Gravitational Waves

Volker Perlick (volker.perlick@zarm.uni-bremen.de)

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Mo 16–18: NW1 S1330 (Lectures)
Fr 12–14: NW1 S1330 (Lectures/Tutorials)

Complementary Reading

The following standard text-books contain useful chapters on gravitational waves:

S. Weinberg: "Gravitation and Cosmology" Wiley (1972)

C. Misner, K. Thorne, J. Wheeler: "Gravitation" Freeman (1973)

H. Stephani: "Relativity" Cambridge University Press (2004)

L. Ryder: "Introduction to General Relativity" Cambridge University Press (2009)

N. Straumann: "General Relativity" Springer (2012)

For regularly updated online reviews see the *Living Reviews on Relativity*, in particular

B. Sathyaprakash, B. Schutz: "Physics, astrophysics and cosmology with gravitational waves" `http://www.livingreviews.org/lrr-2009-2`

Contents:

1 Historic introduction

1905 A. Einstein establishes special relativity. According to this theory, signals cannot propagate faster than light in vacuum. This makes it necessary to modify the Newtonian theory of gravity. According to the latter, gravity is an action-at-a-distance: joggling a mass here is immediately felt as an action on a mass there. Einstein (and others) worked for ten years on a modified theory of gravity according to which the gravitational action propagates at a finite speed, similarly to electromagnetic waves.

1915 A. Einstein establishes the field equation of general relativity.

1916 A. Einstein demonstrates that the *linearised* vacuum field equation admits wavelike solutions which are rather similar to electromagnetic waves.

1918 A. Einstein derives the *quadrupole formula* according to which gravitational waves are produced by a time-dependent mass quadrupole moment.

1925 H. Brinkmann finds a class of *exact* wavelike solutions to the vacuum field equation with plane wavefronts; they were independently rediscovered in the 1950s by by J. Ehlers and W. Kundt who called them pp-waves ("plane-fronted waves with parallel rays")

1925 G. Beck finds a class of *exact* wavelike solutions to the vacuum field equation with cylindrical wavefronts; they were independently rediscovered by A. Einstein and N. Rosen, see next two items, and are now usually (though unjustly) called Einstein-Rosen waves.

1936 A. Einstein submits, together with N. Rosen, a manuscript to *Physical Review* in which they claim that gravitational waves do not exist.

1937 After receiving a critical referee report, A. Einstein withdraws the manuscript with the erroneous claim and publishes, together with N. Rosen, a strongly revised manuscript where they present what is now called Einstein-Rosen waves in the *Journal of the Franklin Institute.*

1957 F. Pirani gives an invariant (i.e., coordinate-independent) characterisation of gravitational radiation.

1960 I. Robinson and A. Trautman discover a class of exact solutions to Einstein's vacuum field equation that describe outgoing gravitational radiation.

1960 J. Weber starts his (unsuccessful) search for gravitational waves with the help of resonant bar detectors ("Weber cylinders").

1962 M. E. Gertsenshtein and V. I. Pustovoit publish a seminal paper on using interferometers as gravitational wave detectors.

1974 R. Hulse and J. Taylor (Nobel prize 1993) discover the binary pulsar PSR B1913+16 and interpret the energy loss of the system as an indirect proof of the existence of gravitational waves.

2002 The first laser interferometric gravitational wave detectors go into operation (GEO66, LIGO, VIRGO,...).

2014 The BICEP2 team erroneously announces the discovery of primordial gravitational waves.

2015 On 14 September 2015 the LIGO detectors in the USA register a gravitational wave signal that perfectly fits the theoretical predictions of the merger of two black holes of about 30 Solar masses each; the discovery is announced in February 2016.

2 Brief review of general relativity

A general-relativistic spacetime is a pair (M, g) where:

M is a four-dimensional manifold; local coordinates will be denoted (x^0, x^1, x^2, x^3) and Einstein's summation convention will be used for greek indices $\mu, \nu, \sigma, \ldots = 0, 1, 2, 3$ and for latin indices $i, j, k, \ldots = 1, 2, 3$.

g is a Lorentzian metric on M, i.e. g is a covariant second-rank tensor field, $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, that is

(a) symmetric, $g_{\mu\nu} = g_{\nu\mu}$, and

(b) non-degenerate with Lorentzian signature, i.e., for any $p \in M$ there are coordinates defined near p such that $g|_p = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

We can, thus, introduce contravariant metric components by

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \, .$$

We use $g^{\mu\nu}$ and $g_{\sigma\tau}$ for raising and lowering indices, e.g.

$$g_{\rho\tau} A^\tau = A_\rho \, , \qquad B_{\mu\nu} g^{\nu\tau} = B_\mu{}^\tau \, .$$

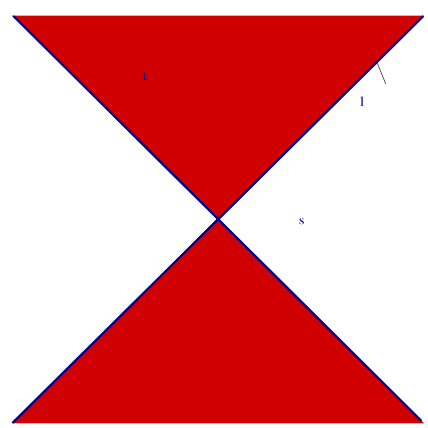
The metric contains all information about the spacetime geometry and thus about the gravitational field. In particular, the metric determines the following.

- The causal structure of spacetime:

  A curve $s \mapsto x(s) = \big(x^0(s), x^1(s), x^2(s), x^3(s)\big)$ is called

  $$\left.\begin{array}{c}\text{spacelike}\\ \text{lightlike}\\ \text{timelike}\end{array}\right\} \iff g_{\mu\nu}\big(x(s)\big)\dot{x}^\mu(s)\dot{x}^\nu(s) \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array}\right.$$

  Timelike curves describe motion at subluminal speed and lightlike curves describe motion at the speed of light. Spacelike curves describe motion at superluminal speed which is forbidden for signals.



For a timelike curve, we usually use proper time τ for the parameter which is defined by

$$g_{\mu\nu}\big(x(\tau)\big)\dot{x}^{\mu}(\tau)\,\dot{x}^{\mu}(\tau) \;=\; -\,c^2 \;.$$

A clock that shows proper time along its worldline is called a *standard clock*. All experiments to date are in agreement with the assumptions that atomic clocks are standard clocks.

The motion of a material continuum, e.g. of a fluid, can be described by a vector field $U = U^{\mu}\partial_{\mu}$ with $g_{\mu\nu}U^{\mu}U^{\nu} = -c^2$. The integral curves of U are to be interpreted as the worldlines of the fluid elements parametrised by proper time.

- Geodesics and covariant derivative:

  By definition, the geodesics are the solutions to the Euler-Lagrange equations

$$\frac{d}{ds}\frac{\partial\mathcal{L}(x,\dot{x})}{\partial\dot{x}^{\mu}} \;-\; \frac{\partial\mathcal{L}(x,\dot{x})}{\partial x^{\mu}} \;=\; 0$$

  of the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) \;=\; \frac{1}{2}\,g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu} \;.$$

  These Euler-Lagrange equations take the form

$$\ddot{x}^{\mu} \;+\; \Gamma^{\mu}{}_{\nu\sigma}(x)\dot{x}^{\nu}\dot{x}^{\sigma} \;=\; 0$$

  where

$$\Gamma^{\mu}{}_{\nu\sigma} \;=\; \frac{1}{2}\,g^{\mu\tau}\left(\,\partial_{\nu}g_{\tau\sigma} \,+\, \partial_{\sigma}g_{\tau\nu} \,-\, \partial_{\tau}g_{\nu\sigma}\,\right)$$

  are the so-called Christoffel symbols.

  The Lagrangian $\mathcal{L}(x,\dot{x})$ is constant along a geodesic (see Worksheet 1), so we can speak of timelike, lightlike and spacelike geodesics. Timelike geodesics ($\mathcal{L} < 0$) are to be interpreted as the worldlines of freely falling particles, and lightlike geodesics ($\mathcal{L} = 0$) are to be interpreted as light rays.

  The Christoffel symbols define a *covariant derivative* that makes tensor fields into tensor fields,

$$\nabla_{\nu}U^{\mu} \;=\; \partial_{\nu}U^{\mu} \,+\, \Gamma^{\mu}{}_{\nu\tau}U^{\tau} \;,$$

$$\nabla_{\nu}A_{\mu} \;=\; \partial_{\nu}A_{\mu} \,-\, \Gamma^{\rho}{}_{\nu\mu}A_{\rho} \;,$$

  and so on. For each upper index there is one Γ term with a plus sign and for each lower index there is a Γ term with a minus sign.

  In Minkowski spacetime (i.e., in the "flat" spacetime of special relativity), we can choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ on the whole spacetime, where we have used the standard abbreviation $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$. In this coordinate system, the Christoffel symbols obviously vanish. Conversely, vanishing of the Christoffel symbols on an open neighbourhood implies that the $g_{\mu\nu}$ are constants; one can then perform a linear coordinate transformation such that $g_{\mu\nu} = \eta_{\mu\nu}$.

- The curvature:

  The Riemannian curvature tensor is defined, in coordinate notation, by

$$R^{\tau}{}_{\mu\nu\sigma} \;=\; \partial_\mu \Gamma^{\tau}{}_{\nu\sigma} \;-\; \partial_\nu \Gamma^{\tau}{}_{\mu\sigma} \;+\; \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\tau}{}_{\mu\rho} \;-\; \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\tau}{}_{\nu\rho} \,.$$

  This defines, indeed, a tensor field, i.e., if $R^{\tau}{}_{\mu\nu\sigma}$ vanishes in one coordinate system, then it vanishes in any coordinate system. The condition $R^{\tau}{}_{\mu\nu\sigma} = 0$ is true if and only if there is a local coordinate system, around any one point, such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\mu}{}_{\nu\sigma} = 0$ on the domain of the coordinate system.

  The Riemannian curvature tensor has the symmetry properties

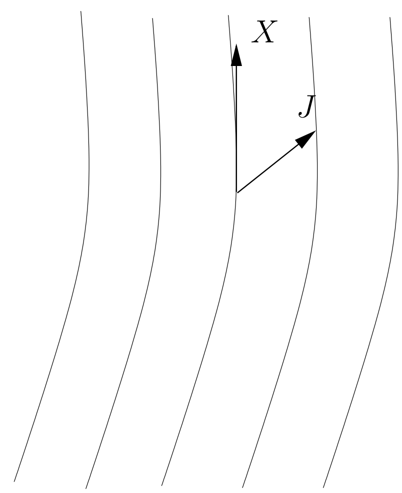
$$R^{\tau}{}_{\mu\nu\sigma} = -R^{\tau}{}_{\nu\mu\sigma} \,, \quad R_{\tau\mu\nu\sigma} = -R_{\sigma\nu\mu\tau} \,,$$

  and satisfies the *Bianchi identities*

$$R^{\tau}{}_{\mu\nu\sigma} + R^{\tau}{}_{\nu\sigma\mu} + R^{\tau}{}_{\sigma\mu\nu} = 0 \,, \quad \nabla_\rho R^{\tau}{}_{\mu\nu\sigma} + \nabla_\mu R^{\tau}{}_{\nu\rho\sigma} + \nabla_\nu R^{\tau}{}_{\rho\mu\sigma} = 0 \,.$$

  The curvature tensor determines the relative motion of neighbouring geodesics: If $X = X^\mu \partial_\mu$ is a vector field whose integral curves are geodesics, and if $J = J^\nu \partial_\nu$ connects neighbouring integral curves of X (i.e., if the Lie bracket between X and J vanishes), then the *equation of geodesic deviation* or *Jacobi equation* holds:

$$\big(X^\mu \nabla_\mu\big)\big(X^\nu \nabla_\nu\big) J^\sigma \;=\; R^{\sigma}{}_{\mu\nu\rho} X^\mu J^\nu X^\rho \,.$$



  If the integral curves of X are timelike, they can be interpreted as worldlines of freely falling particles. In this case the curvature term in the Jacobi equation gives the *tidal force* produced by the gravitational field.

  If the integral curves of X are lightlike, they can be interpreted as light rays. In this case the curvature term in the Jacobi equation determines the influence of the gravitational field on the shapes of light bundles.

- Einstein's field equation:

  The fundamental equation that relates the spacetime metric (i.e., the gravitational field) to the distribution of energy is Einstein's field equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; \kappa \, T_{\mu\nu}$$

  where

  – $G_{\mu\nu} = R_{\mu\nu} - \dfrac{R}{2} g_{\mu\nu}$ is the *Einstein tensor*;
  – $R_{\mu\nu} = R^{\sigma}{}_{\sigma\mu\nu}$ is the *Ricci tensor*;

- $R = R_{\mu\nu} g^{\mu\nu}$ is the *Ricci scalar*;
- $T_{\mu\nu}$ is the *energy-momentum tensor* which gives the energy density $T_{\mu\nu} U^\mu U^\nu$ for any observer field with 4-velocity U^μ normalised to $g_{\mu\nu} U^\mu U^\nu = -c^2$;
- Λ is the cosmological constant;
- κ is Einstein's gravitational constant which is related to Newton's gravitational constant G through $\kappa = 8\pi G/c^4$.

Einstein's field equation can be justified in the following way: One looks for an equation of the form $(\mathcal{D}g)_{\mu\nu} = T_{\mu\nu}$ where $\mathcal{D}$ is a differential operator acting on the metric. One wants to have $\mathcal{D}g$ satisfying the following two properties:

(A) $\mathcal{D}g$ contains partial derivatives of the metric up to second order.

(B) $\nabla^\mu (\mathcal{D}g)_{\mu\nu} = 0$.

Condition (A) is motivated by analogy to the Newtonian theory: The Poisson equation is a second-order differential equation for the Newtonian gravitational potential ϕ, and the metric is viewed as the general-relativistic analogue to ϕ . Condition (B) is motivated in the following way: For a closed system, in special relativity the energy-momentum tensor field satisfies the conservation law $\partial^\mu T_{\mu\nu} = 0$ in inertial coordinates. By the rule of minimal coupling, in general relativity the energy-momentum tensor field of a closed system should satisfy $\nabla^\mu T_{\mu\nu} = 0$. For consistency, the same property has to hold for the left-hand side of the desired equation.

D. Lovelock has shown in 1972 that these two conditions (A) and (B) are satisfied if and only if $\mathcal{D}g$ is of the form

$$(\mathcal{D}g)_{\mu\nu} = \frac{1}{\kappa}\Big(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} \Big)$$

with some constants Λ and κ, i.e., if and only if the desired equation has indeed the form of Einstein's field equation.

For vacuum ($T_{\mu\nu} = 0$), Einstein's field equation reads

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \, .$$

By contraction with $g^{\mu\nu}$ this implies $R = 4\Lambda$, so the vacuum field equation reduces to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Present-day cosmological observations suggest that we live in a universe with a positive cosmological constant whose value is $\Lambda \approx (10^{26}\text{m})^{-2} \approx (10^{16}\text{ly})^{-2}$. As the diameter of our galaxy is approximately 10^5ly, for any distance d within our galaxy the quantity $d^2 \Lambda < 10^{-22}$ is negligibly small. As a consequence, the Λ term can be safely ignored for considerations inside our galaxy. Then the vacuum field equation takes the very compact form

$$R_{\mu\nu} = 0$$

which, however, is a complicated system of ten non-linear second-order partial differential equations for the ten independent components of the metric.

Gravitational waves travelling through empty space are wavelike solutions of the equation $R_{\mu\nu} = 0$.

3 Gravitational waves in the linearised theory about flat spacetime

In 1916 Einstein predicted the existence of gravitational waves, based on his linearised vacuum field equation. In 1918 he derived his famous quadrupole formula which relates emitted gravitational waves to the quadrupole moment of the source. In this chapter we will review this early work on gravitational waves which is based on the *linearised* Einstein theory about flat spacetime. As a consequence, the results are true only for gravitational waves whose amplitudes are small. We will see that, to within this approximation, the theory of gravitational waves is very similar to the theory of electromagnetic waves.

3.1 The linearisation of Einstein's field equation

We consider a metric that takes, in an appropriate coordinate system, the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, .$$

In the following we will linearise Einstein's field equation with respect to the $h_{\mu\nu}$ and their derivatives. This gives a valid approximation of Einstein's theory of gravity if the $h_{\mu\nu}$ and their derivatives are small, i.e., if the spacetime is very close to the spacetime of special relativity.

Our assumptions fix the coordinate system up to transformations of the form

$$x^\mu \mapsto \tilde{x}^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu + f^\mu(x) \qquad \text{(C)}$$

where $(\Lambda^\mu{}_\nu)$ is a Lorentz transformation, $\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta_{\mu\rho} = \eta_{\nu\sigma}$, and the f^μ and their derivatives are so small that terms of second or higher order in these quantities can be neglected.

We agree that, in this chapter, greek indices are lowered and raised with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, respectively. As an abbreviation, we write

$$h := h_{\mu\nu} \, \eta^{\mu\nu} = h_\mu{}^\mu = h^\nu{}_\nu \, .$$

Then the inverse metric is of the form

$$g^{\nu\rho} = \eta^{\nu\rho} - h^{\nu\rho} \, .$$

Proof: $\big(\eta_{\mu\nu} + h_{\mu\nu} \big) \big(\eta^{\nu\rho} - h^{\nu\rho} \big) = \eta_{\mu\nu}\eta^{\nu\rho} + h_{\mu\nu}\eta^{\nu\rho} - \eta_{\mu\nu}h^{\nu\rho} + \ldots = \delta^\rho_\mu + h_\mu{}^\rho - h_\mu{}^\rho = \delta^\rho_\mu \, ,$ where the ellipses stand for a second-order term that is to be neglected, according to our assumptions. □

We will now derive the linearised field equation. As a first step, we have to calculate the Christoffel symbols. We find

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} \, g^{\rho\sigma} \big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \big) = \frac{1}{2} \, \eta^{\rho\sigma} \big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \big) + \ldots$$

Thereupon, we can calculate the components of the Ricci tensor:.

$$R_{\mu\nu} \,=\, \partial_\mu \Gamma^\rho{}_{\rho\nu} \,-\, \partial_\rho \Gamma^\rho{}_{\mu\nu} \,+\, \ldots \,=\, \frac{1}{2}\,\eta^{\rho\sigma}\,\partial_\mu\left(\cancel{\partial_\rho h_{\sigma\nu}} \,+\, \partial_\nu h_{\sigma\rho} \,-\, \partial_\sigma h_{\rho\nu}\right)$$

$$-\,\frac{1}{2}\,\eta^{\rho\sigma}\,\partial_\rho\left(\cancel{\partial_\mu h_{\sigma\nu}} \,+\, \partial_\nu h_{\sigma\mu} \,-\, \partial_\sigma h_{\mu\nu}\right) \,=\, \frac{1}{2}\big(\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu}\big)\;.$$

Here, $\Box$ denotes the wave operator (d'Alembert operator) that is formed with the Minkowski metric,

$$\Box \,=\, \eta^{\mu\nu}\partial_\mu\partial_\nu \,=\, \partial^\nu\partial_\nu\;.$$

From the last expression we can calculate the scalar curvature:

$$R \,=\, g^{\mu\nu}R_{\mu\nu} \,=\, \eta^{\mu\nu}R_{\mu\nu} \,+\, \ldots \,=\, \frac{1}{2}\,\eta^{\mu\nu}\left(\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu}\right)$$

$$=\,\frac{1}{2}\left(\Box h \,-\, \partial^\nu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial^\mu h_{\sigma\mu} \,+\, \Box h\right) \,=\, \Box h \,-\, \partial^\sigma\partial^\mu h_{\sigma\mu}\,.$$

Hence, the linearised version of Einstein's field equation (without a cosmological constant)

$$2\,R_{\mu\nu} \,-\, R\,g_{\mu\nu} \,=\, 2\,\kappa\,T_{\mu\nu}\;, \qquad \kappa = \frac{8\pi G}{c^4}$$

reads

$$\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu} \,-\, \eta_{\mu\nu}\left(\Box h \,-\, \partial^\sigma\partial^\tau h_{\sigma\tau}\right) \,=\, 2\,\kappa\,T_{\mu\nu}\;. \quad (*)$$

This is a system of linear partial differential equations of second order for the $h_{\mu\nu}$. It can be rewritten in a more convenient form after substituting for $h_{\mu\nu}$ the quantity

$$\gamma_{\mu\nu} \,=\, h_{\mu\nu} \,-\, \frac{h}{2}\,\eta_{\mu\nu}\;.$$

As the relation between $h_{\mu\nu}$ and $\gamma_{\mu\nu}$ is linear, the $h_{\mu\nu}$ are small of first order if and only if the $\gamma_{\mu\nu}$ are small of first order. In order to express the $h_{\mu\nu}$ in terms of the $\gamma_{\mu\nu}$, we calculate the trace,

$$\gamma \,:=\, \eta^{\mu\nu}\gamma_{\mu\nu} \,=\, h \,-\frac{1}{2}\,4\,h \,=\, -\,h\;, \qquad h_{\mu\nu} \,=\, \gamma_{\mu\nu} \,-\, \frac{\gamma}{2}\,\eta_{\mu\nu}\;.$$

Upon inserting this expression into the linearised field equation $(*)$, we find

$$-\cancel{\partial_\mu\partial_\nu\gamma} \,-\, \partial_\mu\partial^\rho\gamma_{\rho\nu} \,+\, \cancel{\frac{1}{2}\eta_{\rho\nu}\partial_\mu\partial^\rho\gamma} \,-\, \partial^\sigma\partial_\nu\gamma_{\sigma\mu} \,+\, \cancel{\frac{1}{2}\eta_{\sigma\mu}\partial^\sigma\partial_\nu\gamma} \,+$$

$$+\,\Box\gamma_{\mu\nu} \,-\, \cancel{\frac{1}{2}\eta_{\mu\nu}\Box\gamma} \,-\, \eta_{\mu\nu}\Big(-\,\cancel{\Box\gamma} \,-\, \partial^\sigma\partial^\tau\gamma_{\sigma\tau} \,+\, \cancel{\frac{1}{2}\eta_{\sigma\tau}\partial^\sigma\partial^\tau\gamma}\Big) \,=\, 2\,\kappa\,T_{\mu\nu}\;,$$

$$\Box\gamma_{\mu\nu} - \partial_\mu\partial^\rho\gamma_{\rho\nu} \,-\, \partial_\nu\partial^\rho\gamma_{\rho\mu} \,+\, \eta_{\mu\nu}\,\partial^\sigma\partial^\tau\gamma_{\sigma\tau} \,=\, 2\,\kappa\,T_{\mu\nu}\;. \quad (**)$$

This equation can be simplified further by a coordinate transformation (C) with $a^\mu = 0$ and $\Lambda^\mu{}_\nu = \delta^\mu_\nu$,

$$x^\mu \,\mapsto\, x^\mu \,+\, f^\mu(x)$$

where the f^μ are small of first order. For such a coordinate transformation, we have obviously

$$dx^\mu \mapsto dx^\mu + \partial_\rho f^\mu dx^\rho$$

and thus

$$\partial_\sigma \mapsto \partial_\sigma - \partial_\sigma f^\tau \partial_\tau \,.$$

Proof: $\big(dx^\mu + \partial_\rho f^\mu dx^\rho\big)\big(\partial_\sigma - \partial_\sigma f^\tau \partial_\tau\big) = dx^\mu(\partial_\sigma) + \partial_\rho f^\mu dx^\rho(\partial_\sigma) - \partial_\sigma f^\tau dx^\mu(\partial_\tau) + \ldots = \delta^\mu_\sigma + \cancel{\partial_\rho f^\mu \delta^\rho_\sigma} - \cancel{\partial_\sigma f^\tau \delta^\mu_\tau}$. $\square$

With the help of these equations, we can now calculate how the $g_{\mu\nu}$, the $h_{\mu\nu}$, and the $\gamma_{\mu\nu}$ behave under such a coordinate transformation:

$$g_{\mu\nu} = g\big(\partial_\mu, \partial_\nu\big) \mapsto g\big(\partial_\mu - \partial_\mu f^\tau \partial_\tau, \partial_\nu - \partial_\nu f^\sigma \partial_\sigma\big) = g_{\mu\nu} - \partial_\mu f^\tau g_{\tau\nu} - \partial_\nu f^\sigma g_{\mu\sigma} \,,$$

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \mapsto g_{\mu\nu} - \partial_\mu f^\tau g_{\tau\nu} - \partial_\nu f^\sigma g_{\mu\sigma} - \eta_{\mu\nu} = h_{\mu\nu} - \partial_\mu f^\tau \eta_{\tau\nu} - \partial_\nu f^\sigma \eta_{\mu\sigma} + \ldots$$

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \mapsto h_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu - \frac{1}{2}\eta_{\mu\nu}\big(h - 2\partial_\tau f^\tau\big) = \gamma_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu + \eta_{\mu\nu}\partial_\tau f^\tau \,.$$

For the divergence of $\gamma_{\mu\nu}$, which occurs three times in $(**)$, this gives the following transformation behaviour:

$$\partial^\mu \gamma_{\mu\nu} \mapsto \partial^\mu \gamma_{\mu\nu} - \partial^\mu \partial_\mu f_\nu - \cancel{\partial^\mu \partial_\nu f_\mu} + \cancel{\eta_{\mu\nu} \partial^\mu \partial_\tau f^\tau} = \partial^\mu \gamma_{\mu\nu} - \Box f_\nu \,.$$

This shows that, if it is possible to choose the f_ν such that

$$\Box f_\nu = \partial^\mu \gamma_{\mu\nu} \,,$$

then $\partial^\mu \gamma_{\mu\nu}$ is transformed to zero. Such a choice is, indeed, possible as the wave equation on Minkowski spacetime,

$$\Box f_\nu = \Phi_\nu \,,$$

has solutions for any Φ_ν. This is well-known from electrodynamics. (If Φ_ν is compactly supported or falls off sufficiently fast, a solution is provided by the retarded potentials, see below. In any case, a solution can be found by prescribing arbitrary initial values for f_ν and $\partial_0 f_\nu$ on a hypersurface $x^0 = \text{constant}$ and then solving the Cauchy problem for the inhomogeneous wave equation.)

We have thus shown that, by an appropriate coordinate transformation, we can put the linearised field equation $(**)$ into the following form:

$$\Box \gamma_{\mu\nu} = 2\,\kappa\, T_{\mu\nu} \,.$$

Now the $\gamma_{\mu\nu}$ have to satisfy the additional condition

$$\partial^\mu \gamma_{\mu\nu} = 0$$

which is known as the *Hilbert gauge*. The transformation of $\gamma_{\mu\nu}$ under a change of coordinates is analogous to a gauge transformation of the four-potential A_μ in electrodynamics. Even after imposing the Hilbert gauge condition, there is still the freedom to make coordinate transformations (C) with $\Box f^\mu = 0$. In particular, the theory is invariant under Lorentz transformations.

The linearised Einstein theory is a Lorentz invariant theory of the gravitational field on Minkowski spacetime. It is very similar to Maxwell's vacuum electrodynamics, which is a (linear) Lorentz invariant theory of electromagnetic fields on Minkowski spacetime. The table illustrates the analogy. Here "electrodynamics" stands for "electrodynamics on Minkowski spacetime in vacuum, $G_{\mu\nu} = \mu_0^{-1} F_{\mu\nu}$". Roughly speaking, the main difference is in the fact that the gravitational equations have one index more.

lin. Einstein theory	electrodynamics
$\gamma_{\mu\nu}$	A_μ
$T_{\mu\nu}$	J_μ
Hilbert gauge $\partial^\mu \gamma_{\mu\nu} = 0$	Lorenz gauge $\partial^\mu A_\mu = 0$
$\Box \gamma_{\mu\nu} = 2\kappa T_{\mu\nu}$	$\Box A_\mu = \mu_0^{-1} J_\mu$

Of course, one has to keep in mind that the linearised Einstein theory is only an approximation; an *exact* Lorentz invariant theory of gravity on Minkowski spacetime cannot be formulated.

The linearised Einstein theory has been used as the starting point for developing a quantum theory of gravitation, in analogy to quantum electrodynamics which is the fairly well understood quantised version of Maxwell's electrodynamics on Minkowski spacetime. While the quanta associated with the field A_μ are called *photons*, the quanta associated with the field $\gamma_{\mu\nu}$ (or $h_{\mu\nu}$) are called *gravitons*. The fact that A_μ is a tensor field of rank one while $\gamma_{\mu\nu}$ is a tensor field of rank two has the consequence that photons have spin one while gravitons have spin two. Apart from the fact that it is far from clear if quantising the *linearised* theory is a reasonable way of quantising gravity, one encounters technical problems related to the fact that the coupling constant of gravity has a dimension (whereas in the electromagnetic case we have the dimensionless fine structure constant).

Here we are interested only in the classical aspects of the linearised Einstein theory. In the next section we discuss wavelike solutions to the source-free linearised field equation.

3.2 Plane-harmonic-wave solutions to the linearised vacuum field equation without sources

In this section we consider the linearised vacuum field equation without sources (i.e., in regions where $T_{\mu\nu} = 0$) in the Hilbert gauge,

$$\Box \gamma_{\mu\nu} = 0 \, , \qquad \partial^\mu \gamma_{\mu\nu} = 0 \, .$$

In analogy to the electrodynamical theory, we can write the general solution as a superposition of plane harmonic waves. In our case, any such plane harmonic wave is of the form

$$\gamma_{\mu\nu}(x) = \mathrm{Re}\big\{ A_{\mu\nu} e^{i k_\rho x^\rho} \big\}$$

with a real wave covector k_ρ and a complex amplitude $A_{\mu\nu} = A_{\nu\mu}$.

Such a plane harmonic wave satisfies the linearised vacuum field equation if and only if

$$0 = \eta^{\sigma\tau}\partial_\sigma\partial_\tau\gamma_{\mu\nu}(x) = \mathrm{Re}\big\{\eta^{\sigma\tau}A_{\mu\nu}ik_\sigma ik_\tau e^{ik_\rho x^\rho}\big\}\,.$$

This holds for all x, with $(A_{\mu\nu}) \neq (0)$, if and only if

$$\eta^{\sigma\tau}k_\sigma k_\tau = 0\,.$$

In other words, (k_0, k_1, k_2, k_3) has to be a lightlike covector with respect to the Minkowski metric. This result can be interpreted as saying that, to within the linearised Einstein theory, gravitational waves propagate on Minkowski spacetime at the speed c, just as electromagnetic waves in vacuum.

Our plane harmonic wave satisfies the Hilbert gauge condition if and only if

$$0 = \eta^{\mu\tau}\partial_\tau\gamma_{\mu\nu}(x) = \mathrm{Re}\big\{\eta^{\mu\tau}A_{\mu\nu}ik_\tau e^{ik_\rho x^\rho}\big\}$$

which is true, for all $x = (x^0, x^1, x^2, x^3)$, if and only if

$$k^\mu A_{\mu\nu} = 0 \qquad \text{(H)}\,.$$

For a given k_μ, the Hilbert gauge condition restricts the possible values of the amplitude $A_{\mu\nu}$, i.e., it restricts the possible polarisation states of the gravitational wave. For electromagnetic waves, it is well known that there are two polarisation states ("left-handed and right-handed", or "linear in x-direction and linear in y-direction") from which all possible polarisation states can be formed by way of superposition. We will see that also for gravitational waves there are two independent polarisation states; however, they are of a different geometric nature which has its origin in the fact that $\gamma_{\mu\nu}$ has two indices while the electromagnetic four-potential A_μ has only one.

