On the dimensionality of space–time

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Within constructive axiomatics of space–time geometry based on elements of quantum mechanics it can be shown that by means of the study of the dynamical behavior of general matter fields in a geometry-free way one can give reasons for the dimension of space–time to be four.

I. INTRODUCTION

The notion of the dimensionality of space–time is a fact which lies at the very foundations of geometry[1] and physics.[2] We of course are convinced that the dimension of our physical space–time is four. One reason for this is that we are used to taking four numbers to fix naively an event in space–time: three space coordinates and one time coordinate. However, it has been shown by Cantor,[3] that there is a bijective mapping $f:\mathbb{R}^1 \to \mathbb{R}^2$, and, in addition, Peano[4] proved that there is a continuous mapping $g:\mathbb{R}^1 \supset [0,1] \to [0,1] \times [0,1] \subset \mathbb{R}^2$. These examples show that coordinatization is not a good means to give to some set a certain number of dimensions. This difficulty was overcome by the definition of the dimension by Urysohn, Menger, and Brower (see Hurewicz and Wallman, Ref. 5) whereby the latter proved the topological invariance of this definition. Therefore, by means of this notion, the four dimensionality of our physical space–time can be regarded as a matter of fact.

However, in some physical theories other dimensions have been considered. In some cases they are used as a purely mathematical trick as it is done, for example, in the dimensional regularization procedure in quantum field theory (see, e.g., Ref. 6). In other cases the additional dimensions are taken to describe physical reality as it is in Klein–Kaluza theories (see, e.g., Ref. 7). In an approach of Zeilinger and Svozil[8] who use a modified Hausdorff definition of dimension, the space–time dimension in the microphysical domain is regarded to be a little bit less than four.

If we now accept that the dimension of our physical space–time is four, we may ask, why it is so. The question we are trying to answer is: *Are there fundamental grounds for the four dimensionality of our space–time*?

In the literature one can find considerations giving reasons for the four dimensionality of space–time which are related to effects calculated for some physical laws (for a review see Ref. 2). Thereby the considered physical laws are the formal extensions to arbitrary dimensions of laws established in three dimensions. Among those are considerations concerning the stability of planetary or electron orbits, Huygen's principle for the wave equation,[9] or the occurrence of Bose–Einstein condensation.[10]

In principle their argumentation roughly runs as follows: At first one takes physical laws formulated in three dimensions and generalizes them to an arbitrary number of dimensions. Then dynamical consequences or effects, which can be distinguished from effects which occur

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in three dimensions, are calculated by means of this new theory, whereby the notion of these effects does not depend on the dimensionality. In the case of Newton gravity in higher dimensions, for example, one demands that (as in three dimensions) the force between two bodies depends on the direction and the distance between the two bodies, and that Gauss' law should hold. Thus one physical consequence of this new theory is the nonstability of planetary orbits. The notion of stability is a dynamical effect and does not depend on the dimension. Therefore *dynamical observations* which do not depend on the dimensionality of space, force us to conclude that space is three dimensional. However, in all examples mentioned above, the considered physical laws in higher dimensions are essentially obtained as a *formal* generalization to higher dimensions of physical laws which one has established in three dimensions.

In this article we want to give, along a different way of reasoning, another argument for the space–time dimension to be four. The difference to earlier considerations lies in the fact that we *derive* our dynamical equation from a few first principles (see below) which do not depend in any way on the underlying space–time dimension or space–time geometry. Then this dynamical equation is taken to calculate observable dynamical effects. We are not just generalizing some theory given in three dimensions to an arbitrary number of dimensions.

For doing so we take arguments from new constructive space–time axiomatics based on elements of quantum theory (for an extensive account on the relevance of this approach and a complete exposition see Ref. 11). Thereby we refer to the propagation phenomena of quantum matter only. In doing so we use quantum mechanics in the "position representation," that is, represented by classical field theory. These quantum objects are considered as the "test field," that is, as the field analog to the usual test point particles. In the following we describe the behavior of these quantum objects in particular basic experiences. These basic experiences are then formalized in postulates. Thereby it is possible to formulate the postulates and to perform the underlying experiments *without making reference to geometry*. The introduction of a space–time geometry, which will consist in a Riemannian metric, in a Riemann–Cartan connection and in the space–time dimension, will be the *result* of our constructive axiomatic scheme.

In our approach we use very general propagation phenomena, which we take as our basic experiences, like deterministic evolution, the superposition principle, and finite propagation speed to derive a linear first order system of partial differential equations as the dynamical equation for quantum matter (see Ref. 12). Then we demand that this system possesses some properties concerning the propagation of singularities and of Wentzel–Kramers–Brillouin (WKB) states. This propagation equation as well as the requirements are completely *geometry-free* statements, they do not depend on any geometrical notion like the metric, the connection, or the dimensionality of space–time. Nevertheless, we can derive a Clifford algebra with 4×4 matrices which is only possible in five dimensions at most. An additional argument using the conservation of chirality excludes less than four and five dimensions and leaves us with a four-dimensional space–time (Fig. 1). This last step can be alternatively replaced by using Huygen's principle.

For completeness, in the following we give a short summary of the derivation of the field equation we are dealing with. Then we state our requirements which lead us to the Clifford algebra which in turn gives us the space–time dimension. Therefore all these quantities will prove to be *consequences of our purely dynamical and geometry-free approach*. The metric and the connection is introduced in Refs. 13 and 14 by means of a field equation which is characterized by means of a deterministic linear evolution with finite propagation speed. In this article we want to derive within this scheme the dimensionality of space–time.

