

On the shadows of black holes and of other compact objects

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1. Schwarzschild spacetime

mass m

photon sphere at $r = 3m$

shadow ('escape cones'): J. Synge, 1966

2. Kerr spacetime

mass m , spin a

photon region of 'spherical geodesics'

shadow: J. Bardeen, 1973

3. Plebański-Demiański spacetime

mass m , spin a , electric charge q_e , magnetic charge q_m ,

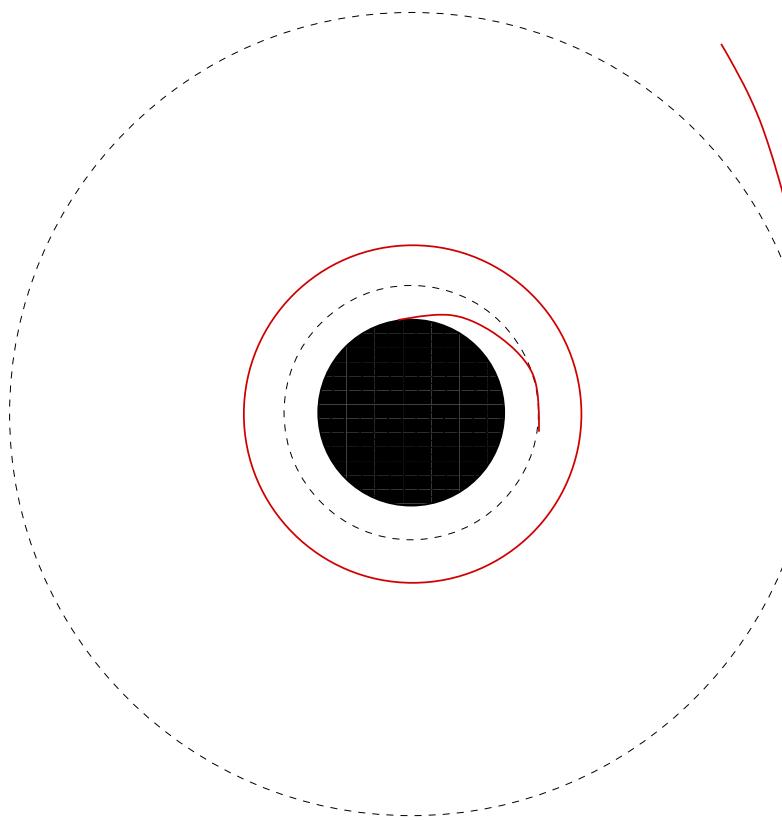
NUT parameter ℓ , cosmological constant Λ

photon region of 'spherical geodesics'

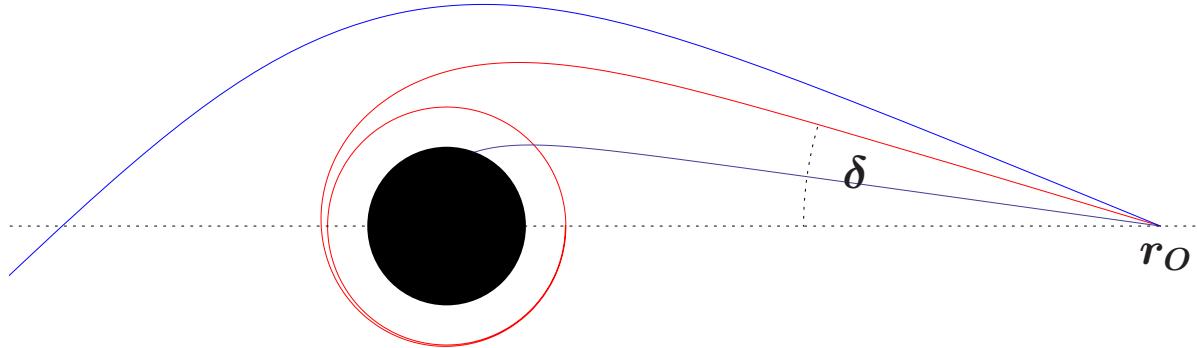
shadow: A. Grenzebach, 2013

1. Schwarzschild spacetime

$$g = - \left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad r_s = 2m$$

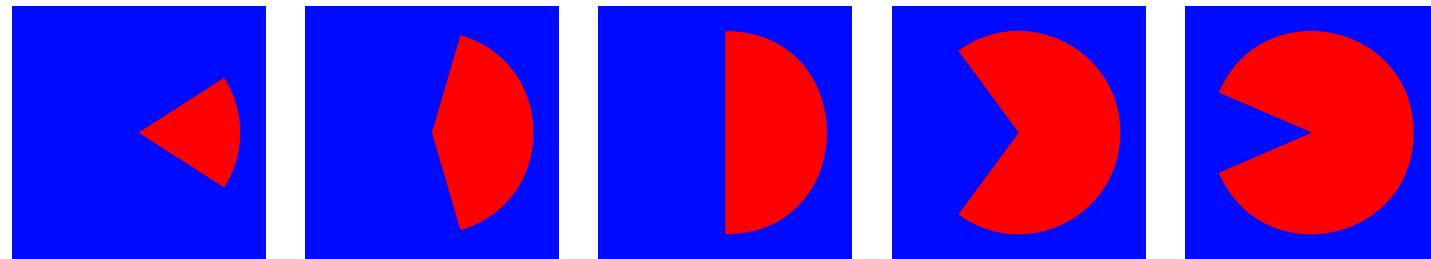


Unstable photon sphere at $r = 3m = 1.5r_s$



Angular radius δ of the “shadow”: $\sin^2 \delta = \frac{27 r_s^2 (r_O - r_s)}{4 r_O^3}$

J. Synge, “The escape of photons from gravitationally intense stars” Mon. Not. R. Astron. Soc. 131, 463 (1966) – Escape cones:



$$r_O = 1.05 r_s \quad r_O = 1.3 r_s \quad r_O = 3 r_s/2 \quad r_O = 2.5 r_s \quad r_O = 6 r_s$$

S. Virbhadra and G. Ellis, “Gravitational lensing by naked singularities” Phys. Rev. D, 65, 103004 (2002)

A naked singularity is called “weakly naked” if surrounded by an unstable photon sphere, and “strongly naked” if not.

2. Kerr spacetime

What happens to the photon sphere at $r = 3m$ and with the shadow if the spin a is switched on?

Shadow of a Kerr black hole:

J. Bardeen, “Timelike and null geodesics in the Kerr metric” in C. DeWitt and B. DeWitt (eds.): “Black Holes” Gordon and Breach (1973), p. 215

Shadow of a Kerr(-Newman) naked singularity:

A. deVries, “The apparent shape of a rotating charged black hole, closed photon orbits and the bifurcation set A4” Class. Quantum Grav. 17, 123 (2000)

C. Bambi and K. Freese, “Apparent shape of super-spinning black holes” Phys. Rev. D 79, 043002 (2009)

If a is switched on, the “photon sphere” at $r = 3m$ turns into a “photon region” filled with “spherical null geodesics” .