In order to find all possible polarisation states of a gravitational wave, we begin by counting the independent components of the amplitude: The $A_{\mu\nu}$ form a (4×4)-matrix which has 16 entries. As $A_{\mu\nu} = A_{\nu\mu}$, only 10 of them are independent; the Hilbert gauge condition (H) consists of 4 scalar equations, so one might think that there are 6 independent components and thus six independent polarisation states. This, however, is wrong. The reason is that we can impose additional conditions onto the amplitudes, even after the Hilbert gauge has been chosen: The Hilbert gauge condition is preserved if we make a coordinate transformation of the form

$$x^\mu \mapsto x^\mu + f^\mu(x) \quad \text{mit} \quad \Box f^\mu = 0\,.$$

We can use this freedom to impose additional conditions onto the amplitudes $A_{\mu\nu}$.

Claim: Assume we have a plane-harmonic-wave solution

$$\gamma_{\mu\nu}(x) = \mathrm{Re}\big\{A_{\mu\nu}e^{ik_\rho x^\rho}\big\}$$

of the linearised vacuum field equation in the Hilbert gauge. Let (u^μ) be a constant four-velocity vector, $\eta_{\mu\nu}u^\mu u^\nu = -c^2$. Then we can make a coordinate transformation such that the Hilbert gauge condition is preserved and such that

$$u^\mu A_{\mu\nu} = 0\,, \qquad \text{(T1)}$$

$$\eta^{\mu\nu}A_{\mu\nu} = 0\,, \qquad \text{(T2)}$$

in the new coordinates (TT *gauge*, *transverse-traceless gauge*).

Proof: We perform a coordinate transformation

$$x^\mu \mapsto x^\mu + f^\mu(x) \,, \qquad f^\mu(x) = \mathrm{Re}\big\{ i\, C^\mu e^{ik_\rho x^\rho} \big\}$$

with the wave covector (k_ρ) from our plane harmonic wave solution and with some complex coefficients C^μ. Then we have $\Box f^\mu = 0$, i.e., the Hilbert gauge condition is satisfied in the new coordinates as well. We want to choose the C^μ such that in the new coordinates (T1) and (T2) hold true. As a first step, we calculate how the amplitudes $A_{\mu\nu}$ transform. We start out from the transformation behaviour of the $\gamma_{\mu\nu}$ which was calculated above,

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu + \eta_{\mu\nu} \partial_\rho f^\rho \,,$$

hence

$$\mathrm{Re}\big\{ A_{\mu\nu} e^{ik_\rho x^\rho} \big\} \mapsto \mathrm{Re}\big\{ \big(A_{\mu\nu} - i\, i\, k_\mu C_\nu - i\, i\, k_\nu C_\mu + \eta_{\mu\nu} i\, i\, k_\rho\, C^\rho \big)\, e^{ik_\rho x^\rho} \big\} \,,$$

$$A_{\mu\nu} \mapsto A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}\, k_\rho\, C^\rho \,.$$

We want to choose the C_μ such that the equations

$$0 = u^\mu \big(A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}\, k_\rho\, C^\rho \big) \,, \qquad \text{(T1)}$$

$$0 = \eta^{\mu\nu} \big(A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}\, k_\rho\, C^\rho \big) = \eta^{\mu\nu} A_{\mu\nu} - 2\, k_\rho\, C^\rho \qquad \text{(T2)}$$

hold. To demonstrate that such a choice is possible, we choose the coordinates such that

$$\big(u^\mu \big) = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

This can be done by a Lorentz transformation which, as a linear coordinate transformation, preserves all the relevant properties of the coordinate system. Then the spatial part of the desired condition (T1) reads:

(T1) for $\nu = j$: $\quad 0 = A_{0j} + k_0 C_j + k_j C_0 \quad \Longleftrightarrow \quad C_j = -\, k_0^{-1} (A_{0j} + k_j C_0) \,.$

These equations show that the C_j are determined by C_0. We have thus only to determine C_0 in such a way that the temporal part of (T1) holds:

(T1) for $\nu = 0$: $\quad 0 = A_{00} + 2\, k_0\, C_0 + \eta^{\rho\sigma}\, k_\rho\, C_\sigma = A_{00} + \cancel{2}\, k_0\, C_0 - \cancel{k_0\, C_0} + \eta^{ij}\, k_i\, C_j$

$$= A_{00} + k_0\, C_0 - \eta^{ij}\, k_i\, k_0^{-1} \big(A_{0j} + k_j C_0 \big) = A_{00} + k_0\, C_0 - \eta^{ij}\, k_i\, k_0^{-1}\, A_{0j} + \eta^{00}\, k_0\, \cancel{k_0}\, \cancel{k_0^{-1}}\, C_0$$

$$\Longleftrightarrow \quad 0 = -\, k_0\, A_{00} + \eta^{ij}\, k_i\, A_{0j} = \eta^{\mu\nu} k_\mu\, A_{0\nu} \,.$$

The last expression vanishes, because of the Hilbert gauge condition (H) that is satisfied by assumption, so the $\nu = 0$ component of (T1) is identically satisfied if the C^j are chosen as required by the $\nu = j$ components of (T1). This leaves C_0 arbitrary. We now turn to the second desired condition (T2).

$$\text{(T2)}: \; 0 \,=\, \eta^{\mu\nu} A_{\mu\nu} \,+\, 2\,k_0\,C_0 \,-\, 2\,\eta^{ij}\,k_i\,C_j \,=\, A_\mu{}^\mu \,+\, 2\,k_0\,C_0 \,+\, 2\,\eta^{ij}\,k_i\,k_0^{-1}\big(A_{0j} + k_j C_0\big)$$

$$=\, A_\mu{}^\mu \,+\, 2\,k_0\,C_0 \,+\, 2\,\eta^{ij}\,k_i\,k_0^{-1}\,A_{0j} \,-\, 2\,\eta^{00}\,k_0\,\cancel{k_0^{-1}}\,\cancel{k_0}\,C_0 \,=\, A_\mu{}^\mu \,+\, 4\,k_0\,C_0 \,+\, 2\,\eta^{ij}\,k_i\,k_0^{-1}\,A_{0j}$$

$$\iff \quad C_0 \,=\, \frac{-\,A_s\mu^\mu\,k_0 \,-\, 2\,\eta^{ij}\,k_i\,A_{0j}}{4\,k_0^2}\;.$$

If we choose C_0 according to this equation, and then the C_j as required above, (T1) and (T2) are indeed satisfied in the new coordinates. □

In the TT gauge we have $\gamma = 0$ and thus $h_{\mu\nu} = \gamma_{\mu\nu}$. As a consequence, the metric is of the form

$$g_{\mu\nu} \,=\, \eta_{\mu\nu} \,+\, \gamma_{\mu\nu}\;, \qquad \gamma_{\mu\nu} \,=\, \mathrm{Re}\big\{A_{\mu\nu}e^{ik_\rho x^\rho}\big\}$$

and the amplitudes are restricted by the conditions

$$k^\mu A_{\mu\nu} \,=\, 0\;, \qquad u^\mu A_{\mu\nu} \,=\, 0\;, \qquad \eta^{\mu\nu} A_{\mu\nu} \,=\, 0\;.$$

If we choose the coordinates such that

$$\big(u^\mu\big) \,=\, \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \big(k^\rho\big) \,=\, \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ \omega/c \end{pmatrix}$$

which can be reached by a Lorentz transformation, the amplitudes $A_{\mu\nu}$ satisfy

$$\text{(H)} \qquad 0 \,=\, k^\mu\,A_{\mu\nu} \,=\, \frac{\omega}{c}\,(A_{0\nu} + A_{3\nu})\;,$$

$$\text{(T1)} \qquad 0 \,=\, u^\mu\,A_{\mu\nu} \,=\, c\,A_{0\nu}\;,$$

$$\text{(T2)} \qquad 0 \,=\, \eta^{\mu\nu}\,A_{\mu\nu} \,=\, -\,A_{00} \,+\, A_{11} \,+\, A_{22} \,+\, A_{33}$$

in the TT gauge. The first two conditions together imply that $A_{0\nu} = 0$ and $A_{3\nu} = 0$ vor all ν. The last condition then requires $A_{22} = -A_{11}$ and the symmetry of the metric requires $A_{12} = A_{21}$. So in this representation there are only two non-zero (compex!) components of $A_{\mu\nu}$,

$$A_{11} \,=\, -\,A_{22} \,=:\, A_+ \,=\, \big|A_+\big|\,e^{i\varphi}\;,$$

$$A_{12} \,=\, A_{21} \,=:\, A_\times \,=\, \big|A_\times\big|\,e^{i\psi}\;.$$

The fact that only the 1- and the 2-components are non-zero demonstrates that gravitational waves are *transverse*. There are only two independent polarisation states, the *plus mode* (+) and the *cross mode* (×).

For the physical interpretation of these two modes we need the following result.

Claim: The x^0-lines, i.e. the worldlines $x^\mu(\tau)$ with $\dot{x}^\mu(\tau) = u^\mu$, are geodesics.

Proof: From $\dot{x}^\mu(\tau) = u^\mu$ we find $\ddot{x}^\mu(\tau) = 0$. The Christoffel symbols read

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2} g^{\mu\tau} \big(\partial_\nu g_{\tau\sigma} + \partial_\sigma g_{\tau\nu} - \partial_\tau g_{\nu\sigma} \big) = \frac{1}{2} \eta^{\mu\tau} \big(\partial_\nu \gamma_{\tau\sigma} + \partial_\sigma \gamma_{\tau\nu} - \partial_\tau \gamma_{\nu\sigma} \big) =$$

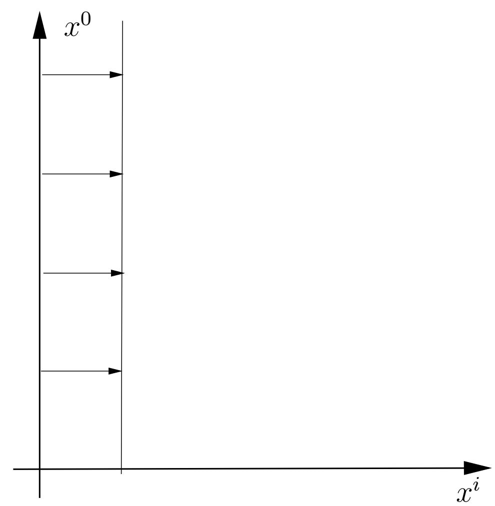
$$= \frac{1}{2} \eta^{\mu\tau} \, \mathrm{Re}\big\{ \big(i\, k_\nu\, A_{\tau\sigma} + i\, k_\sigma\, A_{\tau\nu} - i\, k_\tau\, A_{\nu\sigma} \big) e^{i k_\rho x^\rho} \big\} .$$

This implies that

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\nu = 0 + \frac{1}{2} \eta^{\mu\tau} \, \mathrm{Re}\big\{ \big(i\, k_\nu \underbrace{A_{\tau\sigma} u^\sigma}_{=0} u^\nu + i\, k_\sigma \underbrace{A_{\tau\nu} u^\nu}_{=0} u^\sigma - i\, k_\tau \underbrace{A_{\nu\sigma} u^\sigma}_{=0} u^\nu \big) e^{i k_\rho x^\rho} \big\} = 0 .$$

□

In other words, the x^0-lines are the worldlines of freely falling particles. For any such particle the (x^1, x^2, x^3)-coordinates remain constant. This does, of course, *not* mean that the gravitational wave has no effect on freely falling particles. The distance, as it is measured with the metric, between neighbouring x^0-lines is not at all constant. We calculate the square of the distance between an x^0-line at the spatial origin $(0,0,0)$ and at spatial (x^1, x^2, x^3) for the case that the x^i are so small that the metric can be viewed as constant between $(x^0, 0, 0, 0)$ and (x^0, x^1, x^2, x^3):



$$g_{ij}(x^0,0,0,0)\, x^i x^j = \big(\delta_{ij} + \gamma_{ij}(x^0,0,0,0) \big)\, x^i x^j .$$

If we introduce new spatial coordinates

$$y^i = x^i + \frac{1}{2} \gamma^i{}_k (x^0, 0, 0, 0) x^k ,$$

we see that $\delta_{ij} y^i y^j = \big(\delta_{ij} + \gamma_{ij}(x^0,0,0,0)\big) x^i x^j$, i.e., for a particle at constant y^i the distance from the origin remains constant.

We calculate

$$\delta_{ij} y^i y^j = \delta_{ij} x^i x^j + \mathrm{Re}\big\{ A_+ \big((x^1)^2 - (x^2)^2 \big) e^{-i\omega t} \big\} + \mathrm{Re}\big\{ 2\, A_\times\, x^1\, x^2 e^{-i\omega t} \big\} =$$

$$= \delta_{ij} x^i x^j + \big| A_+ \big| \big((x^1)^2 - (x^2)^2 \big) \cos\big(\varphi - \omega t \big) + 2 \big| A_\times \big|\, x^1\, x^2 \cos\big(\psi - \omega t \big) .$$

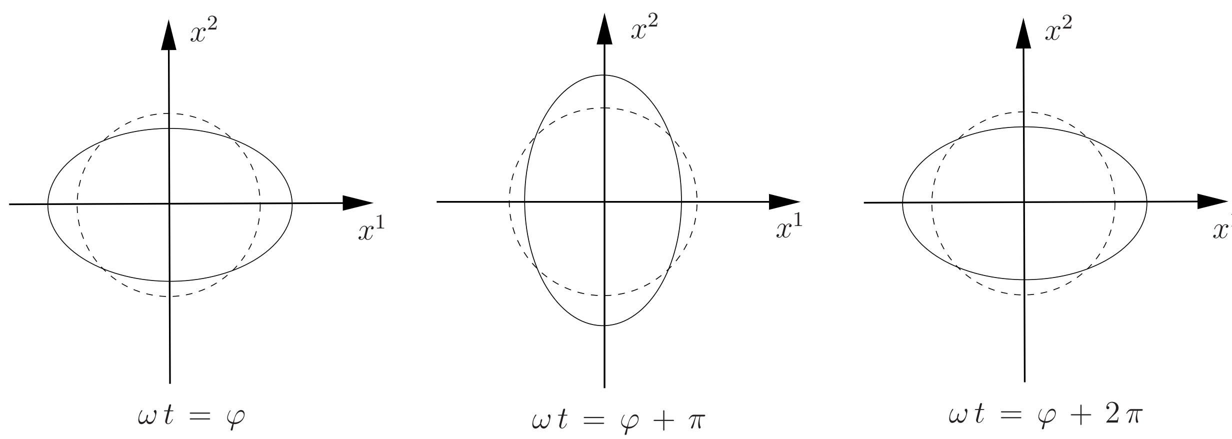
This equation tells how for freely falling particles (i.e., for particles with constant x^i) the distance from the origin changes in dependence of the time t. We demonstrate this in a $y^1 - y^2$–diagram. The pictures on the next page illustrate what happens to particles that are arranged on a small spherical shell and then released to free fall: Both the plus mode and the cross mode of the gravitational wave produce a time-periodic elliptic deformation in the plane perpendicular to the propagation direction. For the plus mode, the main axes of the ellipse coincide with the coordinate axes, for the cross mode they are rotated by 45 degrees. This explains the origin of the names “plus mode” and “cross mode”.

For an animation of the effect of the plus mode on freely falling particles see

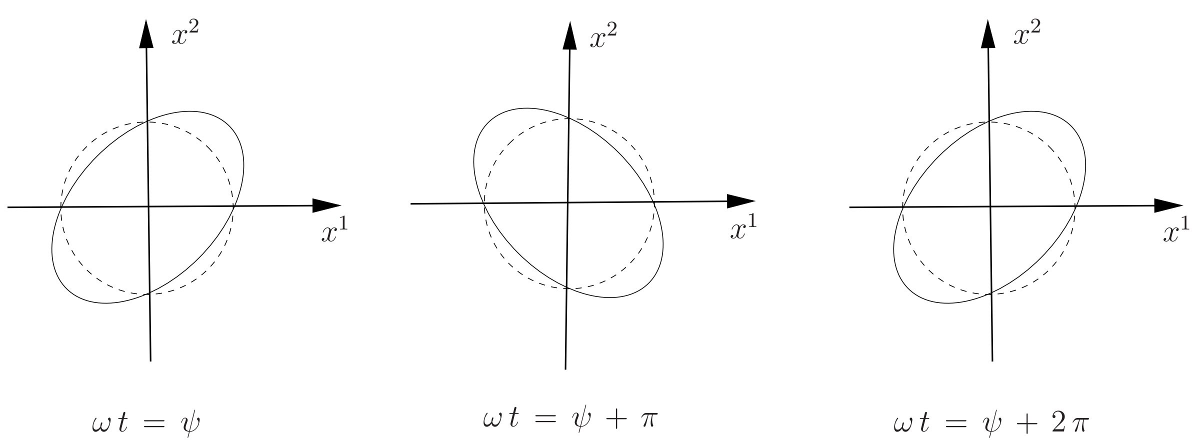
`https://en.wikipedia.org/wiki/Gravitational_wave`

Of course, what is called the plus mode and what is called the cross mode depends on the chosen coordinates. If the coordinate system is rotated by 45°, the two modes interchange. This is in analogy to electromagnetic waves, where there are waves linearly polarised in x direction and waves linearly polarised in y direction; if we rotate the coordinate system by 90°, they interchange their role.

Plus mode ($A_+ \neq 0\,,\ A_\times = 0$):



Cross mode ($A_+ = 0\,,\ A_\times \neq 0$):



We have thus found, as our main result, that a gravitational wave produces a change of the distances between freely falling particles in the plane perpendicular to the propagation direction.

There are two types of gravitational wave detectors that try to measure this effect :

Bar detectors: The first gravitational wave detectors of this type were developed by J. Weber in the 1960s. They were aluminium cylinders of about 1.5 m length. A gravitational wave of an appropriate frequency would excite a resonant oscillation of such a cylinder. With the uprise of interferometric gravitational wave detectors, the bar detectors have lost their relevance. However, some of them are still used.

Interferometric gravitational wave detectors: They are Michelson interferometers with an effective arm length of a few hundred meters at least. The mirrors of such instruments are suspended in a way that they can freely move in the horizontal direction. An incoming gravitational wave would change the distances between the mirrors and the beam splitter (i.e., the travel times of the laser beams) and thus produce a change in the interference pattern. Several such detectors are in operation since the early 2000s. The biggest ones are the two LIGO detectors in the US which made the first direct discovery of a gravitational wave signal on 14 September 2015. This will be discussed in detail below.

3.3 Relating gravitational waves to the source

We will now discuss what sort of sources would produce a gravitational wave. We will see that, in the far-field approximation, the gravitational wave field is determined by the second time-derivative of the quadrupole moment of the source. In other words, gravitational radiation predominantly is quadrupole radiation. By contrast, it is well known that electromagnetic radiation predominantly is dipole radiation.

We now have to consider the linearised field equation with a non-vanishing source term, $T_{\mu\nu} \neq 0$. Again, we choose the Hilbert gauge, so we have to solve the equations

$$\Box \gamma^{\mu\nu} = 2\,\kappa\, T^{\mu\nu} \,, \qquad \partial_\mu \gamma^{\mu\nu} = 0 \,.$$

Clearly, these two equations require the energy-momentum tensor to satisfy the condition

$$\partial_\mu T^{\mu\nu} = \frac{1}{2\,\kappa}\,\partial_\mu \Box\, \gamma^{\mu\nu} = \frac{1}{2\,\kappa}\,\Box\, \partial_\mu \gamma^{\mu\nu} = 0 \,. \qquad \text{(CL)}$$

Recall that in the full non-linear theory of general relativity it is the *covariant* divergence of the energy-momentum tensor that vanishes. In the linearised version it is the *ordinary* divergence, formed with the partial derivatives, that vanishes, as in special relativity for a closed system. In contrast to the covariant divergence condition, the one with the partial derivative can be integrated over so that the usual “pill-box argument” gives an integrated conservation law: The temporal change of the energy within a 3-dimensional volume is given by the flow of the energy over the boundary. The conservation law (CL) is crucial for the linearised theory of gravitational waves.

For given $T_{\mu\nu}$, the general solution to the inhomogeneous wave equation $\Box\gamma^{\mu\nu} = 2\,\kappa\,T^{\mu\nu}$ is the general solution to the homogeneous wave equation (superposition of plane harmonic waves) plus a particular solution to the inhomogeneous equation. Such a particular solution can be written down immediately by analogy with the retarded potentials from electrodynamics:

$$\gamma^{\mu\nu}\big(t,\vec{r}\,\big) \;=\; \frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3} \frac{2\,\kappa\,T^{\mu\nu}\Big(t\,-\,\dfrac{|\vec{r}\,'-\vec{r}\,|}{c}\,,\,\vec{r}\,'\Big)\,dV'}{|\vec{r}\,'-\vec{r}\,|}\,. \qquad \text{(RP)}$$

Here and in the following we write

$$x^0 \,=\, c\,t\,, \qquad (x^1,x^2,x^3) \,=\, \vec{r}\,, \qquad r \,=\, |\vec{r}\,|$$

and dV' is the volume element with respect to the primed coordinates, $dV' \,=\, dx'^1\,dx'^2\,dx'^3$.

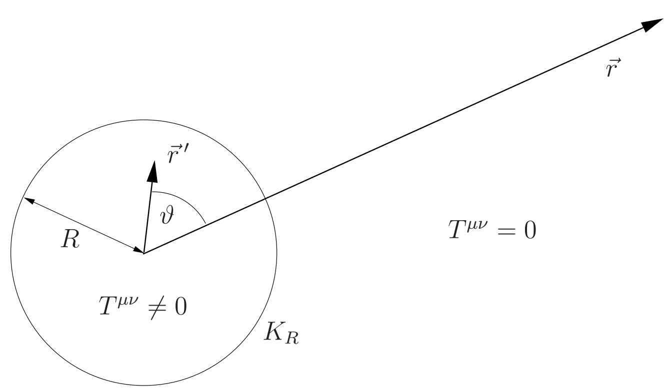
As in electrodynamics one shows by differentiating twice that the $\gamma^{\mu\nu}$ from (RP) satisfy, indeed, the equation $\Box\gamma^{\mu\nu} \;=\; 2\,\kappa\,T^{\mu\nu}$ and that the Hilbert gauge condition holds true provided that the energy-momentum tensor satisfies the conservation law (CL).

The general solution to the inhomogeneous wave equation is given by adding an arbitrary superposition of plane-harmonic waves that satisfy the homogeneous equation, see Section 3.2. If there are no waves coming in from infinity, (RP) alone gives the physically correct solution.

We will now discuss this solution far away from the sources. To that end, we assume that $T^{\mu\nu}$ is different from zero only in a compact region of space. We can then surround this region by a sphere K_R of radius R around the origin, such that

$$T^{\mu\nu}(t,\vec{r}\,) = 0 \quad \text{if}\, r \geq R\,.$$

We are interested in the field $\gamma^{\mu\nu}$ at a point $\vec{r}$ with $r \gg R$.



Then

$$|\vec{r}\,'-\vec{r}\,| \,=\, \sqrt{\big(\vec{r}\,'-\vec{r}\,\big)\cdot\big(\vec{r}\,'-\vec{r}\,\big)} \;=\; \sqrt{\vec{r}\,'\cdot\vec{r}\,' \,+\, \vec{r}\cdot\vec{r} \,-\, 2\,\vec{r}\,'\cdot\vec{r}} \;=$$

$$=\, \sqrt{r'^2 \,+\, r^2 \,-\, 2\,r'\,r\cos\vartheta} \;=\, r\sqrt{1 \,-\, 2\,\frac{r'}{r}\cos\vartheta \,+\, \frac{r'^2}{r^2}} \;=\, r\,\Big(1 \,+\, O(r'/r)\Big)\,.$$

Inserting the result into (RP) yields

$$\gamma^{\mu\nu}\big(t,\vec r\big) \;=\; \frac{\kappa}{2\,\pi}\int\limits_{K_R} \frac{T^{\mu\nu}\Big(t\,-\,\dfrac{r}{c}\,\big(1+O(r'/r)\big)\,,\,\vec r\,'\Big)\,dV'}{r\,\big(1+O(r'/r)\big)}\,.$$

If $r \gg R$, the $O(r'/r)$-terms can be neglected, as $r' \le R$ on the whole domain of integration. This is known as the far-field approximation,

$$\gamma^{\mu\nu}\big(t,\vec r\big) \;=\; \frac{\kappa}{2\,\pi\,r}\int\limits_{K_R} T^{\mu\nu}\Big(t\,-\,\frac{r}{c}\,,\,\vec r\,'\Big)\,dV'\,. \qquad (FF)$$

In this approximation, the $\gamma^{\mu\nu}$ depend on $\vec r$ only in terms of its modulus $r = |\vec r|$, i.e., the wave fronts are spheres, $r =$ constant. As the radii of these spheres are large, they can be approximated as planes on a sufficiently small neighbourhood of any point $\vec r$. This means that, on any such neighborhood, our gravitational wave resembles a plane wave. If the time-dependence of the source is harmonic, it resembles a plane harmonic wave of the type we have studied in Sec. 3.2.

We will now investigate which properties of the source determine the spatial components γ^{ij} in the far-field approximation. To that end we introduce the multipole moments of the source. They are defined in analogy to electrodynamics, with the charge density replaced by the energy density $T_{00} = -T_0{}^0 = T^{00}$.

$$M(t) \;=\; \int\limits_{K_R} T^{00}\big(t,\vec r\big)\,dV \qquad \text{(monopole moment)}\,,$$

$$D^k(t) \;=\; \int\limits_{K_R} T^{00}\big(t,\vec r\big)\,x^k\,dV \qquad \text{(dipole moment)}\,,$$

$$Q^{k\ell}(t) \;=\; \int\limits_{K_R} T^{00}\big(t,\vec r\big)\,x^k\,x^\ell\,dV \qquad \text{(quadrupole moment)}\,,$$

$$\dots$$

Instead of $Q^{k\ell}$ one often uses the trace-free part

$$\mathbb{Q}^{k\ell} = Q^{k\ell} - \frac{1}{3}Q_i{}^i\delta^{k\ell}$$

which is known as the *reduced quadrupole moment.*

We calculate the first and second time derivative of the quadrupole moments. To that end, use the conservation law (CL). We find

$$\frac{d}{dt}Q^{k\ell}(t) \;=\; \int\limits_{K_R} c\,\partial_0 T^{00}(t,\vec r\,)\,x^k\,x^\ell\,dV \;=\; -\,c\int\limits_{K_R} \partial_i T^{i0}(t,\vec r\,)\,x^k\,x^\ell\,dV \;=$$

$$=\;-\,c\int\limits_{K_R}\Big(\partial_i\big(T^{i0}(t,\vec r\,)\,x^k\,x^\ell\big)\;-\;T^{i0}(t,\vec r\,)\,\delta^k_i\,x^\ell\;-\;T^{i0}(t,\vec r\,)\,x^k\,\delta^\ell_i\Big)\,dV\,.$$

The first integral can be rewritten, with the Gauss theorem, as a surface integral over the boundary ∂K_R of K_R,

$$\int\limits_{K_R} \partial_i \big(T^{i0}(t,\vec{r}\,)\, x^k\, x^\ell \big) dV \;=\; \int\limits_{\partial K_R} T^{i0}(t,\vec{r}\,)\, x^k\, x^\ell dS_i$$

where dS_i is the surface element on ∂K_R. As the sphere K_R surrounds all sources, $T^{\mu\nu}$ is equal to zero on ∂K_R, so the last integral vanishes. Hence

$$\frac{d}{dt} Q^{k\ell}(t) \;=\; c \int\limits_{K_R} \big(T^{k0}(t,\vec{r}\,)\, x^\ell \;+\; T^{\ell 0}(t,\vec{r}\,)\, x^k \,\big)\, dV\,.$$

Analogously we calculate the second derivative.

$$\frac{d^2}{dt^2} Q^{k\ell}(t) \;=\; c^2 \int\limits_{K_R} \Big(\,\partial_0 T^{k0}(t,\vec{r}\,)\, x^\ell \;+\; \partial_0 T^{\ell 0}(t,\vec{r}\,)\, x^k \Big)\, dV \;=$$

$$=\; c^2 \int\limits_{K_R} \Big(-\, \partial_i T^{ki}(t,\vec{r}\,)\, x^\ell \;-\; \partial_i T^{\ell i}(t,\vec{r}\,)\, x^k \Big)\, dV \;=$$

$$=\; c^2 \int\limits_{K_R} \Big(-\, \partial_i \big(T^{ki}(t,\vec{r}\,)\, x^\ell \big) \;+\; T^{ki}(t,\vec{r}\,)\, \delta_i^\ell \;-\; \partial_i \big(T^{\ell i}(t,\vec{r}\,)\, x^k \,\big) \;+\; T^{\ell i}(t,\vec{r}\,)\, \delta_i^k \Big)\, dV \;=$$

$$=\; 0 \;+\; c^2 \int\limits_{K_R} T^{k\ell}(t,\vec{r}\,)\, dV \;-\; 0 \;+\; c^2 \int\limits_{K_R} T^{\ell k}(t,\vec{r}\,)\, dV \;=\; 2\, c^2 \int\limits_{K_R} T^{k\ell}(t,\vec{r}\,'\,)\, dV'\,.$$

Upon inserting this result into (FF) we find that, in the far-field approximation

$$\gamma^{k\ell}(t,\vec{r}\,) \;=\; \frac{\kappa}{2\,\pi\, r} \int\limits_{\mathbb{R}^3} T^{k\ell}\Big(t \,-\, \frac{r}{c}\,, \vec{r}\,'\Big)\, dV' \;=\; \frac{\kappa}{2\,\pi\, r}\, \frac{1}{2\, c^2}\, \frac{d^2 Q^{k\ell}}{dt^2}\Big(t \,-\, \frac{r}{c}\Big)\,.$$

If Einstein's gravitational constant is expressed with the help of Newton's gravitational constant, $\kappa = 8\pi G/c^4$, the result reads

$$\gamma^{k\ell}(t,\vec{r}\,) \;=\; \frac{2\, G}{c^6\, r}\, \frac{d^2 Q^{k\ell}}{dt^2}\Big(t \,-\, \frac{r}{c}\Big)\,.$$

Recall from Sec. 3.2 that, far away from the sources, a gravitational wave detector responds to the temporal change of the spatial components $\gamma^{k\ell}$ transverse to the propagation direction of the wave. We have just calculated that these are given by the second time derivative of the quadrupole moment at a retarded time. In this sense, gravitational radiation is quadrupole radiation. By contrast, electromagnetic radiation is dipole radiation: A calculation analogous to the above relates the electromagnetic four-potential to the *first* time derivative of the *dipole* moment of the charge distribution at a retarded time. The difference has, of course, its origin in the fact that $\gamma^{\mu\nu}$ and $T^{\mu\nu}$ have two indices, while the analogous quantities A^μ and J^μ in electrodynamics have only one index.

Note that, according to our results on the preceding page, only the *spatial* components $\gamma^{k\ell}$ are given by the second time derivative of the quadrupole moment. What about the time-time and the time-space components?

Claim: For a source $T^{\mu\nu}$ that is confined to a finite sphere for all times, in the far-field the components γ^{i0} vanish and the component γ^{00} is time-independent and falls off like r^{-1}.

Proof: See Worksheet 3.

For this reason, the components $\gamma^{0\mu}$ give no contribution to the radiation field in the far zone. The time-independent γ^{00} contribution to the far field is, of course, just the Newtonian gravitational field of the source.

In the next two sections we calculate the loss of energy of a system that emits gravitational waves.

3.4 Energy and momentum of a gravitational wave

The question of how to assign energy and momentum to a gravitational wave is conceptually subtle. According to general relativity, the gravitational field is *not* to be considered as a field on the spacetime, it is coded in the geometry of the spacetime itself. The energy-momentum tensor on the right-hand side of Einstein's field equation comprises everything *with the exception of the gravitational field*. An energy-momentum tensor of the gravitational field is not defined and cannot be defined. This is in correspondence with the equivalence principle according to which the gravitational field (coded in the Christoffel symbols which act as the "guiding field" for test particles and light) can be transformed to zero in any one point. As a non-zero tensor is non-zero in *any* coordinates, this is a clear indication that something like an energy-momentum tensor of the gravitational field cannot exist.