Together with the results obtained in Refs. 11–14 we can state: *If space–time geometry is the entity that prescribes to test fields their typical behavior as seen in the basic experiences, then space–time geometry is mathematically described by a Riemann–Cartan geometry with axial torsion in four space–time dimensions.*

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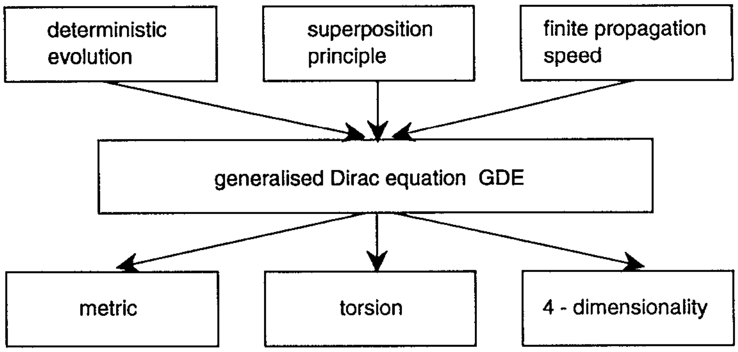

FIG. 1. Scheme of constructive space–time axiomatics based on elements of quantum theory.

II. THE FIELD EQUATION

At first we want to repeat the way of how to arrive in a geometry-free manner at some field equation in a similar way as in Ref. 12. Thereby "geometry-free" means that we do not rely on any geometric notion, e.g., the metric, the connection, or even the dimension. All these entities will be a consequence of our scheme.

According to Ref. 12 we assume that on a differentiable $(n+1)$-dimensional manifold $\mathcal{M}$ there is given a field $\varphi:\mathcal{M} \rightarrow \mathbb{C}^s : x \mapsto \varphi(x)$ whereby each $\varphi(x) \in \mathbb{C}^s$ is a complex vector. The inner degrees of freedom as represented by the complex vector valuedness of the field φ will be essential in the following scheme.

At first we shall derive a general partial differential equation governing the dynamics of the considered matter field. For doing so we first postulate a deterministic evolution of the field with respect to a "time"-parameter t associated with a $(n+1)$ slicing of the manifold $\mathcal{M}$. This means that, given a field on some hypersurface labeled with a parameter t', the field will be determined uniquely on a "later" hypersurface with parameter $t > t'$. All hypersurfaces for which this is true will be called spacelike. Note that n refers to the dimension of the spacelike hypersurface, and will shortly be called the *space dimension*.

Next we require a superposition principle to hold. This property of quantum matter is verified in a lot of experiments (see, e.g., Ref. 15). (Of course there are equations for which the superposition principle does not hold. Here we are not concerned with any of such equations like Einstein's equation or Yang–Mills equation. In demanding the superposition principle to hold we are essentially restricted to source-free matter fields (i.e., fields which are propagating in a source free space–time region) which feel no self interaction. Since matter fields like the Klein–Gordon or the Dirac field fulfill this demand, this represents no limitation with respect to our approach. Thereby we require that the evolution of an arbitrary sum of initial data results in the same field as the sum of separately propagated initial data. One derives an abstract Cauchy problem $d\varphi_t/dt = G_t\varphi_t$, where φ_t is the field φ for fixed t and where G_t is the generator of the dynamical evolution.

According to our experience signals cannot propagate with infinite velocity. Therefore we postulate next that the propagation of all initial data with compact support results in a field which still has compact support. This requirement implies that the generator G_t is local. As a consequence, the generator G_t has to be a partial differential operator implying that the abstract Cauchy problem reduces to a first order system of partial differential equations $(\mu = 0,1,...,n)$

$$i\tilde{\gamma}^{\mu}(x)\partial_{\mu}\varphi(x) + M(x)\varphi(x) = 0, \tag{1}$$

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where the $\tilde{\gamma}^{\mu}$ and M are complex coordinate dependent $s\times s$ matrices. We call this equation the *generalized Dirac equation* (GDE). For coordinate dependent transformations $\varphi\mapsto\varphi'=S\varphi$, $S\in \mathrm{Gl}(\mathbb{C},s)$, $\tilde{\gamma}^{\mu}$ transforms homogeneously, $\tilde{\gamma}^{\mu}\mapsto\tilde{\gamma}'^{\mu}=S\tilde{\gamma}^{\mu}S^{-1}$, while M transforms inhomogeneously, $M\mapsto M'=SMS^{-1}-iS\tilde{\gamma}^{\mu}\partial_{\mu}S^{-1}$. Note that $\tilde{\gamma}^{\mu}$ are not necessarily the usual Dirac matrices. In general they do not fulfill any Clifford algebra.

Next we introduce the probability interpretation of quantum mechanics which must be based on a real vector current j^{μ} which is bilinear in the fields φ. Its zeroth component j^0 is interpreted as the probability density for finding a particle at a certain location. The only object in our theory which carries a vector index is $\tilde{\gamma}^{\mu}$. Therefore we require $\varphi^{+}\beta\tilde{\gamma}^{\mu}\varphi$ to be real for some matrix β and all φ. This implies that $\beta\tilde{\gamma}^{\mu}$ must be Hermitian: $(\beta\tilde{\gamma}^{\mu})^{+}=\beta\tilde{\gamma}^{\mu}$. At this point, the Klein–Gordon equation is ruled out. However, the Dirac equation still satisfies all the requirements above. Indeed, the fact that the Dirac equation leads to a positive definite current was one of the main successes of this equation.

In addition, this first order system can be easily shown to be weakly hyperbolic, that is, the spacelike hypersurfaces are noncharacteristic and all zeros of the characteristic equation $H_c(x,k):=\det(\tilde{\gamma}^{\mu}k_{\mu})=0$ are real. The reality of the zeros can be inferred from the Hermiticity of the matrices $\beta\tilde{\gamma}^{\mu}$. In addition, the spacelike hypersurfaces can be proven to be necessarily noncharacteristic, because otherwise it is not possible to pose arbitrary initial data.

To sum up, a deterministic evolution which is linear and propagates with finite velocity results in a first order system of partial differential equations (1). A probability interpretation then requires the coefficients of the principal part to be Hermitian.

Now one can derive the geometrical content of this partial differential equation, that is, metric, connection, and dimensionality, whereby in the following we would be able to show that within this axiomatic scheme one can easily characterize the four dimensionality of space–time.