Kerr metric in Boyer–Lindquist coordinates $(r, \vartheta, \varphi, t)$:

$$g = \varrho(r, \vartheta)^2 \left(\frac{dr^2}{\Delta(r)} + d\vartheta^2 \right) + \frac{\sin^2 \vartheta}{\varrho(r, \vartheta)^2} \left(a dt - (r^2 + a^2) d\varphi \right)^2 - \frac{\Delta(r)}{\varrho(r, \vartheta)^2} \left(dt - a \sin^2 \vartheta d\varphi \right)^2$$

$$\varrho(r, \vartheta)^2 = r^2 + a^2 \cos^2 \vartheta, \quad \Delta(r) = r^2 - 2mr + a^2 .$$

Lightlike geodesics:

$$\varrho(r, \vartheta)^2 \dot{t} = a (L - E a \sin^2 \vartheta) + \frac{(r^2 + a^2) ((r^2 + a^2) E - a L)}{\Delta(r)},$$

$$\varrho(r, \vartheta)^2 \dot{\varphi} = \frac{L - E a \sin^2 \vartheta}{\sin^2 \vartheta} + \frac{(r^2 + a^2) a E - a^2 L}{\Delta(r)},$$

$$\varrho(r, \vartheta)^4 \dot{\vartheta}^2 = K - \frac{(L - E a \sin^2 \vartheta)^2}{\sin^2 \vartheta} =: \Theta(\vartheta),$$

$$\varrho(r, \vartheta)^4 \dot{r}^2 = -K \Delta(r) + ((r^2 + a^2) E - a L)^2 =: R(r).$$

Lightlike geodesics:

$$\varrho(r, \vartheta)^2 \dot{t} = a(L - Ea \sin^2 \vartheta) + \frac{(r^2 + a^2)((r^2 + a^2)E - aL)}{\Delta(r)},$$

$$\varrho(r, \vartheta)^2 \dot{\varphi} = \frac{L - Ea \sin^2 \vartheta}{\sin^2 \vartheta} + \frac{(r^2 + a^2)aE - a^2L}{\Delta(r)},$$

$$\varrho(r, \vartheta)^4 \dot{\vartheta}^2 = K - \frac{(L - Ea \sin^2 \vartheta)^2}{\sin^2 \vartheta} =: \Theta(\vartheta),$$

$$\varrho(r, \vartheta)^4 \dot{r}^2 = -K \Delta(r) + ((r^2 + a^2)E - aL)^2 =: R(r).$$

Spherical lightlike geodesics exist in the region where

$$R(r) = 0, \quad R'(r) = 0, \quad \Theta(\vartheta) \geq 0.$$

$$a \frac{L}{E} = r^2 + a^2 - \frac{2r\Delta(r)}{r - m}, \quad \frac{K}{E^2} = \frac{4r^2\Delta(r)}{(r - m)^2}.$$

$$(2r\Delta(r) - (r - m)\varrho(r, \vartheta)^2)^2 \leq 4a^2r^2\Delta(r) \sin^2 \vartheta$$

(unstable if $R''(r) \geq 0$)

Pictures of spherical lightlike geodesics:

E. Teo, “Spherical photon orbits around a Kerr black hole” Gen. Rel. Grav. 35, 1909 (2003)

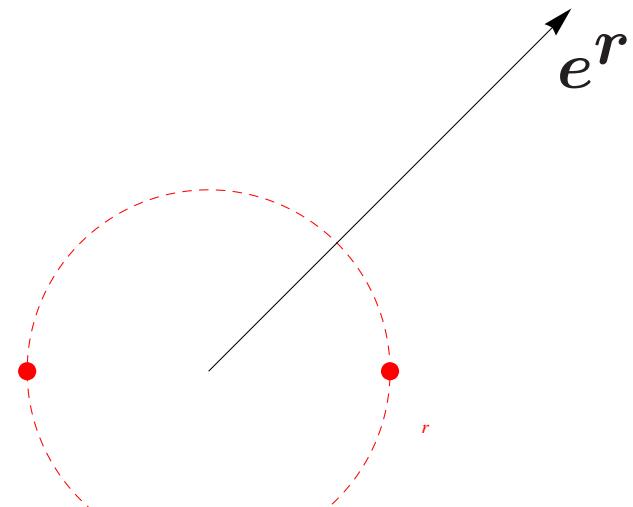
Pictures of the photon region:

W. Hasse and VP, “A Morse-theoretical analysis of gravitational lensing by a Kerr-Newman black hole” J. Math. Phys. 47, 042503 (2006)

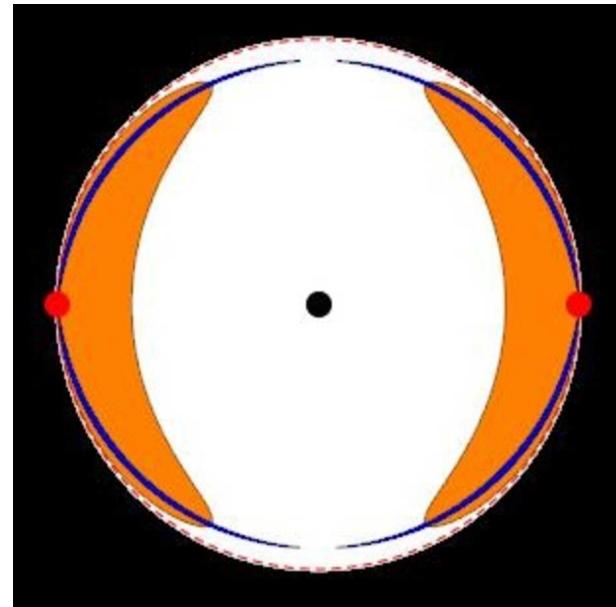
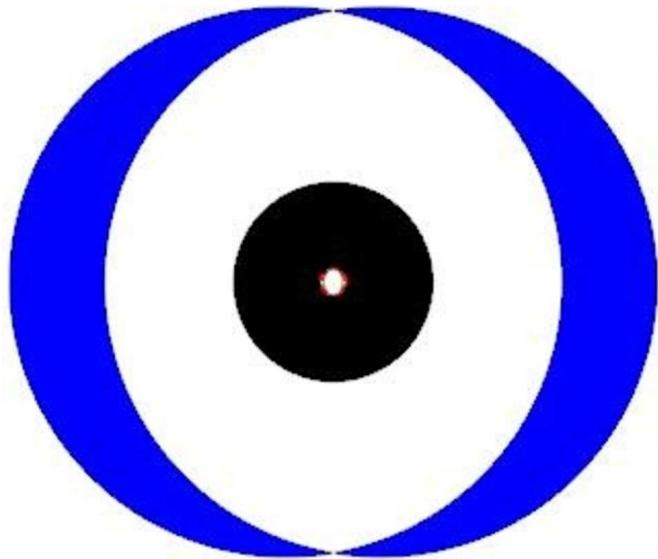
VP: “Gravitational lensing from a spacetime perspective” Living Rev. Relativity 7, (2004), 9