However, a (non-tensorial) quantity that describes energy and momentum of a gravitational field can be defined with respect to a background metric. We assume that we have a spacetime metric which takes, in the chosen coordinates the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \ .$$

For the time being, we do *not* assume that the $h_{\mu\nu}$ are small. The coordinates are then fixed up to Lorentz transformations

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \ , \qquad \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta_{\mu\nu} = \eta_{\sigma\tau} \ . \qquad \text{(LT)}$$

The Ricci tensor of the metric g is then of the form

$$R_{\mu\nu}(h) = R^{(1)}_{\mu\nu}(h) + R^{(2)}_{\mu\nu}(h) + \ \dots$$

where $R^{(n)}_{\mu\nu}(h)$ comprises all terms of n^{th} order in the $h_{\rho\sigma}$ and their first and second derivatives.

Similarly, the Einstein tensor is of the form

$$G_{\mu\nu}(h) \;=\; R_{\mu\nu}(h) \;-\; \frac{1}{2}\, g_{\mu\nu}\, g^{\rho\,\sigma} R_{\rho\sigma}(h) \;=\; G^{(1)}_{\mu\nu}(h) + G^{(2)}_{\mu\nu}(h) + \;\ldots$$

where

$$G^{(1)}_{\mu\nu}(h) = R^{(1)}_{\mu\nu}(h) - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} R^{(1)}_{\rho\sigma}(h)\,,$$

$$G^{(2)}_{\mu\nu}(h) = R^{(2)}_{\mu\nu}(h) - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} R^{(2)}_{\rho\sigma}(h) - \frac{1}{2}h_{\mu\nu}\eta^{\rho\sigma} R^{(1)}_{\rho\sigma}(h) + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma} R^{(1)}_{\rho\sigma}(h)\,,$$

and so on. Here we have used that $g^{\rho\sigma} = \eta^{\rho\sigma} - h^{\rho\sigma} + \ldots$

We assume that our metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies Einstein's field equation, with a source term $T_{\mu\nu}$, *exactly*,

$$G_{\mu\nu}(h) = \kappa T_{\mu\nu}\,.$$

We rewrite this equation by keeping only the first-order terms on the left-hand side and shifting all higher-order terms to the right-hand side,

$$G^{(1)}_{\mu\nu}(h) = \kappa\big(T_{\mu\nu} + t_{\mu\nu}\big)\,, \qquad \text{(FEB)}$$

$$t_{\mu\nu} \;=\; -\,\frac{1}{\kappa}\,\big(G_{\mu\nu}(h) - G^{(1)}_{\mu\nu}(h)\big) \;=\; -\,\frac{1}{\kappa}\,\big(G^{(2)}_{\mu\nu}(h) \;+\; \ldots\,\big)$$

According to (FEB), h satisfies the *linearised* field equation with a source term $T_{\mu\nu} + t_{\mu\nu}$. Of course, this is still the same Einstein equation which is non-linear. We have just renamed the non-linear terms into $t_{\mu\nu}$ and re-interpreted them as additional sources.

The $t_{\mu\nu}$ are not the components of a tensor; it is easy to check that the $G^{(1)}_{\mu\nu}$ and hence the $t_{\mu\nu}$ transform as tensor components under Lorentz transformations (LT), but not under arbitrary coordinate changes. $t_{\mu\nu}$ is called the *energy-momentum pseudotensor* of the gravitational field.

The following observation is crucial.

Claim: The combined source term $T_{\mu\nu} + t_{\mu\nu}$ satisfies the continuity equation

$$\partial^{\mu}\big(T_{\mu\nu} + t_{\mu\nu}\big) = 0\,. \qquad \text{(CL)}$$

Proof: See Worksheet 3.

This conservation law is not a covariant equation. It holds only in the special coordinates in which our background metric has components $\eta_{\mu\nu}$. However, it really gives rise to a conservation law in integral form if integrated over a spacetime region ("the change of the energy content inside a spatial volume equals the energy flux over the boundary"). By contrast, the covariant divergence law $\nabla^{\mu}T_{\mu\nu}$, which is satisfied by our true matter source, is only a conservation law in "infinitesimally small regions"; it does not give rise to a conservation law in integral form.

If our matter source loses energy, exactly the same amount of energy must be carried away in the form of gravitational waves according to the conservation law (CL). This is the way in which the observations of the Hulse-Taylor pulsar are interpreted (which will be discussed in detail below): One observes that the system loses energy and one concludes that this energy is caried away in the form of gravitational waves.

Linearising the field equation with respect to the $h_{\mu\nu}$ and their derivatives is tantamount to setting $t_{\mu\nu}$ equal to zero. In this approximation, $T_{\mu\nu}$ alone satisfies the conservation law (CL). We have to go at least to the second order if we want to have a non-trivial $t_{\mu\nu}$. In the second-order theory, we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots$$

where $h^{(1)}_{\mu\nu}$ is a solution to the linearised field equation. The $h^{(1)}_{\mu\nu}$ are small of first order while the $h^{(2)}_{\mu\nu}$ are small of second order. In other words, terms linear in the $h^{(2)}_{\mu\nu}$ are treated at the same footing as terms quadratic in the $h^{(1)}_{\mu\nu}$.

Expanding both sides of (FEB) up to second order results in

$$G^{(1)}_{\mu\nu}(h^{(1)} + h^{(2)}) \,=\, \kappa \left(T_{\mu\nu} + t_{\mu\nu}\right), \qquad t_{\mu\nu} \,=\, -\frac{1}{\kappa}\, G^{(2)}_{\mu\nu}(h^{(1)})\,.$$

In other words, we get the energy-momentum pseudotensor of a gravitational field in its lowest non-trivial approximation if we insert the corresponding solution to the linearised field equation $h^{(1)}_{\rho\sigma}$ into $G^{(2)}_{\mu\nu}$. We will now carry out this calculation for a plane-harmonic wave of the kind we have considered in Section 3.2. On the basis of this result, we will then determine the power that is radiated away from a source that is confined to a finite sphere, as we have considered in Section 3.3.

As before, we raise and lower indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. We need to calculate $R^{(2)}_{\mu\nu}(h)$ which is a bit tedious. We begin with the Christoffel symbols

$$\Gamma^{\rho}{}_{\mu\nu} \,=\, \frac{1}{2}\, g^{\rho\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big) \,=\, \frac{1}{2}\,\big(\eta^{\rho\sigma} - h^{\rho\sigma} + \dots\big)\big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\big)\,.$$

The Ricci tensor is

$$R_{\mu\nu} = R^{\rho}{}_{\rho\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\mu \Gamma^{\rho}{}_{\rho\nu} + \Gamma^{\rho}{}_{\sigma\rho}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\rho\nu}\,,$$

hence

$$\begin{aligned} R^{(2)}_{\mu\nu}(h) \,=\, &-\frac{1}{2}\,\partial_\rho\big(h^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})\big) + \frac{1}{2}\,\partial_\mu\big(h^{\rho\sigma}(\cancel{\partial_\rho h_{\sigma\nu}} + \partial_\nu h_{\sigma\rho} - \cancel{\partial_\sigma h_{\rho\nu}})\big) \\ &+ \frac{1}{4}\,\eta^{\rho\tau}\big(\partial_\sigma h_{\tau\rho} + \cancel{\partial_\rho h_{\tau\sigma}} - \cancel{\partial_\tau h_{\sigma\rho}}\big)\eta^{\sigma\lambda}\big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\big) \\ &- \frac{1}{4}\,\eta^{\rho\tau}\big(\partial_\sigma h_{\tau\mu} + \partial_\mu h_{\tau\sigma} - \partial_\tau h_{\sigma\mu}\big)\eta^{\sigma\lambda}\big(\partial_\rho h_{\lambda\nu} + \partial_\nu h_{\lambda\rho} - \partial_\lambda h_{\rho\nu}\big)\,. \end{aligned}$$

The energy current density of the gravitational wave is given by the time-space components of the energy-momentum pseudotensor

$$t_{0j} = -\frac{1}{\kappa}\, G^{(2)}_{0j}(h^{(1)})\,.$$

We want to calculate this expression for a plane-harmonic wave in the TT gauge,

$$h^{(1)}_{\mu\nu}(x) = \gamma_{\mu\nu}(x) = \mathrm{Re}\big\{A_{\mu\nu} e^{ik_\rho x^\rho}\big\}$$

where $k_\mu k^\mu = 0\,,\ \gamma_{\mu\nu}(x)k^\nu = 0\,,\ \gamma_\mu{}^\mu(x) = 0\,,\ \gamma_{0\nu}(x) = 0\,.$

We find

$$-\kappa\, t_{0j} \;=\; R^{(2)}_{0j}(h^{(1)}) \;+\; 0 \;+\; 0 \;+\; 0 \;=\; -\,\frac{1}{2}\,\partial_\rho\big(\gamma^{\rho\sigma}(\partial_0\gamma_{\sigma j} + \partial_j \underbrace{\gamma_{\sigma 0}}_{=0} - \partial_\sigma \underbrace{\gamma_{0j}}_{=0})\big)$$

$$+\,\frac{1}{2}\,\partial_0\big(\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho}\big) \;+\; \frac{1}{4}\,\eta^{\rho\tau}\partial_\sigma\gamma_{\tau\rho}\eta^{\sigma\lambda}\big(\partial_0\gamma_{\lambda j} +\; \partial_j \underbrace{\gamma_{\lambda 0}}_{=0} - \partial_\lambda \underbrace{\gamma_{0j}}_{=0}\big)$$

$$-\,\frac{1}{4}\,\eta^{\rho\tau}\big(\partial_\sigma \underbrace{\gamma_{\tau 0}}_{=0} + \partial_0\gamma_{\tau\sigma} - \partial_\tau \underbrace{\gamma_{\sigma 0}}_{=0}\big)\eta^{\sigma\lambda}\big(\partial_\rho\gamma_{\lambda j} + \partial_j\gamma_{\lambda\rho} - \partial_\lambda\gamma_{\rho j}\big)$$

$$=\; -\,\frac{1}{2}\,\partial_\rho\big(\gamma^{\rho\sigma}\partial_0\gamma_{\sigma j}\big) \;+\; \frac{1}{2}\,\partial_0\big(\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho}\big) \;+\; \frac{1}{4}\,\partial_\sigma \underbrace{\gamma_\tau{}^\tau}_{=0}\partial_0\gamma^\sigma{}_j \;-\; \frac{1}{4}\,\partial_0\gamma^{\rho\lambda}\big(\cancel{\partial_\rho\gamma_{\lambda j}} + \partial_j\gamma_{\lambda\rho} - \cancel{\partial_\lambda\gamma_{\rho j}}\big)$$

$$=\; 0 \;-\; \frac{1}{2}\underbrace{\gamma^{\rho\sigma}\partial\rho\partial_0\gamma_{\sigma j}}_{\sim\,\gamma^{\rho\sigma}k_\rho\,=\,0} \;+\; \frac{1}{2}\,\partial_0\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho} \;+\; \frac{1}{2}\,\gamma^{\rho\sigma}\partial_0\partial_j\gamma_{\sigma\rho} \;-\; \frac{1}{4}\,\partial_0\gamma^{\rho\lambda}\partial_j\gamma_{\lambda\rho}$$

$$=\; \frac{1}{2}\,\gamma^{\rho\sigma}\partial_0\partial_j\gamma_{\sigma\rho} \;+\; \frac{1}{4}\,\partial_0\gamma^{\rho\lambda}\partial_j\gamma_{\lambda\rho} \;=\; \frac{1}{2}\,\gamma^{k\ell}\partial_0\partial_j\gamma_{k\ell} \;+\; \frac{1}{4}\,\partial_0\gamma^{k\ell}\partial_j\gamma_{k\ell}$$

$$=\; \frac{1}{2}\,\mathrm{Re}\big\{A^{k\ell}e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{-A_{k\ell}k_0k_je^{ik_\rho x^\rho}\big\} \;+\; \frac{1}{4}\,\mathrm{Re}\big\{A^{k\ell}ik_0e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{A_{k\ell}ik_je^{ik_\rho x^\rho}\big\}$$

$$=\; -\,\frac{k_0k_j}{8}\,\big(A^{k\ell}\,e^{ik_\rho x^\rho} + \overline{A^{k\ell}}\,e^{-ik_\rho x^\rho}\big)\big(A_{k\ell}\,e^{ik_\rho x^\rho} + \overline{A_{k\ell}}\,e^{-ik_\rho x^\rho}\big)$$

$$+\,\frac{k_0k_j}{16}\big(A^{k\ell}\,i\,e^{ik_\rho x^\rho} - \overline{A^{k\ell}}\,i\,e^{-ik_\rho x^\rho}\big)\big(A_{k\ell}\,i\,e^{ik_\rho x^\rho} - \overline{A_{k\ell}}\,i\,e^{-ik_\rho x^\rho}\big)$$

$$=\; -\,\frac{3k_0k_j}{16}\Big(A^{k\ell}A_{k\ell}\,e^{2ik_\rho x^\rho} \;+\; \overline{A^{k\ell}A_{k\ell}}\,e^{-2ik_\rho x^\rho}\Big) \;-\; \frac{k_0k_j}{8}A^{k\ell}\,\overline{A_{k\ell}}$$

$$=\; -\,\frac{3k_0k_j}{8}\Big(\mathrm{Re}\big\{A^{k\ell}A_{k\ell}\big\}\cos\big(2k_\rho x^\rho\big) \;-\; \mathrm{Im}\big\{A^{k\ell}A_{k\ell}\big\}\sin\big(2k_\rho x^\rho\big)\Big) \;-\; \frac{k_0k_j}{8}A^{k\ell}\,\overline{A_{k\ell}}\,.$$

where an overbar means complex conjugation. The first two terms, which are proportional to $\cos(2k_\rho x^\rho) = \cos(2k_ix^i - 2\omega t)$ and $\sin(2k_\rho x^\rho) = \sin(2k_ix^i - 2\omega t)$, respectively, vary periodically with time around zero. If we consider the time-average, denoted by $\langle\cdot\rangle$, they drop out,

$$\kappa\;\langle t_{0j}\rangle \;=\; \frac{k_0k_j}{8}A^{k\ell}\,\overline{A_{k\ell}}\,.$$

3.5 Einstein's quadrupole formula

With the help of the results of the preceding section we will now derive Einstein's famous quadrupole formula which relates the total power radiated away by a source of gravitational waves to the third time derivative of its quadrupole moment.

We have seen that in lowest non-trivial approximation for a plane-harmonic gravitational wave in TT gauge,

$$\gamma_{\mu\nu} = \mathrm{Re}\big\{A_{\mu\nu} e^{ik_\sigma x^\sigma}\big\} ,$$

the time average of the time-space component of the energy-momentum pseudotensor is given by the formula

$$\kappa \, \langle t_{0j} \rangle \; = \; \frac{k_0 k_j}{8} A^{k\ell} \, \overline{A_{k\ell}} \, .$$

We introduce the covector

$$n_j = - \frac{k_j}{k_0}$$

which is parallel to the spatial wave covektor k_j and normalised, because

$$n_j n^j \; = \; \frac{k_j k^j}{k_0^2} \; = \; \frac{k_\rho k^\rho - k_0 k^0}{k_0^2} \; = \; \frac{0 \, + \, k_0^2}{k_0^2} \; = \; 1 \; .$$

Then the time average of the energy current density of the gravitational wave

$$s_j = - u^\rho t_{\rho j} = - u^0 t_{0j} = - c t_{oj}$$

reads

$$\langle s_j \rangle = - \, c \, \langle t_{0j} \rangle \; = \; \frac{c \, k_0^2 n_j}{8 \kappa} A^{k\ell} \, \overline{A_{k\ell}} \, .$$

In the theory of gravitational waves, s_j plays the same role as the Poynting vector in the theory of electromagnetic waves. The last expression can be rewritten as

$$\langle s_j \rangle \; = \; \frac{c \, n_j}{4 \, \kappa} \, \langle \partial_0 \gamma^{k\ell} \partial_0 \gamma_{k\ell} \rangle \tag{EC}$$

as follows from comparison with

$$\big\langle \partial_0 \gamma^{k\ell} \partial_0 \gamma_{k\ell} \big\rangle \; = \; \big\langle \mathrm{Re}\big\{ A^{k\ell} \, i \, k_0 \, e^{ik_\rho x^\rho} \big\} \mathrm{Re}\big\{ A_{k\ell} \, i \, k_0 \, e^{ik_\sigma x^\sigma} \big\} \big\rangle$$

$$= \; \frac{k_0^2}{4} \Big\langle \big(A^{k\ell} \, i \, e^{ik_\rho x^\rho} - \overline{A^{k\ell}} \, i \, e^{-ik_\rho x^\rho} \big) \big(A_{k\ell} \, i \, e^{ik_\sigma x^\sigma} - \overline{A_{k\ell}} \, i \, e^{-ik_\sigma x^\sigma} \big) \Big\rangle \; = \; \frac{k_0^2}{2} A^{k\ell} \, \overline{A_{k\ell}} \, .$$

We now turn back to the situation of an energy-momentum tensor field which has support inside a sphere of radius R, for all time, see picture on p.17. We know from Section 3.3 that the solution to the linearised field equation in the Hilbert gauge satisfies in the far zone

$$\gamma^{k\ell}(t, \vec{r} \,) \; = \; \frac{\kappa}{4 \, \pi \, r \, c^2} \, \frac{d^2 Q^{k\ell}}{dt^2} \Big(t \, - \, \frac{r}{c} \Big) \, .$$

(γ^{0k} and γ^{00} give no contribution to the *wave* field, recall Worksheet 3.) This $\gamma^{k\ell}$may be viewed, in a sufficiently small neighbourhood of any one point in the far zone, as a superposition of plane-harmonic waves propagating in the direction n^j, where n^j is the unit vector in the radial direction. Here it is important to realise that our expression of $\gamma^{k\ell}$ satisfies the Hilbert gauge but not in general the TT gauge. Therefore, we cannot apply (EC) directly for calculating the time-averaged energy current of this gravitational wave, because (EC) holds only for a plane-harmonic wave in the TT gauge. We have to project onto the transverse-traceless part of $\gamma^{k\ell}$ first.

Here "transverse" means "orthogonal to the propagation direction n^j", i.e., the transverse part of $\gamma_{k\ell}$ is

$$\gamma^T_{k\ell} = P_k{}^i P_\ell{}^j \gamma_{ij}$$

where P_i^j is the orthogonal projection onto the orthocomplement of n^j,

$$P_i^j = \delta_i^j - n_i n^j \, .$$

Note that P_i^j satisfies the projection property ("idempotency")

$$P_i^j P_j{}^k = \big(\delta_i^j - n_i n^j\big)\big(\delta_j^k - n_j n^k\big) = \delta_i^k - n_i n^k - \cancel{n_i n^k} + \cancel{n_i n^k} = P_i{}^k$$

and the symmetry property

$$P^{rs} = \delta^{rs} - n^r n^s = P^{sr} \, .$$

$\gamma^T_{k\ell}$ is a second-rank tensor field on the 2-space orthogonal to n^j. We have to subtract the trace part of this tensor field to get the transverse-traceless part of $\gamma_{k\ell}$,

$$\gamma^{TT}_{k\ell} = \gamma^T_{k\ell} - \frac{1}{2} P_{k\ell}\, P^{rs} \gamma^T_{rs} = P_k{}^i P_\ell{}^j \gamma_{ij} - \frac{1}{2} P_{k\ell} P^{rs} P_r{}^i P_s{}^j \gamma_{ij} = P_k{}^i P_\ell{}^j \gamma_{ij} - \frac{1}{2} P_{k\ell} P^{ij} \gamma_{ij} \, .$$

Here we have used that on the 2-space orthogonal to n^j the tensor $P_{k\ell}$ plays the role of the metric (which corresponds to the fact that on thsi 2-space P_i^j is the identity operator).

For applying (EC) to our gravitational field in the far zone we need to calculate

$$\partial_0 \gamma^{TTk\ell} \partial_0 \gamma^{TT}_{k\ell} \;=\; \partial_0 \big(P^{km} P^{\ell n} \gamma_{mn} - \frac{1}{2} P^{k\ell} P^{mn} \gamma_{mn}\big) \partial_0 \big(P_k{}^r P_\ell{}^s \gamma_{rs} - \frac{1}{2} P_{k\ell} P^{rs} \gamma_{rs}\big)$$

$$= \big(P^{km} P^{\ell n} P_k{}^r P_\ell{}^s - \frac{1}{2} P^{km} P^{\ell n} P_{k\ell} P^{rs} - \frac{1}{2} P^{k\ell} P^{mn} P_k{}^r P_\ell{}^s + \frac{1}{4} P^{k\ell} P^{mn} P_{k\ell} P^{rs}\big) \partial_0 \gamma_{mn} \partial_0 \gamma_{rs}$$

$$= \big(P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs} - \frac{1}{2} P^{rs} P^{mn} + \frac{1}{4} P_k{}^k P^{mn} P^{rs}\big) \partial_0 \gamma_{mn} \partial_0 \gamma_{rs}$$

$$= \big(P^{mr} P^{ns} - P^{mn} P^{rs} + \frac{1}{4} \underbrace{(\delta_k^k - n_k n^k)}_{=2} P^{mn} P^{rs}\big) \partial_0 \gamma_{mn} \partial_0 \gamma_{rs} \;=\; \big(P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs}\big) \partial_0 \gamma_{mn} \partial_0 \gamma_{rs}$$

Note that n^j is the unit vector in radial direction, so the P^{mr} depend on $\vec{r}$ but not on t. Therefore, when we calculate the time average the P^{mr} are not affected. When time-averaging $\partial_0 \gamma_{rs}$ we have to be careful because these expressions are not usually harmonic in time. E.g., in a binary system the two constituents orbit each other with a frequency Ω but this frequency is constant only approximately over sufficiently short time intervals. On a larger time scale the frequency increases because the system loses energy in the form of gravitational waves, so the

two constituents spiral towards each other. When we do the time-averaging we do this over sufficiently short intervals, keeping the ("adiabatic") time dependence which results from the loss of energy. In the following, time-averaging is always meant in this sense. (This problem does not occur if we consider systems where, with the help of external forces, a strictly periodic motion is maintained. We consider two problems of the latter kind in Worksheet 4. For binary systems, which will be treated in the next section, however, the loss of energy is a crucial feature.)
We find for the time-averaged energy current density

$$\langle s_j \rangle (t, \vec{r}) \, = \, \frac{c\, n_j}{4\kappa} \big(P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs} \big) \, \langle \partial_0 \gamma_{mn} \partial_0 \gamma_{rs} \rangle \, (t, \vec{r}) \, .$$

With

$$P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs} = (\delta^{mr} - n^m n^r)(\delta^{ns} - n^n n^s) - \frac{1}{2} (\delta^{mn} - n^m n^n)(\delta^{rs} - n^r n^s)$$

$$= \delta^{mr} \delta^{ns} - \frac{\delta^{mn} \delta^{rs}}{2} - 2 \delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2}$$

and

$$\partial_0 \gamma_{k\ell} (t, \vec{r}\,) \, = \, \partial_0 \Big\{ \frac{\kappa}{4\,\pi\, r\, c^2} \, \frac{d^2 Q_{k\ell}}{dt^2} \Big(t \, - \, \frac{r}{c} \Big) \Big\} \, = \, \frac{\kappa}{4\,\pi\, r\, c^3} \, \frac{d^3 Q_{k\ell}}{dt^3} \Big(t \, - \, \frac{r}{c} \Big)$$

this results in

$$\langle s_j \rangle (t, \vec{r}) \, =$$

$$\frac{\kappa n_j}{64 \pi^2 r^2 c^5} \Big(\delta^{mr} \delta^{ns} - \frac{\delta^{mn} \delta^{rs}}{2} - 2 \delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2} \Big) \left\langle \frac{d^3 Q_{mn}}{dt^3} \, \frac{d^3 Q_{rs}}{dt^3} \right\rangle \Big(t - \frac{r}{c} \Big) \, .$$

This equation holds at any point $\vec{r}$ in the far zone where n^j denotes the unit vector in radial direction at this point. We can integrate this equation over a sphere of radius $r (\gg R)$ to get the radiated power (energy per time) that passes through this sphere

$$P(t, r) = \int_0^{2\pi} \int_0^{\pi} \langle s_j \rangle \, (t, \, \vec{r}) r^2 n^j \sin \vartheta \, d\vartheta \, d\varphi \, =$$

$$\int_0^{2\pi} \!\! \int_0^{\pi} \Big(\delta^{mr} \delta^{ns} - \frac{\delta^{mn} \delta^{rs}}{2} - 2 \delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2} \Big) \frac{\kappa n_j \cancel{r^2} n^j \sin \vartheta \, d\vartheta \, d\varphi}{64 \pi^2 \cancel{r^2} c^5} \left\langle \frac{d^3 Q_{mn}}{dt^3} \frac{d^3 Q_{rs}}{dt^3} \right\rangle \Big(t - \frac{r}{c} \Big)$$

and, with $n_j n^j = 1$,

$$P(r, t) =$$

$$\frac{\kappa}{64 \pi^2 c^5} \int_0^{2\pi} \!\! \int_0^{\pi} \Big(\delta^{mr} \delta^{ns} - \frac{\delta^{mn} \delta^{rs}}{2} - 2 \delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2} \Big) \sin \vartheta \, d\vartheta \, d\varphi \left\langle \frac{d^3 Q_{mn}}{dt^3} \frac{d^3 Q_{rs}}{dt^3} \right\rangle \Big(t - \frac{r}{c} \Big) .$$

For evaluating the integrals we need the following results.

Claim: $\int\limits_0^{2\pi}\int\limits_0^{\pi} n^k n^\ell \sin\vartheta\, d\vartheta\, d\varphi = \dfrac{4\pi}{3}\delta^{k\ell}$ and $\int\limits_0^{2\pi}\int\limits_0^{\pi} n^k n^\ell n^r n^s \sin\vartheta\, d\vartheta\, d\varphi = \dfrac{4\pi}{15}(\delta^{k\ell}\delta^{rs}+\delta^{kr}\delta^{\ell s}+\delta^{ks}\delta^{r\ell})$.

Proof: We calculate for all $(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$,

$$\int\limits_0^{\pi}\int\limits_0^{2\pi} \xi_i\xi_j n^i n^j \sin\vartheta\, d\varphi\, d\vartheta \;=\; \int_0^{\pi}\int_0^{2\pi}\big(\xi_1\sin\vartheta\cos\varphi+\xi_2\sin\vartheta\sin\varphi+\xi_3\cos\vartheta\big)^2\sin\vartheta\, d\varphi\, d\vartheta$$

$$=\;\xi_1^2\int\limits_0^{\pi}\sin^3\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\cos^2\varphi\, d\varphi}_{=\pi}+\xi_2^2\int\limits_0^{\pi}\sin^3\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\sin^2\varphi\, d\varphi}_{=\pi}+\xi_3^2\int\limits_0^{\pi}\cos^2\vartheta\sin\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}d\varphi}_{=2\pi}\,.$$

For all other terms the φ integration gives zero. Hence

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\xi_i\xi_j n^i n^j d\varphi\, d\vartheta \;=\; \big(\pi\,\xi_1^2+\pi\,\xi_2^2\big)\underbrace{\int\limits_0^{\pi}\sin^3\vartheta\, d\vartheta}_{=4/3}\;+\;2\,\pi\,\xi_3^2\underbrace{\int\limits_0^{\pi}\cos^2\vartheta\sin\vartheta\, d\vartheta}_{=2/3}$$

$$=\;\frac{4\pi}{3}\big(\xi_1^2+\xi_2^2+\xi_3^2\big)\;=\;\frac{4\pi}{3}\delta^{ij}\xi_i\xi_j\,.$$

Similarly,

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\xi_i\xi_j\xi_k\xi_\ell n^i n^j n^k n^\ell \sin\vartheta\, d\varphi\, d\vartheta \;=\; \int_0^{\pi}\int_0^{2\pi}\big(\xi_1\sin\vartheta\cos\varphi+\xi_2\sin\vartheta\sin\varphi+\xi_3\cos\vartheta\big)^4\sin\vartheta\, d\varphi\, d\vartheta$$

$$=\;\xi_1^4\int\limits_0^{\pi}\sin^5\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\cos^4\varphi\, d\varphi}_{=3\pi/4}+\xi_2^4\int\limits_0^{\pi}\sin^5\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\sin^4\varphi\, d\varphi}_{=3\pi/4}+6\xi_1^2\xi_2^2\int\limits_0^{\pi}\sin^5\vartheta\underbrace{\int\limits_0^{2\pi}\cos^2\varphi\sin^2\varphi\, d\varphi}_{=\pi/4}$$

$$+6\xi_1^2\xi_3^2\int\limits_0^{\pi}\cos^2\vartheta\sin^3\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\cos^2\varphi\, d\varphi}_{=\pi}+6\xi_2^2\xi_3^2\int\limits_0^{\pi}\cos^2\vartheta\sin^3\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}\sin^2\varphi\, d\varphi}_{=\pi}+\xi_3^4\int\limits_0^{\pi}\cos^4\vartheta\sin\vartheta\, d\vartheta\underbrace{\int\limits_0^{2\pi}d\varphi}_{=2\pi}$$

$$=\;\frac{3\pi}{4}\big(\xi_1^4+\xi_2^4+2\xi_1^2\xi_2^2\big)\underbrace{\int\limits_0^{\pi}\sin^5\vartheta\, d\vartheta}_{=16/15}\;+\;6\pi(\xi_1^2+\xi_2^2)\xi_3^2\underbrace{\int\limits_0^{\pi}\cos^2\vartheta\sin^3\vartheta\, d\vartheta}_{=4/15}+2\,\pi\,\xi_3^4\underbrace{\int\limits_0^{\pi}\cos^4\vartheta\sin\vartheta\, d\vartheta}_{=2/5}$$

$$=\;\frac{4\pi}{5}\big(\xi_1^2+\xi_2^2+\xi_3^2\big)^2\;=\;\frac{4\pi}{5}\delta^{ij}\delta^{k\ell}\xi_i\xi_j\xi_k\xi_\ell\,.$$

Symmetrisation of the coefficients gives the desired result. $\square$

Hence

$$P(t,r) =$$

$$\frac{\kappa}{16\pi c^5}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{2}-\frac{2\delta^{mr}\delta^{ns}}{3}+\frac{\delta^{mn}\delta^{rs}}{3}+\frac{\delta^{mn}\delta^{rs}+\delta^{mr}\delta^{ns}+\delta^{ms}\delta^{rn}}{30}\Big)\Big\langle\frac{d^3Q_{mn}}{dt^3}\frac{d^3Q_{rs}}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \frac{\kappa}{16\pi c^5}\Big(\frac{2\delta^{mr}\delta^{ns}}{5}-\frac{2\delta^{mn}\delta^{rs}}{15}\Big)\Big\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \frac{\kappa}{40\pi c^5}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{3}\Big)\Big\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \frac{\kappa}{40\pi c^5}\Big\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q^{mn}}{dt^3}-\frac{1}{3}\frac{d^3{Q_m}^m}{dt^3}\,\frac{d^3{Q_r}^r}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big).$$

This can be rewritten more conveniently if we introduce the *reduced quadrupole moment* $\mathbb{Q}_{k\ell}$ which is defined as the trace-free part of $Q_{k\ell}$,

$$\mathbb{Q}_{k\ell} = Q_{k\ell} - \frac{1}{3}\delta_{k\ell}{Q_j}^j\,.$$

Then the energy flux through the sphere of radius r reads

$$P(t,r) \;=\; \frac{\kappa}{40\pi c^5}\Big\langle\frac{d^3}{dt^3}\Big(\mathbb{Q}_{mn}+\frac{1}{3}\delta_{mn}{Q_k}^k\Big)\frac{d^3}{dt^3}\Big(\mathbb{Q}^{mn}+\frac{1}{3}\delta^{mn}{Q_\ell}^\ell\Big)-\frac{1}{3}\frac{d^3{Q_m}^m}{dt^3}\,\frac{d^3{Q_r}^r}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \frac{\kappa}{40\pi c^5}\Big\langle\frac{d^3\mathbb{Q}_{mn}}{dt^3}\frac{d^3\mathbb{Q}^{mn}}{dt^3}+0+0+\cancel{\frac{1}{9}\,3\,\frac{d^3{Q_k}^k}{dt^3}\frac{d^3{Q_\ell}^\ell}{dt^3}}-\cancel{\frac{1}{3}\frac{d^3{Q_k}^k}{dt^3}\frac{d^3{Q_\ell}^\ell}{dt^3}}\Big\rangle\Big(t-\frac{r}{c}\Big)$$

which eventually gives us Einstein's *quadrupole formula*

$$P(t,r) \;=\; \frac{\kappa}{40\pi c^5}\Big\langle\frac{d^3\mathbb{Q}_{mn}}{dt^3}\frac{d^3\mathbb{Q}^{mn}}{dt^3}\Big\rangle\Big(t-\frac{r}{c}\Big).$$

This formula allows us to calculate the power that is radiated away by a time-dependent matter source. In electrodnamics, a similar formula holds where instead of the (reduced) energy quadrupole moment one has the charge dipole moment and instead of third derivatives one has second derivatives.