III. THE PROPAGATION OF SINGULARITIES

The propagation of singularities is a first consequence of our field equation (1). Singularities are jumps resp. discontinuities in solutions of Eq. (1) or in one of its derivatives which can occur only on certain hypersurfaces.[16] In general relativity they are related to the notion of the light cone because for all physical theories of matter the characteristics are identical with the usual light cones describing also the causal behavior of the fields. The jumps of lowest order along a hypersurface $\Phi=0$ obey $0=\tilde{\gamma}^{\mu}k_{\mu}a$ with the *normal* $k_{\mu}=\partial_{\mu}\Phi$ related to the hypersurface Φ. Since multiplication of this equation with a nonsingular matrix, e.g., β, does not alter the solutions for k_{μ} and a we use instead

$$0=\beta\tilde{\gamma}^{\mu}k_{\mu}a. \tag{2}$$

The advantage of this form is that the coefficient matrix $\beta\tilde{\gamma}^{\mu}k_{\mu}$ is Hermitian.

Vectors $a\in\mathbb{C}^s$ solving Eq. (2) are called *jump amplitudes* on the hypersurface. At the moment these jump amplitudes are just the set of vectors which describe possible jumps in the solutions of Eq. (1). The jump amplitudes solving Eq. (2) are defined only along the hypersurface where the discontinuity of the solution occurs.

The solvability condition of Eq. (2) is given by the characteristic polynomial

$$H_c(x,k):=\det(\beta(x)\tilde{\gamma}^{\mu}(x)k_{\mu})=g^{\mu_1\cdots\mu_s}(x)k_{\mu_1}\cdots k_{\mu_s}=0, \tag{3}$$

whereby the tensor $g^{\mu_1\cdots\mu_s}$ of rank s is real. Equation (3) is a scalar partial differential equation of the Hamilton–Jacobi-type for the function Φ which always possesses a solution. These solutions are called *characteristic surfaces* or *null surfaces* which are the only hypersurfaces

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where discontinuities of solutions can occur. The set of k_μ solving Eq. (3) defines the *normal cone*. These characteristic surfaces generalize the usual null surfaces appearing in relativistic field equations (note that $k_0 \neq 0$ for all $k_{\hat{\mu}}$, $\hat{\mu}=1,\ldots,n$).

Now we demand according to physical experience that there is only one null cone (that is, one past and one future cone) and there are only two jump amplitudes. (If this requirement is not fulfilled one is led to Lorentz noninvariant theories whose physical consequences are discussed in Ref. 17. Estimates on possible Lorentz noninvariance parameters from experimental observation are also given.)

The latter requirement implies that the multiplicity of the zeros of the characteristic polynomial H_c must be two. This in turn implies that there is another polynomial H_0, so that

$$H_c(x,k) = (H_0(x,k))^2. \tag{4}$$

That there is only one future and one past null cone implies (see Appendix A)

$$H_0(x,k) = \overset{c}{g}{}^{\mu\nu}(x) k_\mu k_\nu. \tag{5}$$

Accordingly, $s=4$, that is, φ has four components, and $\tilde{\gamma}^\mu$ and M are 4×4 matrices. This means that there is a second rank tensor $\overset{c}{g}{}^{\mu\nu}$ which is determined up to a positive scalar function.

It is easy to see that this tensor is nonsingular and has (according to our convention to be chosen) the signature $n-1$: Since $\overset{c}{g}{}^{\mu\nu}$ is a symmetric second rank tensor, it can be diagonalized by means of a coordinate transformation. For meeting the above requirements, i.e., $k_0 \in \mathbb{R}$, it is necessary that for the special case of $k_{\hat{\mu}} = \delta^1_{\hat{\mu}}$ we have nonvanishing $\overset{c}{g}{}^{00}$ and $\overset{c}{g}{}^{11}$ with opposite signs. We make the convention $\overset{c}{g}{}^{00} > 0$. Then $\overset{c}{g}{}^{11} < 0$. Analogously $\overset{c}{g}{}^{22} < 0$, $\overset{c}{g}{}^{33} < 0$, and so on. Therefore we have $\det \overset{c}{g}{}^{\mu\nu}(x) \neq 0$ and also that the tensor $\overset{c}{g}{}^{\mu\nu}$ has the correct signature $1-n$. This also means that the GDE(1) is a first order system of partial differential equations of normal hyperbolic-type.

Because $\overset{c}{g}{}^{\mu\nu}$ is nonsingular, it possesses an inverse $\overset{c}{g}_{\mu\nu}(x)$. This *metrical tensor* defines a *conformal structure* in the usual sense (Ref. 11) in $n+1$ dimensions. One can introduce the following notions: a vector w is called time-, null-, or spacelike if $\overset{c}{g}_{\mu\nu} w^\mu w^\nu$ is >0, <0, or $=0$, respectively.

It can be shown that the vector field $v^\mu := \overset{c}{g}{}^{\mu\nu} k_\nu$ is characterized by $\overset{c}{g}_{\mu\nu} v^\mu v^\nu = 0$ and obeys a geodesic equation $v^\nu \partial_\nu v^\mu + \{{}^{\mu}_{\nu\sigma}\}^c v^\nu v^\sigma \sim v^\mu$ with the Christoffel symbol $\{{}^{\mu}_{\nu\sigma}\}^c := \frac{1}{2} \overset{c}{g}{}^{\mu\rho} (\partial_\nu \overset{c}{g}_{\rho\sigma} + \partial_\rho \overset{c}{g}_{\nu\sigma} - \partial_\sigma \overset{c}{g}_{\nu\rho})$. These properties justify calling $\overset{c}{g}_{\mu\nu}$ a metric. Solutions of the geodesic equation are null trajectories, that is, at each point along the trajectory the tangent is a null vector. The set of points which can be connected by null trajectories starting from a certain point x_0 is called the *null cone* ν_{x_0} at x_0. The *interior of the null cone*, that is, all points which can be reached by a timelike curve, is denoted by J_{x_0}. The future null cone and its interior are denoted by $\nu^+_{x_0}$ and $J^+_{x_0}$, respectively.

