

# Gravitational lensing beyond the weak-field approximation

Volker Perlick

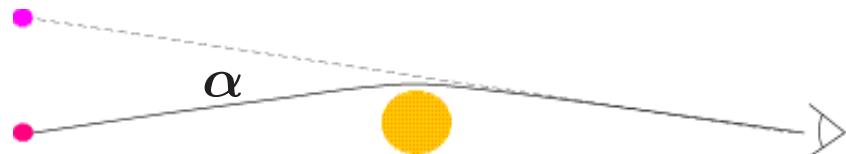
ZARM, Univ. Bremen, Germany

- phenomenology of lensing
- lensing in the weak-field approximation
- exact lens map for spherically symmetric and static spacetimes
- shadow of a Schwarzschild black hole
- shadow of a Kerr black hole

VP: "Gravitational Lensing from a Spacetime Perspective", Living Rev. Relativity 7, (2004), <http://www.livingreviews.org/lrr-2004-9>

## Light deflection at the Sun

(1919, Eddington)



$$\alpha \approx \frac{4 G M}{c^2 R} \approx 1.73''$$

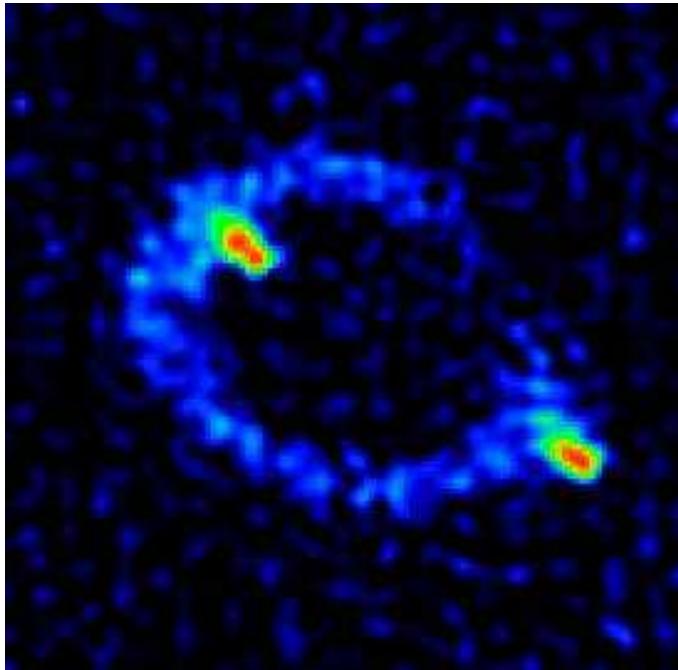
**Multiple Quasars:**



**Twin Quasar 0957+561:**

**D. Walsh, R. Carlswell, R. Weyman (1979)**

Rings:

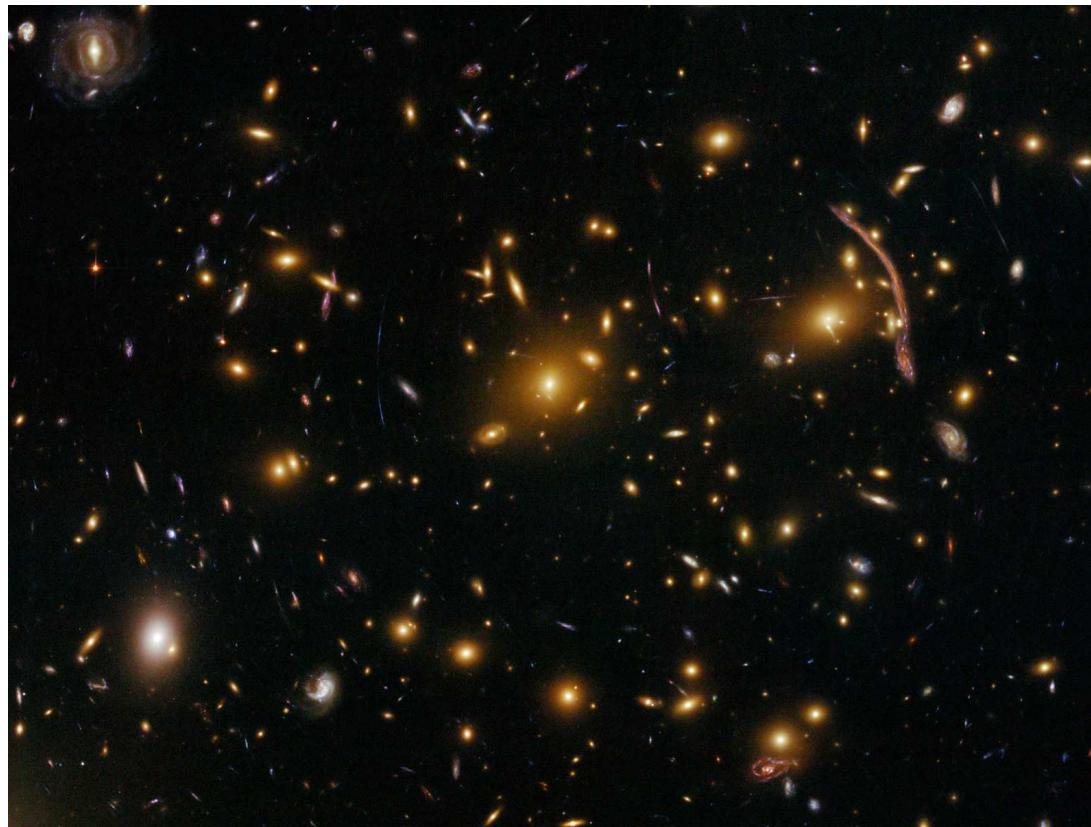


MG1131+0456: Jacqueline Hewitt et al. (1988)