For pictures of the
photon region, use e^r
as the radial coordinate

red solid: singularity
red dashed: “throat”



$$a = 0.15 m$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

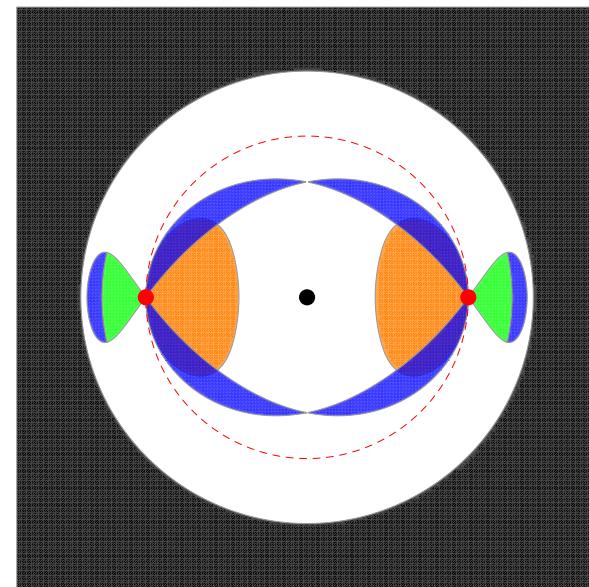
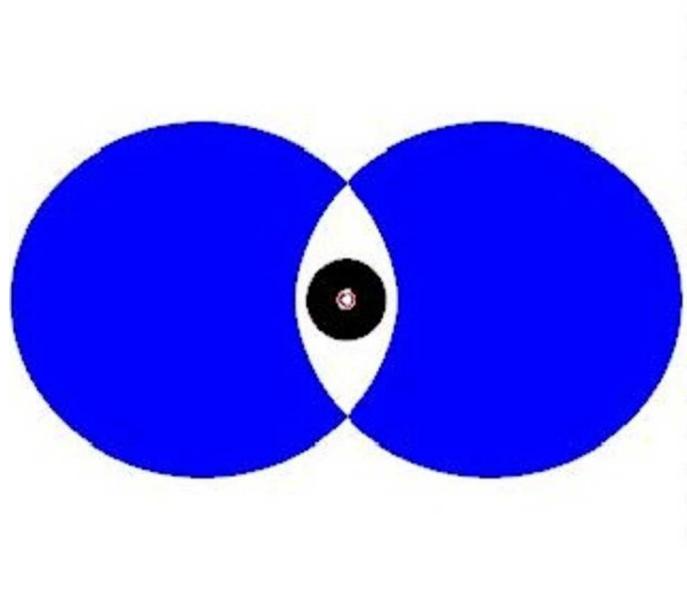
orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

$$a = 0.75 m$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

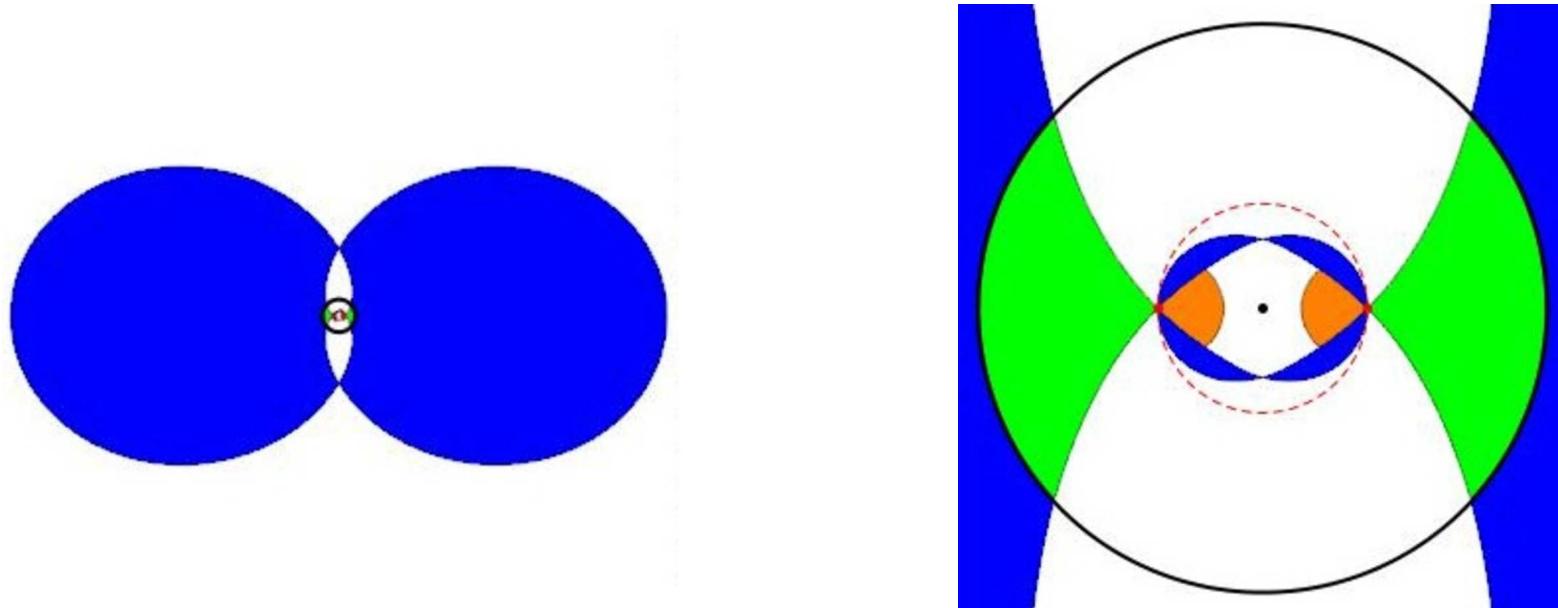
orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