IV. THE PROPAGATION OF WKB STATES

Next we want to consider approximate solutions of Eq. (1). The solutions we are looking for are approximate plane wave solutions. We make the ansatz

$$\varphi(x) = a(x) e^{iS(x)}, \tag{6}$$

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with a real function S. We demand that derivatives of the amplitude a are small compared with derivatives of the phase $S:\|\tilde{\gamma}^{\mu}\partial_{\mu}a\| \ll \|\tilde{\gamma}^{\mu}(\partial_{\mu}S)a\|$. Here $\|\cdot\|$ is some norm in the vector space $\mathbb{C}^4$. Inserting Eq. (6) into Eq. (1) we arrive at

$$0 = -\tilde{\gamma}^{\mu}\partial_{\mu}Sa - Ma + i\tilde{\gamma}^{\mu}\partial_{\mu}a. \tag{7}$$

The last term is small compared with the first one. The second term must be split into a covariant one $M^{(0)}$ and a second one $M^{(1)}$ which transforms in the same way as M. That is, $M = M^{(0)} + M^{(1)}$ (a formal derivation of this result is given in Ref. 14). The covariance of the resulting equations requires this kind of splitting. We are not interested in any special form of $M^{(0)}$ and we must not know it. $M^{(0)}$ can be any complex 4×4 matrix.

If we use this splitting, then we get from Eq. (7)

$$0 = (-\tilde{\gamma}^{\mu}\partial_{\mu}S - M^{(0)})a + i\tilde{\gamma}^{\mu}\partial_{\mu}a + M^{(1)}a. \tag{8}$$

In a first step, the last two terms can be neglected in comparison to the first term, so that we have

$$0 = (\tilde{\gamma}^{\mu}p_{\mu} - M^{(0)})a. \tag{9}$$

If this equation is fulfilled, we additionally get

$$0 = i\tilde{\gamma}^{\mu}\partial_{\mu}a - M^{(1)}a. \tag{10}$$

The vectors a solving Eq. (9) are called *spin states* and $p_{\mu} := -\partial_{\mu}S$ are the *momenta* of the approximate plane waves. Any solution for S and a of Eqs. (9) and (10) are approximately plane matter waves.

If there is a solution of the form of an approximately plane matter wave, then a solvability condition $\det(\beta(\tilde{\gamma}^{\mu}p_{\mu} - M^{(0)}))$ for the first equation (9), which is algebraic, must be fulfilled [we multiplied Eq. (10) with β for the same reason as in Sec. III]. This results in a Hamilton–Jacobi equation which is a polynomial of forth order

$$\tilde{H}(x,p) := \det(\beta\tilde{\gamma}^{\mu}p_{\mu} - \beta M^{(0)}) = (\overset{c}{g}{}^{\mu\nu}p_{\mu}p_{\nu})^2 + \bar{g}^{\mu\nu\rho}p_{\mu}p_{\nu}p_{\rho} + \bar{g}^{\mu\nu}p_{\mu}p_{\nu} + \bar{g}^{\mu}p_{\mu} + \bar{g} = 0,$$

with coefficients $\bar{g}^{\mu\nu\rho},\ldots,\bar{g}$ which are determined by $\tilde{\gamma}^{\mu}$ and $M^{(0)}$.

Now we demand that for each solution p of the above equation there are two spin states and that there is at least one timelike group velocity $v^{\mu} := \partial\tilde{H}/\partial p_{\mu}$. This implies that $\tilde{H}$ is the square of another polynomial $\tilde{H}(x,p) = (H'(x,p))^2$ leading to the solvability condition

$$H'(x,p) = \overset{c}{g}{}^{\mu\nu}p_{\mu}p_{\nu} + g^{\mu}p_{\mu} + g = 0. \tag{11}$$

A redefinition of p: $P_{\mu} := p_{\mu} + \frac{1}{2}\overset{c}{g}_{\mu\nu}g^{\nu}$ and a subsequent rescaling of Eq. (11) leads to an equation of the structure

$$H(x,P) = g^{\mu\nu}P_{\mu}P_{\nu} - 1 = 0, \tag{12}$$

with $g^{\mu\nu} := (\frac{1}{4}\overset{c}{g}_{\rho\sigma}g^{\rho}g^{\sigma} - g)^{-1}\overset{c}{g}{}^{\mu\nu}$. $g^{\mu\nu}$ now plays the role of a Riemannian metric on the manifold $\mathscr{M}$. The fact that the rescaling in Eq. (12) leads to the term -1 (and not to a zero), resp. the fact, that the denominator $\frac{1}{4}\overset{c}{g}_{\rho\sigma}g^{\rho}g^{\sigma} - g > 0$, is due to the requirement that there is at least one timelike group velocity. From Eq. (12) one can derive an equation of motion for the respective group velocity $v^{\mu} := g^{\mu\nu}P_{\nu}$ showing that the Riemannian metric $g^{\mu\nu}$ as well as

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the coefficient g^{μ} (leading to a Lorentz-type force) prescribes for the wave packets their paths on $\mathscr{M}$. (The discussion of the equation of motion for the group velocity is not subject of this article and will be postponed to a later publication. All following results are independent of this.) In addition, from Eq. (10) a Riemann–Cartan connection with axial torsion which describes the motion of the spin vector, can be derived. This has been done in Refs. 11 and 13.

Combining the redefinition and rescaling we have with $\hat{\gamma}^{\mu} := (\frac{1}{4}\overset{c}{g}_{\rho\sigma} g^{\rho} g^{\sigma} - g)^{-1/2}\tilde{\gamma}^{\mu}$ and $\widetilde{M} := (\frac{1}{4}\overset{c}{g}_{\rho\sigma} g^{\rho} g^{\sigma} - g)^{-1/2}(M^{(0)} + \frac{1}{2}\overset{c}{g}_{\mu\nu} g^{\nu}\tilde{\gamma}^{\mu})$

$$\det(\beta\hat{\gamma}^{\mu}P_{\mu} - \beta\widetilde{M}) = (g^{\mu\nu}P_{\mu}P_{\nu} - 1)^2. \tag{13}$$

This now is the equation we are using to derive the Clifford algebra of the matrices γ^{μ} which in turn gives us restrictions of the dimension of space–time.