**“Giant Luminous Arcs”:**

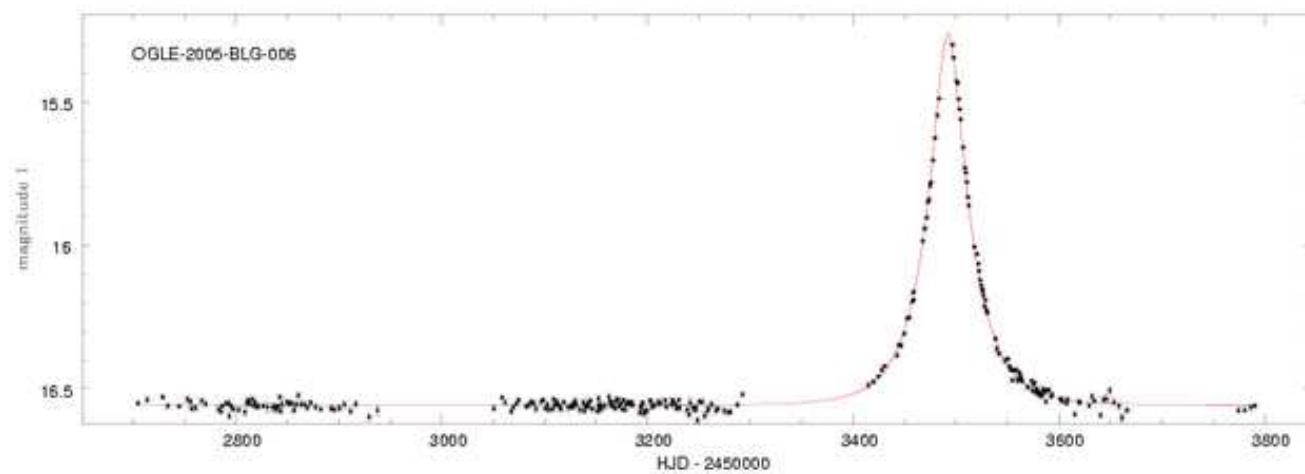
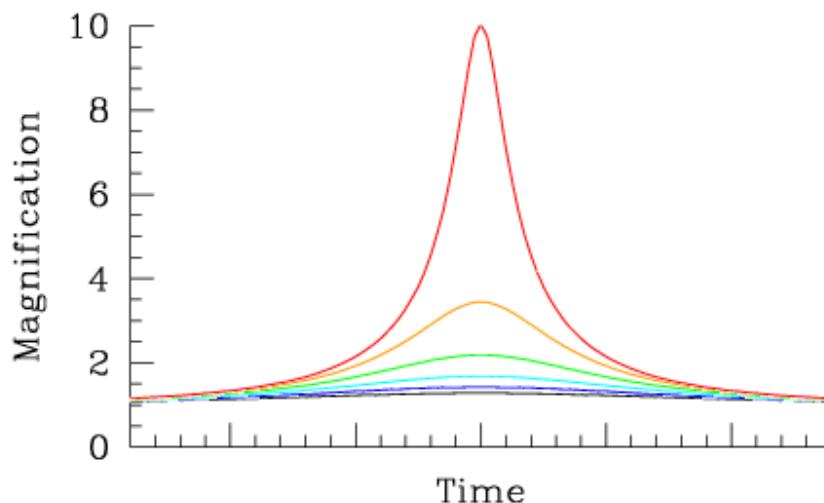
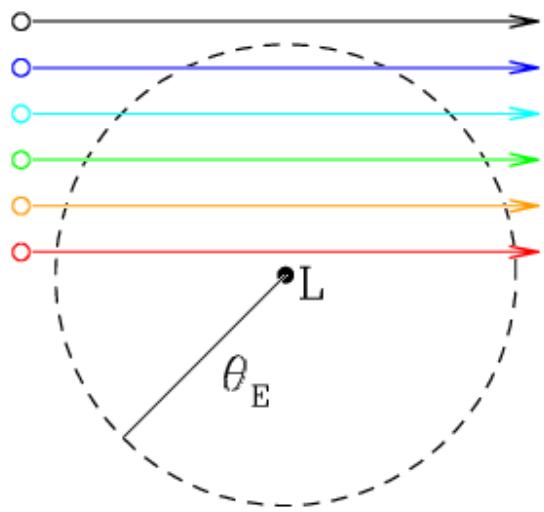
**R. Lynds, V. Petrosian (1986)**

**G. Soucail et al. (1987)**

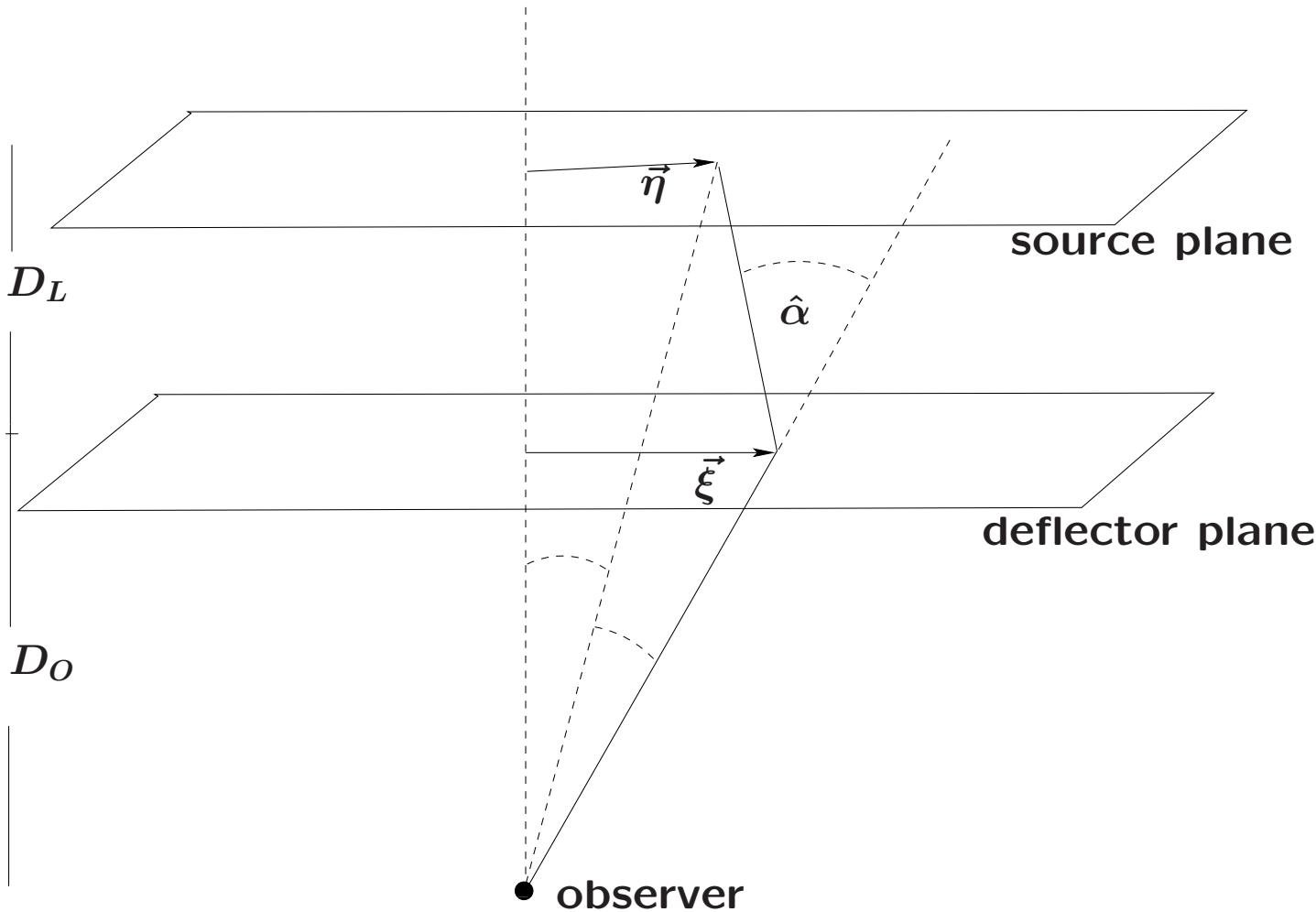


**Abell 370**

# Microlensing



## Lens map of the weak-field formalism (S. Refsdal, 1963)

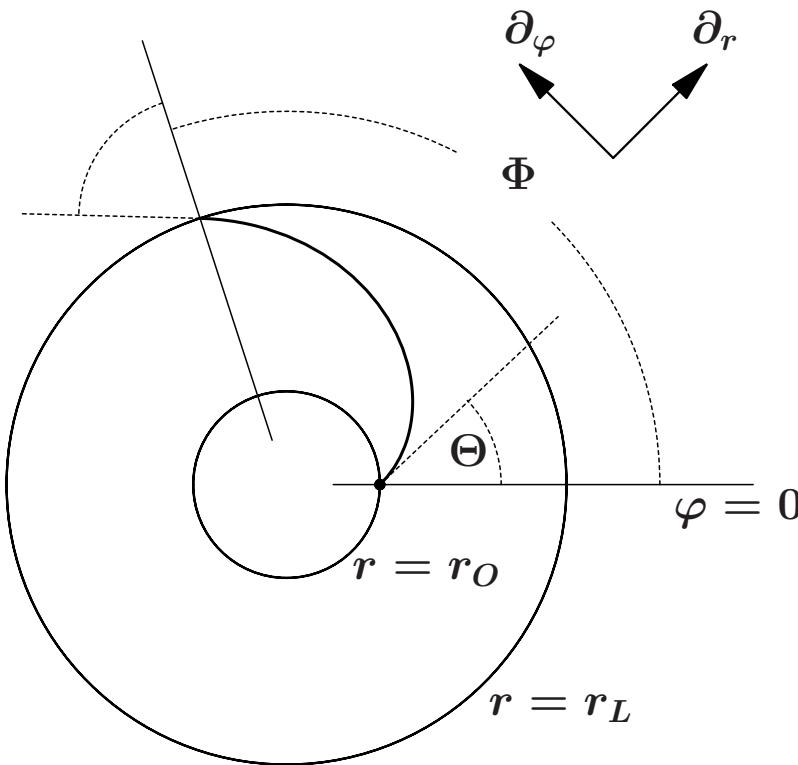


$$\vec{\eta} = \frac{D_L + D_O}{D_O} \vec{\xi} - D_L \vec{\alpha}, \quad \vec{\alpha} = \frac{4G}{c^2} \int_{\mathbb{R}^2} \frac{(\vec{\xi} - \vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} \Sigma(\vec{\xi}') d^2 \vec{\xi}' .$$