If we want to apply Einstein's quadrupole formula, we need to know the reduced energy quadrupole moment of the source,

$$\mathbb{Q}_{k\ell} = Q_{k\ell} - \frac{1}{3}\delta_{k\ell}{Q_j}^j\,,\quad Q_{k\ell}(t) = \int_{\mathbb{R}^3} T_{00}(t,\vec{r})x_kx_\ell d^3\vec{r}\,.$$

This, in turn, requires to know the energy density T_{00}. The latter contains the whole energy content of the source which is difficult to determine. For slowly moving bodies the biggest contribution to the energy density comes from the mass density $\mu(t, \vec{r})$. As long as the source involves only motions that are slow in comparison to the speed of light, we can write as a valid approximation

$$T_{00}(t, \vec{r}) \approx c^2 \mu(t, \vec{r})\,.$$

We can then replace the reduced energy quadrupole moment $\mathbb{Q}_{k\ell}$ by the reduced mass quadrupole moment

$$\mathbb{I}_{k\ell} = I_{k\ell} - \frac{1}{3}\delta_{k\ell} I_j{}^j\,,$$

$$I_{k\ell}(t) = \int_{\mathbb{R}^3} \mu(t, \vec{r}) x_k x_\ell d^3\vec{r}.$$

With the aproximation

$$\mathbb{Q}_{k\ell} \approx c^2\, \mathbb{I}_{k\ell}$$

Einstein's quadrupole formula reads

$$P(t, r) \;=\; \frac{\kappa}{40\,\pi\,c} \left\langle \frac{d^3\mathbb{I}_{mn}}{dt^3}\frac{d^3\mathbb{I}^{mn}}{dt^3} \right\rangle \left(t - \frac{r}{c}\right)$$

or, with $\kappa = 8\pi G/c^4$,

$$P(t, r) \;=\; \frac{G}{5\,c^5} \left\langle \frac{d^3\mathbb{I}_{mn}}{dt^3}\frac{d^3\mathbb{I}^{mn}}{dt^3} \right\rangle \left(t - \frac{r}{c}\right).$$

This is the form in which the formula is usually applied. In this version, the quadrupole formula involves the following approximations.

- The energy-momentum pseudotensor was calculated only to within second order (which is the lowest non-trivial order); the solution to the field equation that is needed to calculate this order is a first-order solution, i.e., a solution to the linearised field equation.
- The formula holds in the far zone, i.e., it gives the energy flux per time through a sphere of radius r which is big in comparison to the radius R of the sphere to which the matter source $T_{\mu\nu}$ is confined.
- The formula is based on the assumption that all motions inside the source are slow in comparison to the speed of light.

In addition, the formula involves a time-averaging over an interval that covers the periods of all Fourier components that contribute to the gravitational wave but is short enough so that the loss of energy during this time interval can be neglected.

In the next section we will apply Einstein's quadrupole formula to binary systems. This will allow us to calculate the energy loss that has been observed with the Hulse-Taylor pulsar. For this observation, which is generally accepted as an indirect detection of gravitational waves, Hulse and Taylor won the Nobel prize in 1994.

In Worksheet 4 we will consider the gravitational radiation for two simpler systems, where we will also give an estimate demonstrating that gravitational waves of detectable strength cannot be produced in the laboratory.

3.6 Gravitational waves from a binary source

We want to calculate, with the help of Einstein's quadrupole formula, the gravitational radiation produced by a binary system. A typical example we have in mind is the Hulse-Taylor pulsar and his companion which will be discussed below. In this case we have two neutron stars which are very close in comparison to binary systems that consist of ordinary (main sequence) stars, but still so far apart that the orbits may be calculated in terms of Newtonian gravity. We will recall the solution to the Newtonian two-body problem and then use this solution for calculating the mass quadrupole moment of the binary system which is needed for Einstein's quadrupole formula.

The Newtonian two-body problem can be given in terms of the Lagrangian

$$\mathcal{L} = \frac{M_1}{2}\big|\dot{\vec{r}}_1\big|^2 + \frac{M_1}{2}\big|\dot{\vec{r}}_1\big|^2 + \frac{G\,M_1 M_2}{\big|\vec{r}_1 - \vec{r}_2\big|}$$

where M_1 and M_2 are the masses and $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two bodies in an inertial system. Introducing the position vector of the centre of mass

$$\vec{r}_0 = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2}$$

and the relative position vector

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

allows to express the position vectors of the two bodies as

$$\vec{r}_1 = \vec{r}_0 + \frac{M_2\,\vec{r}_{12}}{M_1 + M_2}\,, \quad \vec{r}_2 = \vec{r}_0 - \frac{M_1\,\vec{r}_{12}}{M_1 + M_2}\,,$$

hence

$$\begin{aligned}\mathcal{L} &= \frac{M_1}{2}\left|\dot{\vec{r}}_0 + \frac{M_2\,\dot{\vec{r}}_{12}}{M_1 + M_2}\right|^2 + \frac{M_2}{2}\left|\dot{\vec{r}}_0 - \frac{M_1\,\dot{\vec{r}}_{12}}{M_1 + M_2}\right|^2 + \frac{G\,M_1 M_2}{\big|\vec{r}_{12}\big|} \\ &= \frac{M_1 + M_2}{2}\,\big|\dot{\vec{r}}_0\big|^2 + \frac{M_1 M_2\,\cancel{(M_1 + M_2)}}{2\,(M_1 + M_2)^{\cancel{2}}}\,\big|\dot{\vec{r}}_{12}\big|^2 + \frac{G\,M_1 M_2}{\big|\vec{r}_{12}\big|}\,.\end{aligned}$$

The Euler-Larange equations with respect to the components of $\vec{r}_0$ give $\ddot{\vec{r}}_0 = 0$, i.e., the centre of mass is in uniform rectilinear motion. It is, thus, possible to transform to another inertial system in which the centre of mass is at rest, $\vec{r}_0 = 0$. Then

$$\vec{r}_1 = \frac{M_2\,\vec{r}_{12}}{M_1 + M_2}\,, \quad \vec{r}_2 = -\frac{M_1\,\vec{r}_{12}}{M_1 + M_2}\,, \qquad (R1)$$

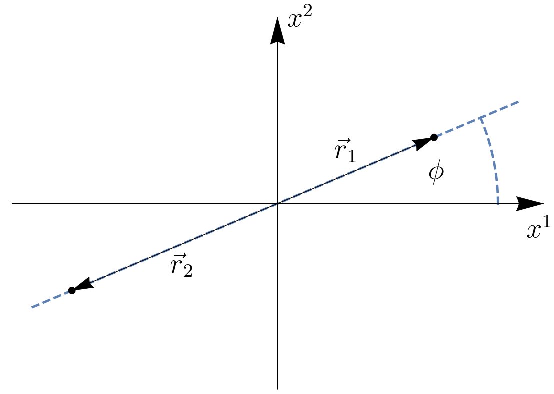
and

$$\mathcal{L} = \frac{M_1 M_2}{2\,(M_1 + M_2)}\,\big|\dot{\vec{r}}_{12}\big|^2 + \frac{G\,M_1 M_2}{\big|\vec{r}_{12}\big|}\,.$$

We choose the coordinate axes such that

$$\vec{r}_{12} = r_{12} \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} , \qquad (R2)$$

see the picture. Then



$$\dot{\vec{r}}_{12} = \dot r_{12} \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} + r_{12}\,\dot\phi \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} ,$$

and the Lagrangian reads

$$\mathcal{L} = \frac{M_1 M_2}{2\,(M_1+M_2)} \left(\dot r_{12}^2 + r_{12}^2\,\dot\phi^2\right) + \frac{G\,M_1 M_2}{r_{12}} .$$

The Euler-Lagrange equations with respect to r_{12} and ϕ are

$$\frac{M_1 M_2}{(M_1+M_2)}\,\ddot r_{12} - \frac{M_1 M_2}{(M_1+M_2)}\,r_{12}\,\dot\phi^2 + \frac{G\,M_1 M_2}{r_{12}^2} = 0 ,$$

$$\frac{M_1 M_2}{(M_1+M_2)}\,r_{12}^2\,\dot\phi = L = \text{constant} . \qquad (K2)$$

Inserting the second equation, which is just Kepler's second law for the relative position vector, into the first one yields

$$\frac{M_1 M_2}{(M_1+M_2)}\,\ddot r_{12} - \frac{\cancel{M_1}\cancel{M_2}\,\cancel{r_{12}}\,L^2(M_1+M_2)^{\cancel{2}}}{\cancel{(M_1+M_2)}M_1^{\cancel{2}}M_2^{\cancel{2}}r_{12}^{\cancel{4}3}} + \frac{G\,M_1 M_2}{r_{12}^2} = 0$$

We rewrite the first term with the chain rule,

$$\frac{M_1 M_2}{(M_1+M_2)}\,\dot\phi\frac{d}{d\phi}\Big(\dot\phi\frac{dr_{12}}{d\phi}\Big) - \frac{L^2(M_1+M_2)}{M_1 M_2 r_{12}^3} + \frac{G\,M_1 M_2}{r_{12}^2} = 0$$

Re-expressing $\dot\phi$ in terms of the conserved angular momentum L with the help of (K2) results in

$$\frac{\cancel{M_1}\cancel{M_2}\,L\,\cancel{(M_1+M_2)}}{\cancel{(M_1+M_2)}\cancel{M_1}\cancel{M_2}\,\cancel{r_{12}^2}}\frac{d}{d\phi}\Big(\frac{L(M_1+M_2)}{M_1M_2r_{12}^2}\,\frac{dr_{12}}{d\phi}\Big) - \frac{L^2(M_1+M_2)}{M_1M_2r_{12}^{\cancel{3}}} + \frac{G\,M_1M_2}{\cancel{r_{12}^2}} = 0 ,$$

$$-\frac{L^2(M_1+M_2)}{M_1M_2}\,\frac{d^2}{d\phi^2}\Big(\frac{1}{r_{12}}\Big) - \frac{L^2(M_1+M_2)}{M_1M_2r_{12}} + G\,M_1M_2 = 0 ,$$

$$\frac{d^2}{d\phi^2}\Big(\frac{1}{r_{12}}\Big) + \frac{1}{r_{12}} = \frac{G\,M_1^2M_2^2}{L^2(M_1+M_2)} .$$

This is an inhomogeneous linear differential equation. The general solution to the homogeneous equation is

$$\frac{1}{r_{12}} = A\cos\phi + B\sin\phi$$

and a particular solution to the inhomogeneous equation is

$$\frac{1}{r_{12}} = \frac{G\,M_1M_2}{L^2(M_1+M_2)}\,.$$

We choose the initial conditions such that $B = 0$. Then the solution is

$$\frac{1}{r_{12}} = A\cos\phi + \frac{G\,M_1M_2}{L^2(M_1+M_2)} = \frac{G\,M_1M_2}{L^2(M_1+M_2)}\left(1 + \frac{A\,L^2(M_1+M_2)}{G\,M_1M_2}\cos\phi\right),$$

hence

$$r_{12} = \frac{a\,(1-\varepsilon^2)}{1+\varepsilon\cos\phi} \qquad (K1)$$

where

$$a(1-\varepsilon^2) = \frac{L^2(M_1+M_2)}{G\,M_1^2M_2^2} \qquad \text{(E1)}$$

and

$$\varepsilon = \frac{A\,L^2(M_1+M_2)}{G\,M_1^2M_2^2}\,.$$

(K1) is the equation of an ellipse with semi-major axis a and eccentricity ε where A has to be chosen such that $0 \le \varepsilon \le 1$. The equation (K1) gives Kepler's first law for the relative position vector in the centre-of-mass system.

(K1) and (K2) allow us to express the angular momentum as

$$L = \frac{M_1M_2}{(M_1+M_2)}\,r_{12}^2\dot{\phi} = \frac{M_1M_2}{(M_1+M_2)}\,\frac{a^2(1-\varepsilon)^2}{(1+\varepsilon\cos\phi)^2}\,\dot{\phi}\,.$$

Integration over one period results in

$$\int_0^T L\,dt = \frac{M_1M_2}{(M_1+M_2)}\,a^2(1-\varepsilon)^2\int_0^{2\pi}\frac{d\phi}{(1+\varepsilon\cos\phi)^2}$$

$$= \frac{M_1M_2}{(M_1+M_2)}\,a^2(1-\varepsilon)^2\,\frac{2\,\pi}{(1-\varepsilon^2)^{3/2}}\,,$$

$$L\,T = \frac{M_1M_2a^2\,2\,\pi\,\sqrt{1-\varepsilon^2}}{(M_1+M_2)}\,. \qquad \text{(E2)}$$

With (E1) we find

$$a\,\cancel{(1-\varepsilon^2)} = \frac{\cancel{(M_1+M_2)}\cancel{M_1^2}\cancel{M_2^2}a^4 4\,\pi^2\cancel{(1-\varepsilon^2)}}{G\,\cancel{M_1^2}\cancel{M_2^2}T^2(M_1+M_2)^{\cancel{2}}}$$

hence

$$\frac{T^2}{a^3} = \frac{4\,\pi^2}{G\,(M_1+M_2)} \qquad (K3)$$

which is Kepler's third law. This allows us to eliminate T from (E2),

$$L = \frac{M_1M_2a^2\,\cancel{2\pi}\sqrt{1-\varepsilon^2}\sqrt{G(M_1+M_2)}}{(M_1+M_2)a^{3/2}\,\cancel{2\pi}} = \frac{M_1M_2\sqrt{a\,G}\sqrt{1-\varepsilon^2}}{\sqrt{M_1+M_2}}\,.$$

As a consequence, (K2) may be rewritten as

$$\dot\phi = \frac{L\,(M_1+M_2)}{M_2M_2r_{12}^2} = \frac{\cancel{M_1}\cancel{M_2}\sqrt{a\,G}\,\sqrt{1-\varepsilon^2}\,(M_1+M_2)(1+\varepsilon\cos\phi)^2}{\sqrt{M_1+M_2}\,\cancel{M_1}\cancel{M_2}a^2(1-\varepsilon^2)^2}\,,$$

hence

$$\dot\phi = \frac{\sqrt{(M_1+M_2)\,G}}{\sqrt{a^3(1-\varepsilon^2)^3}}\,(1+\varepsilon\cos\phi)^2\,. \qquad \text{(E3)}$$

We will make use of this equation in what follows.

We now calculate the mass quadrupole tensor in the co-rotating system, that is, in the body-fixed coordinate system. Then we transform into the non-rotating observer system. At last, the third time-derivative of this mass quadrupole tensor has to be inserted into the radiation formula. From that we can calculate the change of the orbital parameters of the system due to the loss of energy.

We denote quantities in the body-fixed coordinate system by a prime. Then (R1) and (R2) give

$$\vec r\,'_1 = \frac{M_2\,\vec r\,'_{12}}{M_1+M_2}\,,\quad \vec r\,'_2 = -\frac{M_1\,\vec r\,'_{12}}{M_1+M_2}\,,\quad \vec r\,'_{12} = r_{12}\begin{pmatrix}1\\0\\0\end{pmatrix}\,.$$

Calculating the mass quadrupole tensor

$$I'_{k\ell} = \int_{\mathbb{R}^3}\mu(\vec r\,')\,x'_kx'_\ell\,d^3\vec r\,' = \int_{\mathbb{R}^3}\Big(M_1\delta\big(\vec r\,'-\vec r\,'_1\big)+M_2\delta\big(\vec r\,'-\vec r\,'_2\big)\Big)\,x'_k\,x'_\ell\,d^3\vec r\,'$$

yields

$$I'_{11} = M_1\,\frac{M_2^2\,r_{12}^2}{(M_1+M_2)} + M_2\,\frac{M_1^2\,r_{12}^2}{(M_1+M_2)} = \frac{M_1M_2\cancel{(M_1+M_2)}\,r_{12}^2}{(M_1+M_2)^{\cancel{2}}}$$

$$= \frac{M_1M_2a^2(1-\varepsilon^2)^2}{(1+\varepsilon\cos\phi)^2(M_1+M_2)} =: I_1\,.$$

All the other components are equal to zero,

$$\left(I'_{k\ell}\right) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Transformation in the observer system gives

$$\left(I_{k\ell}\right) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_1\cos\phi & I_1\sin\phi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I_1\cos^2\phi & I_1\cos\phi\sin\phi & 0 \\ I_1\cos\phi\sin\phi & I_1\sin^2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{I_1}{2} \begin{pmatrix} 1+\cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & 1-\cos(2\phi) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The traceless part is

$$\left(\mathbb{I}_{k\ell}\right) = \frac{I_1}{2} \begin{pmatrix} 1+\cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & 1-\cos(2\phi) & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{I_1}{\cancel{2}} \cancel{2}$$

$$= \frac{I_1}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \frac{I_1}{2} \begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

For the quadrupole formula we need the expression

$$\dddot{\mathbb{I}}_{k\ell}\,\dddot{\mathbb{I}}^{k\ell} = \left(\dddot{\mathbb{I}}_{11}\right)^2 + \left(\dddot{\mathbb{I}}_{22}\right)^2 + \left(\dddot{\mathbb{I}}_{33}\right)^2 + 2\left(\dddot{\mathbb{I}}_{12}\right)^2$$

$$= \left|\dddot{\mathbb{I}}_{11} + i\,\dddot{\mathbb{I}}_{12}\right|^2 + \left|\dddot{\mathbb{I}}_{22} + i\,\dddot{\mathbb{I}}_{12}\right|^2 + \left(\dddot{\mathbb{I}}_{33}\right)^2$$

$$= \left(\frac{I_1}{6} + \frac{I_1}{2}e^{2i\phi}\right)^{\cdots} \left(\frac{I_1}{6} + \frac{I_1}{2}e^{-2i\phi}\right)^{\cdots} + \left(\frac{I_1}{6} - \frac{I_1}{2}e^{-2i\phi}\right)^{\cdots} \left(\frac{I_1}{6} - \frac{I_1}{2}e^{2i\phi}\right)^{\cdots} + \frac{\left(\dddot{I}_1\right)^2}{9}$$

$$= \frac{2}{36}\left(\dddot{I}_1\right)^2 + \frac{2}{4}\left|\left(I_1 e^{2i\phi}\right)^{\cdots}\right|^2 + \frac{\left(\dddot{I}_1\right)^2}{9} = \frac{1}{6}\left(\dddot{I}_1\right)^2 + \frac{1}{2}\left|\left(I_1 e^{2i\phi}\right)^{\cdots}\right|^2 . \qquad (**)$$

We see that we need to know the third time derivatives of I_1 and of $I_1 e^{2i\phi}$. With the help of (E3) we calculate

$$I_1 = \frac{M_1 M_2 a^2 (1-\varepsilon^2)^2}{\big(M_1+M_2\big)\big(1+\varepsilon\cos\phi\big)^2}$$

$$\dot I_1 = \frac{M_1 M_2 a^2 (1-\varepsilon^2)^2\, 2\,\varepsilon\sin\phi\,\dot\phi}{\big(M_1+M_2\big)\big(1+\varepsilon\cos\phi\big)^3} = \frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}\,\varepsilon\sin\phi}{\sqrt{M_1+M_2}\,\big(1+\varepsilon\cos\phi\big)}\ ,$$

$$\ddot I_1 = \frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}\,\varepsilon\,\big(\cos\phi+\varepsilon\big)\dot\phi}{\sqrt{M_1+M_2}\,\big(1+\varepsilon\cos\phi\big)^2} = \frac{2M_1M_2G\,\varepsilon\,\big(\cos\phi+\varepsilon\big)}{a\,(1-\varepsilon^2)}\ ,$$

$$\dddot I_1 = \frac{-2M_1M_2G\,\varepsilon\sin\phi\,\dot\phi}{a\,(1-\varepsilon^2)}\ ,$$

and

$$\big(I_1e^{2i\phi}\big)^{\cdot} = e^{2i\phi}\big(\dot I_1 - 2i\dot\phi I_1\big)$$

$$= e^{2i\phi}\left(\frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}\,\varepsilon\sin\phi}{\sqrt{M_1+M_2}\,\big(1+\varepsilon\cos\phi\big)} + \frac{2iM_1M_2a^2(1-\varepsilon^2)^2\sqrt{G(M_1+M_2)}\,\cancel{\big(1+\varepsilon\cos\phi\big)^2}}{(M_1+M_2)\,\cancel{\big(1+\varepsilon\cos\phi\big)^2}\,\sqrt{a^3(1-\varepsilon^2)^3}}\right)$$

$$= \frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}}{\sqrt{M_1+M_2}}\,e^{2i\phi}\left(\frac{\varepsilon\sin\phi}{1+\varepsilon\cos\phi} + i\right)$$

$$\big(I_1e^{2i\phi}\big)^{\cdot\cdot} = \frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}}{\sqrt{M_1+M_2}}\,e^{2i\phi}\,\dot\phi\,\left(\frac{\varepsilon\big(\cos\phi+\varepsilon\big)}{\big(1+\varepsilon\cos\phi\big)^2} + \frac{2\,i\,\varepsilon\sin\phi}{1+\varepsilon\cos\phi} - 2\right)$$

$$= \frac{2M_1M_2\sqrt{Ga(1-\varepsilon^2)}\sqrt{G}\,\cancel{\sqrt{M_1+M_2}}\,\big(1+\varepsilon\cos\phi\big)^2}{\cancel{\sqrt{M_1+M_2}}\,\sqrt{a^3(1-\varepsilon^2)^3}}\,e^{2i\phi}\left(\frac{\varepsilon\big(\cos\phi+\varepsilon\big)}{\big(1+\varepsilon\cos\phi\big)^2} + \frac{2\,i\,\varepsilon\sin\phi}{1+\varepsilon\cos\phi} - 2\right)$$

$$= \frac{2M_1M_2G}{a(1-\varepsilon^2)}\,e^{2i\phi}\,\Big(\varepsilon\cos\phi + \varepsilon^2 - 2(1+\varepsilon\cos\phi)^2 + 2\,i\,\varepsilon\sin\phi(1+\varepsilon\cos\phi)\Big)$$

$$\big(I_1e^{2i\phi}\big)^{\cdot\cdot\cdot} = \frac{2M_1M_2G}{a(1-\varepsilon^2)}\,e^{2i\phi}\,\dot\phi\,\Big(-\varepsilon\sin\phi + \cancel{4\,\varepsilon\sin\phi\big(1+\varepsilon\cos\phi\big)} + 2\,i\,\varepsilon\cos\phi\big(1+\varepsilon\cos\phi\big)$$

$$-2\,i\,\varepsilon^2\sin^2\phi + 2\,i\,\varepsilon\cos\phi + 2\,i\,\varepsilon^2 - 4\,i\big(1+\varepsilon\cos\phi\big)^2 - \cancel{4\,\varepsilon\sin\phi\big(1+\varepsilon\cos\phi\big)}\Big)$$

$$= \frac{2M_1M_2G}{a(1-\varepsilon^2)}\,e^{2i\phi}\,\dot\phi\,\Big(-\varepsilon\sin\phi - 4\,i\,\big(1+\varepsilon\cos\phi\big)\Big)\ .$$

Upon inserting these results into $(**)$, we find

$$\dddot{\mathbb{I}}_{jk}\,\dddot{\mathbb{I}}^{jk} = \frac{4M_1^2M_2^2G^2\varepsilon^2\sin^2\!\phi\,\dot{\phi}^2}{6\,a^2(1-\varepsilon^2)^2} + \frac{2M_1^2M_2^2G^2\dot{\phi}^2}{a^2(1-\varepsilon^2)^2}\Big(\varepsilon^2\sin^2\!\phi + 16\big(1+\varepsilon\cos\phi\big)^2\Big)$$

$$= \frac{2M_1^2M_2^2G^2\dot{\phi}^2}{3\,a^2(1-\varepsilon^2)^2}\Big(\varepsilon^2\sin^2\!\phi + 3\,\varepsilon^2\sin^2\!\phi + 48\big(1+\varepsilon\cos\phi\big)^2\Big)$$

$$= \frac{8M_1^2M_2^2G^2}{3\,a^2(1-\varepsilon^2)^2}\,\frac{\sqrt{G(M_1+M_2)}\,\dot{\phi}\,\big(1+\varepsilon\cos\phi\big)^2}{\sqrt{a^3(1-\varepsilon^2)^3}}\Big(\varepsilon^2\sin^2\!\phi + 12\big(1+\varepsilon\cos\phi\big)^2\Big)$$

Time averaging over a period yields

$$\langle\dddot{\mathbb{I}}_{jk}\,\dddot{\mathbb{I}}^{jk}\rangle \;=$$

$$= \frac{8M_1^2M_2^2\sqrt{G^5(M_1+M_2)}}{3\,\sqrt{a^7(1-\varepsilon^2)^7}}\frac{1}{T}\int_0^T \big(1+\varepsilon\cos\phi\big)^2\Big(\varepsilon^2\big(1-\cos^2\!\phi\big) + 12\big(1+\varepsilon\cos\phi\big)^2\Big)\,\dot{\phi}\,dt$$

$$= \frac{8M_1^2M_2^2\sqrt{G^5(M_1+M_2)}}{3\,\sqrt{a^7(1-\varepsilon^2)^7}}\,\frac{\sqrt{G(M_1+M_2)}}{2\,\pi\,\sqrt{a^3}}\int_0^{2\pi}\Big(\varepsilon^2 + 12 + \big(71\varepsilon^2+\varepsilon^4\big)\cos^2\!\phi + 11\varepsilon^4\cos^4\!\phi\Big)d\phi\;.$$

In the last step we have used Kepler's third law, and we have considered only even powers of $\cos\phi$ under the integral because odd powers of $\cos\phi$ give zero when integrated over a full period. Upon calculating the integrals we find

$$\langle\dddot{\mathbb{I}}_{jk}\,\dddot{\mathbb{I}}^{jk}\rangle \;=\; \frac{4M_1^2M_2^2G^3(M_1+M_2)}{3\,a^5(1-\varepsilon^2)^{7/2}\cancel{\pi}}\Big((\varepsilon^2+12)2\cancel{\pi} + \big(71\varepsilon^2+\varepsilon^4\big)\cancel{\pi} \,+\, 11\varepsilon^4\frac{3\cancel{\pi}}{4}\Big)$$

$$= \frac{32M_1^2M_2^2G^3(M_1+M_2)}{a^5(1-\varepsilon^2)^{7/2}}\Big(1\,+\,\frac{73\varepsilon^2}{24}+\frac{37\varepsilon^4}{96}\Big)\;.$$

The power that is radiated through a sphere of (big) radius r at time t is given, to within our approximations, by Einstein's quadrupole formula

$$P(t,r) = \frac{G}{5c^5}\langle\dddot{\mathbb{I}}_{jk}\,\dddot{\mathbb{I}}^{jk}\rangle\,\Big(t-\frac{r}{c}\Big)$$

$$= \frac{32M_1^2M_2^2G^4(M_1+M_2)}{5c^5a^5(1-\varepsilon^2)^{7/2}}\Big(1\,+\,\frac{73\varepsilon^2}{24}+\frac{37\varepsilon^4}{96}\Big)\,\Big(t-\frac{r}{c}\Big)\;.$$

Note that G^4/c^5 is a very small number in conventional units. As the masses are in the numerator and the semi-major axis is in the denominator, the latter even with a power of 5, we see that a measurable effect can be expected only for compact binaries that are close together, i.e., not for main sequence stars or planets, but for neutron stars or black holes. The eccentricity has an influence on the order of magnitude only if it is extremely close to 1.

In the last equation for $P(t,r)$ we have to assume that the orbital parameters a and ε vary (slowly) with time and that they have to be taken on the right-hand side at the retarded time: As nobody feeds energy from the outside into a binary system, the energy conservation law $\partial^\mu(T_{\mu\nu}+t_{\mu\nu})$ (recall Worksheet 3) can be satisfied only if the radiated energy is compensated for by the fact that the binary system loses energy. If we exclude the possibility that (rest) mass might be converted into some other kind of energy (i.e., that M_1 or M_2 might change), the time-dependence must be in a and/or ε.
In order to specify this time-dependence we have to calculate how the energy of the binary depends on the orbital parameters. From the Lagrangian given above we can read the expressions for kinetic and potential energy which gives us the total energy as

$$E=\frac{M_1M_2}{2(M_1+M_2)}\big(\dot r_{12}^2+r_{12}^2\dot\phi^2\big)-\frac{GM_1M_2}{r_{12}}\,.$$

With (K1) for r_{12} and (E3) for $\dot\phi$ this can be rewritten as

$$E=\frac{M_1M_2\dot r_{12}^2}{2(M_1+M_2)}+\frac{GM_1M_2a(1-\varepsilon^2)}{2r_{12}^2}-\frac{GM_1M_2}{r_{12}}\,.$$

Evaluating this equation at the periastron, $\dot r_{12}=0$ and $r_{12}=a(1+\varepsilon)$, yields

$$E=\frac{GM_1M_2a(1-\varepsilon^2)}{2a^2(1+\varepsilon)^2}-\frac{GM_1M_2}{a(1+\varepsilon)}=\frac{GM_1M_2}{a}\Big(\frac{\cancel{(1+\varepsilon)}(1-\varepsilon)}{2(1+\varepsilon)^{\cancel{2}}}-\frac{1}{1+\varepsilon}\Big)=\frac{-\,G\,M_1M_2}{2\,a}\,.$$

So we see that the energy depends on a but not on ε, hence the assumed (slow) time-variation of E corresponds to a (slow) time-variation of a according to

$$\frac{dE(t)}{dt}=\frac{GM_1M_2}{2\,a(t)^2}\,\frac{da(t)}{dt}\,.$$

As a consequence, the equation

$$P(t,r)=-\frac{dE}{dt}\Big(t-\frac{r}{c}\Big)$$

gives us the time-dependence of the semi-major axis

$$\frac{da(t)}{dt}=\frac{2\,a(t)^2}{GM_1M_2}\frac{dE(t)}{dt}=\frac{-64M_1M_2G^3(M_1+M_2)}{5\,c^5a(t)^3(1-\varepsilon(t)^2)^{7/2}}\Big(1+\frac{73\varepsilon(t)^2}{24}+\frac{37\varepsilon(t)^4}{96}\Big)\,.\qquad\text{(D1)}$$

Here we have to face the problem that not only the semi-major axis a but also the eccentricity ε changes (slowly) over time, so we have to combine the last equation with a first-order differential equation for ε to get a well-posed initial-value problem. For deriving this differential equation for ε one has to determine the loss of *angular momentum* of the system. The calculation is analogous to the calculation of the loss of energy, but even more tedious, and will not be given here. One finds that

$$-\frac{dL}{dt}=\frac{32M_1^2M_2^2G^{7/2}(M_1+M_2)^{1/2}}{5\,c^5a^{7/2}(1-\varepsilon^2)^2}\Big(1+\frac{7\varepsilon^2}{8}\Big)\,.$$

The formula for dE/dt was derived by P. C. Peters and J. Mathews [Phys. Rev. 131, 435 (1963)] and the one for dL/dt by P. C. Peters [Phys. Rev. 136, 1224 (1964)]. We will use the formula for dL/dt for deriving the differential equation for the eccentricity in Worksheet 5. We will see that an orbit that is initially circular (i.e., $\varepsilon = 0$ at time $t = 0$) will remain circular (i.e., $\varepsilon = 0$ for all times t). Anticipating this result, we will consider circular orbits in the rest of this section. They will give us a good idea about the time scales over which the size of the orbit and the period change, even for eccentric orbits as long as the eccentricity is not extremely close to 1.