V. THE CLIFFORD ALGEBRA

Given some matrix, its determinant is given by the multiplication of this matrix with its minor (see, e.g., Ref. 18). Therefore, there is a matrix $\widetilde{B}$, so that $\widetilde{B}(\hat{\gamma}^{\mu}P_{\mu} - \widetilde{M}) = (g^{\mu\nu}P_{\mu}P_{\nu} - 1)^2$. In addition, one can show (see, e.g., Ref. 18) that for multiple zeros of a certain degree d the minor is proportional to the $(d-1)$st power of this zero. In our case there is another matrix B with $\widetilde{B} = (g^{\mu\nu}P_{\mu}P_{\nu} - 1)B$. Therefore we get from Eq. (13)

$$B(\hat{\gamma}^{\mu}P_{\mu} - \widetilde{M}) = g^{\mu\nu}P_{\mu}P_{\nu} - 1. \tag{14}$$

It is not difficult to show that, because of the fact that the right hand side is a polynomial of order 2 in P, B must be a polynomial of order 1 (see Appendix B)

$$B = B(x,P) = B^{\mu}(x)P_{\mu} + B^{0}(x). \tag{15}$$

Insertion of this B into Eq. (14) and equating the coefficients of equal powers of P_{μ} gives $B^0 = \widetilde{M}^{-1}$, $B^{\mu} = B^0\hat{\gamma}^{\mu}B^0$ and therefore the Clifford algebra

$$\tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}) = g^{\mu\nu}, \tag{16}$$

with $\gamma^{\mu} := B^0\hat{\gamma}^{\mu}B^0$. Here the γ^{μ} are 4×4 matrices.

On the other hand we know that only for four and five space–time dimensions there is a faithful realization of a Clifford algebra with 4×4 matrices. Because our γ matrices are derived to be 4×4 matrices we conclude at first that space–time can be at most five dimensional: It is well-known[19] that for $\dim \mathscr{M} = n+1$ a Clifford algebra possesses a representation in a $2^{[(n+1)/2]}$-dimensional complex vector space ($[(n+1)/2] = (n+1)/2$ for n odd, $= n/2$ for n even). Since we do not assume irreducibility for our case we can conclude $s = 4 \geqslant 2^{[(n+1)/2]}$ only. This means that only space–time dimensions $n+1 \leqslant 5$ are possible. Thereby $n=0$ means that there is no space dimension and only one time dimension which is nonphysical. We exclude $n=1$ because in this case two solutions of Eq. (5) cannot be continuously connected by solutions of Eq. (5) which should always be possible according to our physical experience. We also exclude $n=2$, that is, three-dimensional space–time, because in this case in the same way as it is in five dimensions (see below) the chirality of particles is not conserved. In addition, lower-dimensional physics, especially gravity, is said to be nonphysical.[20]

In principle one also should consider fields with more than four components, that is, fields with higher spin. In this case, the representation of the Clifford algebra may also be higher dimensional which may lead to other conclusions than above, namely, to the possibility of more than four space–time dimensions. However, beside the fact that it is possible also in this case

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to modify appropriately the requirements to arrive at the same conclusion, there is an argument which shows that it is not necessary to consider these fields: At first we note that a field with the properties required in our axioms does exist in nature, namely, the Dirac field which is realized, e.g., by neutrons or electrons. On the other side, other fields may lead to more than four space–time dimensions. However, because all fields live in the same space–time manifold we have to take this very dimensionality which is derived from that field which is most restrictive for the number of dimensions. Therefore, treating the four-component GDE is sufficient for our conclusion.

VI. CONSERVATION OF CHIRALITY

It is possible to distinguish four dimensions from five by means of the following observation:

In five dimensions a representation of the Clifford algebra (up to similarity transformations) is given by γ^μ, $\mu=0,\ldots,4$ with $\gamma^0,\ldots,\gamma^3$ as the usual γ matrices in four dimensions and $\gamma^4:=i\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$.

This means that the vector space $\mathbb{C}^4$ cannot be split in any way into two subspaces so that the dynamical evolution given by the GDE (1) leaves the solution in these subspaces. (In the usual special relativistic field theories this property is referred to as the conservation of chirality.)

For proving this proposition we reformulate Eq. (1) as an evolution equation (here we redefined M and φ in an obvious manner according to the replacement of $\tilde{\gamma}\mu$ with $\gamma\mu$)

$$i\partial_0\varphi=-i(\gamma^0)^{-1}\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\varphi+(\gamma^0)^{-1}M\varphi, \tag{17}$$

whereby x^0 acts as a "time" coordinate and the surfaces $x^0=\text{const}$ are spacelike hypersurfaces Σ_{x^0}, and $\hat{\mu}$ runs from 1 to 4 denoting the coordinates within Σ_{x_0}.

By means of Eq. (17) the field on a hypersurface with the parameter $x^0+\delta x^0$ is then given by

$$\varphi=\varphi_0+\delta x^0(-i(\gamma^0)^{-1}\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\varphi_0+(\gamma^0)^{-1}M\varphi_0), \tag{18}$$

where φ_0 is the initial value on the hypersurface at x^0.

If a function ϕ is given which depends on the coordinates of Σ_{x^0} only, then each initial value given by $\varphi_0=e^{i\phi}\tilde{\varphi}_0$ with $\|\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\tilde{\varphi}_0\|\ll\|\gamma^{\hat{\mu}}(\partial_{\hat{\mu}}\phi)\tilde{\varphi}_0\|$ and $\|M\tilde{\varphi}_0\|\ll\|\gamma^{\hat{\mu}}(\partial_{\hat{\mu}}\phi)\tilde{\varphi}_0\|$ is called an *oscillatory initial value* (see, e.g., Ref. 9) for the GDE. Physically one may interpret these functions as fields with high momentum. Inserting the oscillatory initial values into Eq. (18) we get with $\alpha^{\hat{\mu}}:=(\gamma^0)^{-1}\gamma^{\hat{\mu}}$

$$\varphi=\varphi_0-i\delta x^0\alpha^{\hat{\mu}}(\partial_{\hat{\mu}}\phi)\varphi_0. \tag{19}$$

Now there are experimental observations with respect to the propagation of oscillatory initial values: There is a representation of the γ matrices so that in this representation the dynamics of oscillatory initial values decouple (we need not know the explicit form of the γ matrices for this representation). This means that in the representation under consideration there are projection operators $P_+=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $P_-=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ where 0 and 1 are the 2×2 zero and unity matrices, so that for a field prepared in an oscillatory initial state with $P_\pm\tilde{\varphi}_0=\tilde{\varphi}_0$, the propagated state also fulfills $P_\pm\tilde{\varphi}=\tilde{\varphi}$. In other words, if an oscillatory initial state is prepared with $P_+\tilde{\varphi}_0=\tilde{\varphi}_0$, then it always remains a state with this property. There is no dynamical mixing between states fulfilling $P_+\varphi=\varphi$ and those fulfilling $P_-\varphi=\varphi$. This we now take as our last postulate. (For Dirac's theory this is equivalent to the conservation of helicity for massless particles.)