# Exact lens map for spherically symmetric and static spacetimes

VP: Phys. Rev. D 69, 064917 (2004)

$$g = e^{2f(r)} \left( -c^2 dt^2 + S(r)^2 dr^2 + R(r)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right)$$

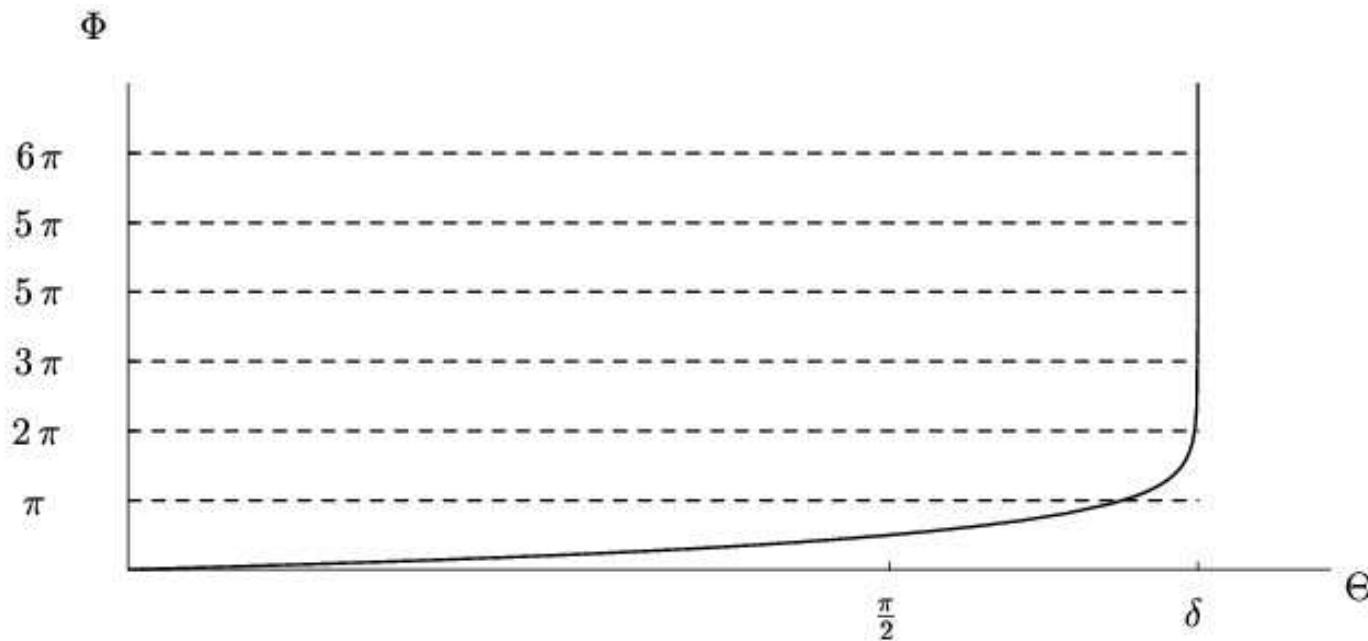


$$\Phi = R(r_O) \sin \Theta \int_{r_O}^{r_L} \frac{S(r) dr}{R(r) \sqrt{R(r)^2 - R(r_O)^2 \sin^2 \Theta}}$$

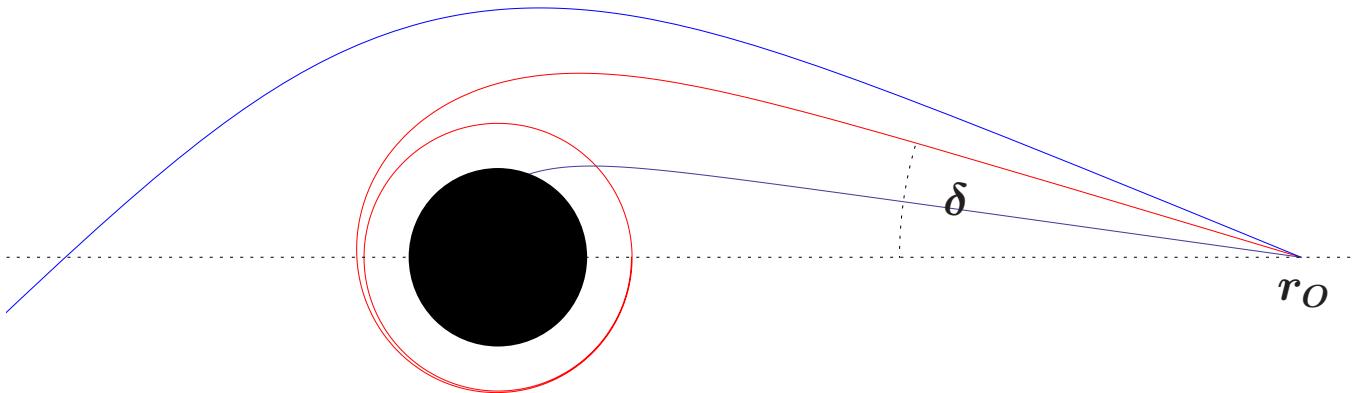
## Example 1: Schwarzschild spacetime

$$g = - \left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) , \quad r_s = \frac{2GM}{c^2}$$

$$S(r)^{-1} = 1 - \frac{r_s}{r}, \quad R(r) = \frac{r}{\sqrt{1 - \frac{r_s}{r}}}$$

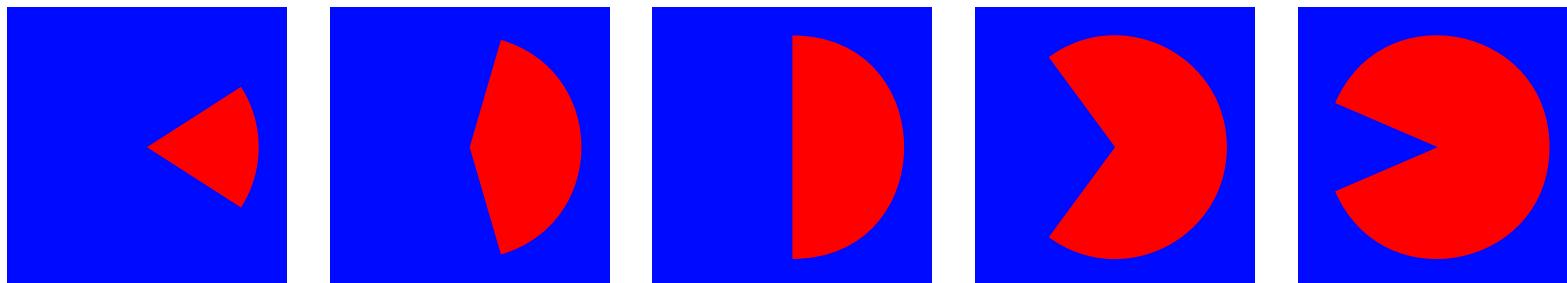


Lens map  $\Theta \mapsto \Phi$  for  $r_O = 2.5 r_s$  and  $r_L = 5 r_s$



**Angular radius  $\delta$  of the “shadow” of a Schwarzschild black hole:**

$$\sin^2 \delta = \frac{27 r_S^2 (r_O - r_S)}{4 r_O^3}, \quad r_S = \frac{2GM}{c^2}$$