For a circular orbit we have

$$r_{12} = a\,, \quad \dot{\phi} = \frac{\sqrt{G(M_1 + M_2)}}{a^{3/2}} =: \Omega\,.$$

Equation (D1), which gives the (slow) variation of the semi-major axis, simplifies to

$$a(t)^3 \frac{da(t)}{dt} = -\frac{64\, G^3 M_1 M_2 (M_1 + M_2)}{5\, c^5} =: -A$$

which can be easily integrated,

$$\frac{1}{4} \frac{da(t)^4}{dt} = -A\,, \quad a(t)^4 = a_0^4 - 4\,A\,t$$

where $a_0 = a(0)$. This can be rewritten, if we introduce the *inspiral time*

$$t_{\mathrm{sp}} = \frac{a_0^4}{4\,A} = \frac{5\,c^5 a_0^4}{256\, G^3 M_1 M_2 (M_1 + M_2)}\,,$$

as

$$a(t) = a_0 \left(1 - \frac{t}{t_{\mathrm{sp}}} \right)\,. \qquad \text{(D2)}$$

t_{sp} is the time the system needs to inspiral completely. Of course, for the final stage of the inspiralling process our simple model of two Newtonian point masses is no longer valid: Neutron stars would deform each other when their surfaces come very close together, and in the case of black holes the horizons would merge. Nonetheless, t_{sp} gives a good estimate of the lifetime of binary systems. For a typical system of two neutron stars, M_1 and M_2 are in the order of a Solar mass and a_0 is in the order of a few Solar radii. In such cases the inspiral time equals several hundred million years, see the next section and Worksheet 5.

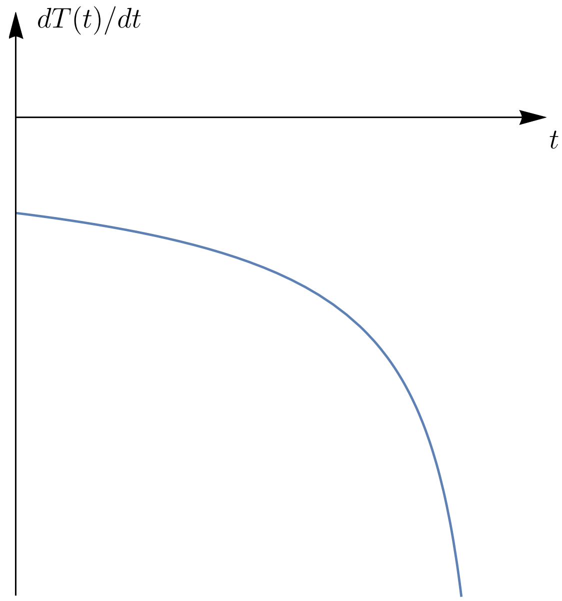
If we combine (D2) for the decay of the radius with Kepler's third law (K3) we get an equation for the (slow) time variation of the period T,

$$T(t) = \frac{2\,\pi\, a(t)^{3/2}}{\sqrt{G\,(M_1 + M_2)}} = \frac{2\,\pi\, a_0^{3/2}}{\sqrt{G\,(M_1 + M_2)}} \left(1 - \frac{t}{t_{\mathrm{sp}}} \right)^{3/8}$$

which implies

$$\frac{dT(t)}{dt} = -\frac{3\,\pi\, a_0^{3/2}}{4\,\sqrt{G\,(M_1 + M_2)}\, t_{\mathrm{sp}}} \left(1 - \frac{t}{t_{\mathrm{sp}}} \right)^{-5/8}\,.$$

The quantity $dT(t)/dt$ is plotted against t in the diagram below. A similar time-dependence of the period holds for eccentric orbits. It was this time-dependence that was observed with the Hulse-Taylor pulsar, see the next section.



The period T immediately gives the frequency

$$\Omega(t) = \frac{2\,\pi}{T(t)} = \frac{\sqrt{G\,(M_1+M_2)}}{a_0^{3/2}}\left(1-\frac{t}{t_{\rm sp}}\right)^{-3/8} . \tag{D3}$$

The time derivative of the frequency,

$$\frac{d\Omega(t)}{dt} = \frac{3\,\sqrt{G\,(M_1+M_2)}}{8\,a_0^{3/2}t_{\rm sp}}\left(1-\frac{t}{t_{\rm sp}}\right)^{-11/8} ,$$

is often called the *chirp*. Obviously, this refers to the analogy to sound waves where a positive time derivative of the frequency corresponds to an increasing pitch.

For circular orbits, the quadrupole moment can be written as

$$\big(I_{k\ell}(t)\big) = \frac{I_1}{2}\begin{pmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1-\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad I_1 = \frac{M_1 M_2\, a^2}{M_1+M_2} .$$

In this case it is easy to calculate the second time derivative,

$$\big(\dot I_{k\ell}(t)\big) = I_1\Omega\begin{pmatrix} -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\big(\ddot I_{k\ell}(t)\big) = 2\,I_1\Omega^2\begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Thereupon, we find the gravitational field in the far zone

$$\big(\gamma_{k\ell}(t,\vec{r}\,)\big) = \frac{2\,G}{c^4 r}\big(\ddot{I}_{k\ell}(t-r/c)\big)$$

$$= \frac{4\,G\,M_1 M_2 a^2}{c^4 r\,(M_1+M_2)}\,\frac{4\,\pi^2}{T^2}\begin{pmatrix} -\cos\big(2\,\Omega\,(t-r/c)\big) & -\sin\big(2\,\Omega\,(t-r/c)\big) & 0 \\ -\sin\big(2\,\Omega\,(t-r/c)\big) & \cos\big(2\,\Omega\,(t-r/c)\big) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With Kepler's third law (K3) this can be rewritten as

$$\big(\gamma_{k\ell}(t,\vec{r}\,)\big) = \frac{4\,G^2 M_1 M_2}{c^4 r\,a}\begin{pmatrix} -\cos\big(2\,\Omega\,(t-r/c)\big) & -\sin\big(2\,\Omega\,(t-r/c)\big) & 0 \\ -\sin\big(2\,\Omega\,(t-r/c)\big) & \cos\big(2\,\Omega\,(t-r/c)\big) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For an observer on the 3-axis,

$$\vec{r} = r\,\vec{n}\,, \quad \vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we verify that obviously $\gamma_{k\ell} = \gamma_{k\ell}^{TT}$ in the notation used on p. 25. For such an observer we can thus immediately apply our results for waves in the TT gauge. We see that the wave is the superposition of a plus mode

$$-\frac{4\,G^2 M_1 M_2}{c^4 r\,a}\cos\big(2\Omega t - 2\Omega r/c\big)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and a cross mode

$$-\frac{4\,G^2 M_1 M_2}{c^4 r\,a}\cos\big(2\Omega t - 2\Omega r/c - \pi/2\big)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Frequency and wave vector are given, for both modes, by

$$\omega = 2\,\Omega\,, \quad \vec{k} = \frac{2\,\Omega}{c}\,\vec{n}\,,$$

and the amplitudes are

$$A_+ = -\frac{4\,G^2 M_1 M_2}{c^4 r\,a}\,, \quad A_\times = A_+ e^{-i\pi/2}\,.$$

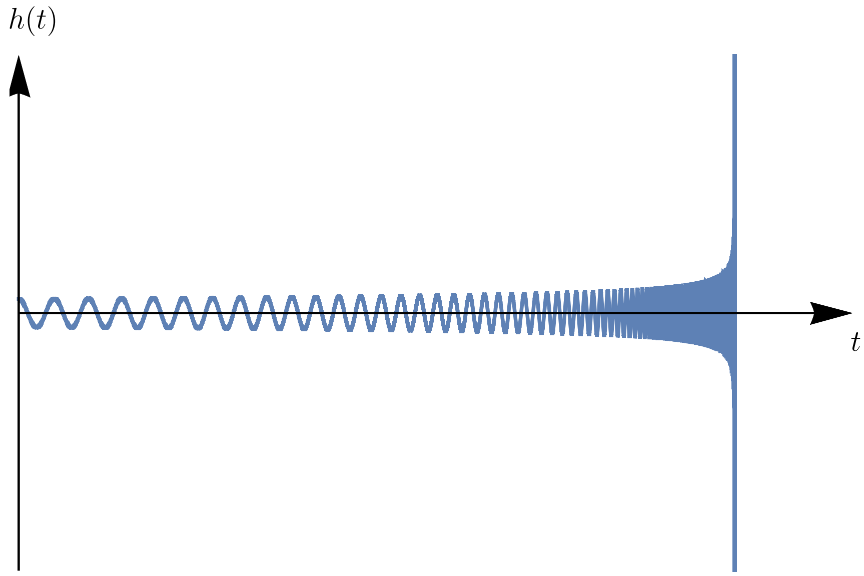
If we take the (slow) time variation of a and Ω into account, we see that both modes give a signal at the position of the observer that varies with time according to

$$h(t) = -\frac{4\,G^2 M_1 M_2}{c^4 r\,a(t)}\cos\big(2\Omega(t)t + \psi\big)$$

where ψ is time-independent. With $a(t)$ and $\Omega(t)$ from (D2) and (D3), respectively, this can be rewritten as

$$h(t) = h_0\big(1 - t/t_{\rm sp}\big)^{-1/4}\cos\!\big(\beta\,(1 - t/t_{\rm sp})^{-3/8}t + \psi\big)$$

where h_0 and β are time-independent. This is a typical *chirp signal*, see plot below: Both the amplitude and the frequency grow with time, i.e., in the acoustical analogy the sound becomes louder and higher pitched. In our over-idealised setting, both the amplitude and the frequency even go to infinity if t approaches $t_{\rm sp}$.

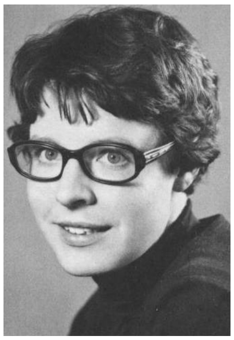


For realistically calculating the final stage of the *merger* of two neutron stars, or of two black holes, it is of course necessary to go beyond our over-idealised model of two Newtonian point particles. Then, instead of growing to infinity, the gravitational wave signal shows a *ring-down* until the system settles in an (approximately) stationary state. The usual method of calculating such mergers is to use the Newtonian equations we have derived here as a zeroth-order approximation and then to add post-Newtonian (PN) correction terms up to a certain order. Here one speaks of the "$k^{\rm th}$ order PN approximation" if one takes all terms up to order c^{-2k} into account. As odd powers of c do occur, k is in general a half-integer. PN calculations have been worked out for spinning binaries up to order 3.5 which is extremely challenging. These analytical approximation methods are complemented by numerical studies. The templates that have been calculated for gravitational waves produced by coalescing black holes or neutron stars with prescribed properties are the results of such combined efforts.

3.7 Indirect evidence for gravitational waves from binary pulsars

Before coming to binary pulsars, we will briefly recall what pulsars are and how they were discovered.

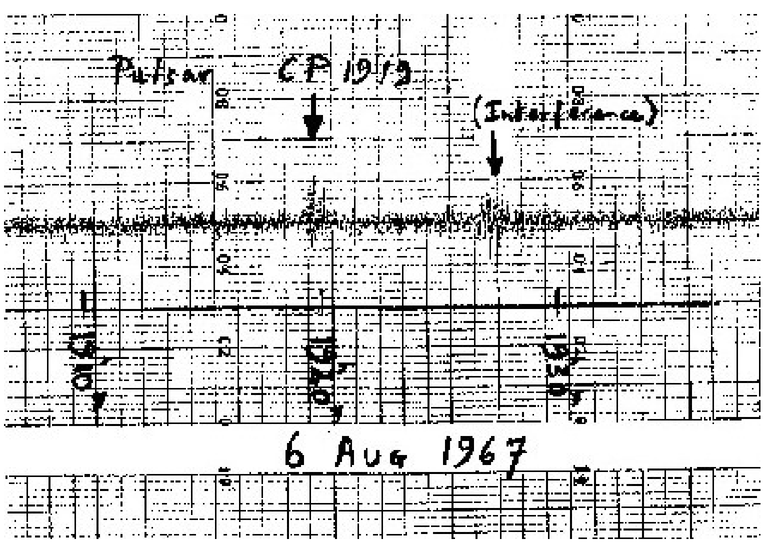
Pulsars were discovered in 1967 by Jocelyn Bell, later Jocelyn Bell-Burnell, then a PhD student in the group of Antony Hewish at Cambridge University. The picture below shows Hewish in front of the do-it-yourself radio telescope with which the discovery was made.



Jocelyn Bell (1941 -)
from cwp.library.ucla.edu/

Antony Hewish (1924 -)
from www.mrao.cam.ac.uk/

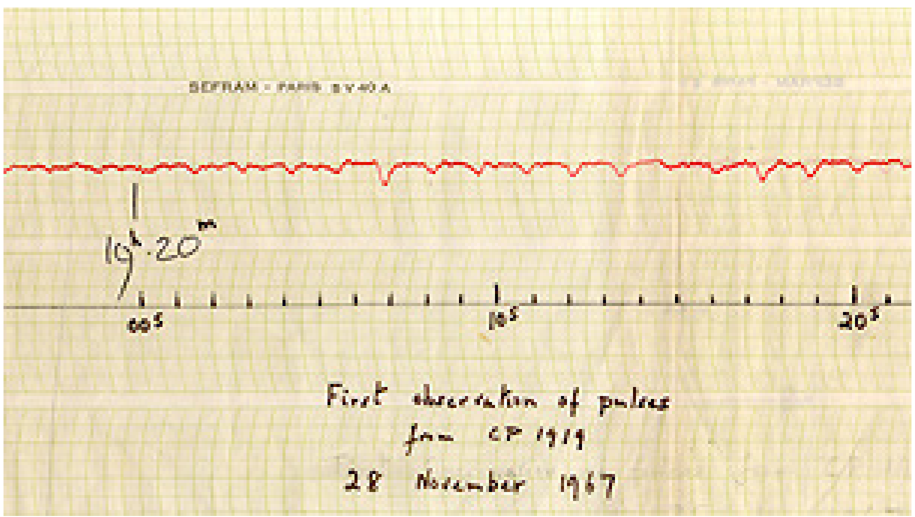
After having constructed the radio telescope, together with other students, with her own hands, Jocelyn Bell concentrated in her PhD work on the search for quasars with the scintillation method. The observation was often affected by interferences caused by terrestrial sources such as cars. On 6 August 1967 Jocelyn Bell observed some "scruff", as she later put it, that appeared to be different from these usual interferences, see picture below. She discussed the observation with her supervisor. After having verified that the source remained fixed with respect to the stars it seemed certain that it was an astronomical object. Hewish and Bell decided to look at it more closely.



Discovery of the mysterious signal on 6 August 1967
from pulsar.ca.astro.it/

On 28 November 1967 Jocelyn Bell observed the mysterious object at a different frequency with a higher time resolution, see the picture below. It showed highly regular pulses with a period of 1.337 seconds. Whereas the shape of the pulses changed considerably, the period remained stable with an incredible accuracy. It was seriously discussed in the group whether the signal could come from an alien civilisation, and it was only half-jocular that the object was initially called LGM-1, with LGM standing for *Little Green Men*. Later, the object was given the systematic name PSR B1919+21. Here PSR stands for *Pulsating Source of Radio emission*, which was soon abbreviated as *pulsar* and the numbers give the celestial coordinates of the source, a point in the constellation Vulpecula: $19^{\rm h}19^{\rm m}$ is the right ascension and $+21^{\rm o}$ is the declination; the letter B is added for coordinates refering to the epoch 1950 while a letter J is added for the epoch 2000.

Within a few weeks the Cambridge group found three more similar objects. In early 1968, they published their observations, see A. Hewish, J. Bell, J. Pilkington, P. Scott and R. Collins ["Observation of a rapidly pulsating radio source" Nature 217, 709 (1968)].

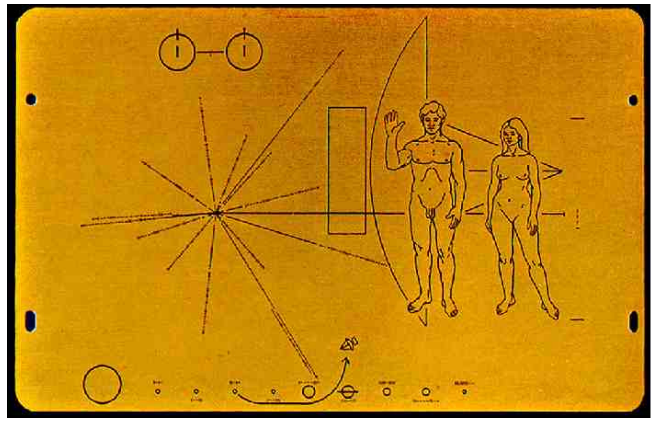


Signals from the pulsar PSR B1919+21 on 28 November 1968

from www.bbc.co.uk/

Passionate discussions started about the nature of the radiation. A majority, including Hewish, first thought that it might come from radial oscillations of a white dwarf. However, it turned out that not even a white dwarf, let alone a main sequence star, could perform oscillations with such a high frequency. After about a year, it was the prevailing opinion that the radiation comes from a rotating neutron star. Thomas Gold was the first to suggest such a model in 1968 [T. Gold: "Rotating neutron stars as the origin of the pulsating radio sources" Nature 218, 731-732 (1968)], which was initially ridiculed by many astrophysicists. The idea was that the neutron star has a magnetic field that is not aligned with the rotation axis. Radiation is emitted in a cone around the magnetic field axis, and this cone rotates like the beacon of a lighthouse. The observer registers a pulse whenever the cone hits the Earth. Neutron stars had been introduced, as a theoretical possibility, in 1934 by Walter Baade and Fritz Zwicky, but up to the discovery of pulsars there was no indication that they actually exist in Nature. An animation of the lighthouse model can be found in Section 2.1 of D. Lorimer ["Binary and Millisecond Pulsars", Living Rev. Relativity 11, (2008), http://www.livingreviews.org/lrr-2008-8].

Within a few years after the discovery of PSR B1919+21, several dozens of pulsars were found. 14 of them are shown in the plaques that are on board the spacecraft Pioneer 10 and 11. They were launched in 1972 and 1973 and are the first spacecraft to leave the Solar system. The positions of the pulsars are shown, relative to the Earth, in the diagram in the left part of the plaque, see picture. This should tell an extraterrestrial civilisation where the spacecraft came from, in case that Pioneer 10 or 11 is intercepted by them.



Plaque on board Pioneer 10 and 11
from en.wikipedia.org

The best known example of a pulsar is the neutron star at the centre of the Crab Nebula. It is the remnant of a supernova that was observed from the Earth in 1054. It is also visible in the optical and X-ray parts of the spectrum. There are also some pulsars that emit gamma rays, e.g. the Vela pulsar.

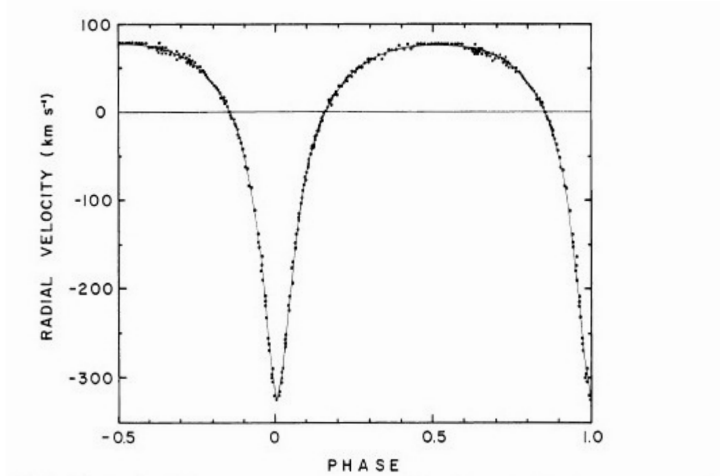
In 1974 Hewish received the Nobel Prize for the discovery of pulsars. Some people thought that it would have been fair if Jocelyn Bell had shared the prize. By now more than 2000 radio pulsars are known. Most of them are within our galaxy, but there are also a few in the Magellanic Clouds. The periods vary from a few milliseconds to about 10 seconds.

After these remarks on pulsars in general, we turn now to binary pulsars. This term refers to binary systems in which at least one partner is a pulsar. About 10 % of all known pulsars have a companion. The first binary pulsar, PSR B1913+16, was discovered in 1974. It was again the work of a PhD student and a supervisor, this time Russell Hulse and Joseph Taylor from Cornell University. In contrast to the earlier story, both were awarded the Nobel Prize in 1993.

Russel Hulse (1950 -) and Joseph Taylor (1941 -)
from th.physik.uni-frankfurt.de/

The discovery was made with the 305-meter Arecibo radio telescope, see picture, which is placed in a natural karst sinkhole and is not movable. PSR B1913+16 is a pulsar with a period of 59 milliseconds. Evidence for the existence of a companion, which is dark and mute, came from the fact that the arrival time of the pulses varied periodically. If this is interpreted as a Doppler effect, it means that the pulsar is moving towards us, then away from us, then again towards us, and so on.

The Arecibo telescope in Puerto Rico
from en.wikipedia.org



from Hulse and Taylor, loc. cit.

The plot of the radial velocity above is taken from the original paper by R.Hulse and J. Taylor ["Discovery of a pulsar in a binary system" Astrophys. J. 195, L51 (1975)]. After correcting for the motion of the Earth, for dispersion in the intergalactic medium and for other effects, Hulse and Taylor fitted the observed time dependence of the radial velocity to a Kepler orbit. There is a certain degeneracy, i.e., not all orbital elements can be uniquely determined, but the following parameters of the system were found. The numbers are taken, again, from the paper by Hulse and Taylor but adapted to our notation.

T	7.75 hours
ε	0.61
$\dfrac{a\,M_2}{M_1+M_2}\sin i$	$1.0\,R_\odot$
$\dfrac{(M_2 \sin i\,)^3}{(M_1+M_2)^2}$	$0.13\,M_\odot$

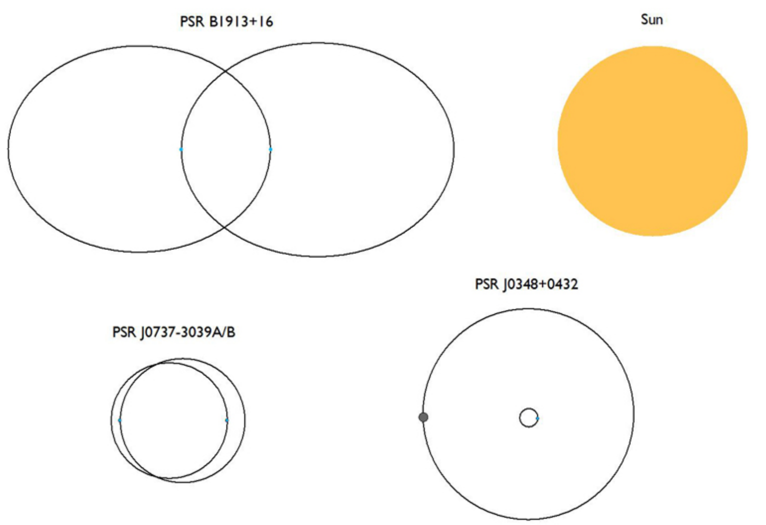
Here an index 1 stands for the pulsar and an index 2 stands for the invisible companion. i is the inclination angle. From the Newtonian analysis one cannot determine the individual masses M_1 and M_2. However, this is possible with the help of relativistic corrections, using the post-Newtonian (PN) approximation which was mentioned alreday at the end of the preceding section. Roughly speaking, this is an expansion in powers of v/c. If relativistic effects are taken into account, in particular the transverse Doppler effect and the gravitational Doppler effect, the individual masses and all orbital parameters can be determined. The method, which was worked out by V. Brumberg, Y. Zeldovich, I. Novikov and N. Shakura ["Determination of the component masses and inclination of a binary system containing a pulsar from relativistic effects", Sov. Astr. Lett. 1, 2 (1975)], is sketched in Straumann's book.

One finds

M_1	$1.44\,M_\odot$
M_2	$1.39\,M_\odot$
a	$2.80\,R_\odot$
i	45^{o}
periastron shift	$4.2\,^{\mathrm{o}}/\mathrm{yr} = 1.7''/T$

For the sake of comparison, we recall that the perihelion shift of Mercury is $43''/\mathrm{cty} = 0.1''/T$. With reference to the orbital period T (7.75 hours in the case of the binary pulsar and 88 days in the case of Mercury), the difference is not very big.

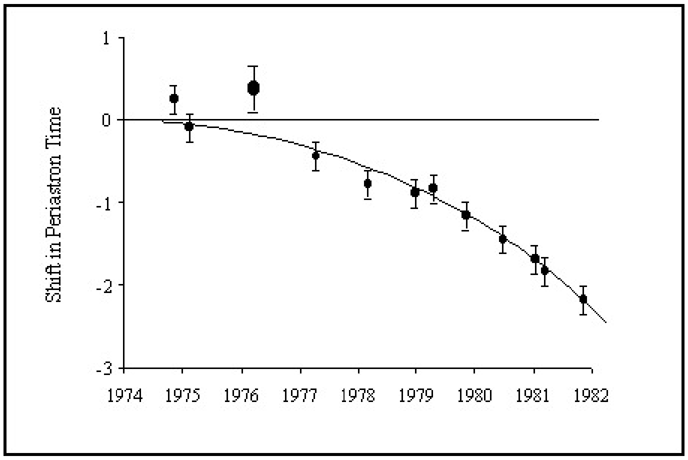
From the orbital elements one finds that at periastron the separation of the two stars is only 1.1 Solar radii, while at apastron it is 4.5 Solar radii. The companion is thought to be a neutron star as well. We do not know the radii of the two stars precisely, but typically neutron stars have radii between 10 and 20 km. The picture below compares the orbits of PSR B1913+16 and two other binary pulsars to the size of the Sun.



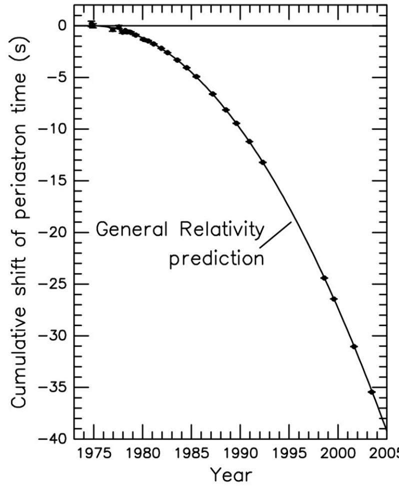
from http://www.usra.edu

From the orbital elements and the two masses we can calculate the inspiralling time t_{sp}. (Recall that we have worked out a formula for t_{sp} only for the case of a circular orbit; however, the binary pulsar's eccentricity is not so close to 1 that it would have a significant impact.) We find that t_{sp} is about 300 million years.

Already in the original Hulse-Taylor paper it is remarked that the system should be a highly promising candidate for testing general relativity. In the above-mentioned paper by Brumberg et al. it was noted that it could provide indirect evidence for the existence of gravitational waves: With the masses and the orbital elements known, one could check if the period T depends on time according to the formula derived from general relativity.



from Taylor and Weisberg, loc. cit.



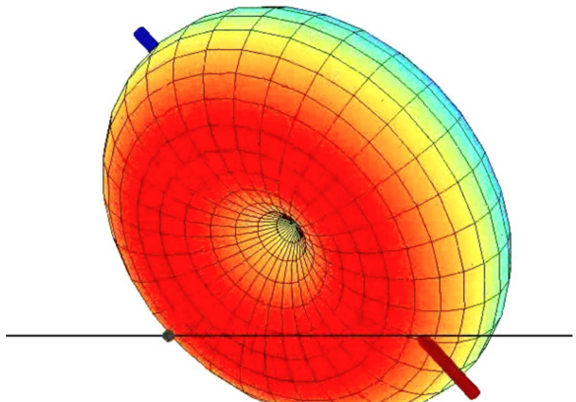
from www.ast.cam.ac.uk

Such a dependence of T on time was reported by J. Taylor, L. Fowler and P. McCulloch ["Measurements of general relativistic effects in the binary pulsar PSR 1913+16" Nature 277, 437 (1979)] and confirmed, on the basis of more data, by J. Taylor and J. Weisberg ["A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16" Astrophys. J. 253, 908 (1982)]. The plot on the left is taken from the latter paper. It clearly shows the decrease of the orbital period. The solid line gives the prediction according to general relativity, on the basis of the determined orbital parameters. In the course of time, the agreement between observation and theory became very impressive, see plot on the right. It is to be emphasised that this is *not* a fit: The curve is calculated with Einstein's quadrupole formula from the orbital elements and the masses as they have been found from the observed Doppler redshifts. There is no free parameter to which the data points could be fitted. This agreement between theoretical prediction and observation is one of the best confirmation of general relativity. It was generally accepted as (indirect) proof of the existence of gravitational waves beyond any reasonable doubt.

After the discovery of the Hulse-Taylor pulsar, several other binary pulsars were detected. They are used on a regular basis for testing general relativity and alternative theories of gravity. Up to now, general relativity has passed all tests with flying colours, whereas severe restrictions have been found for many alternative theories.

In addition to the Hulse-Taylor pulsar, there are some other binary pulsars that deserve special attention.

- In 2003, Marta Burgay et al. found a *double pulsar*, PSR J0737-3039A and PSR J0737-3039B, i.e., a binary system in which both stars are pulsars. This allows for even more precise tests of general relativity. Pulsar A has a period of 23 Milliseconds, pulsar B of 2.8 seconds. The masses are $M_A = 1.34 M_\odot$ and $M_B = 1.25 M_\odot$. The period is only 2.4 hours. Correspondingly, the separation of the two stars is even smaller than for PSR B1913+16 and its companion; the whole system would fit within the Sun. As the orbital plane is seen almost edge-on, there are eclipses. The apparent irregularity of the eclipses caused a puzzle for a while.



from Breton et al., loc. cit.

A model that could solve this puzzle was brought forward by R. Breton, V. Kaspi, M. McLaughlin, M. Lyutikov, M. Kramer, I. Stairs, S. Ransom, R. Ferdman, F. Camilo and A. Possenti ["The double pulsar eclipses. I. Phenomenology and multi-frequency analysis" Astrophys. J. 747, 89, (2012)]. According to this model, pulsar B is surrounded by a doughnut-shaped magnetosphere which, in the course of its rotation, sometimes eclipses pulsar A. The picture above is taken from the paper by Breton et al. Since March 2008 the radio pulses from pulsar B are invisible because, as a consequence of the precession of the spin axis, the beam misses the Earth.