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With respect to the operator $\alpha^{\hat{\mu}}\partial_{\hat{\mu}}\phi$ in Eq. (19) governing the time evolution of oscillatory initial values this means that it should commute with the projection operators $[P_{\pm}, \alpha^{\hat{\mu}}\partial_{\hat{\mu}}\phi]=0$. This in turn implies that in this representation the matrix $\alpha^{\hat{\mu}}$ must have diagonal form. This, however, is not possible in four space dimensions as can be seen by the following observation: In a special orthonormal tetrad, or a special coordinate system for which $g^{\mu\nu}$ has Minkowskian form, we infer from the Clifford algebra (16) $(\gamma^0)^2=1$ [that is $(\gamma^0)^{-1}=\gamma^0$], $\gamma^0\alpha^{\hat{\mu}}+\alpha^{\hat{\mu}}\gamma^0=0$, and

$$\alpha^{(\hat{\mu}}\alpha^{\hat{\nu})}=\delta^{\hat{\mu}\hat{\nu}}. \tag{20}$$

If now the $\alpha^{\hat{\mu}}$ matrices are of diagonal form, say

$$\alpha^{\hat{\mu}}=\begin{pmatrix} \alpha_1^{\hat{\mu}} & 0 \\ 0 & \alpha_2^{\hat{\mu}} \end{pmatrix},$$

where $\alpha_1^{\hat{\mu}}$ and $\alpha_2^{\hat{\mu}}$ are 2×2 matrices, then relation (20) splits into two relations

$$\alpha_1^{(\hat{\mu}}\alpha_1^{\hat{\nu})}=\delta^{\hat{\mu}\hat{\nu}}, \quad \alpha_2^{(\hat{\mu}}\alpha_2^{\hat{\nu})}=\delta^{\hat{\mu}\hat{\nu}}.$$

However, in four space dimensions (that is, $\hat{\mu}$, $\hat{\nu}=1,\ldots,4$) this kind of relation, namely, a Clifford algebra with a positive definite metric, cannot be fulfilled with 2×2 matrices. This is only possible for at most three space dimensions.

This kind of reasoning is related to the fact that in four space–time dimensions for massless particles one can define the chirality as the eigenvalue of the matrix γ_5 [then $P_{\pm}=\frac{1}{2}(1+\gamma_5)$]. The chirality represents a conserved quantity. Therefore, for massless particles γ_5 should commute with the Hamilton operator $\alpha^{\hat{\mu}}i\partial_{\hat{\mu}}$. First, in four space dimensions it is not even possible to define any helicity, and second, there is no matrix which commutes with all $\alpha^{\hat{\mu}}$ as one can easily prove using Eq. (16).

Therefore, by this last step we have fully characterized the four dimensionality of space–time on the level of the dynamics of matter fields by means of geometry-free postulates resp. requirements.

VII. HUYGEN'S PRINCIPLE

One may think of an alternative way to arrive at the above conclusion, for example, the consideration of Huygen's principle for the squared version of the GDE (1). Let us take Eq. (1) and act on it with $-i\gamma^\mu(x)\partial_\mu$ we then get, with Eq. (16) and the above redefinition of M and φ,

$$0=g^{\mu\nu}(x)\partial_\mu\partial_\nu\varphi(x)+A^\mu(x)\partial_\mu\varphi(x)+B(x)\varphi(x), \tag{21}$$

with some 4×4 matrices $A^\mu(x)$ and $B(x)$. Each solution fulfilling Eq. (1) must also fulfill Eq. (21). The first part of this equation consists in the wave operator. Locally, there is a coordinate system so that this first term acquires the form $\partial_0^2-\Delta_{(n)}$ with $\Delta_{(n)}$ as the Laplacian in n space dimensions. The other terms usually contain the mass and connection term.

Huygen's principle can be stated in several ways (see, e.g., Refs. 9, 21). One way to say that *Huygen's principle* is fulfilled is, that the (retarded) solution of a hyperbolic equation with a δ-like source term (or with a δ-like initial value) at x_0 is a distribution, the support of which consists in the null cone $\nu_{x_0}^+$ only. This means that there is no "tail."[22] (Another way to state this is to say that the value of the field at x depends only on the initial values on the intersection of the past light cone ν_x^- with some initial hypersurface Σ.)

It is well-known that in flat space–time, that is, for the wave equation $\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi=(\partial_0^2-\Delta_{(n)})\varphi=0$, Huygen's principle holds for odd n, while it does not hold in even space

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dimensions. Thereby the form of the fundamental solution in odd space dimensions is of the form $G^+(x,x_0) \sim \delta^{((n-3)/2)}(s^2)$ with $s^2 := \eta_{\mu\nu}(x-x_0)^\mu(x-x_0)^\nu = (t-t_0)^2 - (\mathbf{x}-\mathbf{x}_0)^2$, while for even space dimensions it has the structure $G^+(x,x_0) \sim \Theta(s^2)s^{1-n}$ where Θ is the step function $\Theta(s^2)=1$ for $s^2>0$ and $\Theta(s^2)=0$ for $s^2<0$ (see Refs. 9, 23, and 24). Therefore the singular support of each fundamental solution consists in the null cone $v^+_{x_0}$. However, the type of the singularities differs for even and odd space dimensions: While in odd space dimensions on both sides of the null cone the fundamental solution is exactly zero, it is nonzero inside the null cone (it declines continuously by progressing from the light cone $v^+_{x_0}$ to its interior $J^+_{x_0}$) for even space dimensions.