$$a = 0.999999 \text{ m}$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

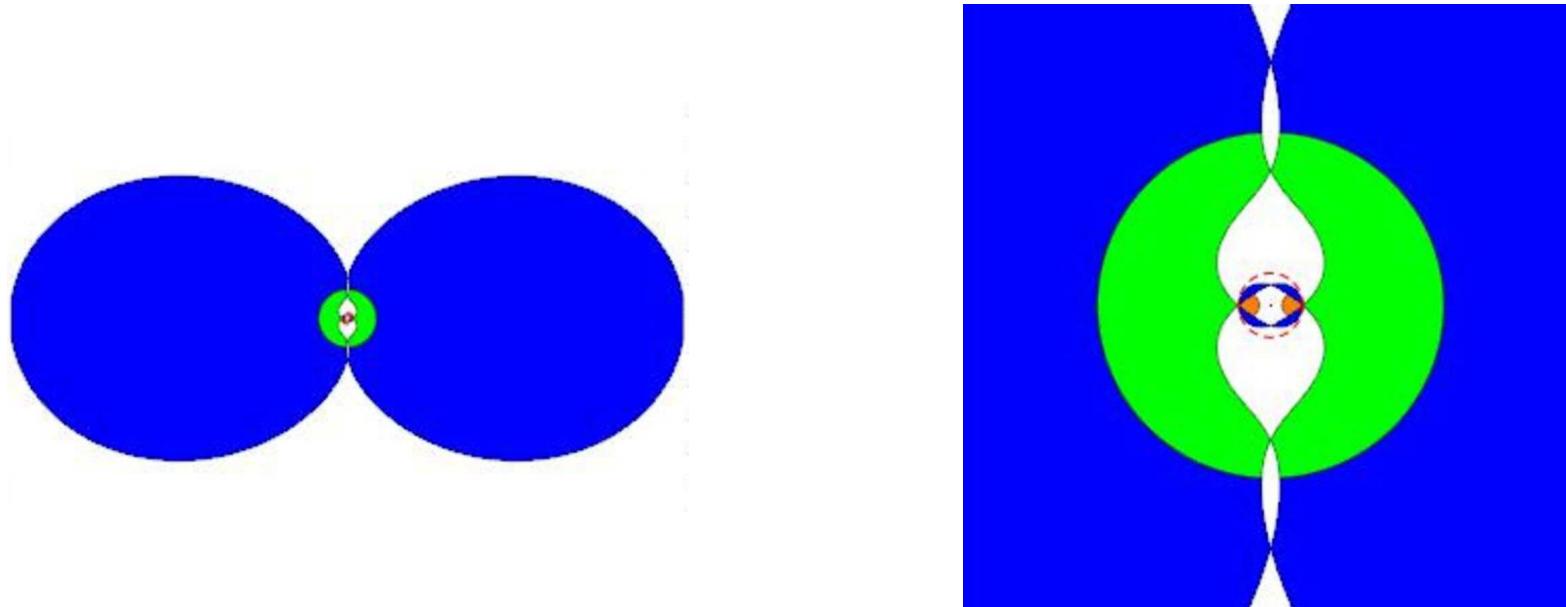
orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

$$a = 1.15 m$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

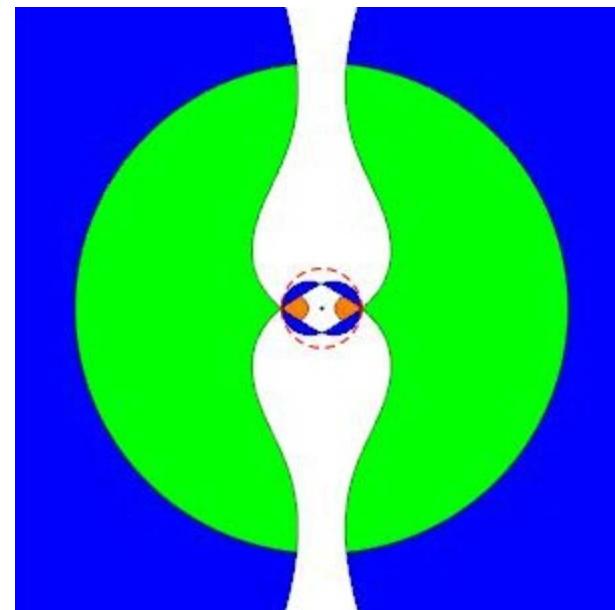
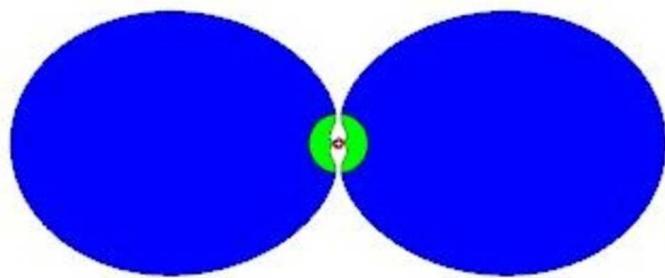
orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

$$a = 1.25 m$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

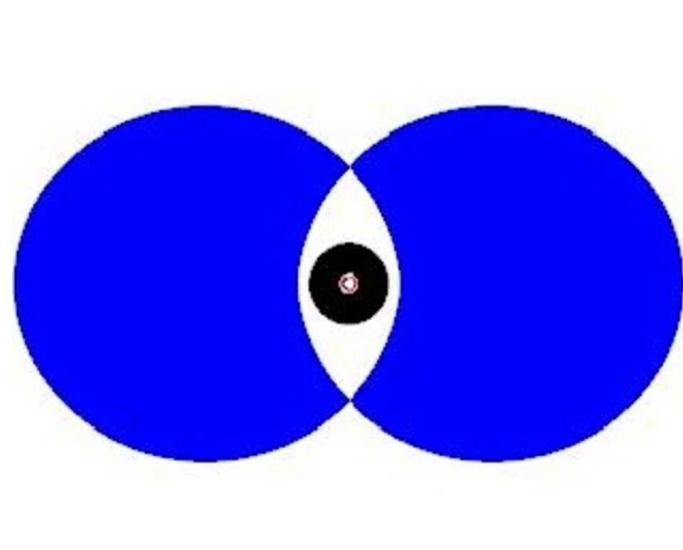
The “shadow” is determined by light rays that approach an unstable spherical lightlike geodesic.

Choose observer at r_O and ϑ_O . Introduce celestial coordinates (θ, ϕ) at the observer. Relation between **celestial coordinates** and **constants of motion**:

$$\sin^2 \theta = \frac{g_{\varphi t}^2 - g_{\varphi\varphi}g_{tt}}{\left(g_{\varphi t}\frac{L}{E} + g_{\varphi\varphi}\right)^2} \left\{ \frac{L^2}{E^2} + \frac{g_{\varphi\varphi}}{r^2 + a^2\cos^2\vartheta} \left(\frac{K}{E^2} - \frac{\left(a\sin^2\vartheta - \frac{L}{E}\right)^2}{\sin^2\vartheta} \right) \right\}_{r_O, \vartheta_O}$$

$$\sin^2 \phi = \frac{a\sin^2\vartheta \frac{L^2}{E^2}}{a\sin^2\vartheta \frac{L^2}{E^2} + g_{\varphi\varphi} \left(\frac{K}{E^2} - \frac{\left(a\sin^2\vartheta - \frac{L}{E}\right)^2}{\sin^2\vartheta} \right)}_{r_O, \vartheta_O}$$