- In 2013, a magnetar (i.e., a neutron star with a very strong magnetic field) was found at an angular distance of only 3 arcseconds from the centre of our galaxy, PSR J1745-2900. Of course, in terms of the Schwarzschild radius of the supermassive black hole at the centre of our galaxy, this is still a fairly large distance; the Schwarzschild radius corresponds to about 10 microarcseconds. Therefore, there is not a strong gravitational coupling of this magnetar to the centre. Finding a pulsar that is in a close orbit around a black hole is considered as the Holy Grail of pulsar research.

- In 2014, a ternary pulsar was discovered, PSR J0337+1715. Both companions are white dwarfs. Already in the late 1990s a pulsar in a triple system had been found, but the separations were quite large with orbital periods of several decades. The newly found system is much closer so that it is a much more promising candidate for additional tests of general relativity.

4 Gravitational wave detectors

In the preceding chapter we have discussed the generation of gravitational waves. As the most important results, we have found that, to within certain approximations, the gravitational field in the far zone is determined by the second time-derivative of the (mass) quadrupole tensor of the source and that the radiated power is determined by the third time-derivative of this quadrupole tensor. In this chapter we will now introduce various types of gravitational wave detectors that have been conceived and we will discuss for what kind of sources they are sensitive.

4.1 Resonant bar detectors

Resonant bar detectors are vibrating systems in which a gravitational wave would excite a resonant oscillation. The idea was brought forward in 1960 by Joseph Weber ["Detection and generation of gravitational waves", Phys. Rev. 117, 306 (1960)]. A few years later, the first resonant bar detectors constructed by Weber went into operation. Some more sophisticated resonant bar detectors are still in use.

To explain the basic idea, we begin by considering the simplest vibrating system that can be used as a gravitational wave detector, namely two masses connected by a spring. This simple example is also treated in the first part of Weber's 1960 paper and it is dicussed in fairly great detail in the book by Misner, Thorne and Wheeler.

We have to recall some of our earlier results. In Worksheet 2 we derived a differential equation for the motion of freely falling particles under the influence of a gravitational wave,

$$\frac{d^2 y^\ell(t)}{c^2 dt^2} = R^\ell{}_{0k0}\big(ct, \vec{0}\big) y^k(t) \, , \tag{J}$$

where the curvature tensor can be expressed as

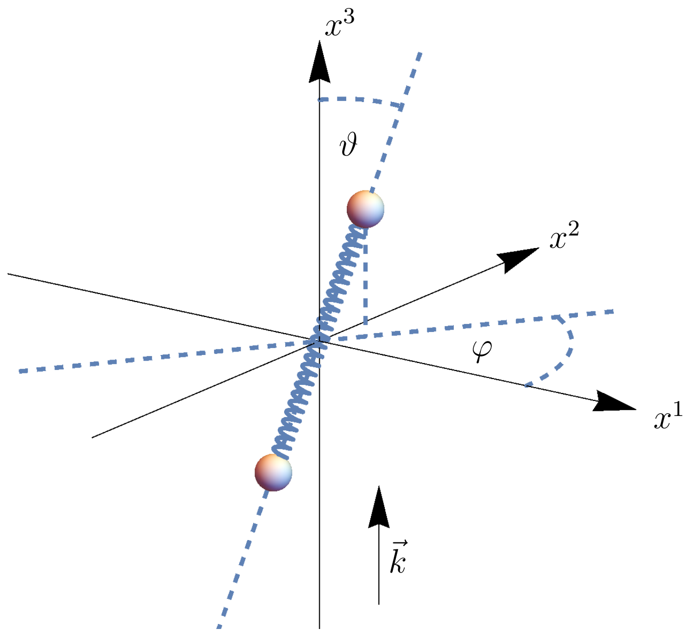
$$R^\ell{}_{0k0}\big(ct, \vec{0}\big) = \frac{1}{2} \partial_0^2 \gamma^\ell{}_k\big(ct, \vec{0}\big) \, .$$

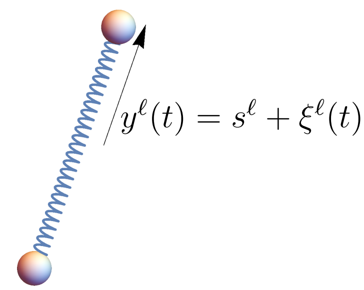
Here $\gamma^\ell{}_k$ is a plane-harmonic gravitational wave in the TT gauge, with the four-velocity u^μ of the chosen observer tangent to the x^0-lines. The coordinates y^k are chosen such that the freely falling particle at $y^k(t)$ has distance $\sqrt{y^k(t) y_k(t)}$ from the freely falling particle at the origin. The differential equation is linearised with respect to $y^k(t)$, i.e., it is valid only as long as this quantity is sufficiently small.

(J) is a version of the Jacobi equation (or equation of geodesic deviation). If looked at with Newtonian eyes, the right-hand side of (J) is to be interpreted as the gravitational force. The solutions to (J) give, for $\gamma^\ell{}_k$ either a plus mode or a cross mode, the familiar patterns from p. 15.

We will now consider a particle with mass m that is acted on by an additional (i.e., non-gravitational) force $f^\ell(t)$. Then we have to replace (J) with the equation of motion

$$\frac{d^2y^\ell(t)}{dt^2} = c^2 {R^\ell}_{0k0}\big(ct,\vec{0}\big)y^k(t) + \frac{1}{m}f^\ell(t)\ .$$





For a system of two masses with $m_1 = m_2 = m$ connected by a spring, the position $y^\ell(t)$ of mass m_1 satisfies this equation with

$$y^\ell(t) = s^\ell + \xi^\ell(t)\,, \qquad f^\ell(t) = -k\,\xi^\ell(t) - \gamma\frac{d\xi^\ell(t)}{dt}\ ,$$

see the figure above. Here s^ℓ gives the position of m_1 in the equilibrium state, $-k\,\xi^\ell(t)$ is the restoring force with a spring constant k, and $-\gamma d\xi^\ell(t)/dt$ is the damping force with a damping constant γ. The equation of motion reads

$$\frac{d^2\xi^\ell(t)}{dt^2} = c^2 {R^\ell}_{0k0}\big(ct,\vec{0}\big)\big(s^k + \xi^k(t)\big) - \frac{k}{m}\xi^\ell(t) - \frac{\gamma}{m}\frac{d\xi^\ell(t)}{dt}\ .$$

If the elongation of the spring from the equilibrium state is small, we can neglect $\xi^k(t)$ in comparison to s^k, i.e.

$$\frac{d^2\xi^\ell(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi^\ell(t)}{dt} + \frac{k}{m}\xi^\ell(t) = c^2 {R^\ell}_{0k0}\big(ct,\vec{0}\big)s^k\ .$$

As given above, we can express the curvature tensor by the second derivative of the ${\gamma^\ell}_k$. With

$${\gamma^\ell}_k\big(ct,\vec{r}\big) = \mathrm{Re}\big\{{A^\ell}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\big\}\ ,$$

$$\partial_0^2{\gamma^\ell}_k\big(ct,\vec{r}\big) = \frac{1}{c^2}\mathrm{Re}\big\{-\omega^2{A^\ell}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\big\}\ ,$$

$$c^2 R^{\ell}{}_{0k0}\big(ct,\vec{0}\big) = \frac{1}{2}\partial_0^2 \gamma^{\ell}{}_{k}\big(ct,\vec{0}\big) \;=\; -\,\frac{\omega^2}{2}\mathrm{Re}\big\{A^{\ell}{}_{k} e^{-i\omega t}\big\}\;.$$

If we assume that the masses at the ends of the spring can be displaced only in the longitudinal direction of the spring, we have

$$\xi^{\ell}(t) = \xi(t)\frac{s^{\ell}}{s}\;,$$

where, according to the figure on p. 50,

$$\big(s^{\ell}\big) = s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\;, \qquad \big(\xi^{\ell}(t)\big) = \xi(t)\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\;.$$

Then the equation of motion reads

$$\frac{s^{\ell}}{s}\Big(\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t)\Big) \;=\; -\,\frac{\omega^2}{2}\mathrm{Re}\big\{A^{\ell}{}_{k}s^{k} e^{-i\omega t}\big\}\;,$$

or, after multiplication with s_ℓ/s,

$$\cancel{\frac{s^{\ell}s_{\ell}}{s^2}}\Big(\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t)\Big) \;=\; -\,\frac{\omega^2}{2}\mathrm{Re}\big\{A^{\ell}{}_{k}s^{k}\frac{s_\ell}{s} e^{-i\omega t}\big\}\;.$$

We evaluate the right-hand side for a pure plus mode. The gravitational wave is assumed to propagate in the x^3 direction, as indicated in the figure on p. 50 by the wave vector $\vec{k}$. We find

$$\big(A^{\ell}{}_{k}\big) = \begin{pmatrix}A_+ & 0 & 0\\ 0 & -A_+ & 0\\ 0 & 0 & 0\end{pmatrix}\;,$$

$$s_\ell A^{\ell}{}_{k}s^{k} = s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}A_+ & 0 & 0\\ 0 & -A_+ & 0\\ 0 & 0 & 0\end{pmatrix} s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

$$= s^2\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}A_+\cos\varphi\sin\vartheta\\ -A_+\sin\varphi\sin\vartheta\\ 0\end{pmatrix} = s^2 A_+\big(\cos^2\!\varphi - \sin^2\!\varphi\big)\sin^2\!\vartheta = s^2 A_+\cos(2\varphi)\sin^2\!\vartheta\,.$$

This gives us the equation of motion in its final form,

$$\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t) = -\,\frac{s\,\omega^2}{2}\,A_+\cos(2\varphi)\sin^2\!\vartheta\cos(\omega t)\;,$$

which is the equation of a one-dimensional damped harmonic oscillator with a driving force. In the last step we have assumed that A_+ is real. This is no loss of generality because a non-zero phase of A_+ can be compensated for by a shift of the zero on the time axis.

Solving this equation is an elementary text-book matter. The general solution to the inhomogeneous ODE is the general solution of the homogeneous ODE plus a particular solution to the inhomogeneous ODE.

To solve the homogeneous ODE,

$$\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t) = 0 \, ,$$

we insert the ansatz

$$\xi(t) = C\, e^{\kappa t} \, .$$

This gives us

$$C\, e^{\kappa t}\Big(\kappa^2 + \frac{\gamma\,\kappa}{m} + \frac{k\,\kappa}{m}\Big) = 0 \, ,$$

hence

$$\kappa_{1/2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}} \, .$$

As long as the damping is undercritical,

$$0 < \frac{k}{m} - \frac{\gamma^2}{4m^2} =: \omega_0^2 \, ,$$

we have

$$\kappa_{1/2} = -\frac{\gamma}{2m} \pm i\,\omega_0 \, ,$$

and the general solution to the homogeneous equation is

$$\xi_{\text{hom}} = C_1 e^{\kappa_1 t} + C_2 e^{\kappa_2 t} = e^{-\gamma t/(2m)}\big(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\big) \, .$$

C_1 and C_2 are determined by initial conditions. Whatever the initial conditions are, the solution dies down in the course of time.

We have now to find one particular solution to the inhomogeneous equation

$$\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t) = -\frac{s\,\omega^2}{2}\cos(2\varphi)\sin^2\!\vartheta\,\mathrm{Re}\big\{A_+\, e^{-i\omega t}\big\} \, .$$

With the ansatz

$$\xi(t) = \mathrm{Re}\big\{a\, e^{-i\omega t}\big\}$$

we get

$$\mathrm{Re}\Big\{a\, e^{-i\omega t}\Big(-\omega^2 - \frac{i\omega\gamma}{m} + \frac{k}{m}\Big)\Big\} = -\frac{s\,\omega^2}{2}\cos(2\varphi)\sin^2\!\vartheta\,\mathrm{Re}\big\{A_+\, e^{-i\omega t}\big\} \, ,$$

$$\mathrm{Re}\Big\{e^{-i\omega t}\Big(a\Big(\omega^2 - \frac{k}{m} + \frac{i\,\omega\,\gamma}{m}\Big) - \frac{s\,\omega^2 A_+}{2}\cos(2\varphi)\sin^2\!\vartheta\Big)\Big\} = 0 \, ,$$

$$a = \frac{s\,\omega^2 A_+ \cos(2\varphi)\sin^2\!\vartheta}{\omega^2 - \frac{k}{m} + \frac{i\,\omega\,\gamma}{m}} \, .$$

Therefore, if we wait until the solution to the homogeneous equation has died down, the oscillation of our spring system driven by the gravitational wave is given by

$$\xi(t) = \mathrm{Re}\big\{a\, e^{-i\omega t}\big\} = s\,\omega^2\cos(2\varphi)\sin^2\!\vartheta\,\mathrm{Re}\Bigg\{\frac{A_+\, e^{-i\omega t}}{\omega^2 - \frac{k}{m} + \frac{i\,\omega\,\gamma}{m}}\Bigg\} \, .$$

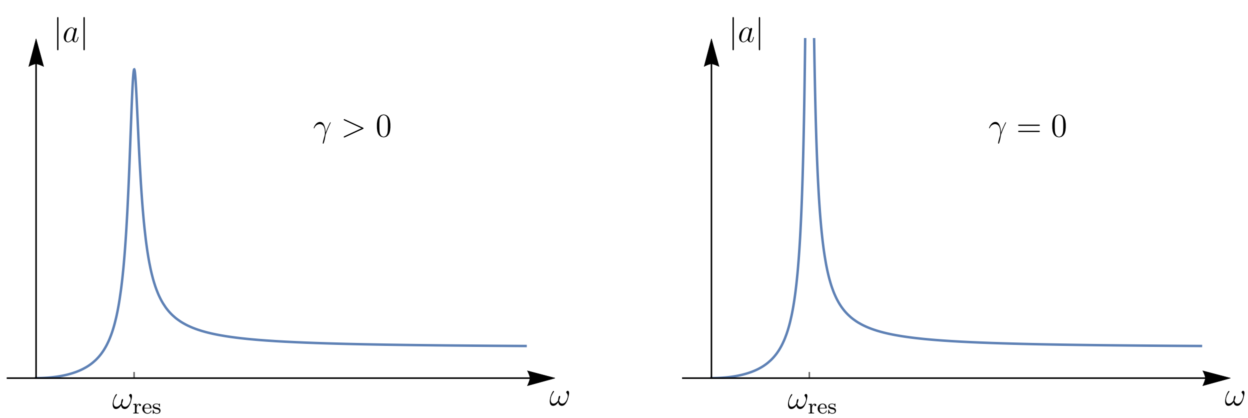
The amplitude

$$|a| = \frac{s\,|A_+|\,|\cos(2\varphi)|\,\sin^2\vartheta\,\omega^2}{\left|\omega^2 - \dfrac{k}{m} + \dfrac{i\,\omega\,\gamma}{m}\right|} = \frac{s\,|A_+|\,|\cos(2\varphi)|\,\sin^2\vartheta\,\omega^2}{\sqrt{\left(\omega^2 - \dfrac{k}{m}\right)^2 + \dfrac{\omega^2\,\gamma^2}{m^2}}}$$

takes, as a function of ω, its maximum at the *resonance frequency*

$$\omega_{\mathrm{res}} = \frac{k}{m}\left(\frac{k}{m} - \frac{\gamma^2}{4m^2}\right)^{-1/2}.$$

In the case of vanishing damping, $\gamma = 0$, the amplitude is even infinite at $\omega_{\mathrm{res}} = \sqrt{k/m}$, see the diagram.



The optimal orientation of the spring is transverse to the direction of the incoming gravitational wave, $\sin^2\vartheta = 1$. In the case of longitudinal orientation, $\sin^2\vartheta = 0$, the amplitude is zero. With respect to the φ dependence, which gives the orientation in the plane perpendicular to the propagation direction of the wave, there is not only a 2π periodicity but even a π periodicity. This reflects the fact that the (linearised) gravitational field has spin 2, cf. Problem 1 of Worksheet 3.

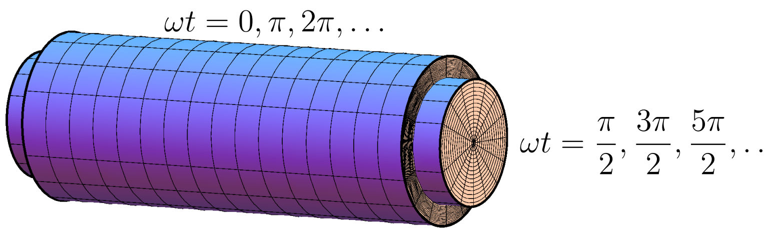
If the damping could be made arbitrarily small, a gravitational wave would produce a signal of arbitrarily large amplitude, at least near the resonance frequency. In practice the damping is of course limited. One often expresses the damping in terms of the Q factor

$$Q = \frac{\sqrt{km}}{\gamma}$$

which, for undercritical damping, ranges from 1/2 to infinity. For oscillating mechanical systems the Q factor is limited by about 10^5.

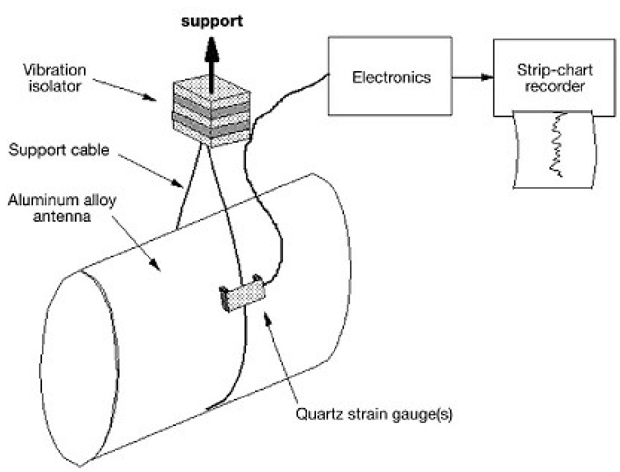
We have used the spring system to explain the basic idea of how to use vibrating systems for detecting gravitational waves. The resonant bar detectors which were built by Weber and others are based on the same idea. However, instead of masses connected by a spring one uses elastic solids, traditionally with a cylindrical shape.

In this case, $y^\ell(t)$ denotes the position vector of an arbitray mass element of the solid with respect to a body-fixed reference point. Again, we write $y^\ell(t) = s^\ell + \xi^\ell(t)$ where s^ℓ gives the position in equilibrium. One introduces a second rank tensor $\varepsilon^\ell{}_k(t)$ by the equation $\xi^\ell(t) = \varepsilon^\ell{}_k(t) s^k$. The antisymmetric part of $\varepsilon_{\ell k}(t)$ describes a rotation of the mass element, while the symmetric part describes expansion and shear. The symmetric part of $\varepsilon_{\ell k}(t)$ is known as the *strain tensor*. By assuming again a linear restoring force (i.e., Hooke's law now in the version of continuum mechanics) and a linear damping, one gets a differential equation for the strain tensor which is very similar to the damped oscillator equation for the spring system. As a consequence, a cylinder that is positioned transverse to a plane-harmonic gravitational wave undergoes periodic deformations as shown in the picture below.



Oscillating Weber cylinder

Weber's first gravitational wave detector was an aluminium cylinder with a weight of 1.5 tons (150 centimeters long, 60 centimeters in diameter). The fundamental resonance frequency was at about 1660 Hertz. Quartz crystals glued to the surface were used for measuring the deformations; as quartz crystals are piezoelectric, they transform a strain into a voltage which can be measured. Below the picture on the left shows the principal method, the picture on the right shows Joseph Weber with one of his resonant bar detectors.

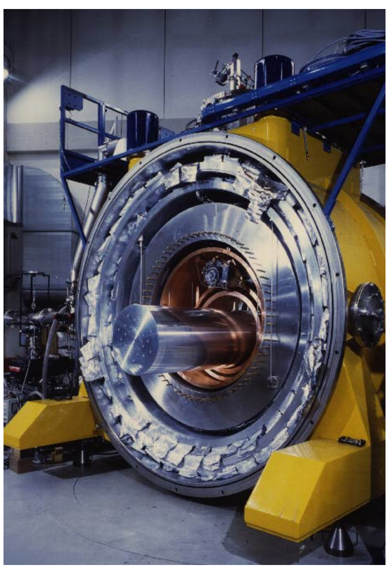


from J. Levine, Phys. Perspect. 6, 42 (2004)

from physics.aps.org

Weber operated his resonant bar detectors in pairs, searching for coincidences. In the beginning he had two detectors on the campus of Maryland University, then he moved one of them to Chicago. He claimed that he had found significant statistical evidence for coincident events which he thought to be gravitational wave signals. Nowadays there is agreement that his detectors were too crude to measure gravitational waves. Weber also had the idea of using the Moon as a resonator. To that end he sent a gravimeter (essentially nothing but a highly sensitive spring balance) to the Moon with the Apollo 17 mission. The Moon is seismically very quiet, and its resonance frequency is at about 10^{-3} Hz, i.e., much lower than that of the Weber cylinders. This would make gravitational waves of low frequencies observable. Apollo 17 indeed placed Weber's gravimeter on the Moon but unfortunately the instrument malfunctioned.

Joseph Weber died in the year 2000. By that time, attempts to detect gravitational waves had shifted to interferometric methods, see next section. However, there are still a few resonant bar detectors in operation. While in the beginning Weber did his observations at room temperature, all modern resonant bar detectors are operated at a temperature of a few millikelvins to reduce thermal noise. The picture below on the left shows the AURIGA instrument near Padova in Italy which was operational until 2009. It was a resonant bar detector of the traditional cylindrical shape, situated inside a tank to keep it at cryogenic temperatures. The picture below on the right shows the MiniGRAIL instrument at the Kamerlingh Onnes Institute in Leiden in the Netherlands. It has a spherical shape, so it can detect gravitational waves from all spatial directions. There is a similar instrument, named after the late physicist Mario Schenberg, in Brazil. These are the only two resonant detectors that are in operation at present.



from www.auriga.lnl.infn.it

from www.minigrail.nl

Resonant bar detectors can detect waves only in a narrow frequency band around the resonance frequency which is above or slightly below 1 kHz. Inspiralling binaries have a considerably lower frequency. Spinning bumpy neutron stars could produce gravitational waves with a frequency close to 1 kHz, see Worksheet 6, but their amplitude would probably be too low for being detected with resonant bar detectors. Therefore, the search with such instruments concentrates on burst sources such as core-collapse supernovae.

For measuring a gravitational wave signal with the help of a resonant bar detector it is not sufficient that the amplitude of the signal is big enough to be observable; it is also necessary that it is bigger than the noise level. For resonant bar detectors, there is in particular seismic noise and thermal noise. For characterising the noise level, which is frequency dependent, one proceeds in the following way. Let us assume that we have a detector just under the influence of noise, i.e., with no real gravitational wave signal coming in. Then we could measure the strain $n(t_1)$ at time t_1 and the strain $n(t_2)$ at time t_2. If the detector and the noise sources have no explicit time dependence, only the time difference $\tau = \left|t_2 - t_1\right|$ will matter. So if we do these measurements very often, always with the same time difference τ, we can form an ensemble average

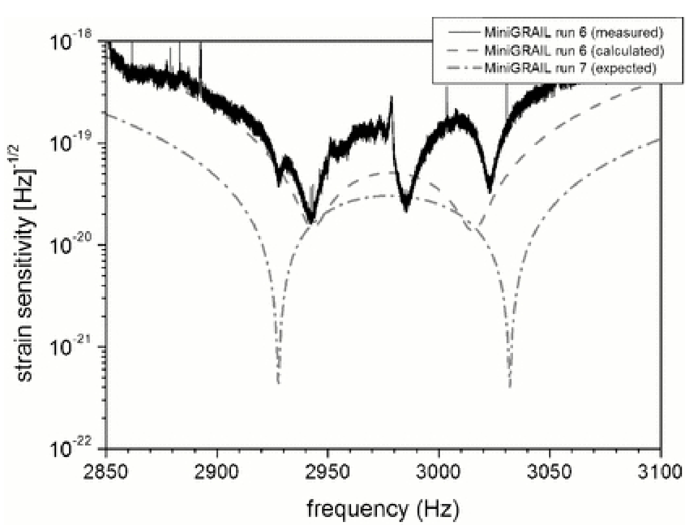
$$\langle n(t_1) n(t_2) \rangle = \kappa(\tau) \,.$$

For determining the frequency dependence one performs a one-sided Fourier expansion, i.e., one defines

$$S_h(\omega) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^\infty \kappa(\tau)\, e^{i\omega\tau}\, d\tau & \text{if } \omega > 0 \\ \\ 0 & \text{if } \omega < 0 \,. \end{cases}$$

The quantity $\sqrt{S_h(\omega)}$ is known as the *power spectral density* of the strain or as the *strain sensitivity*. This quantity, which has the dimension $1/\sqrt{\text{Hz}}$, is usually plotted against ω. A signal must lie above the graph of this function for being observable; if it lies below this graph it is drowned in the noise.

The picture below shows the sensitivity of the MiniGRAIL instrument. One sees that ths instrument can detect a signal only near 3 kHz.

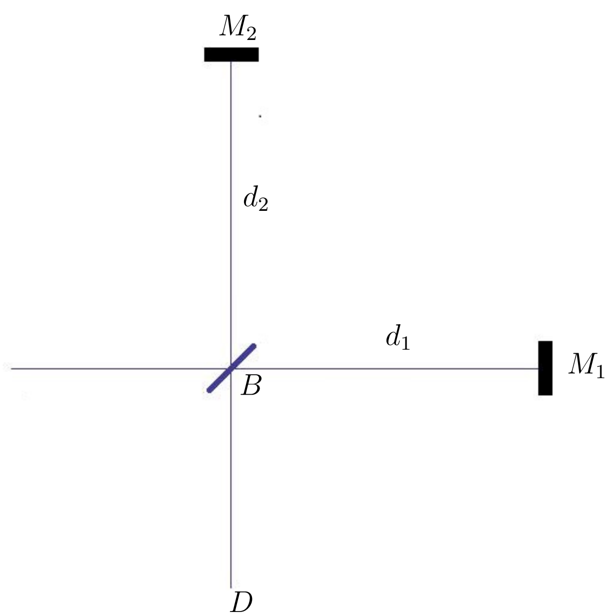


from A. de Waard et al, Class. Quant. Grav. 22, S215 (2005)

4.2 Interferometric gravitational wave detectors

With the help of a Michelson interferometer, tiny distance changes can be measured. The idea to use this well-known fact for the detection of gravitational waves came up in the early 1960s. The first published paper on the subject was by M. Gertsenshtein and V. Pustovoit ["On the detection of low-frequency gravitational waves", Sov. Phys. JETP 16, 433 (1962)]. The idea was strongly supported by V. Braginsky who became the leading figure in gravitational wave research in the Soviet Union and later in Russia, but no powerful gravitational wave detector was ever built there. Concrete plans for constructing an interferometric gravitational wave detector were brought forward in the US and in Western Europe in the 1970s. J. Forward actually built a small model detector in the mid-1970s. The construction of big instruments (GEO600, LIGO, VIRGO etc., see below) started in the 1990s. Many people were instrumental. The GEO600 project was initiated by H. Billing and later advanced by K. Danzmann. The first LIGO directors were R. Weiss, R. Drever and K. Thorne. All of them lived long enough to witness the detection of gravitational waves in 2015, but H. Billing and R. Drever passed away in 2017.

For understanding the basic idea of how an interferometric gravitational wave detector works, we have to recall what a Michelson interferometer is.



A laser beam is sent through the beam splitter B. One beam is reflected at mirror M_1, the other one at mirror M_2. When arriving at the detector D the two beams have a phase difference that can be observed in terms of an interference pattern. If the instrument is operated in vacuo, the phase difference is

$$\Delta\phi = \frac{2\pi}{\lambda}\, 2\,(d_1 - d_2)$$

where λ is the wave length of the laser. The gravitational wave detectors LIGO, VIRGO and GEO600 are operated with an Nd:YAG Laser at $\lambda = 1064\,\mathrm{nm}$ which is in the infrared, just outside of the visible part of the spectrum.

To use such a device as a gravitational wave detector, we think of the beam splitter B, the mirror M_1 and the mirror M_2 as being suspended with the help of files in such a way that they can move freely in the plane of the interferometer. For their motion in this plane, we can thus use the equation of motion for freely falling particles. Under the influence of a gravitational wave whose propagation direction is orthogonal to the plane of the interferometer, they will move according to the patterns from p. 15. Here we should identify the beam splitter with the particle at the centre of the coordinate system, and the mirrors M_1 and M_2 with particles on the x^1 axis and on the x^2 axis, respectively. For determining the time-dependence of the distances d_1 and d_2, and thus of the phase difference, we use our results from p. 14 where we assume, for simplicity, a pure plus mode,

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \big|A_+\big| \big((x^1)^2 - (x^2)^2 \big) \cos\big(\omega t - \varphi\big) \, .$$

If we assume that in the unperturbed state both arms have the same length d_0, for the mirror M_1 we have $x^1 = d_0, x^2 = x^3 = 0$, hence

$$d_1(t)^2 = d_0^2 \Big(1 + \big|A_+\big| \cos\big(\omega t - \varphi\big) \Big) \, ,$$

and for the mirror M_2 we have $x^2 = d_0, x^1 = x^3 = 0$, hence

$$d_2(t)^2 = d_0^2 \Big(1 - \big|A_+\big| \cos\big(\omega t - \varphi\big) \Big) \, .$$

As a consequence, the phase difference reads

$$\Delta\phi(t) = \frac{4\pi}{\lambda} \Big(d_1(t) - d_2(t) \Big) = \frac{4\pi}{\lambda} \, d_0 \, \left(\sqrt{1 + \big|A_+\big| \cos\big(\omega t - \varphi\big)} - \sqrt{1 - \big|A_+\big| \cos\big(\omega t - \varphi\big)} \right)$$

which, according to our general agreement to linearise all expressions with respect to the gravitational wave, simplifies to

$$\begin{aligned} \Delta\phi(t) &= \frac{4\pi}{\lambda} \, d_0 \, \Big(1 + \frac{1}{2} \big|A_+\big| \cos\big(\omega t - \varphi\big) - 1 + \frac{1}{2} \big|A_+\big| \cos\big(\omega t - \varphi\big) + \dots \Big) \\ &= \frac{4\pi}{\lambda} \, d_0 \, \big|A_+\big| \cos\big(\omega t - \varphi\big) \, . \end{aligned}$$

Clearly, the phase difference is proportional to the amplitude $|A_+|$ of the incoming gravitational wave. It is also proportional to the armlength d_0 of the interferometer. This is the reason why gravitational wave detectors need a long armlength, several hundred meters at least. As always with Michelson interferometers, the phase difference is proportional to the inverse of the wave length λ of the laser. λ is not to be confused with the wave length of the gravitational wave. The frequency ω of the gravitational wave enters into the formula for the phase shift only insofar as it gives the periodicity with which the interference pattern changes. In contrast to the resonant bar detectors, interferometric detectors are not restricted to a narrow frequency band. The observable frequency ω is mainly limited by seismic noise which, for ground-based interferometric detectors, will render gravitational wave signals of less than 1 Hz practically unobservable.

In addition to the noise produced by seismic vibrations and by a (time-dependent) gradient of the gravitational acceleration, resulting from the fact that the Earth is not a perfect homogeneous sphere, there are several other sources of noise. Thermal noise has the effect that interferometric gravitational wave detectors have to be cooled down if they are to operate below $\approx 30\,\mathrm{Hz}$. The first-generation detectors (TAMA300, GEO600, LIGO, VIRGO, see below) operate at room temperature, but the next generation of detectors will use cryogenic techniques to reach lower frequencies. At the upper end of the frequency band, quantum noise plays a major role. The elementary theoretical explanation of how an interferometer works is based on a classical wave theory of light. If it is taken into account that, actually, light consists of quantum particles (photons), deviations from the classical interference patterns occur. Roughly speaking, the mirrors in the interferometer are hit not by a classical wave but rather by a stream of photons, similar to a stream of pellets from a shot gun. The resulting deviations from the classical interference pattern are known as *shot noise*. These deviations are small if the laser beam consists of many photons, i.e., if the laser power is high. Noise resulting from the quantum nature of light restricts the existing interferometric wave detectors to frequencies below $\approx 10\,\mathrm{kHz}$.