The fundamental solution will be modified in curved space–times, where the wave equation reads $g^{\mu\nu}(x)\,(\partial_\mu\partial_\nu\varphi(x) - \{^{\sigma}_{\mu\nu}\}\partial_\sigma\varphi(x)) = 0$ with the Christoffel symbol built from the metric $g_{\mu\nu}$. In comparison to flat space–time the fundamental solution will be altered (see Ref. 24): The general structure for odd-dimensional space is

$$G^+(x,x_0) \sim \sum_{m=0}^{(n-3)/2} \delta^{((n-3)/2-m)}(s^2) + V,$$

where V is a regular function with support in $J^+_{x_0}$ and s^2 now is the geodesic distance between x_0 and x. For even space dimensions we have $G^+(x,x_0) \sim Ws^{1-n}$ where W again is a regular function with support in $J^+_{x_0}$. We notice that (i) for even as well as for odd dimensions the singular support, like in the flat case, still consists in $v^+_{x_0}$ and (ii) the change in the solution consists only of the modification by regular functions. Therefore we can again distinguish between even and odd space dimension by noting that in odd space dimensions on both sides of the null cone the fundamental solution is regular and therefore finite, while for even space dimensions it declines continuously from infinity by progressing from the light cone $v^+_{x_0}$ to its interior $J^+_{x_0}$.

In our case of Eq. (21), deviations from the wave operator, that is, for example, mass, in general add to the fundamental solution of the wave operator additional terms (among others) of the form $\sim \Theta(s^2)(\partial A + B^2 s^2)$, etc., which do not modify the singular support. In any case, for an equation of the type (21) A^μ and B cannot lead to terms becoming infinite, for the following two reasons: (i) For the limit $A^\mu \to 0$ and $B \to 0$ the undisturbed fundamental solution must be recovered, and singularities cannot vanish with such a limiting process. (ii) In view of Sec. III we notice that discontinuities of solutions can occur on characteristic surfaces only. Since these surfaces are determined solely and completely by the principal part of the differential operator the additional terms A^μ and B in Eq. (20) cannot influence these surfaces. Therefore these terms cannot produce other singularities so that we are left with those given by the principal part $g^{\mu\nu}\partial_\mu\partial_\nu\varphi$ which we discussed above. This argumentation has been proven in detail for the wave equation by Friedlander,[24] see also DeWitt.[25]

Therefore, violations of Huygens principle which are caused by odd space–time dimensions can be experimentally distinguished from violations which are caused by nonzero A^μ and B. This can be done as follows (see Fig. 2): Assume that the field is caused by a δ-like source at x_0. Then the field has the form of a retarded (we choose appropriate boundary conditions) Green function $G^+(x,x_0)$. An observer (which we assume to be represented by a parametrized world line with monotonically increasing parameter τ) near the source measures the intensity of the field at his position $x(\tau)$. At first (let us say at the moment τ_0) he measures an infinite peak having its origin either in the δ function on the light cone (in the case of odd space dimensions) or in the factor s^{1-n} (in the case of even space dimensions). If for each moment $\tau > \tau_0$ after this moment the intensity is finite [i.e., $\lim_{\tau\searrow\tau_0} G^+(x,x_0) = c < \infty$], then there are only finite non-Huygens terms which necessarily originate from A^μ and B. If, however, the

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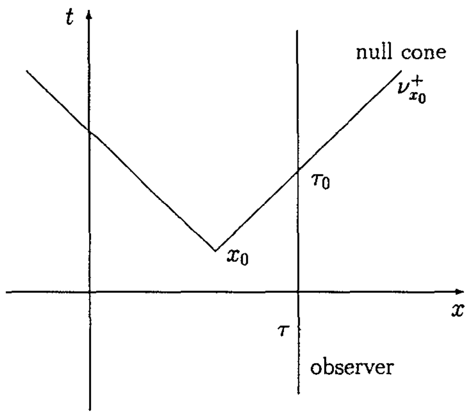

FIG. 2. Observer, parametrized with τ, crossing a null cone defined by the fundamental solution with vertex at x_0.

non-Huygens terms for $\tau \to \tau_0$ with $\tau > \tau_0$ tend to infinity [that is, $\lim_{\tau \searrow \tau_0} G^+(x,x_0) = \infty$], then this infinity can only be caused by a term of the form s^{1-n} which is present in even space dimensions only.

Consequently, if we demand that in nature violations of Huygens principle of the form $\Theta(s)s^{1-n}$ never occur, then we can infer that space–time must have even dimensions, that is, in our case, four dimensions. [One might ask what happens if $n=5$ and if furthermore Huygen's principle in the above form should be valid. In this case one expects according to the results of Ref. 23 that one has to modify the field equation (21) in such a way that it is of a pseudodifferential operator type. Such operators are nonlocal and therefore violate our requirement of finite propagation speed (see Refs. 11 and 26) and are therefore excluded in our axiomatic approach.] In this way we have, in analogy to the last section, characterized the four dimensionality of space–time by means of geometry-free restrictions of the dynamics of matter fields, which are observable in principle.

VIII. CONCLUSION

Within a constructive axiomatic scheme[11–14] we were able to derive the structure of space–time. Thereby the structure is given by the metric, the connection, and the dimensionality. Here we derived within this axiomatic scheme that space–time has to possess four dimensions.

The derivation proceeded in a completely geometry-free way essentially along the following steps whereby each step is based on physical experiences:

(1) A first order system of partial differential equations is derived from the requirement of deterministic linear evolution with finite propagation velocity.

(2) A probability interpretation implies the Hermiticity of the coefficients of the principal part.

(3) According to experience we demand one light cone and two jump amplitudes and two spin states only. This results in the Clifford algebra (16) which exists in at most five space–time dimensions.

(4) The requirement of conservation of chirality during evolution or the validness of a form of Huygen's principle sorts out the four space–time dimensions.

Therefore we succeeded in characterizing the four dimensionality of space–time by means of propagation phenomena only.

The main points which are essential for the characterization of four space–time dimensions are (i) the conservation law which implies that there is a Hermitizing matrix for the γ

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matrices, (ii) the demand that there are two jump amplitudes on each of the two light cones and two spin states on each mass shell which are essentially the facts implying that there is a Clifford algebra with 4×4 matrices, and (iii) the demand of conservation of helicity, respectively, of a form of Huygens principle. Therefore the four dimensionality of our physical configuration space $\mathscr{M}$ is deeply connected with the spin content, that is, with the number of inner degrees of freedom of our field φ (compare the end of Sec. V).

