horizon in the core /Kastor, Traschen ’92/: replace the boundary condition at the origin, \( r = 0 \), by those at the regular horizon, \( r = r_h \). This generalizes YM vortices to black strings /Hartmann ’04/.

For a given \( \kappa \) there can be several YM vortices \( \Rightarrow \) one finds several black string solutions with small \( r_h \). As \( r_h \) increases, these solutions approach each other and finally merge for some maximal value \( r_h^{\text{max}}(\kappa) \). Black strings exists only for a finite domain of the \( \kappa - r_h \) parameter plane. /Brihaye, Hartmann, Radu ’05/.
‘Twisted’ solutions

/Brihaye, Radu ’05/ consider the case of a non-hypersurface orthogonal Killing vector $\partial/\partial x^4$:

$$g_{MN}dx^M dx^N = e^{-\zeta} g_{\mu\nu} dx^\mu dx^\nu - e^{2\zeta} (dx^4 + W_\mu dx^\mu)^2$$

$$A^a_M dx^M = A^a_\mu dx^\mu + H^a (dx^4 + W_\mu dx^\mu)$$

Upon the reduction to D=4 the twist $W_\mu$ becomes a U(1) vector fields $\Rightarrow$ 4D EYM-Higgs-dilaton+U(1) model. When the charge associated to the U(1) field vanishes, the solutions reduce to the YM vortices/black strings.
Deformed solutions

/Brihaye, Hartmann, Radu ’05/ after the reduction to 4D choose the fields to be static, axially symmetric

\[ ds^2 = f(r, \theta)dt^2 - m(r, \theta)(dr^2 + r^2 \sin^2 \theta) - l(r, \theta)d\varphi^2. \]

Within this ansatz they have obtained solutions analogues to multimonomopes and monopole - antimonopole pairs.

Both regular solutions and black strings are considered, the existence of several ‘solution branches’ is detected.

Lorentz boosting the solutions along the \( x^4 \) axis gives stationary spinning configurations.
Non-Abelian braneworlds

Uplifting the D=4 monopole gives a vortex in D=5, a domain wall in D=6 and a 3-brane in D=7. The YM and Higgs field can be chosen to be spherically symmetric,

\[
A^a_k = \epsilon_{akj} x^j W(r), \quad \Phi^a = x^a H(r) / r^2 = x^k x^k / ,
\]

where \( x^k \) are coordinates on the orthogonal space,

\[
ds^2 = A(r) \eta_{\mu\nu} dy^\mu dy^\nu - B(r) \delta_{ik} dx^i dx^k ,
\]

solutions with \( A(0) = 1, A(\infty) = 0 \Rightarrow \text{gravity localization by the monopole} /\text{Shaposhnikov et al. '03}/.

Similar solutions in D dimensions, with \( \Lambda \)-term, and with global monopoles have been analyzed.
Gravitating Yang-Mills fields in string theory.
Heterotic string theory

has the YM in D=10. Heterotic 5-brane contains the YM instanton in the orthogonal 4-space /Strominger ’90/

\[ ds^2 = A(x^\mu)\eta_{MN}dy^Mdy^N - B(x^\nu)\delta_{\mu\nu}dx^\mu dx^\nu \]

\[ A_a^\mu(x^\nu), \quad F_{\mu\nu} = \tilde{F}_{\mu\nu}, \quad \phi(x^\mu) \]

compactifications \(\Rightarrow\) many solitons – heterotic solitons /Duff, Khuri, Liu ’95/
YM via Kaluza-Klein

Type I string theory

\[ L_{10} = \frac{1}{4} \hat{R} - \frac{1}{2} \partial_M \hat{\phi} \partial^M \hat{\phi} - \frac{1}{12} e^{-2\hat{\phi}} \hat{H}_{MNP} \hat{H}^{MNP} \]

fermion SUSY variations

\[ \delta \hat{\psi}_M = \hat{D}_M \hat{\epsilon} - \frac{1}{48} e^{-\hat{\phi}} \left( \hat{\Gamma}^{SPQ}_M + 9 \delta_M \hat{\Gamma}^{PQ} \right) \hat{H}_{SPQ} \hat{\epsilon}, \]

\[ \delta \hat{\chi} = -\frac{1}{\sqrt{2}} (\hat{\Gamma}^M \partial_M \hat{\phi}) \hat{\epsilon} - \frac{1}{12 \sqrt{2}} e^{-\hat{\phi}} \hat{\Gamma}^{SPQ} \hat{H}_{SPQ} \hat{\epsilon}. \]
Reduction on a group manifold