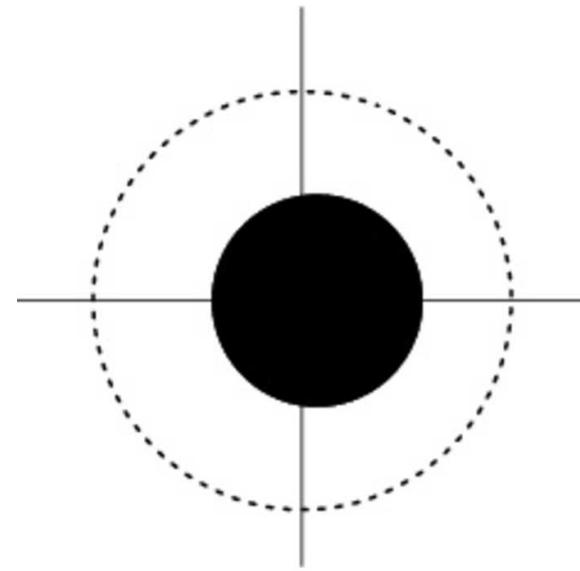
Inserting L/E and K/E^2 as functions of r gives the boundary of the shadow parametrised with r .

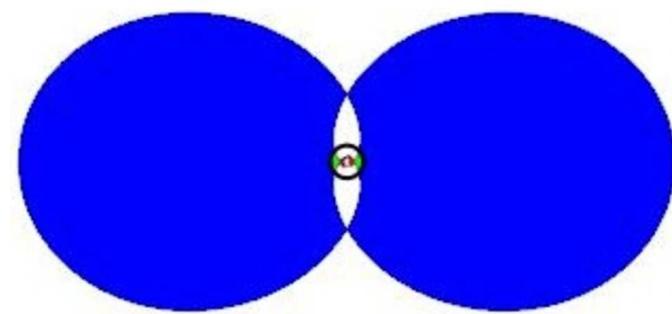


$$a = 0.4 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$

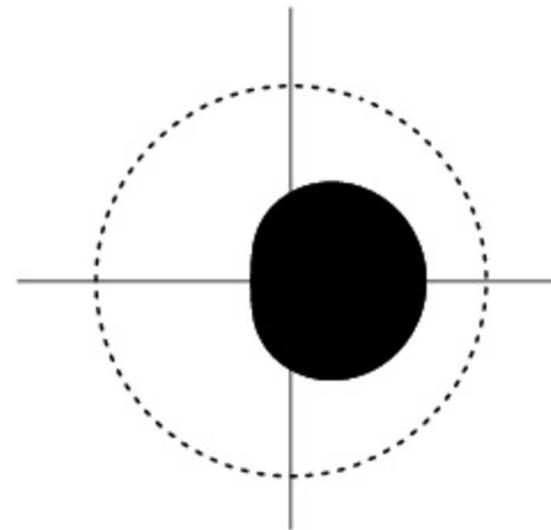


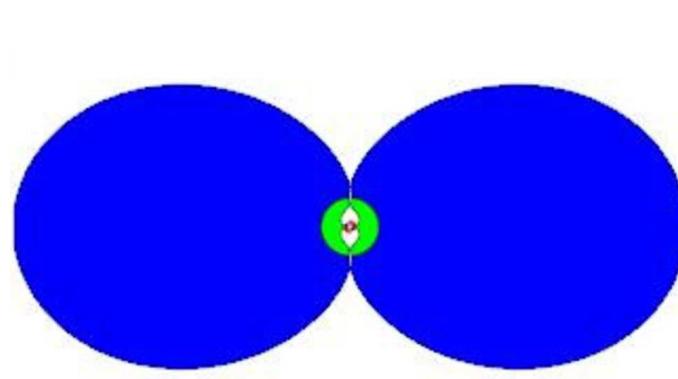


$$a = 0.9999999 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$

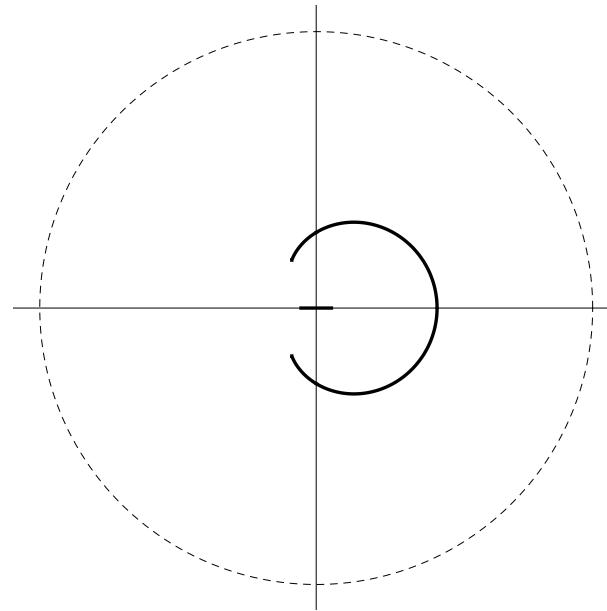


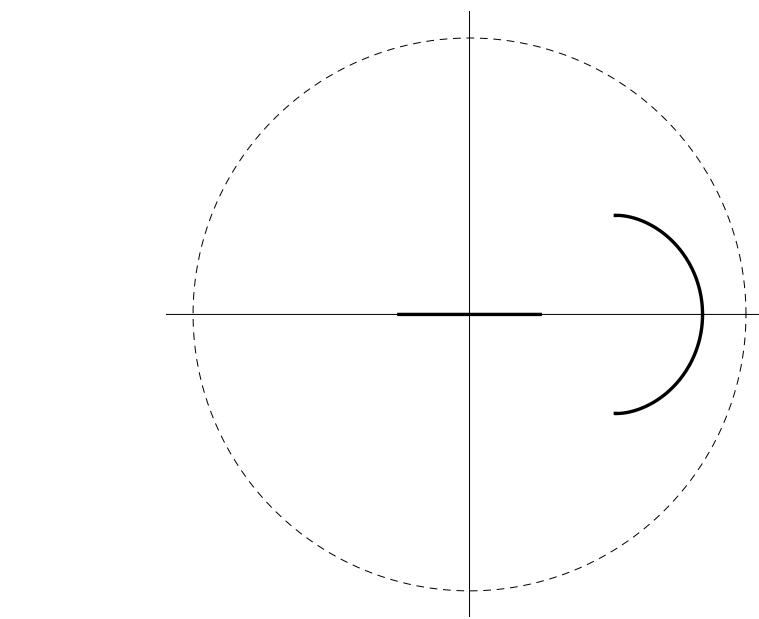
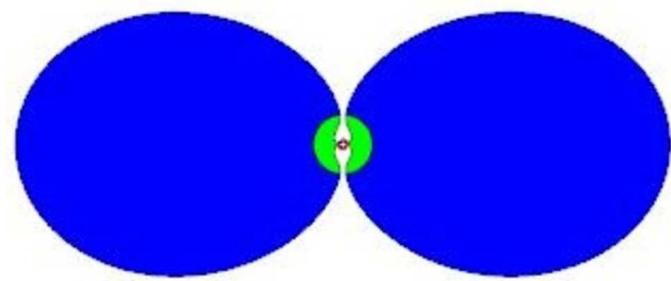