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APPENDIX A: PROOF OF EQ. (5)

According to our assumptions $H_0(x,k)$ has two zeros for given $k_{\hat{\mu}}$, that is, it has the form $H_0(x,k)=\alpha(x)(k_0-h_1(x,k_{\hat{\mu}}))(k_0-h_2(x,k_{\hat{\mu}}))$ with some proportionality factor $\alpha(x)$. On the other hand, $\det(\beta\gamma^{\mu}k_{\mu})=g^{\mu\nu\rho\sigma}k_{\mu}k_{\nu}k_{\rho}k_{\sigma}$ is a polynomial of order four. Therefore the following equality must hold:

$$g^{\mu\nu\rho\sigma}k_{\mu}k_{\nu}k_{\rho}k_{\sigma}=[\alpha(k_0-h_1(x,k_{\hat{\mu}}))(k_0-h_2(x,k_{\hat{\mu}}))]^2$$

and therefore

$$g^{0000}k_0^4+3g^{\hat{\mu}000}k_{\hat{\mu}}k_0^3+6g^{\hat{\mu}\hat{\nu}00}k_{\hat{\mu}}k_{\hat{\nu}}k_0^2+3g^{\hat{\mu}\hat{\nu}\hat{\rho}0}k_{\hat{\mu}}k_{\hat{\nu}}k_{\hat{\rho}}k_0+g^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}k_{\hat{\mu}}k_{\hat{\nu}}k_{\hat{\rho}}k_{\hat{\sigma}}$$
$$=\alpha^2[k_0^4-2(h_1+h_2)k_0^3+(h_1^2+4h_1h_2+h_2^2)k_0^2-2h_1h_2(h_1+h_2)k_0+h_1^2h_2^2].$$

Comparison of the coefficients to powers of k_0 gives $\alpha^2=g^{0000}$ and

$$h_1+h_2=q^{\hat{\mu}}k_{\hat{\mu}}, \tag{A1}$$

$$h_1^2+4h_1h_2+h_2^2=q^{\hat{\mu}\hat{\nu}}k_{\hat{\mu}}k_{\hat{\nu}}, \tag{A2}$$

$$h_1h_2(h_1+h_2)=q^{\hat{\mu}\hat{\nu}\hat{\rho}}k_{\hat{\mu}}k_{\hat{\nu}}k_{\hat{\rho}},$$

$$h_1^2h_2^2=q^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}k_{\hat{\mu}}k_{\hat{\nu}}k_{\hat{\rho}}k_{\hat{\sigma}}$$

for appropriate coefficients $q^{\hat{\mu}},\ldots,q^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$. Inserting Eq. (A1) into Eq. (A2) gives

$$h_1h_2=\tfrac{1}{2}(q^{\hat{\mu}\hat{\nu}}-q^{\hat{\mu}}q^{\hat{\nu}})k_{\hat{\mu}}k_{\hat{\nu}}.$$

Using this result and Eq. (A1) we therefore get

$$\begin{aligned}H_0(x,k)&=\alpha(k_0-h_1)(k_0-h_2)\\&=\alpha(k_0^2-(h_1+h_2)k_0+h_1h_2)\\&=\alpha(k_0^2-q^{\hat{\mu}}k_{\hat{\mu}}k_0+\tfrac{1}{2}(q^{\hat{\mu}\hat{\nu}}-q^{\hat{\mu}}q^{\hat{\nu}})k_{\hat{\mu}}k_{\hat{\nu}})\\&=:\overset{c}{g}{}^{\mu\nu}k_{\mu}k_{\nu}\end{aligned}$$

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for some tensor $\overset{c}{g}{}^{\mu\nu}$. This has the form stated above.

APPENDIX B: PROOF OF EQ. (15)

To show Eq. (15), we rewrite Eq. (14) with $\hat{B}:=B\hat{\gamma}^0$ as

$$\hat{B}(P_0+(\hat{\gamma}^0)^{-1}(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1))=g^{\mu\nu}P_\mu P_\nu-1. \tag{B1}$$

In Ref. 18 it is shown that $\hat{B}$ must be a polynomial of first order in P_0. Therefore we make the ansatz $\hat{B}=\hat{B}^0+\hat{B}^1P_0$ with $\hat{B}^0=B^0\hat{\gamma}^0$ and $\hat{B}^1=B^1\hat{\gamma}^0$, insert this into Eq. (B1), and expand the resulting expression

$$B^1\hat{\gamma}^0P_0^2+(B^1(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)+B^0\hat{\gamma}^0)P_0+B^0(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)=g^{00}P_0^2+2g^{\hat{\mu}0}P_{\hat{\mu}}P_0+g^{\hat{\mu}\hat{\nu}}P_{\hat{\mu}}P_{\hat{\nu}}-1.$$

Comparison of the coefficients to the powers of P_0 then gives

$$B^1\hat{\gamma}^0=g^{00},$$

$$B^1(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)+B^0\hat{\gamma}^0=2g^{\hat{\mu}0}P_{\hat{\mu}},$$

$$B^0(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)=g^{\hat{\mu}\hat{\nu}}P_{\hat{\mu}}P_{\hat{\nu}}-1.$$

From these equations we can infer that B^1 does not depend on $P_{\hat{\mu}}$ and

$$B^1(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)+B^0\hat{\gamma}^0=g^{00}(\hat{\gamma}^0)^{-1}(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)+B^0\hat{\gamma}^0=2g^{\hat{\mu}0}P_{\hat{\mu}}$$

so that $B^0\hat{\gamma}^0=2g^{\hat{\mu}0}P_{\hat{\mu}}-g^{00}(\hat{\gamma}^0)^{-1}(\hat{\gamma}^{\hat{\mu}}P_{\hat{\mu}}-1)$, that is, B^0 is a polynomial of $P_{\hat{\mu}}$. Taking all results together, we see that B is a polynomial of first order in P_μ.

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