\[ ds^2 = e^{\phi} \left\{ \frac{1}{2} e^{-2\phi} g_{\mu\nu} dx^\mu dx^\nu + \sum_{(\sigma)=1,2} \frac{1}{g^2_2(\sigma)} \eta^{(\sigma)}_{ab} \Theta^a(\sigma) \Theta^b(\sigma) \right\} \]

\[ g_{\mu\nu} \sim (s^2, +1, +1, +1), \quad \eta^{(1)}_{ab} = \delta_{ab}, \quad \eta^{(2)}_{ab} = \text{diag}(1, 1, -s^2) \]

\[ \hat{H}_3 = \sum_{(\sigma)=1,2} \left( \frac{1}{g^2_2(\sigma)} \Theta^1(\sigma) \wedge \Theta^2(\sigma) \wedge \Theta^3(\sigma) - \Theta^a(\sigma) \wedge F^a(\sigma) \right) + e^{-4\phi} \ast da \]

\[ \Theta^a(\sigma) = A^a(x) - \theta^a(\sigma)(z), \quad \theta^a(\sigma) + \frac{1}{2} \eta^{(\sigma)}_{ad} \epsilon_{bcd} \theta^b(\sigma) \wedge \theta^c(\sigma) = 0 \]
D=4 theory

\[ L_4 = \frac{R}{4} - \frac{1}{2} (\partial_\mu \phi)^2 + \frac{s^2}{2} e^{-4\phi} (\partial_\mu a)^2 - \frac{1}{4} e^{2\phi} \sum_{(\sigma)=1,2} \frac{\eta_{ab}^{(\sigma)}}{g^2(\sigma)} F_{\mu\nu}^{(\sigma)a} F_{\sigma b\mu\nu} \]

\[ -\frac{1}{2} a \sum_{(\sigma)=1,2} \frac{\eta_{ab}^{(\sigma)}}{g^2(\sigma)} \ast F_{a\mu\nu}^{(\sigma)} F_{b\mu\nu}^{(\sigma)} + \frac{1}{8} \left( g^2_{(1)} - s^2 g^2_{(2)} \right) e^{-2\phi} \]

\[ s^2 = -1 \] gives N=4 SU(2)×SU(2) gauged SUGRA (Freedman-Schwarz) /Chamseddine and M.S.V. ’98/

\[ s^2 = +1 \] gives Euclidean N=4 SU(2)×SU(1,1) SUGRA /M.S.V. ’00/
fermionic SUSY variations

\[ \delta \chi = \left( \frac{1}{\sqrt{2}} \gamma^\mu \partial_\mu \phi - \frac{1}{\sqrt{2} s} e^{-2 \phi} \gamma^5 \gamma^\mu \partial_\mu a \right) \epsilon \]
\[ + \frac{1}{2s} e^\phi \left( s \mathcal{F}^{(1)} - \gamma_5 \mathcal{F}^{(2)} \right) \epsilon + \frac{1}{4s} e^{-\phi} \left( s g^{(1)} - g^{(2)} \gamma_5 \right) \epsilon , \]

\[ \delta \psi_\mu = \left( \partial_\mu + \frac{1}{4} \omega_{\alpha \beta, \mu} \gamma^\alpha \gamma^\beta + \sum_{(\sigma) = 1, 2} K_{ab}^{(\sigma)} T^{(\sigma) a} A_\mu^{(\sigma) b} + \frac{1}{2s} e^{-2 \phi} \gamma_5 \partial_\mu a \right) \epsilon \]
\[ + \frac{1}{2\sqrt{2} s} \ e^\phi \left( s \mathcal{F}^{(1)} + \gamma_5 \mathcal{F}^{(2)} \right) \gamma_\mu \epsilon + \frac{1}{4\sqrt{2} s} e^{-\phi} \left( s g^{(1)} + g^{(2)} \gamma_5 \right) \gamma_\mu \epsilon \]

with \( \mathcal{F}^{(\sigma)} = - \frac{1}{g^{(\sigma)}} \eta_{ab}^{(\sigma)} \gamma^\alpha \gamma^\beta F^{(\sigma) a}_{\alpha \beta} T^{(\sigma) b} \)
Supersymmetry

Setting

$$\delta \chi = \delta \psi_\mu = 0$$

gives equations for the SUSY Killing spinor $\epsilon$.

In most cases these equations are inconsistent.