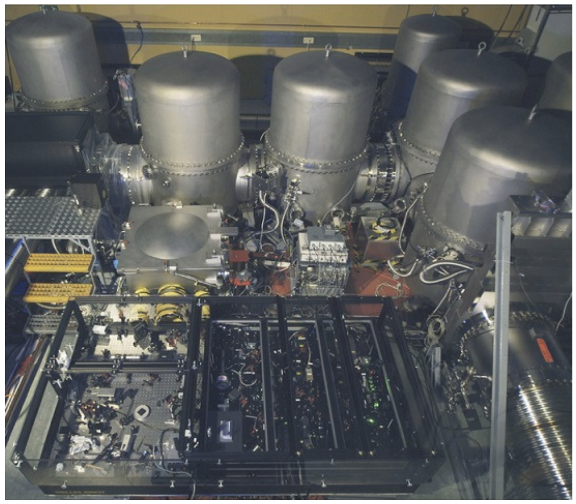
We now give a brief overview on the existing and planned interferometric gravitational wave detectors. The first small model detector of this type was built by J. Forward in Malibu, USA, in 1970. This was followed by a number of similar detectors at a laboratory scale, too small to actually detect gravitational waves but useful for testing the technology, e.g. in Garching, Germany, and in Glasgow, UK. In the mid-nineties the construction of detectors with an armlength of at least a few hundred meters began. In chronological order of the date when they became operational, these are the following.

TAMA300: This was a detector of 300 m arm length, located at the Mitaka Campus in Tokyo, Japan. It was operational from 1999 until 2004. As a comparatively small instrument its main purpose was to develop advanced technologies to be used in bigger detectors.

GEO600: This is a German-British project, originally planned to be realised near Munich. Finally, the detector was built near Ruthe near Sarstedt near Hannover in the middle of nowhere in Northern Germany. It became operational in 2001.

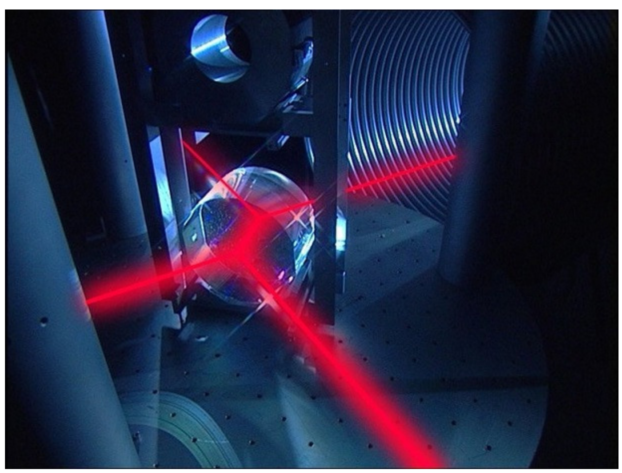
GEO600
from http://www.questhannover.de

The design is quite inconspicuous. In the picture on the preceding page we see the two vacuum tubes around the two arms of the interferometer, each of which has a length of 600 m. The two tubes meet at the main building.



GEO600 vacuum containers
from http://www.2physics.com

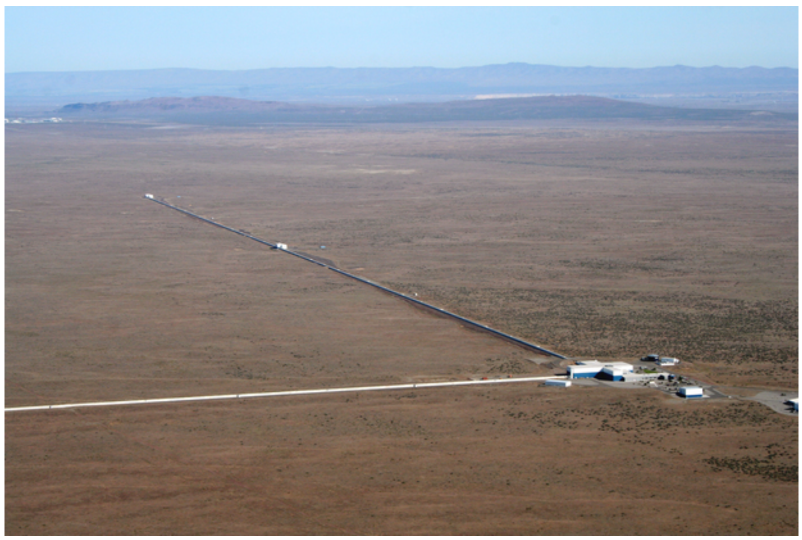
The main building hosts the laser, the beam splitter and several additional mirrors, e.g. for power recyling and for mode cleaning, each in a vacuum container, see figure above. The figure below gives an inside view of the container that houses the beam splitter.



GEO600 beam splitter
from http://u-182-ls004.am10.uni-tuebingen.de

GEO600 is sensitive in the frequency band between 50 Hz and 1.5 kHz. It is operated with an Nd:Yag laser with an output power of 10 W at a wavelength of $\lambda = 1064\,\mathrm{nm}$. With the help of power recycling, the laser power that is actually circulating in the interferometer is much bigger, namely $\approx$ 10 kW. Since 2011 GEO600 uses a second laser that produces squeezed light for reducing quantum noise. This second laser beem is seen in the figure above. Squeezed light is light in a state that minimises Heisenberg's uncertainty relation in such a way that the uncertainty in space is very small while the uncertainty in Fourier space is correspondingly big. The reduction of quantum noise is achieved by feeding this squeezed light into the interferometer (from below in the figure), in addition to the light from the main laser (which comes from the left in the figure). GEO600 is sensitive enough to detect length changes $d_1 - d_2$ in the order of 10^{-18} m. Recall for the sake of comparison that the diameter of a proton is about 10^{-15} m. In contrast to other existing interferometric gravitational wave detectors, GEO600 has no Fabry-Perrot cavities in the arms. During the time when LIGO and VIRGO underwent their upgrade to Advanced LIGO and Advanced VIRGO respectively, GEO600 was operational. It will be upgraded, afterwards, to GEO-HF.

LIGO: There are two LIGO sites, one in Hanford, Washington, USA, and one in Livingston, Louisiana, USA. At each site there is an interferometer with 4 km arm length. At the Hanford site there is a second interferometer with 2 km arm length in the same vacuum tube.



LIGO Hanford (Washington)
from https://www.ligo.caltech.edu

LIGO Livingston (Louisiana)
from https://www.ligo.caltech.edu

LIGO went operational in 2002. The vacuum tubes of 4 km length and 1.2 m diameter are the biggest existing ultra-high vacua. LIGO operates in the range between 30 Hz and 7 kHz. Having two smilar instruments working in parallel allows searching for coincident events. With the help of Fabry-Perot cavities in the arms the effective arm length of the LIGO detectors is raised to about 1600 km. The data of LIGO and GEO600 are pooled and analysed jointly. The data analysis team is known as the LIGO Scientific Collaboration (LSC). Amateurs are included in the data analysis. Within the Einstein@home project, everybody is invited to provide his or her computer for analysing scientific data. Einstein@home was already very succesful in analysing data from the radio telescopes at Arecibo and at Green Banks; more than a dozen new pulsars were found by amateurs. After being upgraded, LIGO became operational again under the name of Adanced LIGO in Summer 2015. At one of the very first runs after the update, which was not even planned as a science run originally, on 14 September 2015 a gravitational wave signal from a black-hole merger was detected, see next section. – A third detector of the Advanced-LIGO type is going to be built in India.

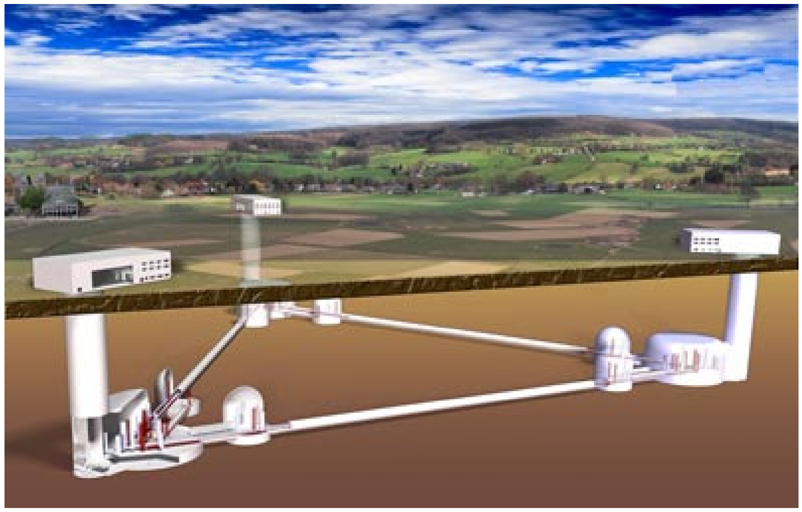
VIRGO: This is an Italian-French gravitational wave detector at Cascina near Pisa in Italy that became operational in 2007. The geometrical arm length is 3 km, but the effective arm length can be extended up to 100 km. VIRGO is operated at frequencies between 10 Hz and 10 kHz. At present, VIRGO is shut down. It will return after being upgraded under the name of Advanced VIRGO later in 2017.

VIRGO
from http://www.ego-gw.it

These are the interferometric gravitational wave detectors that already exist. Future plans include the following two ground-based detectors:

KAGRA: The original name of this Japanese project was LCGT (Large Scale Cryogenic Gravitational Wave Telescope). As suggested by the C in the name, it is a detector that will use cryogenic materials such that it can be operated at low temperatures. The instrument is to be built in tunnels of the Kamioka mine, with an arm length of 3 km. It is planned to become operational in 2018.

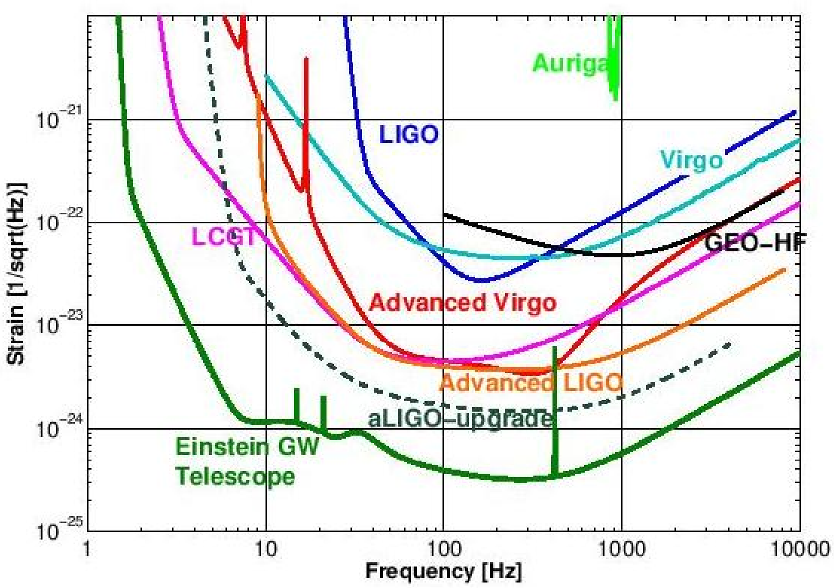
Einstein Telescope: This is a joint project of eight European institutions, including the Albert Einstein Institute in Hannover, Germany. At the moment it is unclear if, when and where the project will be realised.



Einstein Telescope
from http://physicsworld.com

Similarly to KAGRA, it will be an underground detector (at a depth of 100−200 m) and it will use cryogenic materials for low thermal noise.

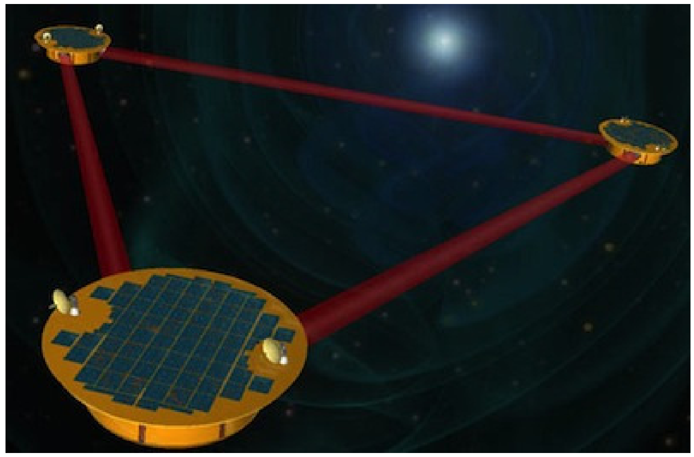
The sensitivity of existing and planned ground-based interferometric gravitational wave detectors is shown in the picture below. The resonant bar detector AURIGA is included for the sake of comparison.



from S. Hild, Class. Quantum Grav. 29, 124006 (2012)

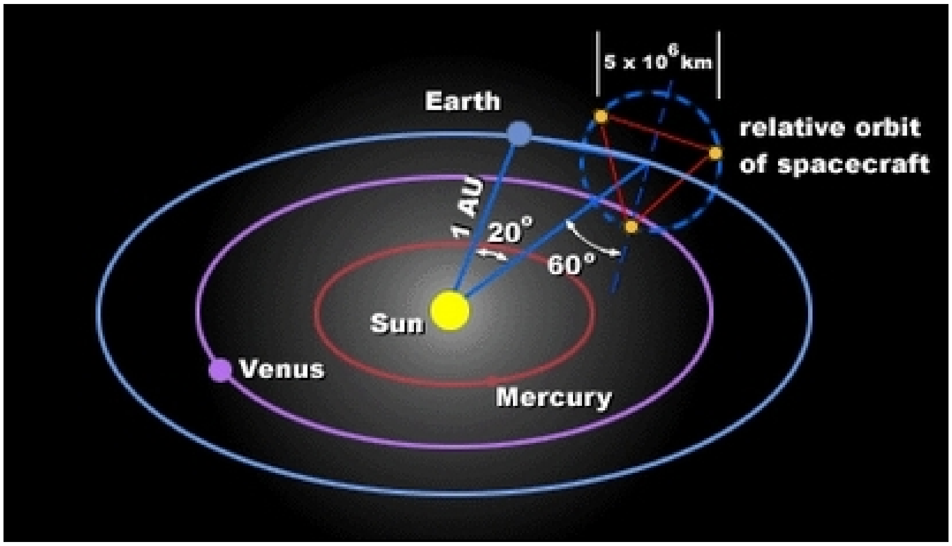
As mentioned above, ground-based interferometers are limited to frequencies above 1 Hz, because of seismic noise. Therefore, e.g. gravitatonal waves emitted by the Hulse-Taylor pulsar (with a frequency of less than $\approx 10^{-4}$ Hz) or by similar binary pulsars are outside of the range of such detectors. There are plans for space-based interferometric gravitational wave detectors that could overcome this limit. They include the following.

LISA: This is a long-standing project, designed already in the 1990s, for a space-based interferometric detector. LISA (Laser Interferometer Space Antenna) was originally planned as a joint project of NASA and ESA.



from http://lisa.nasa.gov

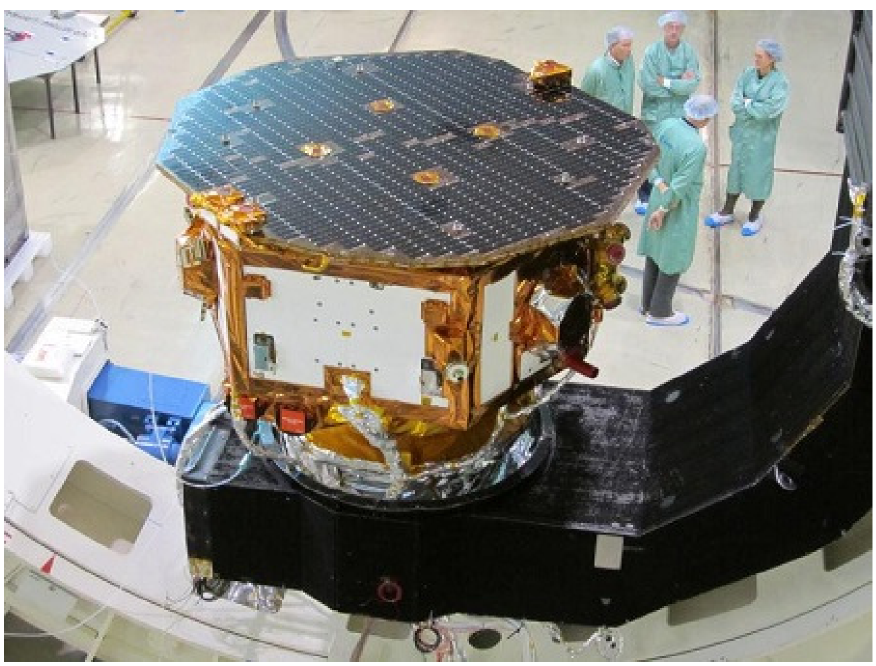
In this original version, LISA should consist of three satellites, see the picture, arranged in an equilateral triangle with a side length of 5 million kilometers. (That's about 12 times the separation of the Earth and the Moon.) This triangular array was supposed to fly along the orbit of the Earth around the Sun, trailing the Earth by 20 degrees. The inclination of the plane of the triangle with respect to the ecliptic was planned to be 60 degress. Each of the three satellites was to host two laser sources and two test masses, so that from each satellite a laser beam could be sent to a test mass in either of the two others. As it is impossible to receive a reflected laser beam with a measurable intensity over a distance of 5 million kilometers, it was planned that each satellite should host two transponders which would send back, after receiving a laser beam from a partner satellite, coherently a laser beam with the same frequency. In 2011, NASA stopped funding for LISA. Since then, it was a European-only project. Under the name NGO (New Gravitational wave Observatory) it entered into ESA's L1 mission selection, together with two competitors: The Jupiter Icy Moon Explorer (JUICE) and the X-ray observatory ATHENA. The winner was JUICE. NGO was re-designed and was elected as an L3 mission under the name eLISA (evolved LISA). It was planned as a system of a mother spacecraft with two daughter spacecraft. The mother would emit laser beams that are sent back from transponders on board the daughters. There would be no laser beam between the two daughters. The separation between the spacecraft was down-sized to 1 million kilometers. With the detection of gravitational waves by LIGO the plan for a space-based interferometric antenna got a strong boost. There are hopes that LISA, in a version close to the original plan with three arms, might eventually fly around 2030. LISA would be sensitive in the range between 0.1 mHz und 1 Hz where ground-based detectors cannot operate.



from http://sci.esa.int/lisa

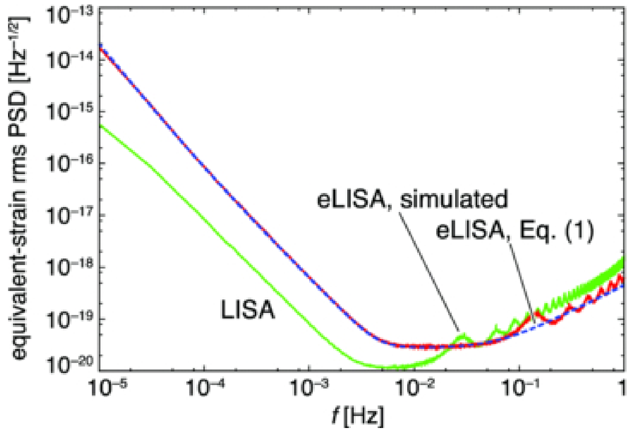
The picture above shows the orbit of LISA, according to the original plan.

As a preparation for the (e)LISA mission, a spacecraft called LISA Pathfinder was launched in December 2015. It houses laser and test masses at a separation of $\approx 40\,\mathrm{cm}$ in one spacecraft. The main purpose of the project is to test the technology for LISA (drag-free control, transponders for laser beams, etc.) under space conditions. The fact that LISA pathfinder, which is still in operation, works extremely well gives further hopes that LISA will eventually be launched around 2030.



LISA Pathfinder
from http://news.softpedia.com

The picture below shows the sensitivity of (e)LISA. Note that this detector could operate at much lower freqencies than the ground-based detectors.



from P. Amaro et al., Class. Quantum Grav. 29, 124016 (2012)

DECIGO: The acronym stands for DECI-Hertz Interferometer Gravitational wave Observatory. It is a proposed Japanese space-based instrument. The name refers to the fact that this detector is planned to operate in the frequency range between 0.1 Hz and 10 Hz (a decihertz). At present, it is unclear if and when this project will be actually realised.

BBO: The Big Bang Observer is a far-future project that has been suggested by physicists from the USA. As the name suggests, its main goal is the detection of gravitational waves that came into existence shortly after the big bang. The proposed instrument consists of 12 spacecraft, arranged into 4 LISA-type triangular patterns. It is written in the stars if BBO wil ever fly.

4.3 Pulsar timing arrays

A pulsar emits radio pulses at a rate that is highly stable. For millisecond pulsars, the stability of the pulse frequency is comparable to the stablity of the best clocks we have. This, however, does not mean that the pulses arrive with a constant frequency here on Earth. Changes in the times of arrival are caused e.g. by the relative motion of the pulsar and the Earth, by the influence of the gravitatonal field of the Sun and of other masses the signal might pass, and by the interstellar medium. All these known influences are taken into account in the socalled timing formulas used by radio astronomers for evaluating their observations. Remaining differences between theory and observation are known as *timing residuals*. A gravitational wave should produce such residuals.

The idea of searching for signatures from gravitational waves in the timing residuals of pulsars was brought forward by M. Sazhin ["Opportunities for detecting ultralong gravitational waves" Astron. Zh. 55, 65 (1978)] and further developed by S. Detweiler ["Pulsar timing measurements and the search for gravitational waves" Astrophys. J. 234, 1100 (1979)].

Here we will give a calculation under highly idealised assumptions, just to outline the basic idea. We treat the pulsar and the Earth as at rest in a Minkowski background, and we ignore the influence of the interstellar medium. The gravitational wave is considered as a perturbation of the Minkowski background within the linearised theory,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,,$$

where $h_{\mu\nu}$ is assumed, for simplicity, to be a pure plus mode,

$$h_{\mu\nu} = \mathrm{Re}\big\{A_{\mu\nu}\, e^{ik_\sigma x^\sigma}\big\}$$

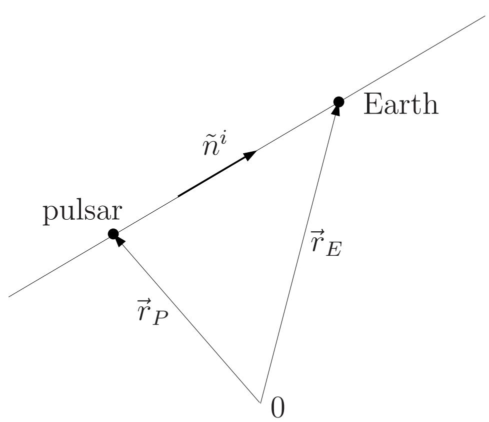
with

$$(k_\mu) = \begin{pmatrix} -\omega/c \\ 0 \\ 0 \\ \omega/c \end{pmatrix}, \quad (A_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & 0 & 0 \\ 0 & 0 & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We assume that the worldlines of the pulsar and of the Earth are both t lines. From the form of the metric,

$$g_{\mu\nu}(x)dx^\mu dx^\nu = -c^2dt^2 + \delta_{ij}dx^i dx^j + h_{ij}(x)x^i dx^j\,,$$

we read that along these worldlines the time coordinate t coincides with proper time. Therefore, we can identify frequencies with respect to the time coordinate t with frequencies with respect to proper time of the pulsar or of the Earth. We assume that the pulsar emits signals at a fixed frequency ν_P. They will arrive at the Earth with a frequency $\nu_E(t_E)$ that depends on the time of arrival, t_E. It is our goal to determine this function $\nu_E(t_E)$.



Along a light ray from the pulsar to the Earth, we must have

$$0 = -c^2dt^2 + \delta_{ij}dx^i dx^j + h_{ij}(x)dx^i dx^j$$

and thus

$$c^2\left(\frac{dt}{d\ell}\right)^2 = 1 + h_{ij}(x)\frac{dx^i}{d\ell}\frac{dx^j}{d\ell}$$

where ℓ denotes arclength with respect to the flat background metric, $d\ell^2 = \delta_{ij}dx^i dx^j$.

Without a gravitational wave, the light ray moves on a straight line from the pulsar to the Earth, i.e., $dx^i/d\ell$ is a constant unit vecor $\tilde{n}^i$. With a gravitational wave, we have

$$\frac{dx^i}{d\ell} = \tilde{n}^i + O(h)$$

and thus

$$c\,\frac{dt}{d\ell} = \sqrt{1 + h_{ij}(x)\tilde{n}^i\tilde{n}^j + O(h^2)}\,,$$

$$d\ell = c\Big(1 + h_{ij}(x)\tilde{n}^i\tilde{n}^j + O(h^2)\Big)^{-1/2} dt = c\Big(1 - \frac{1}{2}h_{ij}(x)\tilde{n}^i\tilde{n}s^j + \dots\Big)dt$$

where the ellipses indicate terms of at least second oorder which, in the linearised theory considered throughout, will be neglected. Integration over the path of the light ray, from its emission time t_P to the arrival time t_E, yields

$$\int_{t_P}^{t_E}\Big(1 - \frac{1}{2}h_{ij}(x)\tilde{n}^i\tilde{n}^j\Big)\,dt = \frac{L}{c}$$

where L is the distance from the pulsar to the Earth measured in the flat background. This equation gives t_E as a function of t_P. Differentiation with respect to t_P yields

$$\frac{dt_E}{dt_P}\Big(1 - \frac{1}{2}h_{k\ell}(ct_E,\,\vec{r}_E)\tilde{n}^k\tilde{n}^\ell\Big) - \Big(1 - \frac{1}{2}h_{ij}(ct_P,\,\vec{r}_P)\tilde{n}^i\tilde{n}^j\Big) = 0\;,,$$

$$\frac{dt_P}{dt_E} = \frac{\Big(1 - \frac{1}{2}h_{k\ell}(ct_E,\,\vec{r}_E)\tilde{n}^k\tilde{n}^\ell\Big)}{\Big(1 - \frac{1}{2}h_{ij}(ct_P,\,\vec{r}_P)\tilde{n}^i\tilde{n}^j\Big)} = 1 - \frac{1}{2}h_{k\ell}(ct_E,\,\vec{r}_E)\tilde{n}^k\tilde{n}^\ell + \frac{1}{2}h_{ij}(ct_P,\,\vec{r}_P)\tilde{n}^i\tilde{n}^j + \dots\,,$$

$$\frac{dt_P}{dt_E} = 1 + \frac{\tilde{n}^i\tilde{n}^j}{2}\Big(h_{ij}(ct_P,\,\vec{r}_P) - h_{ij}(ct_E,\,\vec{r}_E)\Big)\,.$$

We have thus found that the pulses, which are emitted with a constant frequency ν_P, arrive with a frequency $\nu_E(t_E)$ where the redshift is given by

$$\frac{\nu_P - \nu_E(t_E)}{\nu_P} = 1 - \frac{dt_P}{dt_E} = \frac{\tilde{n}^i\tilde{n}^j}{2}\Big(h_{ij}(ct_E,\,\vec{r}_E) - h_{ij}(ct_P,\,\vec{r}_P)\Big)\,.$$

We now insert the special form of h_{ij} which was assumed to be a pure plus mode, propagating in x^3 direction. If we parametrise the unit vector $\tilde{n}^i$ in the usual way by spherical polar coordinates,

$$\big(\tilde{n}^i\big) = \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\,,$$

we find

$$h_{ij}\big(ct_E,\,\vec{r}_E\big) = \mathrm{Re}\big\{A_{ij}e^{ik_\sigma x_E^\sigma}\big\} = \mathrm{Re}\big\{A_{ij}e^{i(-\omega t_E + \vec{k}\cdot\vec{r}_E)}\big\} = \mathrm{Re}\big\{A_{ij}e^{i\frac{\omega}{c}(x_E^3 - ct_E)}\big\}$$

and

$$h_{ij}\big(ct_P,\,\vec{r}_P\big) = \mathrm{Re}\big\{A_{ij}e^{ik_\sigma x_E^\sigma}e^{ik_\sigma(x_P^\sigma - x_E^\sigma)}\big\} = \mathrm{Re}\big\{A_{ij}e^{i\frac{\omega}{c}(x_E^3 - ct_E)}e^{ik_0(x_P^0 - x_E^0) + i\vec{k}\cdot(\vec{r}_P - \vec{r}_E)}\big\}\,.$$

As

$$k_0(x^0_P - x^0_E) = -\frac{\omega}{c}\,(ct_P - ct_E) = \frac{\omega\, L}{c} + O(h)$$

and

$$\vec{k}\cdot\Big(\vec{r}_P - \vec{r}_E\Big) \;=\; -\,L\,k_i\tilde{n}^i \;=\; -\,\frac{\omega\,L}{c}\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} \;=\; -\,\frac{\omega L}{c}\cos\vartheta\,,$$

the latter equation results in

$$h_{ij}\big(ct_P,\,\vec{r}_P\big) \;=\; \mathrm{Re}\Big\{A_{ij}e^{i\frac{\omega}{c}(x^3_E - ct_E)}e^{i\frac{\omega L}{c}(1-\cos\vartheta)}\Big\}\,,$$

hence

$$\frac{\nu_P - \nu_E}{\nu_P} \;=\; \frac{1}{2}\mathrm{Re}\Big\{\tilde{n}^iA_{ij}\tilde{n}^je^{i\frac{\omega}{c}\left(x^3_E - ct_E\right)}\Big(1\;-\;e^{i\frac{\omega L}{c}(1-\cos\vartheta)}\Big)\Big\}\,.$$

With

$$\tilde{n}^iA_{ij}\tilde{n}^j \;=\; \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}A_+ & 0 & 0\\ 0 & -A_+ & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

$$=\; A_+\sin^2\vartheta\big(\cos^2\varphi - \sin^2\varphi\big) \;=\; A_+\sin^2\vartheta\cos(2\varphi)\,,$$

this can be rewritten as

$$\frac{\nu_P - \nu_E}{\nu_P} \;=\; \frac{1}{2}\mathrm{Re}\Big\{A_+\sin^2\vartheta\cos(2\varphi)e^{i\frac{\omega}{c}(x^3_E - ct_E)}\Big(1\;-\;e^{i\frac{\omega L}{c}(1-\cos\vartheta)}\Big)\Big\}\,.$$

With $A_+ = \big|A_+\big|\,e^{i\psi}$ we find

$$\frac{\nu_P - \nu_E}{\nu_P} \;=\; \frac{\big|A_+\big|}{2}\sin^2\vartheta\cos(2\varphi)\,\mathrm{Re}\Big\{e^{i\psi}e^{i\frac{\omega}{c}(x^3_E - ct_E)}e^{i\frac{\omega L}{2c}(1-\cos\vartheta)}\Big(e^{-i\frac{\omega L}{2c}(1-\cos\vartheta)}\;-\;e^{i\frac{\omega L}{2c}(1-\cos\vartheta)}\Big)\Big\}$$

$$=\;-\,\frac{\big|A_+\big|}{\cancel{2}}\sin^2\vartheta\cos(2\varphi)\,\mathrm{Re}\Big\{e^{i\left(\psi + \frac{\omega}{c}(x^3_E - ct_E) + \frac{\omega L}{2c}(1-\cos\vartheta)\right)}\cancel{2}\,i\sin\left(\tfrac{\omega L}{2c}(1-\cos\vartheta)\right)\Big\}\,,$$

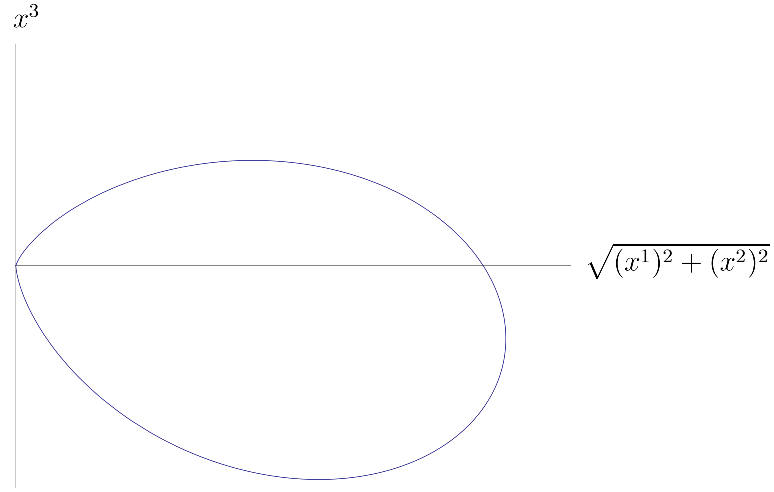
i.e.,

$$\frac{\nu_E}{\nu_P} \;=\; 1\;+\;\big|A_+\big|\sin^2\vartheta\cos(2\varphi)\sin\Big(\tfrac{\omega L}{2c}(1-\cos\vartheta)\Big)\sin\Big(\omega t_E + \tilde{\psi}\Big)\,.$$

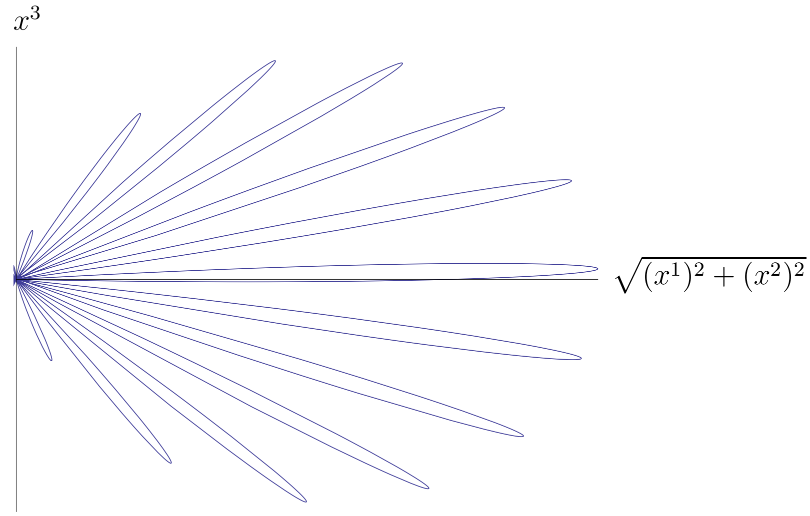
This is a sinusoidal function of t_E. The amplitude

$$A(\vartheta,\varphi) \;=\; \big|A_+\big|\sin^2\vartheta\cos(2\varphi)\sin\Big(\tfrac{\omega L}{2c}(1-\cos\vartheta)\Big)$$

has the usual π-periodic φ-dependence, via a factor of $\cos(2\varphi)$, we are used to. The ϑ-dependence is more complicated. In addition to the usual $\sin^2\vartheta$ term it involves a factor that depends on $\omega L/c$. If $\omega L/c$ is smaller than π, this factor has the only effect of shifting the maximum of the characteristic away from the equatorial plane; if $\omega L/c$ is bigger than π, it changes the qualitative features of the characteristic completely, see pictures on the next page. For gravitational wave searches with pulsar timing arrays we may assume that $\omega > 10^{-10}$Hz and $L > 1\,$kpc, hence $\omega L/c > 10$.