$$a = 1.1 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$





$$a = 2 \text{ m}$$

$$r_O = 8 \text{ m}$$

$$\vartheta_O = \pi/2$$

3. Plebański-Demiański spacetime

$$g = \Sigma \left(\frac{dr^2}{\Delta_r} + \frac{d\vartheta^2}{\Delta_\vartheta} \right) + \left((\Sigma + \textcolor{red}{a}\chi)^2 \Delta_\vartheta \sin^2 \vartheta - \Delta_r \chi^2 \right) \frac{d\varphi^2}{\Sigma}$$

$$+ (\Delta_r \chi - \textcolor{red}{a}(\Sigma + \textcolor{red}{a}\chi) \Delta_\vartheta \sin^2 \vartheta) \frac{2 dt d\varphi}{\Sigma} - (\Delta_r - \textcolor{red}{a}^2 \Delta_\vartheta \sin^2 \vartheta) \frac{dt^2}{\Sigma}$$

where $\Sigma = r^2 + (\ell + \textcolor{red}{a} \cos \vartheta)^2$

$$\chi = \textcolor{red}{a} \sin^2 \vartheta - 2 \textcolor{brown}{l} (\cos \vartheta + \textcolor{brown}{C})$$

$$\Delta_\vartheta = 1 + \textcolor{violet}{\Lambda} \left\{ \frac{4}{3} \textcolor{red}{a} \textcolor{brown}{l} \cos \vartheta + \frac{\textcolor{red}{a}^2}{3} \cos^2 \vartheta \right\}$$

$$\Delta_r = r^2 - 2\textcolor{blue}{m}r + \textcolor{red}{a}^2 - \textcolor{brown}{l}^2 + \textcolor{green}{q_e}^2 + \textcolor{green}{q_m}^2 - \textcolor{violet}{\Lambda} \left\{ (\textcolor{red}{a}^2 - \ell^2) \textcolor{brown}{l}^2 + \left(\frac{\textcolor{red}{a}^2}{3} + 2\textcolor{brown}{l}^2 \right) r^2 + \frac{r^4}{3} \right\}$$

$\textcolor{blue}{m}$: mass

$\textcolor{green}{q_e}$: electric charge

$\textcolor{brown}{l}$: NUT parameter

$\textcolor{violet}{\Lambda}$: cosmological constant

$\textcolor{red}{a}$: spin

$\textcolor{green}{q_m}$: magnetic charge

$\textcolor{brown}{C}$: Manko-Ruiz parameter

Introduction of NUT metric:

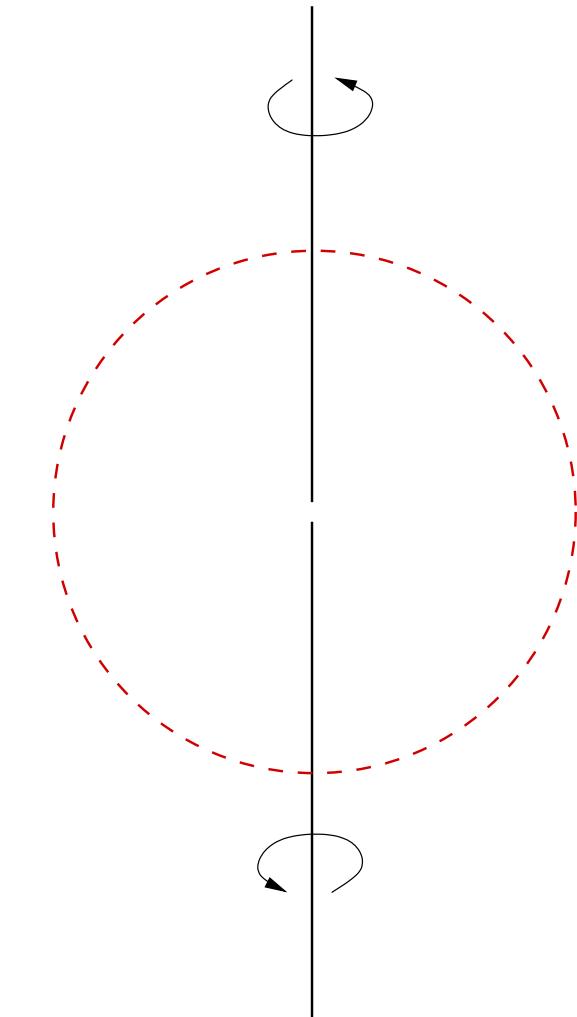
E. Newman, T. Unti, L. Tamburino, “Empty-space generalization of the Schwarzschild metric”
J. Math. Phys. 4, 915 (1963)

Interpretation of NUT parameter ℓ as “gravito-magnetic charge” of a spinning massless string:

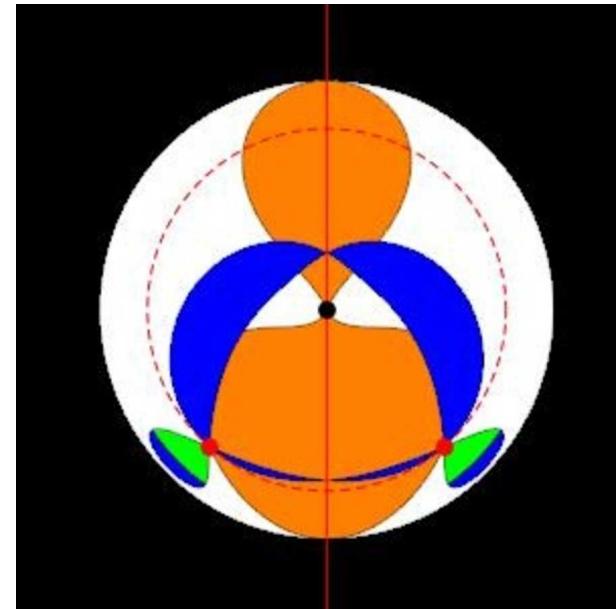
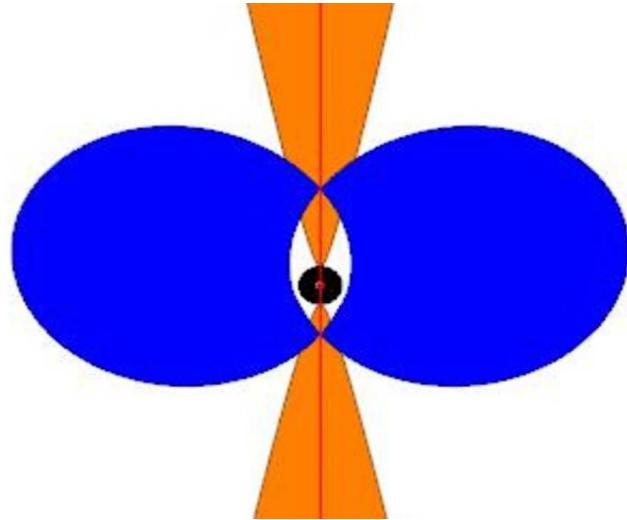
W. B. Bonnor, “A new interpretation of the NUT metric in general relativity” Proc. Camb. Philos. Soc. 66, 145 (1969)