Sometimes their consistency conditions can be reformulated as first order non-linear Bogomol’nyi equations for the bosonic fields $\{g_{\mu\nu}, A^{(\sigma)b}_\mu, \phi, a\}$

Exactly solvable case: $\ g(1) = 1, \ g(2) = 0, \ A^{(2)a}_\mu = 0$
SUGRA monopole

solution of Bogomol’nyi equations

\[ ds^2 = 2 e^{2\phi} \left\{ dt^2 - d\rho^2 - R^2(\rho) \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right\}, \]

\[ \tau^a A^a_\mu dx^\mu = \frac{i}{2} (1 - w(\rho))[T, dT], \quad \phi = \phi(\rho); \quad T = \tau^a n^a \]

\[ w = \pm \frac{\rho}{\sinh \rho}, \quad e^{2(\phi - \phi_0)} = \frac{\sinh \rho}{2R(\rho)}, \quad R(\rho) = \sqrt{2 \rho \coth \rho - w^2 - 1}. \]

/Chamseddine, M.S.V. ’97/

Four independent Killing spinors \( \Rightarrow \) N=1 SUSY
Uplifting to $D=10$

\[ ds^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2 - d\rho^2 - R^2(\rho) \, d\Omega_2^2 - \Theta^a \Theta^a \]

\[ H = \frac{1}{2\sqrt{2}} e^{-\frac{3}{4}\phi} (F^a \wedge \Theta^a + \epsilon_{abc} \Theta^a \wedge \Theta^b \wedge \Theta^c) \]

\[ \Theta^a = A^a - \theta^a \]

According to /Maldacena, Nunez ’01/ this solution describes the NS-NS 5-brane wrapped on $S^2 \Rightarrow$ the dual description of N=1 SYM $\Rightarrow$ the dual description of confinement.

Used to be called ‘Maldacena-Nunez solution’.
Gates-Zwiebach model

SO(4), N=4 gauged SUGRA

\[ \mathcal{L}_4 = \frac{1}{4} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{2\phi} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{8} (e^{-2\phi} + \xi^2 e^{2\phi} + 4\xi). \]

can be obtained by reducing D=11 SUGRA on \( S^7 \)

/Cvetic, Lu, Pope ’00/; /Cvetic, Gibbons, Pope ’04/

\[ \delta \chi = \frac{1}{\sqrt{2}} \gamma^\mu \partial_\mu \phi \epsilon + \frac{1}{2} e^\phi \mathcal{F} \epsilon + \frac{1}{4} (e^{-\phi} - \xi e^\phi) \epsilon, \]

\[ \delta \psi_\mu = \mathcal{D}_\mu \epsilon + \frac{1}{2\sqrt{2}} e^\phi \mathcal{F} \gamma_\mu \epsilon + \frac{1}{4\sqrt{2}} (e^{-\phi} + \xi e^\phi) \gamma_\mu \epsilon. \]

\[ \mathcal{F} = \frac{1}{2} \alpha^a \alpha_\beta \gamma^\alpha \gamma^\beta \]

\[ \delta \chi = \delta \psi_\mu = 0 \Rightarrow \text{consistency conditions} \]

\Rightarrow \text{Bogomol’nyi equations} \Rightarrow /Chamseddine, M.S.V. ’04/
Bogomol’nyi equations

\[ ds^2_{(4)} = -e^{2V(\rho)} dt^2 + e^{2\lambda(\rho)} d\rho^2 + r^2(\rho) d\Omega^2, \]

\[ \tau^a A_\mu^a dx^\mu = \frac{i}{2} (1 - w(\rho))[T, dT], \quad \phi = \phi(\rho); \quad T = \tau^a n^a \]

\[ V' - \phi' = \xi \frac{P}{\sqrt{2N}} e^{\phi + \lambda}, \quad Q = e^{V + \phi} \frac{w}{N}, \]

\[ \phi' = \sqrt{2} \frac{BP}{N} e^\lambda, \quad w' = -\frac{rwB}{N} e^{-\phi + \lambda}, \]

\[ N = \sqrt{w^2 + P^2}, \quad r' = Ne^\lambda. \]

\[ P = e^{\phi} \frac{1-w^2}{\sqrt{2r}} + \frac{r}{2\sqrt{2}} (e^{-\phi} + \xi e^\phi), \quad B = -\frac{P}{\sqrt{2r}} + \frac{1}{2} e^{-\phi}. \]
Solutions

comprise a family labeled by $\xi$, all have N=1 SUSY. $\xi = 0 \Rightarrow$ SUGRA monopole. $\xi > 0$ asymptotically AdS. $\xi < 0$ compact. For $\xi = -2$ static Einstein universe $ds^2 = -dt^2 + 2\theta^a \theta^a$, $A^a = \theta^a$ inv. forms on $S^3$.