$$r_O = 1.05 r_S$$

$$r_O = 1.3 r_S$$

$$r_O = 3 r_S/2$$

$$r_O = 2.5 r_S$$

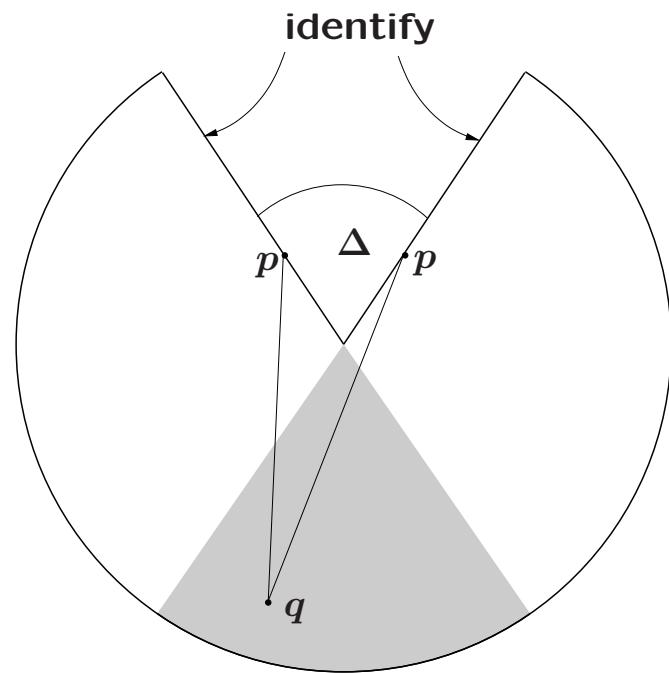
$$r_O = 6 r_S$$

## Example 2: Barriola-Vilenkin monopole

M. Barriola, A. Vilenkin: Phys. Rev. Lett. 63, 341 (1989)

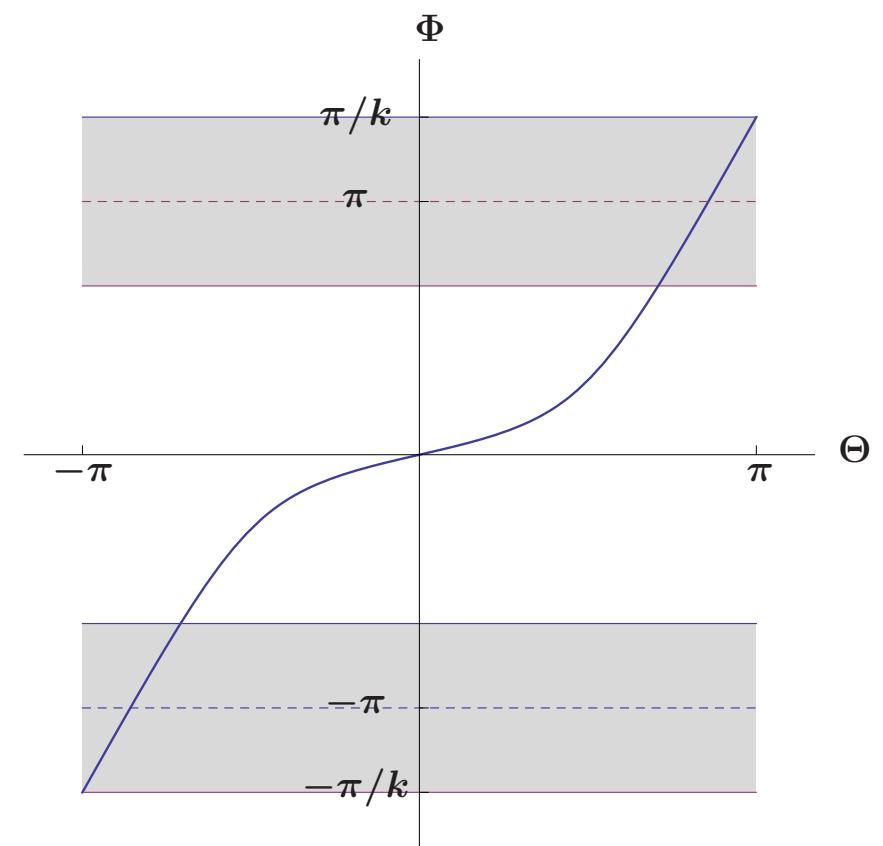
$$g = -c^2 dt^2 + dr^2 + k^2 r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

$$S(r) = 1, \quad R(r) = k r$$



**Deficit angle**

$$\Delta = (1 - k)2\pi$$



$$k = 0.75, \quad r_O = 0.75r_L$$

### Example 3: Ellis wormhole

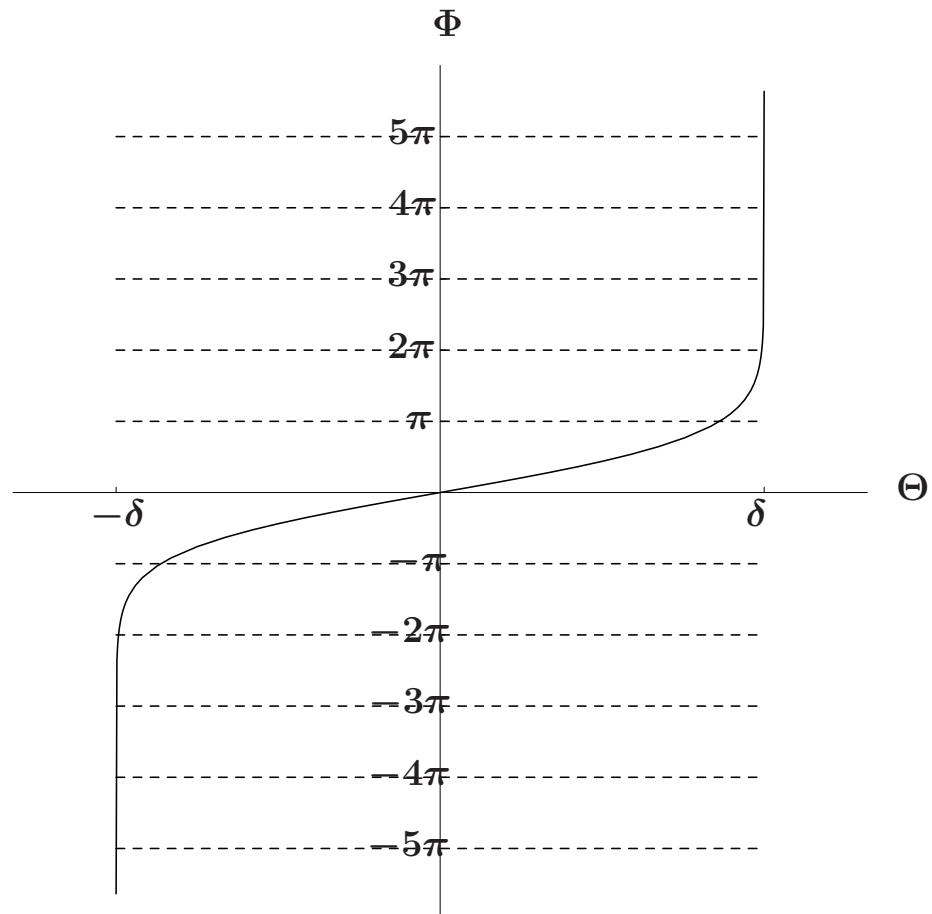
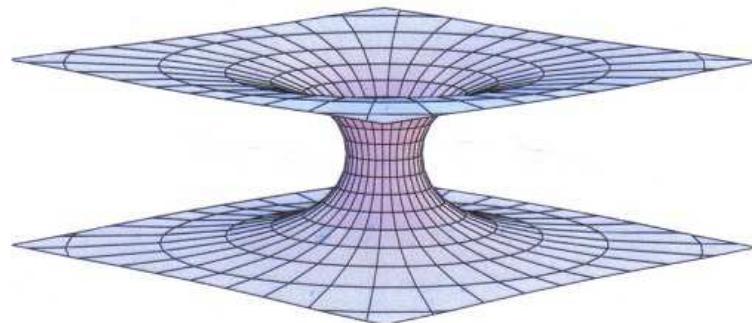
H. Ellis: J. Math. Phys. 14, 104 (1973)