Introduction of the parameter C :

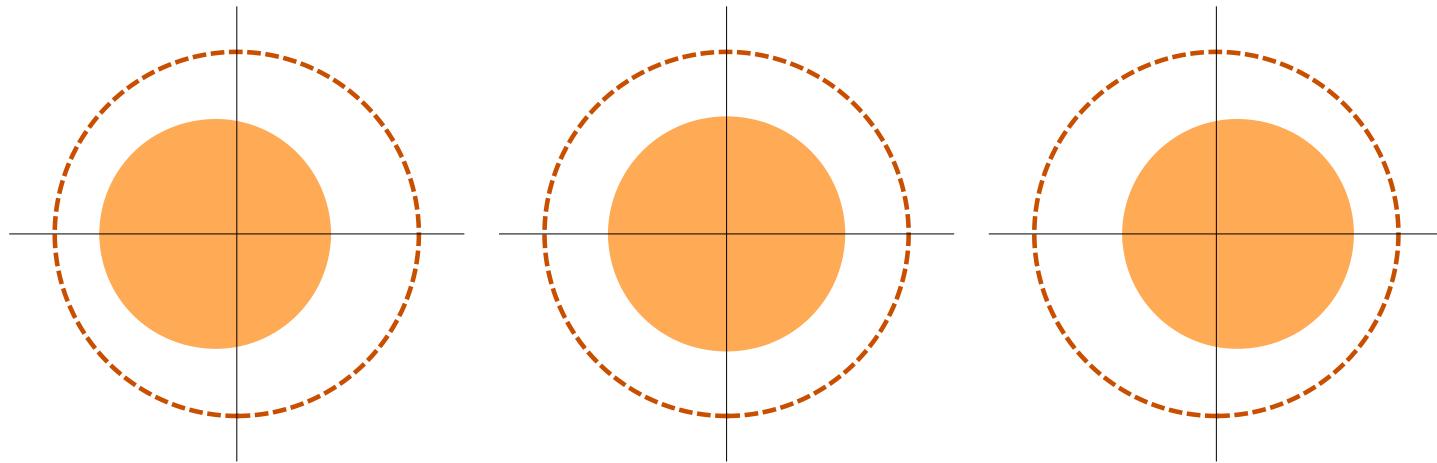
V. Manko, E. Ruiz, “Physical interpretation of the NUT family of solutions” Class. Quantum Grav. 22, 3555 (2005)



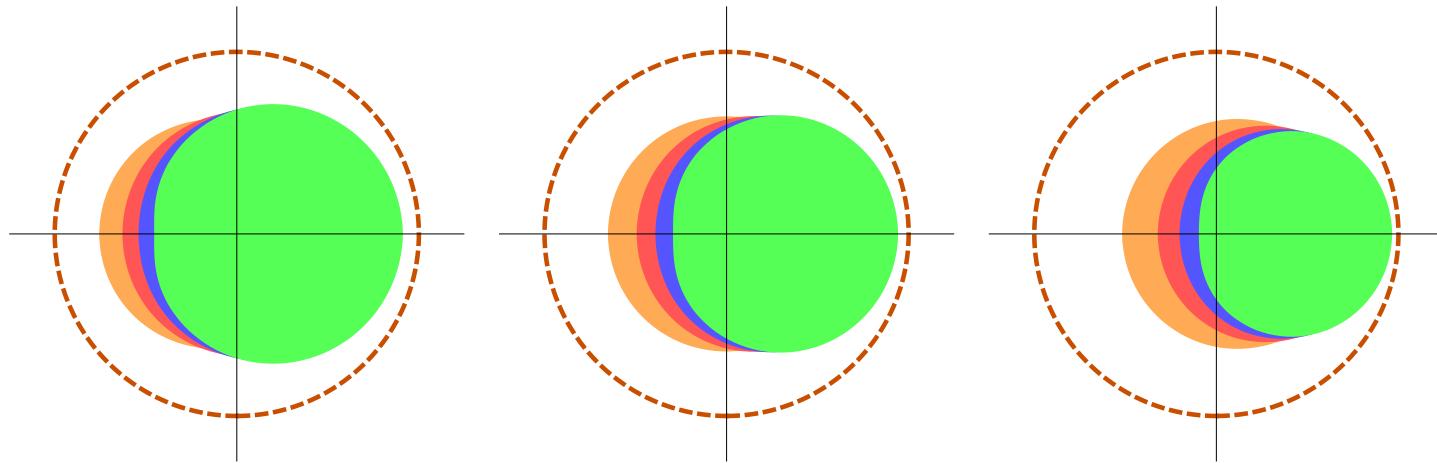
$$a = 0.99 \text{ m}, \ell = 0.75 \text{ m}$$