Oxidizing gives branes in D=11.
Wormholes in Euclidean SUGRA

\[ \mathcal{L}_4 = \frac{R}{4} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{-4\phi} \partial_\mu a \partial^\mu a - \frac{1}{4} e^{2\phi} F_{\mu \nu}^a F^{a \mu \nu} \]

\[-\frac{a}{2} \ast F_{\mu \nu}^a F^{a \mu \nu} + \frac{1}{4} (1 - q^2) e^{-2\phi} \]

reduction on \( S^3 \times AdS_3 \) with the same radii.

Simplest solutions

\[ g_{\mu \nu} = \delta_{\mu \nu}, \quad F_{\mu \nu}^a = \pm \ast F_{\mu \nu}^a, \quad a = \mp \frac{1}{2} e^{2\phi}, \]

\[ \partial_\sigma \partial^\sigma e^{-2\phi} = -F_{\mu \nu}^a F^{a \mu \nu} \]

coincide with the heterotic instantons of Strominger, but are not supersymmetric, their D=10 analogs are different.
Homogeneous and isotropic ansatz

\[ ds^2 = e^{2\nu}d\tau^2 + e^{2\lambda}(d\xi^2 + R(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)) \]

\[ R = \sin \xi, \sinh \xi, \xi \text{ if } K = 1, -1, 0 \]

gauge field /with \( T = \tau^1 \sin \theta \cos \varphi + \tau^2 \sin \theta \sin \varphi + \tau^3 \cos \theta / \)

\[ A = (a_0 d\tau + a_1 d\chi) T + i(1 - W) [T, dT], \]

\[ a_0 = -\frac{w'R}{W^2} \frac{dR}{d\xi}, \quad a_1 = \frac{w(w^2 - K) R^2}{W^2}, \quad W = \sqrt{1 + R^2(w^2 - K)} \]

Here

\[ w = w(\tau), \quad \phi = \phi(\tau), \quad a = a(\tau), \quad \nu = \nu(\tau), \quad \lambda = \lambda(\tau) \]
Wormhole solution

\[ ds^2 = \frac{e^{2\lambda_0}}{\cos (2e^{2\lambda_0}\tau)} \left( \frac{e^{4\lambda_0} d\tau^2}{(\cos (2e^{2\lambda_0}\tau))^2} + d\Omega_3^2 \right) \]

\[ e^{2(\phi_0 - \phi)} = \cos \left( 4(2 + h)e^{2\phi_0}\tau \right) \]

\[ a' = 2e^{4\phi}(3w - w^3 + h), \quad \omega = \pm 1, \quad K = 1 \]

Let \( \tau_0 = \frac{\pi}{4}e^{-2\lambda_0} \). If \( \tau \to \pm \tau_0 \) then \( \cos (2e^{2\lambda_0}\tau) \to 0 \), the metric is flat – interpolates between two flat regions. If

\[ |2 + h| < 1/2 \]

then dilaton is bounded in the interval \((-\tau_0, \tau_0)\). Otherwise it diverges, but this does not spoil the metric.

Supersymmetric wormholes? /Maldacena and Maoz ’05/
Supersymmetry conditions

\[ \frac{1}{Y} \frac{dY}{d\tau} = \left( w + \frac{3Kw - w^3 + h}{Y^2} \right) \cos \Psi + \frac{2(w^2 - K)}{Y} \sin \Psi, \]

\[ \frac{1}{Y} \frac{dw}{d\tau} = \left( w + \frac{3Kw - w^3 + h}{Y^2} \right) \sin \Psi - \frac{2(w^2 - K)}{Y} \cos \Psi, \]

where \( Y = \exp(\lambda - \phi), \)

\[ \tan \frac{\Psi}{2} = -\frac{B + Q}{N + \Lambda}, \quad B = \frac{K - w^2}{Y} + \frac{Y}{2}, \quad Q = -\frac{qY}{2}, \]

\[ \Lambda = \frac{3}{2} w + \frac{3Kw - w^3 + h}{2Y^2}, \quad N = \sqrt{B^2 + \Lambda^2 - Q^2}. \]
Solutions

with the reflection symmetry comprise a 4-parameter family labeled by $q, h, K, w'(0)$
Conclusions

Analysis of gravitating YM fields in General Relativity are useful for accumulating a qualitative understanding of the solution behaviour. Interesting mathematically.

This qualitative experience can be used for constructing non-Abelian solutions in string theory inspired SUGRA models.