$$g = -c^2 dt^2 + dr^2 + (r^2 + a^2) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

$$S(r) = 1$$

$$R(r) = \sqrt{r^2 + a^2}$$

$$\sin^2 \delta = \frac{a^2}{r_O^2 + a^2}$$



$$r_L < 0 < r_O$$

**What happens to the shadow if spherical symmetry is broken?**

**Shadow of Kerr black hole:**

J. Bardeen in C. DeWitt and B. DeWitt (eds.): "Black Holes" Gordon and Breach (1973), p. 215

**Shadow of Kerr(-Newman) naked singularities:**

A. deVries: Class. Quantum Grav. 17, 123 (2000)

**Shadow in Plebański-Demiański spacetime:**

PhD Thesis of Arne Grenzebach (in progress)

**Kerr metric in Boyer–Lindquist coordinates  $(r, \vartheta, \varphi, t)$ :**

$$g = \varrho(r, \vartheta)^2 \left( \frac{dr^2}{\Delta(r)} + d\vartheta^2 \right) + \frac{\sin^2 \vartheta}{\varrho(r, \vartheta)^2} \left( a dt - (r^2 + a^2) d\varphi \right)^2 - \frac{\Delta(r)}{\varrho(r, \vartheta)^2} \left( dt - a \sin^2 \vartheta d\varphi \right)^2$$

$$\varrho(r, \vartheta)^2 = r^2 + a^2 \cos^2 \vartheta, \quad \Delta(r) = r^2 - 2mr + a^2.$$

**Lightlike geodesics:**

$$\varrho(r, \vartheta)^2 \dot{t} = a (L - Ea \sin^2 \vartheta) + \frac{(r^2 + a^2) ((r^2 + a^2)E - aL)}{\Delta(r)},$$

$$\varrho(r, \vartheta)^2 \dot{\varphi} = \frac{L - Ea \sin^2 \vartheta}{\sin^2 \vartheta} + \frac{(r^2 + a^2)aE - a^2 L}{\Delta(r)},$$

$$\varrho(r, \vartheta)^4 \dot{\vartheta}^2 = K - \frac{(L - Ea \sin^2 \vartheta)^2}{\sin^2 \vartheta} =: \Theta(\vartheta),$$

$$\varrho(r, \vartheta)^4 \dot{r}^2 = -K\Delta(r) + ((r^2 + a^2)E - aL)^2 =: R(r).$$

**Spherical lightlike geodesics exist in the region where**

$$R(r) = 0, \quad R'(r) = 0, \quad \Theta(\vartheta) \geq 0.$$

$$(2r\Delta(r) - (r - m)\varrho(r, \vartheta)^2)^2 \leq 4a^2r^2\Delta(r) \sin^2 \vartheta$$

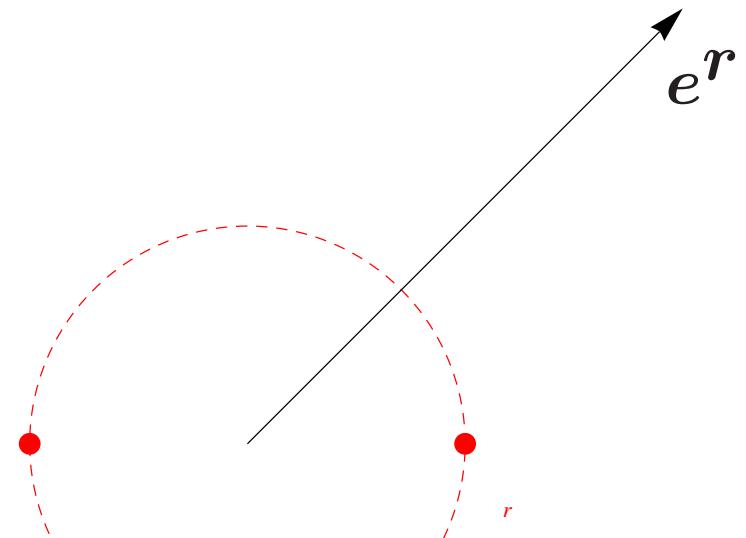
(unstable if  $R''(r) \geq 0$ )

## “Photon region” for Kerr black hole:

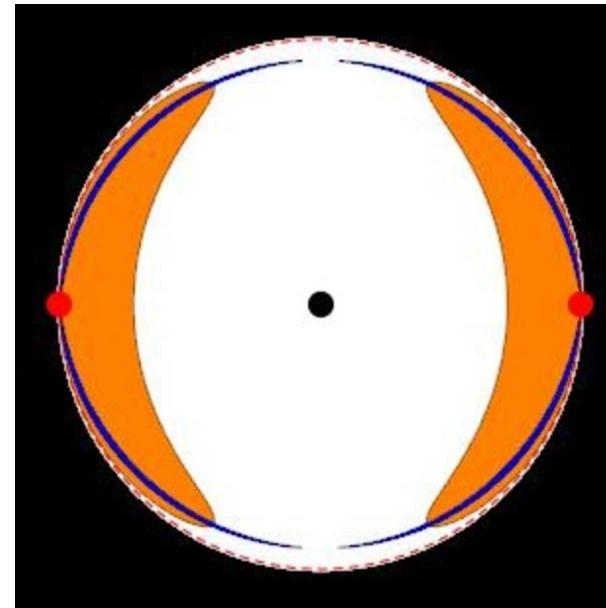
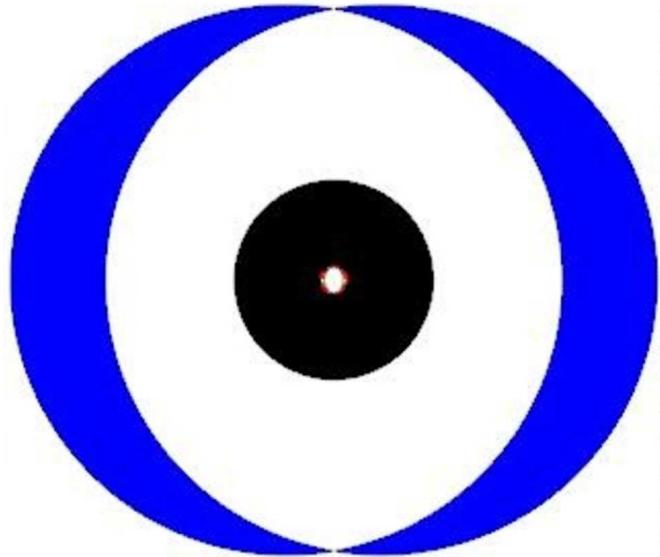
W. Hasse and VP: J. Math. Phys. 47, 042503 (2006)