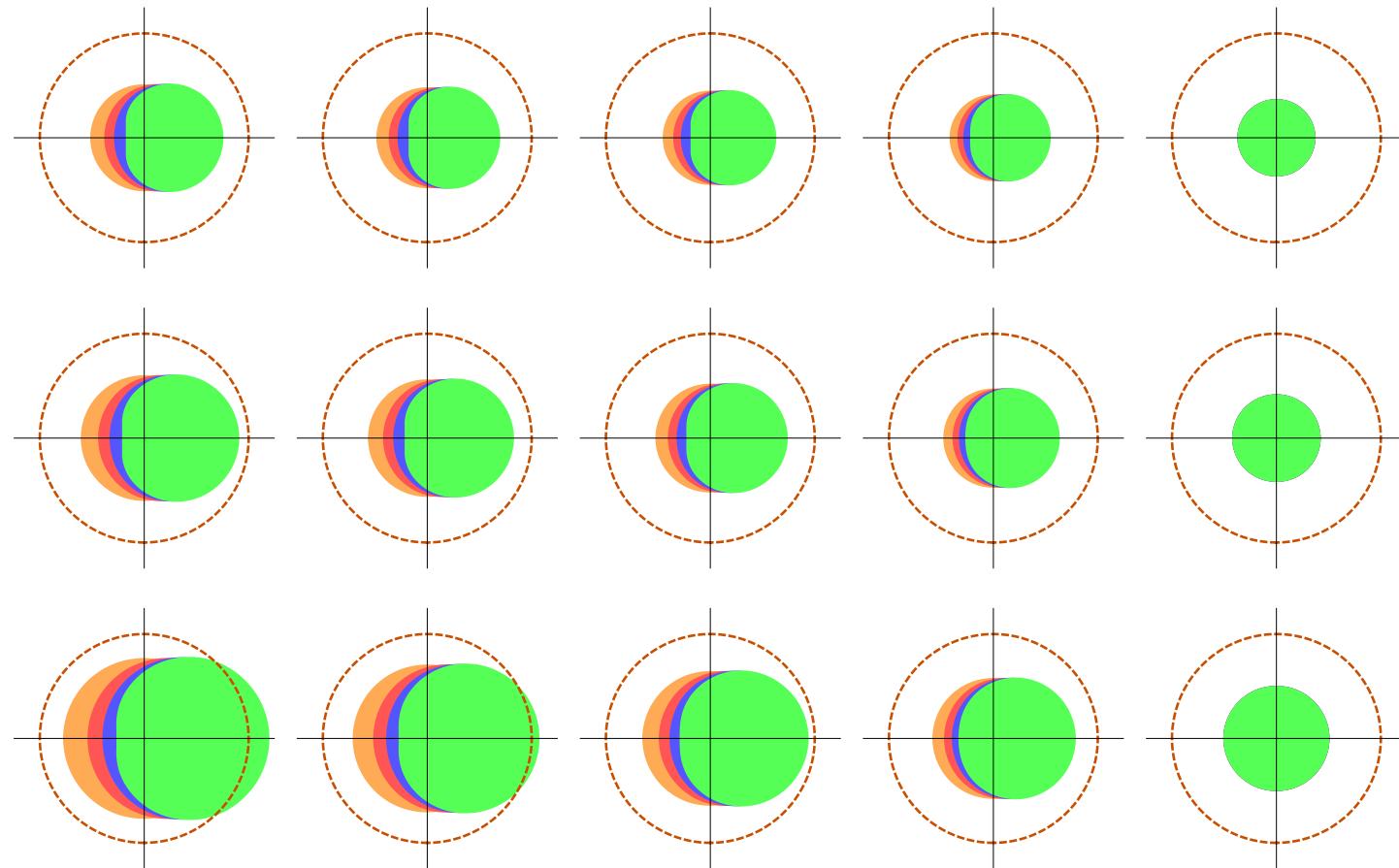
blue: unstable spherical lightlike geodesics
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Dependence of the shadow on C



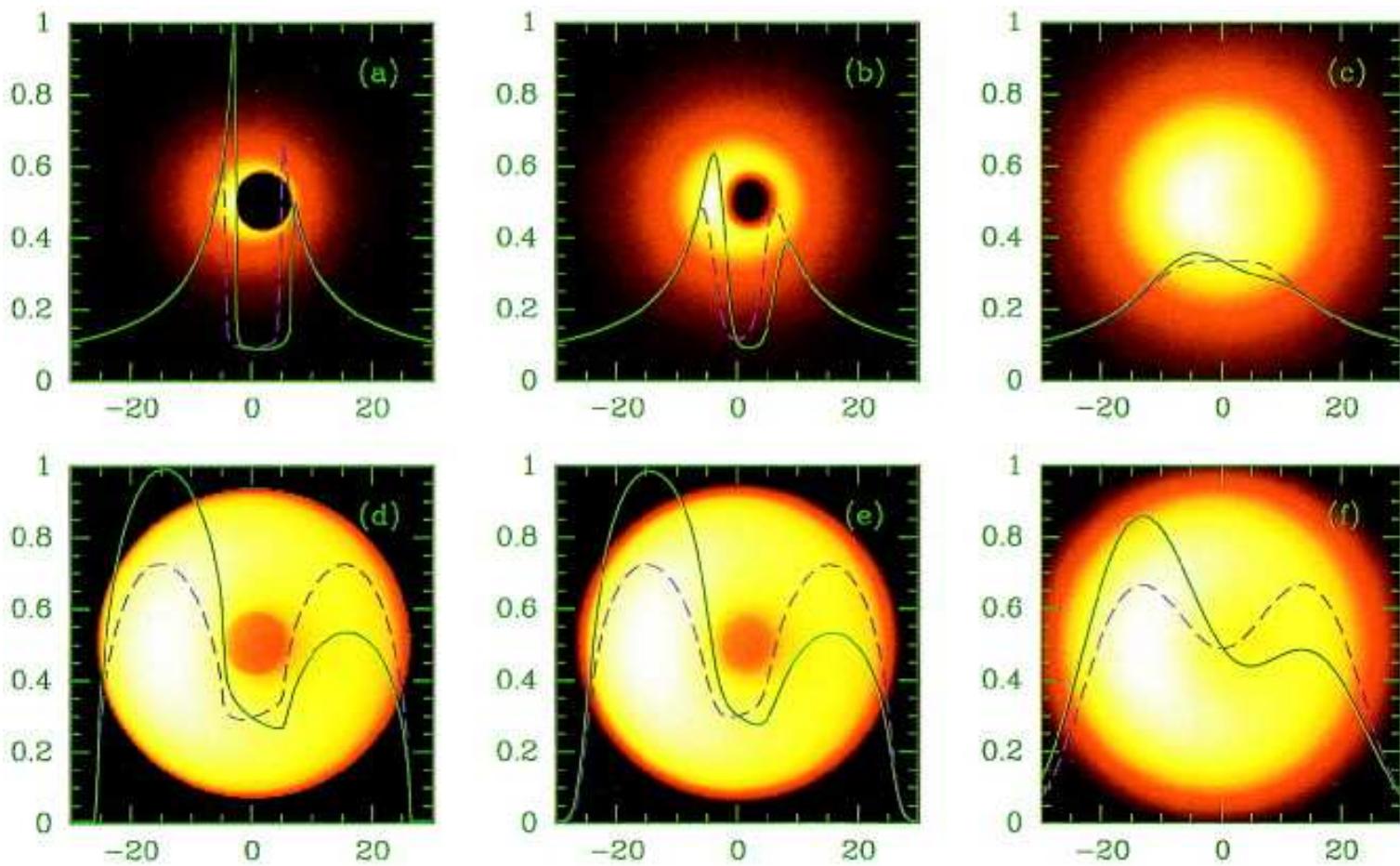
Colour: Increasing a



Colour: increasing a

Left to right: increasing ℓ

Top to bottom: Increasing $q_e^2 + q_m^2$



H. Falcke, F. Melia, E. Agol, “Viewing the shadow of the black hole at the galactic center” *Astrophys J.* 528, L13 (2000)