For pictures of the photon region, use  $e^r$  as the radial coordinate

red solid: singularity  
red dashed: “throat”



$$a = 0.15 m$$



blue: unstable spherical lightlike geodesics

green: stable spherical lightlike geodesics

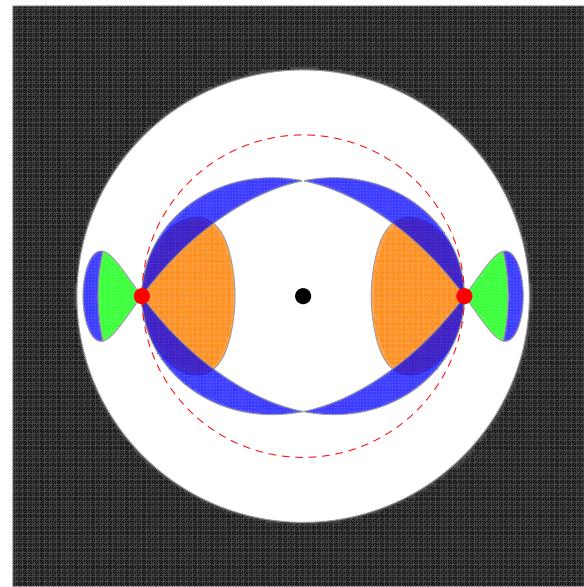
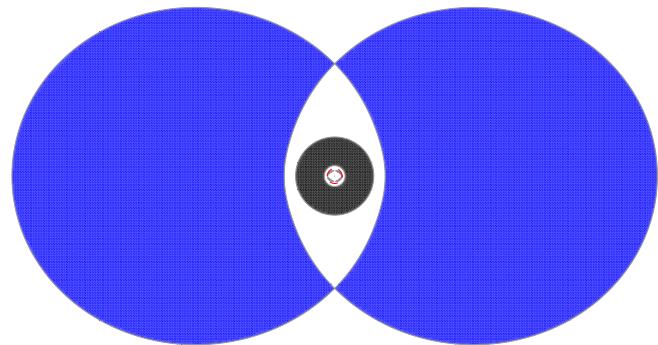
orange: causality violation

black: region between horizons

red solid: singularity

red dashed: “throat”

$$a = 0.75 \text{ m}$$



**blue: unstable spherical lightlike geodesics**

**green: stable spherical lightlike geodesics**

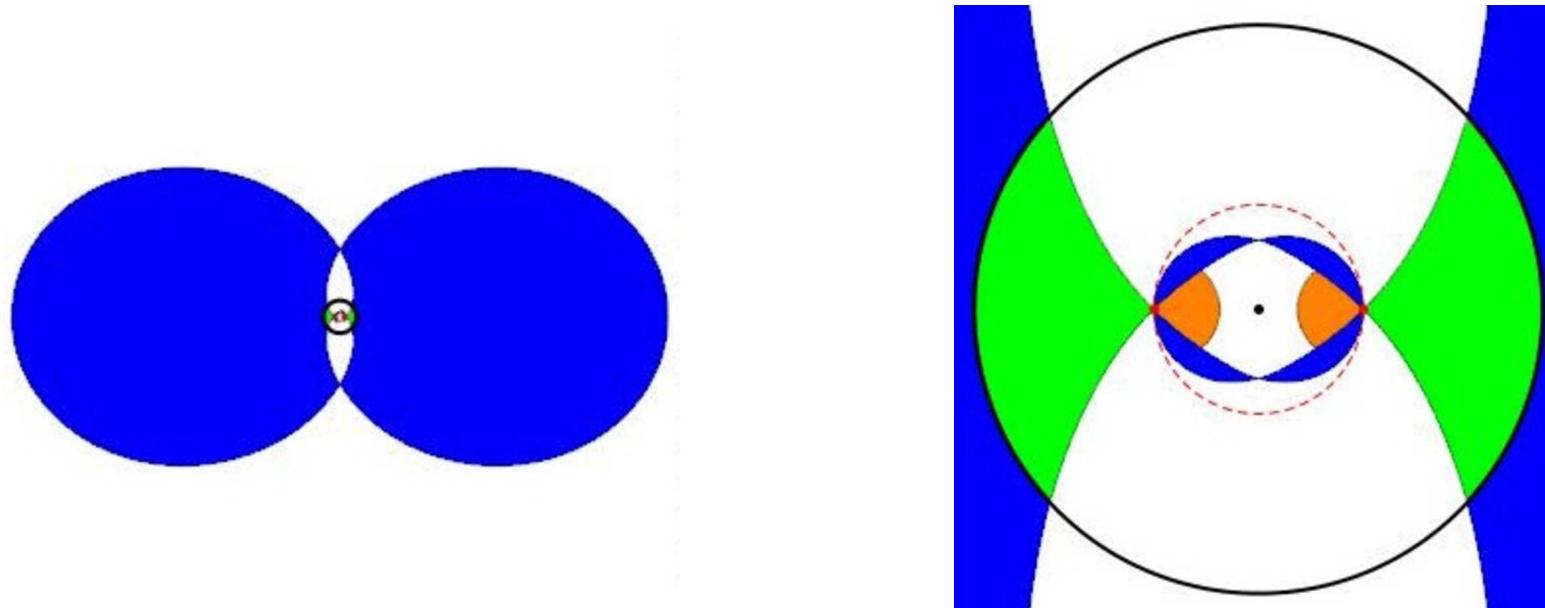
**orange: causality violation**

**black: region between horizons**

**red solid: singularity**

**red dashed: “throat”**

$$a = 0.9999999 m$$



**blue: unstable spherical lightlike geodesics**

**green: stable spherical lightlike geodesics**

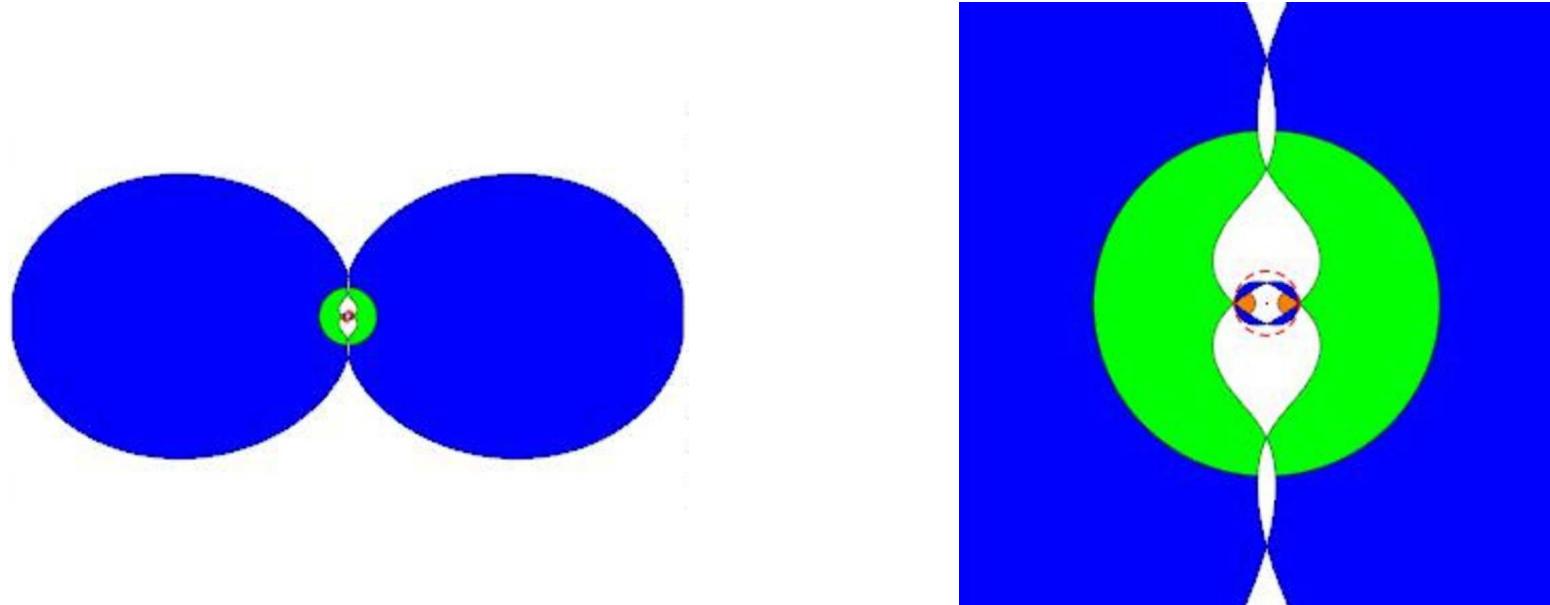
**orange: causality violation**

**black: region between horizons**

**red solid: singularity**

**red dashed: “throat”**

$$a = 1.15 m$$



**blue: unstable spherical lightlike geodesics**

**green: stable spherical lightlike geodesics**

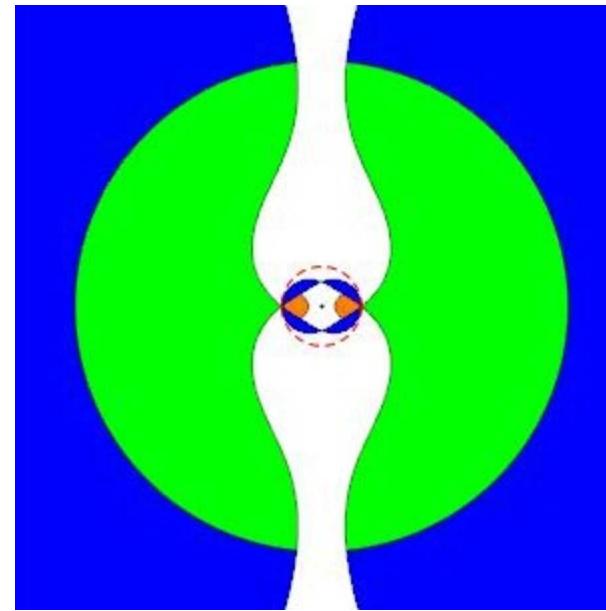
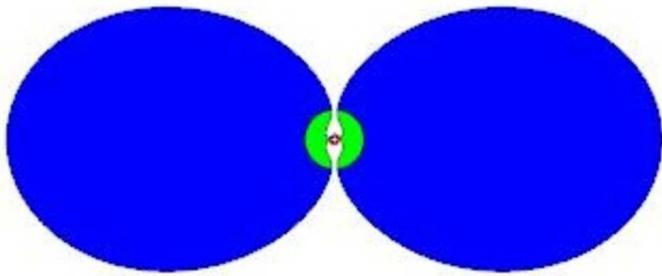
**orange: causality violation**

**black: region between horizons**

**red solid: singularity**

**red dashed: “throat”**

$$a = 1.25 m$$



**blue: unstable spherical lightlike geodesics**

**green: stable spherical lightlike geodesics**

**orange: causality violation**

**black: region between horizons**

**red solid: singularity**

**red dashed: “throat”**

The “shadow” is determined by light rays that approach an unstable spherical lightlike geodesic.

**Choose observer at  $r_O$  and  $\vartheta_O$ .**

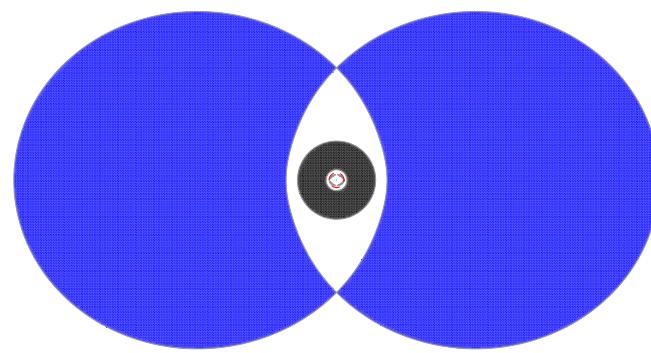
**$(\theta, \phi)$ : Celestial coordinates at the observer**

**Relation between celestial coordinates and constants of motion:**

$\vartheta_O = \pi/2$  :

$$\frac{L}{E} = a + \frac{\left( (r_O(r_O^2 + a^2) \sin \theta \sin \phi - ar_O \sqrt{\Delta(r_O)}) \right)}{r_O \sqrt{\Delta(r_O)} + 2ma \sin \theta \sin \phi},$$

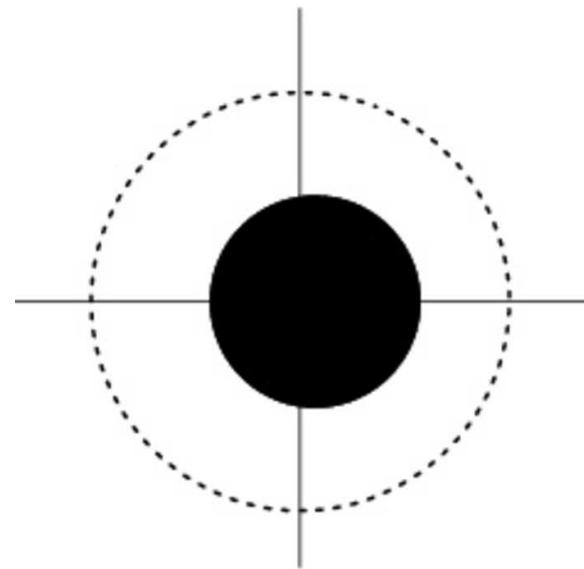
$$\frac{K}{E^2} = \frac{r_O^2 \left( r_O^2 + a^2 - a \sqrt{\Delta(r_O)} \sin \theta \sin \phi \right)^2 - r_O^3 (r_O(r_O^2 + a^2) + 2ma^2) \cos^2 \theta}{\left( r_O \sqrt{\Delta(r_O)} + 2ma \sin \theta \sin \phi \right)^2}.$$

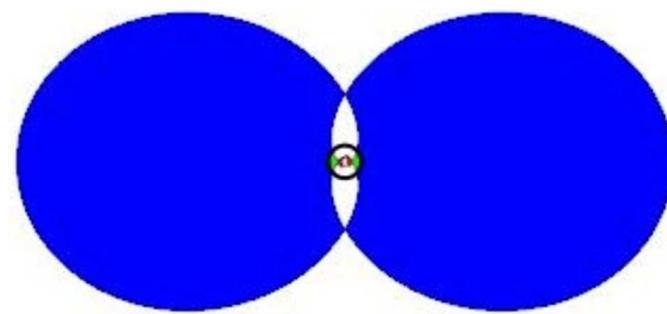


$$a = 0.4 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$

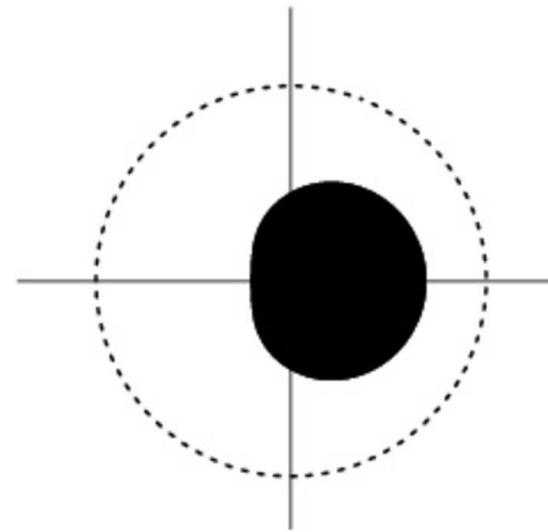


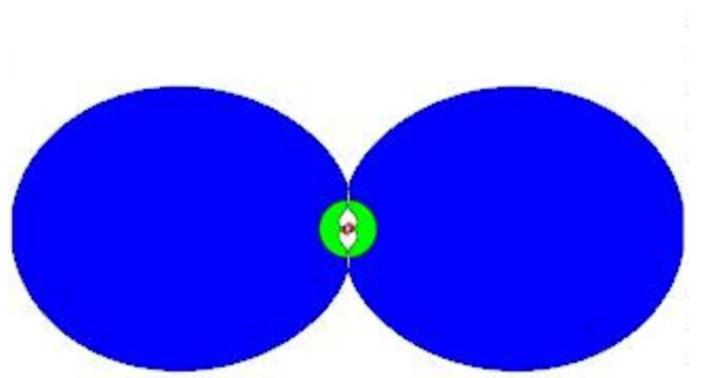


$$a = 0.9999999 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$

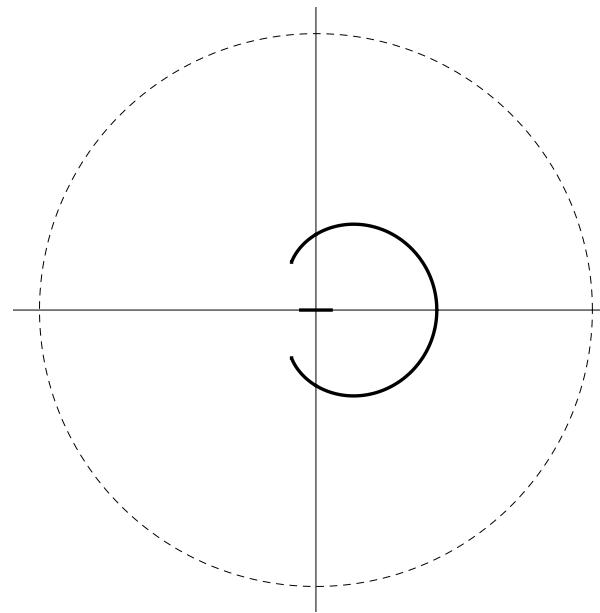


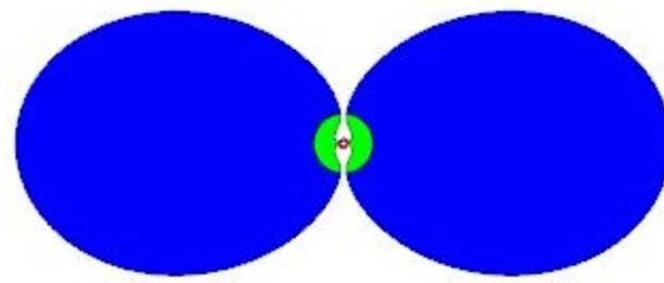


$$a = 1.1 \text{ m}$$

$$r_O = 6 \text{ m}$$

$$\vartheta_O = \pi/2$$

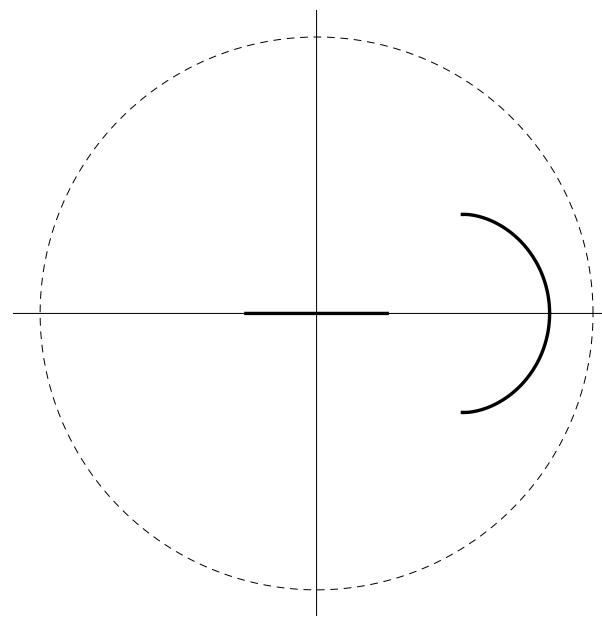




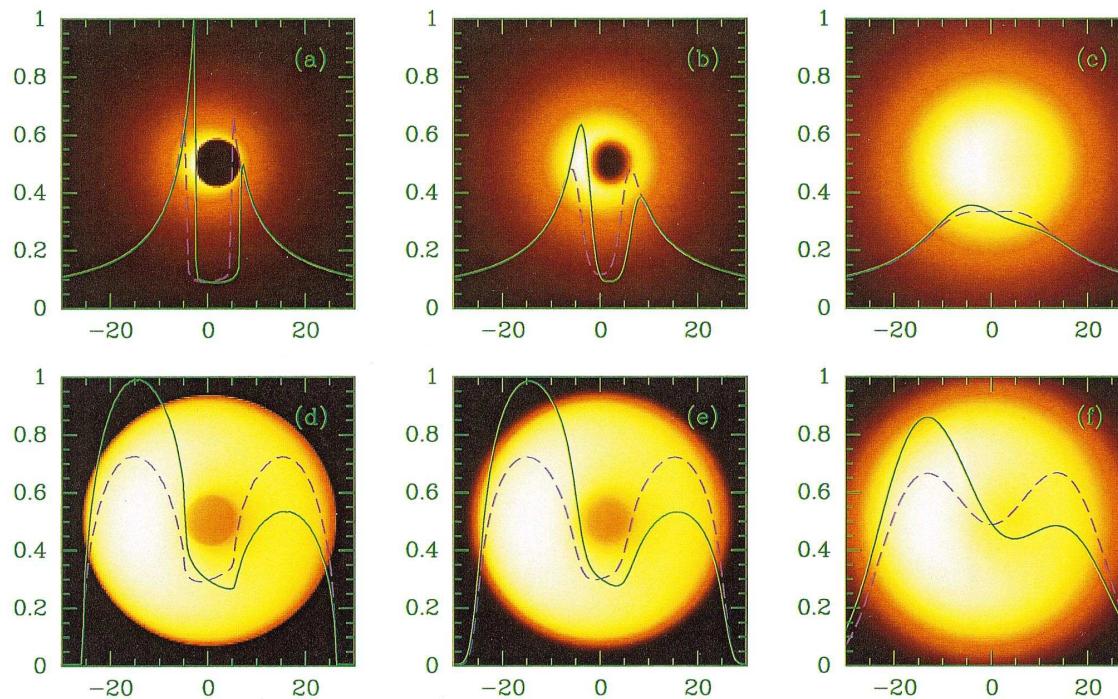
$$a = 2 \text{ m}$$

$$r_O = 8 \text{ m}$$

$$\vartheta_O = \pi/2$$



**Kerr shadow with emission region and scattering taken into account:**



**H. Falcke, F. Melia, E. Agol: Astrophys. J. 528, L13 (2000)**

**Expected angular diameter of the shadow of  
Sagittarius A\*: approx. 30 microarcseconds.**