$\frac{\omega L}{c} = 1$

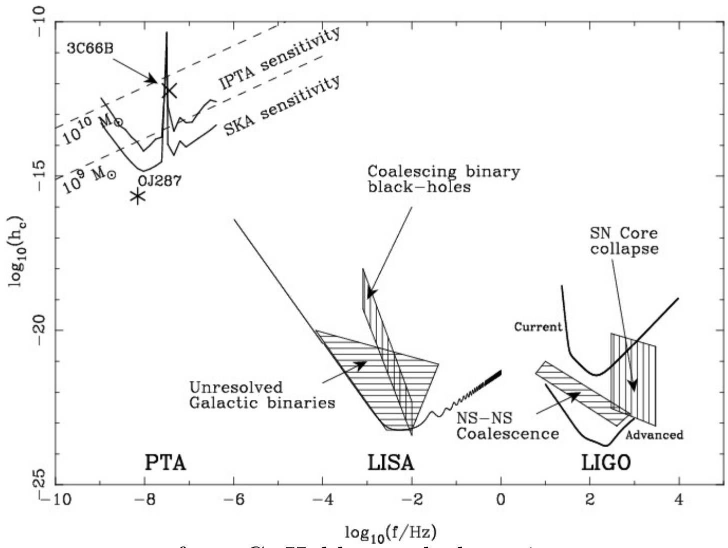


$\frac{\omega L}{c} = 40$

Pulsar timing arrays are used for many applications; the search for gravitational waves is only one of them. Three pulsar timing arrays have been established which routinely observe the times of arrival of signals from many pulsars:

- Parkes Pulsar Timing Array (PPTA): This uses the Parkes Telescope in Australia and takes data since 2005.
- European Pulsar Timing Array (EPTA): This uses data from five radio telescopes in Europe, namely Effelsberg, Jodrell Bank, Westerbork, Nançay, and a new one in Sardinia.
- North American Nanohertz Observatory for Gravitational Waves (NanoGRAV): This is an Americal pulsar timing array using data from Arecibo and Green Bank.

They are now joined into an International Pulsar Timing Array (IPTA), see G. Hobbs et al. [“The International Pulsar Timing Array project: using pulsars as a gravitational wave detector” Class. Quantum Grav. 27, 084013 (2010)] for a review. Great progress with pulsar timing data is expected from the Square Kilometer Array (SKA), an array of radio telescopes in the Southern hemisphere with an effective aperture of a square kilometer that is planned to be operational around 2020.



from G. Hobbs et al., loc. cit.

Pulsar timing arrays are sensitive to gravitational waves at very low frequencies, between 10^{-6} and 10^{-10} Hz; the amplitudes needed are considerably bigger than e.g. for the LIGO detectors, see the picture above. Possible sources that could be detected with this method are supermassive black hole binaries. Two galaxies that may host a supermassive black hole binary at the centre are indicated in the diagram: The BL Lac object OJ287 with an orbital period of about 12 years and the radio galaxy 3C66B with an orbital period of about one year.

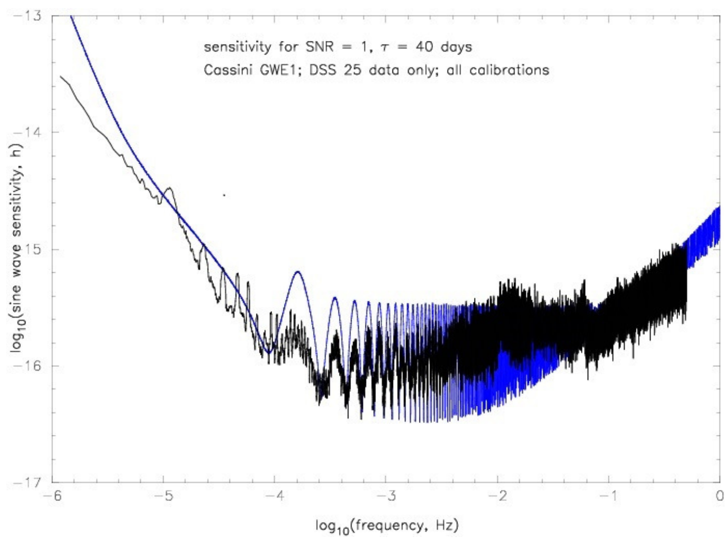
4.4 Doppler tracking of satellites:

Resonant bar detectors and interferometric detectors are instruments that are constructed for the sole purpose of detecting gravitational waves. In particular the advanced interferometric detectors are rather expensive instruments. In this section we will discuss a method of searching for gravitational waves that, similarly to the use of Pulsar Timing Arrays discussed in the preceding section, does not require to build any expensive new instruments but uses equipment that already exists. The method uses spacecraft which have been launched for some other purpose, in particular spacecraft investigating the outer parts of our Solar system like Voyager, Pioneer 10, Pioneer 11 and Cassini.

The path of such a spacecraft is routinely monitored with the help of Doppler tracking. The idea is to search in the Doppler tracking data for signatures of gravitational waves. This is very similar to the search for changes in the frequency of pulsar signals: The difference is that Doppler tracking with satellites is a two-way method: From the Earth a signal is sent to the spacecraft; there a transponder multiplies the frequency with a certain factor and then sends the signal back to the Earth where the change in the frequency is measured. In essence, all the formulas from the preceding section can be taken over; we just have to apply them twice: Once for the outgoing signal and then for the returning signal. Of course, as in the case of pulsar timing all other effect that produce a frequency shift have to be taken into account, in particular the relative motion of the Earth and the Satellite and the gravitational redshift produced by the Sun.

The idea of using Doppler tracking data for detecting gravitational waves came up in the early 1970s. The mathematical formalism was worked out by F. Estabrook and H. Wahlquist [Gen. Rel. Grav. 6, 439 (1975)]. A comprehensive overview of the method can be found in the Living Review by J. Armstrong ["Low-frequency gravitational wave searches using spacecraft Doppler tracking", Living Rev. Relativity 9, (2006), `http://www.livingreviews.org/lrr-2006-1`.]

Searches for gravitational waves with the help of Doppler tracking have been carried through, e.g., with Voyager 1 and 2, with Pioneer 10 and 11, and in particular with the Cassini spacecraft that was launched in 1997 and reached Saturn in 2004. From the picture below we can read the frequencies and the amplitudes of gravitational waves that could have been detected by Doppler tracking of the Cassini spacecraft. We see that this method is sensitive only for frequencies near 10^{-3} Hz (similarly to LISA) and for big amplitudes (similarly to the Pulsar Timing Arrays).



from http://www.livingreviews.org/lrr-2006-1

The radio links with spacecraft in the outer region of our Solar system are established with a system of radio telscopes that is known as the Deep Space Network (DSN). It comprises sites in the USA, in Spain and in Australia such that at any time of the day at least one of the stations can communicate with the spacecraft.

4.5 Influence of gravitational waves on electromagnetic waves

All the methods we have discussed so far were based on measuring the effect of a gravitational wave onto massive bodies, either onto vibrating masses or onto free test masses. Electromagnetic waves were used in some of these methods, but only as a tool for measuring the effect onto the massive bodies.

In this section we briefly discuss the possibility of detecting a gravitational wave by its effect onto an electromagnetic wave. One such method was suggested by M. Cruise ["An interaction between gravitational and electromagnetic waves" Mon. Not. Roy. Astron. Soc. 204, 485 (1983)]. It is based on the observation that a gravitational wave causes a rotation of the polarisation plane of an electromagnetic field.

In an arbitrary general-relativistic spacetime, the polarisation vector Π of a linearly polarised electromagnetic wave in vacuo is parallely transported along each ray. This can be deduced from Maxwell's equations in the high-frequency limit (i.e., in the geometric optics approximation). If we denote the tangent vector to the ray by K, the polarisation vector satisfies the equation

$$\nabla_K \Pi = 0$$

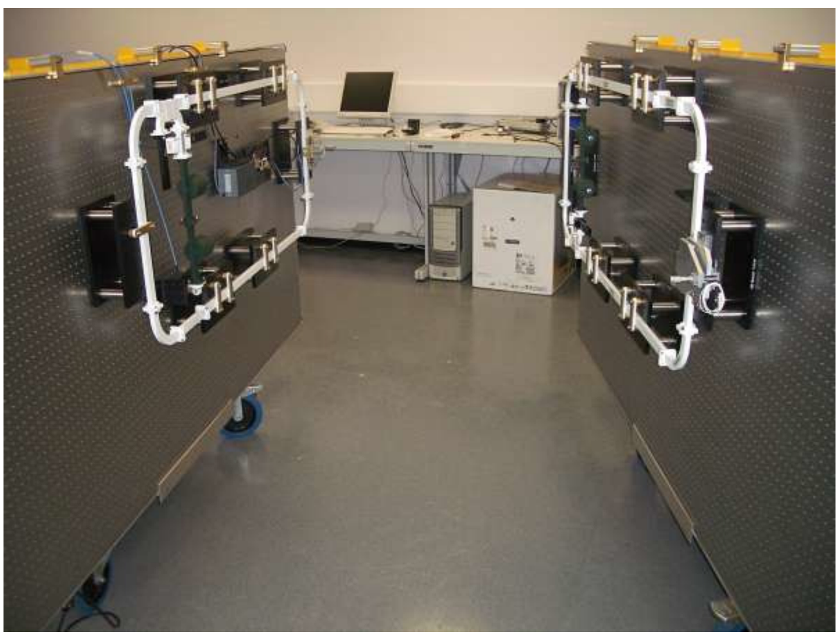
or, in coordinate notation,

$$K^\mu \partial_\mu \Pi^\rho + \Gamma^\rho{}_{\nu\tau} K^\nu \Pi^\tau = 0 \, .$$

The vectors K and Π span the polarisation plane. This plane always contains the direction tangent to the ray, so the only thing it can do is to rotate about this direction. We see that, with respect to the coordinate system used, such a rotation is caused by the Christoffel symbols. For a gravitational wave in TT gauge, we already know that the Christoffel symbols read

$$\Gamma^\rho{}_{\nu\tau} \; = \; \frac{1}{2} \, \eta^{\rho\sigma} \, \mathrm{Re}\big\{ \big(k_\nu A_{\sigma\tau} + k_\tau A_{\sigma\nu} - k_\sigma A_{\nu\tau} \big) \, i \, e^{i k_\mu x^\mu} \big\}$$

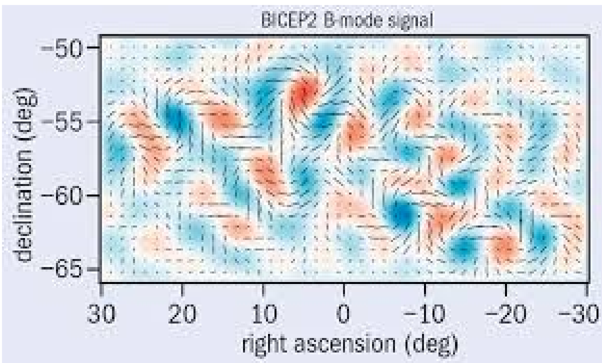
where k_μ is the wave covector of the gravitational wave. According to this equation, a gravitational wave would cause a rotation of the polarisation plane of an electromagnetic wave.



from http://www.sr.bham.ac.uk/gravity

The expected rotation angle is tiny. Therefore, Cruise designed a gravitational wave detector that would enhance this rotation by making use of a resonance effect. The electromagnetic wave is a radio wave in a wave-guide that is bent into a loop. The resonance frequency of the system is 100 MHz. If a gravitational wave with the same frequency comes in, the polarisation plane is periodically kicked by a tiny rotation angle in such a way that these tiny rotations add up. Cruise built two such instruments in Birmingham so that he could look for coincidences, see the figure above. Note that these detectors would be sensitive at a frequency of about 100 MHz, i.e., at an extremely high frequency in comparison to all other gravitational wave detectors.

In addition to the possibility of constructing non-orthodox gravitational wave detectors, the effect of a gravitational wave onto electromagnetic waves is of crucial relevance in view of the cosmic background radiation. In recent years several experiments are analysing the polarisation of the cosmic background radiation. In analogy to decomposing a vector field into rotation-free and divergence-free parts, the Fourier components of the cosmic background radiation are decomposed into electric (E) and magnetic (B) modes. Primordial gravitational waves that have come into existence through quantum fluctuations in the very early universe would produce a specific signature of B modes. These B modes from primordial gravitational waves could have a measurable amplitude only if our universe underwent an inflationary period, i.e., a period in which the universe expanded exponentially.

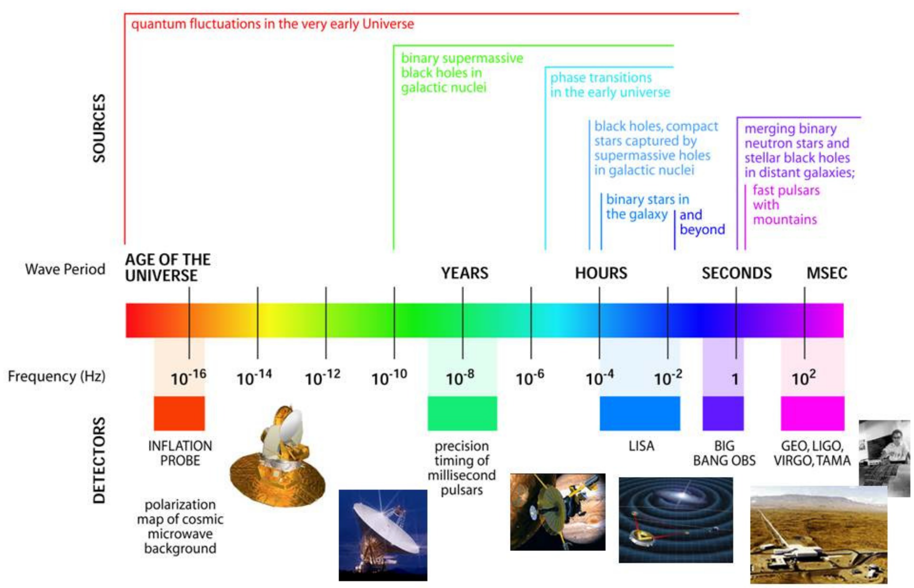


from http://www.astro.gla.ac.uk

In March 2014 it was announced that the BICEP2 experiment had found B modes from primordial gravitational waves in the cosmic background radiation. BICEP2 was a radio telescope near the South Pole that was operational from 2010 to 2012. If true, the BICEP2 result would have given strong support for the idea that quantum fluctuations in the early universe have produced gravitational waves and that there was an inflationary period.

The idea of primordial gravitational waves, resulting from quantum fluctuations, was developed already in the 1970s by L. Grishchuk and others. The idea of an inflationary universe, brought forward by A. Starobinsky, A. Guth, F. Englert, A. Linde and others around 1980, allowed for an increase in the amplitudes of these primordial gravitational waves that could make them measurable. However, it was realised after a while that the BICEP2 team had misinterpreted their observations: The B modes they had found had not been produced by primordial gravitational waves but rather by dust in the foreground.

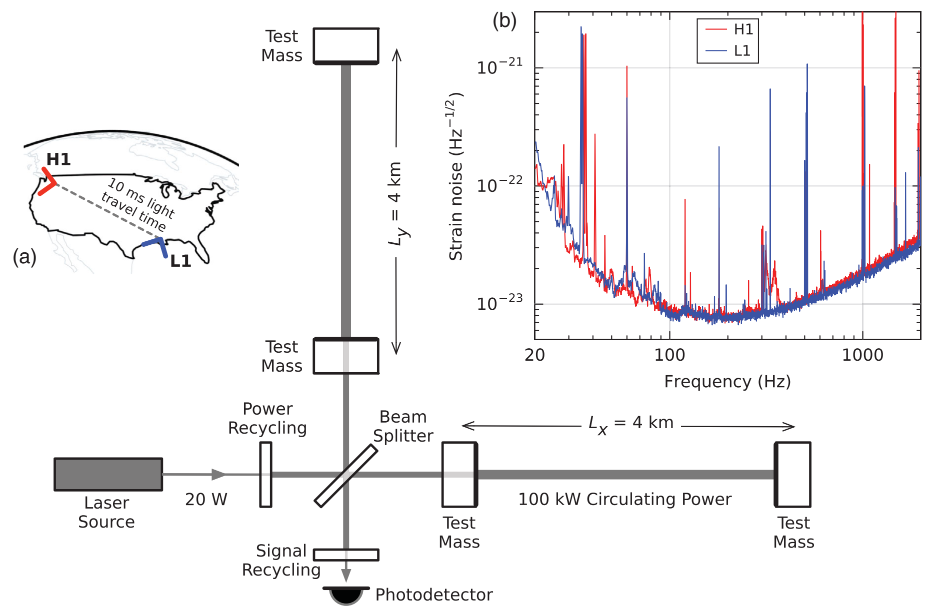
The figure on the right summarises the different types of gravitational wave detectors, the frequency range in which they are sensitive and the types of sources they could detect.



from http://www.astro.gla.ac.uk

5 LIGO detection of gravitational waves

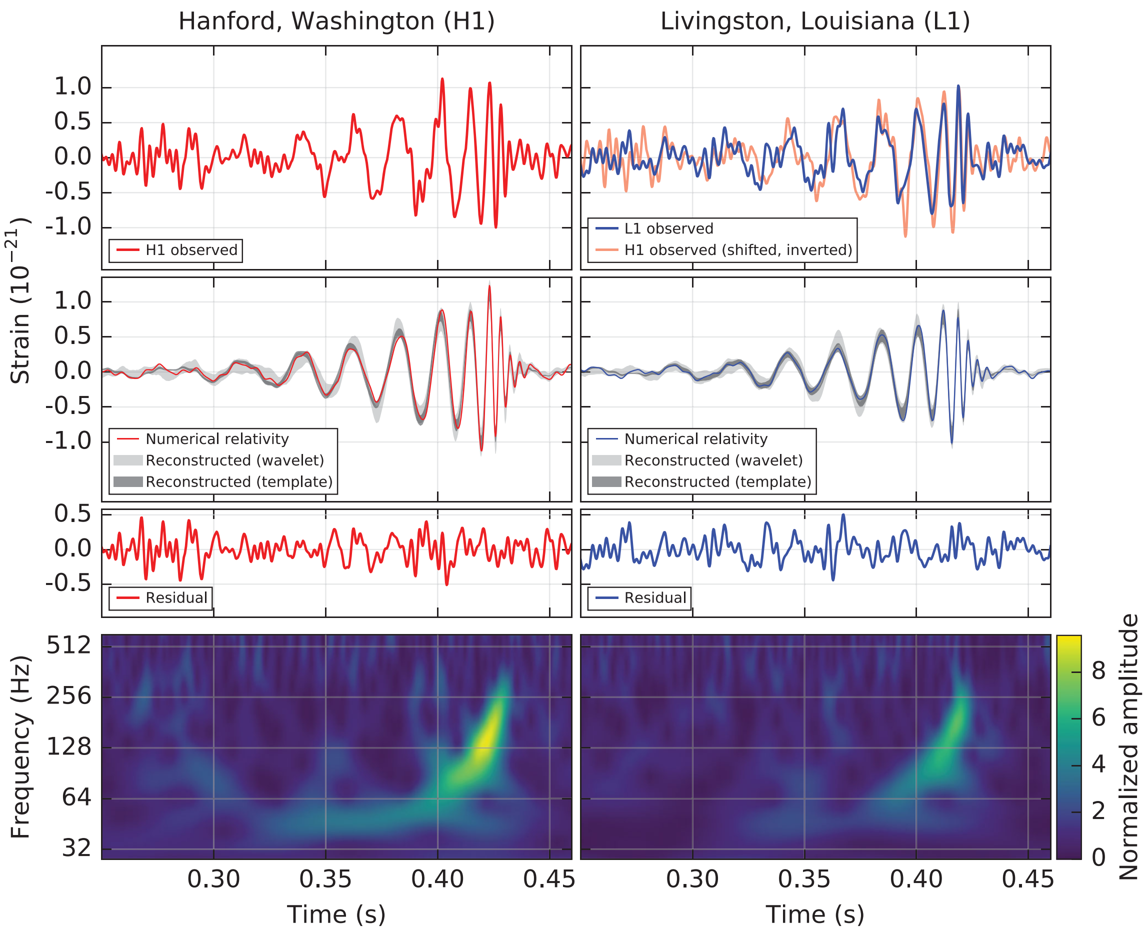
As of June 2017, the two LIGO instruments have registered three signals that were sufficiently high above the noise level so that the LIGO team announced them as detected gravitational waves. They are named GW150914, GW151226 and GW170104, where "GW" stands for "Gravitational Wave" and the numbers give the date: The signals were received on 14 September 2015, 26 December 2015 and 4 January 2017. All three events were most likely produced by the merger of two black holes. Another event, known as LVT151012, was made public but the LIGO team found it too close to the noise level for calling it a gravitational wave signal. (The abbreviation "LVT" stands for "LIGO-Virgo Trigger" and is used for all events that are considered as candidates for a gravitational wave signal.)



from B. Abbott et al., Phys. Rev. Lett. 116, 061102 (2016)

Recall the the LIGO detectors are most sensitive at frequencies of a few hundred Hz. This was exactly the frequency range in which the three confirmed gravitational wave signals have been observed. Only signals that are registered in both LIGO detectors (in Hanford and in Livingston) are taken into account. The two LIGO detectors are approximately 3000 km apart, so a signal that travels at the speed of light needs about 10 ms from one to the other. Therefore, a gravitational wave whose wave vector is parallel to the line that connects the two LIGO sites should be registered in both detectors with a time delay of about 10 ms. If the wave vector makes an angle different from zero with this connection line, the time delay is smaller. Measuring the time delay locates the source on a circle on the sky. As this circle is given only to within a finite accuracy, and as the LIGO detectors cannot receive signals from all directions, the source can actually be located in a banana-shaped region. This region is very big: If one wants to locate the source with a likelihood of 90 %, one gets a banana-shaped region in the sky that is about as big as the constellation Orion. So we do not really know from which direction the three observed gravitational-wave signals have come. If additional detectors have become operational (Virgo, KAGRA, LIGO India, ...), we will be able to locate the sources with much higher accuracy.

We now look at the three signals GW150914, GW151226 and GW170104 one by one. The discovery of GW150914 was announced in February 2016 which was a major media event, with a press conference that can be watched on YouTube. It was the first *direct* observation of gravitational waves. The signal was received during one of the first runs of the two LIGO detectors after the upgrade from original LIGO to advanced LIGO; this run was not even planned as a science run. The signal was unexpectedly strong, and it showed the expected signature of a binary merger so cleanly that many LIGO scientists in the beginning thought that it was an injection. Four members of the LIGO team are supposed to inject mock signals from time to time into the detectors; the idea is that one wants to see if the data analysis works sufficiently well to detect such signals. The rule is that all the evaluation of the signal has to be finished; only then will the responsible member from the injection team confess whether or not it was a mock signal. In the case of GW150914, no such confession was made, and after painstakingly interviewing all possible suspects the LIGO team became convinced that this signal was *not* an injection.



from B. Abbott et al., Phys. Rev. Lett. 116, 061102 (2016)

The picture above shows the signals as they were received in Hanford (left) and in Livingston (right). From the time axis at the bottom we read that everything happened within 0.2 seconds. In the first row, the signals are shown as they were received, with the only modification that frequencies outside an interval from 35 to 350 Hz have been filtered away. (The Earth-bound interferometric detectors are too strongly affected by seismic noise at lower frequencies and by shot noise at higher frequencies.) On the right, the Livingston signal (blue) is overlaid with the Hanford signal (red), but with the latter shifted backwards by 6.9 ms and inverted. The time shift reflects the fact that the signal arrived first in Livingston and then 6.9 ms later in Hanford; this time delay gives the angle $\vartheta = \arccos(6.9/10) \approx 46^0$ under which the signal came in with respect to the connection line of the two detectors. The inversion is necessary because the two detectors are oriented under 90^o with respect to each other. The second row compares the signal with a black-hole merger, calculated with numerical relativity, where the parameters of the two inspiralling black holes have been fitted to the observation as closely as possible. The third row shows the residual, i.e., the signal one has to subtract from the observation for matching it with the theoretical prediction; if the interpretation is correct, this residual is just noise. We see that the noise level is about $0.3 \times 10^{-21}\,\mathrm{Hz}^{-1/2}$. While the signal lies well above the noise level during the merger, the ring-down is practically drowned in the noise. *So we cannot really compare the ringdown phase with the theoretical prediction for a black-hole merger.* The last row shows the frequency as a function of time. We see that we have a typical "chirp signal", i.e., a signal where the frequency grows with time: It begins at about 35 Hz and it reaches a maximum, immediately before the merger, at about 150 Hz. As the frequency of the gravitational wave is always *twice* the frequency of the source motion ($\omega = 2\Omega$ in the notation we have used throughout), this means that the orbital motion was at about 75 Hz immediately before the merger. If translated from gravitational waves to sound waves, the signal is in the audible range, i.e., you can mimic it by a buzz that increases up to a maximum in frequency and in amplitude. A sound bite can be found at https://www.ligo.caltech.edu/video/ligo20160211v2.

Why do we believe that GW150914 was the merger of two black holes? The reason is *not* that the shape of the signal favours a black-hole merger: The inspiralling and merger phases are similar for all types of binaries, and the ringdown phase has not really been observed, as emphasised above. The motivation comes from the masses involved: One can derive, from lowest-order post-Newtonian theory, the following relation between the so-called "chirp mass" M_{chirp} and the frequency $f = \omega/(2\pi)$ of the gravitational wave:

$$M_{\mathrm{chirp}} := \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}} = \frac{c^3}{G} \left(\frac{5\,\dot{f}}{96\,\pi^{8/3}\,f^{11/3}} \right)^{3/5} .$$

In the Newtonian approximation for the source motion, which we have used above, it is impossible to extract from the observed data an expression that involves M_1 and M_2 alone, but in the lowest-order post-Newtonian approximation it is: f and $\dot{f}$ are directly measurable, and from these data one can calculate the chirp mass. For GW150914 one finds that $M_{\mathrm{chirp}} \approx 30\,M_\odot$. As $(M_1 + M_2)^2 \geq 4M_1 M_2$, the chirp mass gives a lower bound for the sum of the two masses,

$$M_1 + M_2 \geq 4^{3/5}\,M_{\mathrm{chirp}} \approx 70\,M_\odot\,.$$

This rules out the possibility that it was a merger of two neutron stars: The heaviest neutron star that has been detected so far has a mass of just over 2 $M_\odot$. Conservative estimates indicate that this is close to the maximal mass a neutron star can have, and even (speculative) theories about exotic equations of state for neutron stars certainly limit their masses to well below 10 $M_\odot$.

Could it have been a merger of a black hole with a neutron star? Then it would have been a black hole of not much less than 70 $M_\odot$. For such a binary, the frequency of the inspiralling before the merger would have been much lower than 75 Hz, so this doesn't work either.

Could it have been a merger of some exotic objects, e.g. the merger of two boson stars? To be honest, this possibility cannot be completely ruled out on the basis of the observed data. However, boson stars (and all other alternatives that might come to mind) are very exotic objects; most astrophysicists do not believe that they exist. By contrast, there is now overwhelming evidence that black holes do exist. Even very down-to-earth observing astronomers speak about black holes as if they were the most natural things in the world. (This was completely different 20 years ago.) Using Occam's razor, it seems therefore well justified to assume that GW150914 was a black-hole merger. If one accepts this idea as a working hypothesis, it is amazing how well everything fits together.

For black-hole mergers, templates for emitted gravitational waves have been produced numerically, for various choices of the parameters. This important numerical work is supported by analytical approximation methods, using post-Newtonian expansions up to a certain order. (Unfortunately, these analytical approximation methods are so awfully involved that I feel unable to present them here.) Comparing the GW150914 data with these templates led to the conclusion that the observations can be very well explained if one assumes that the gravitational wave signal was produced by a black-hole merger with the following data:

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

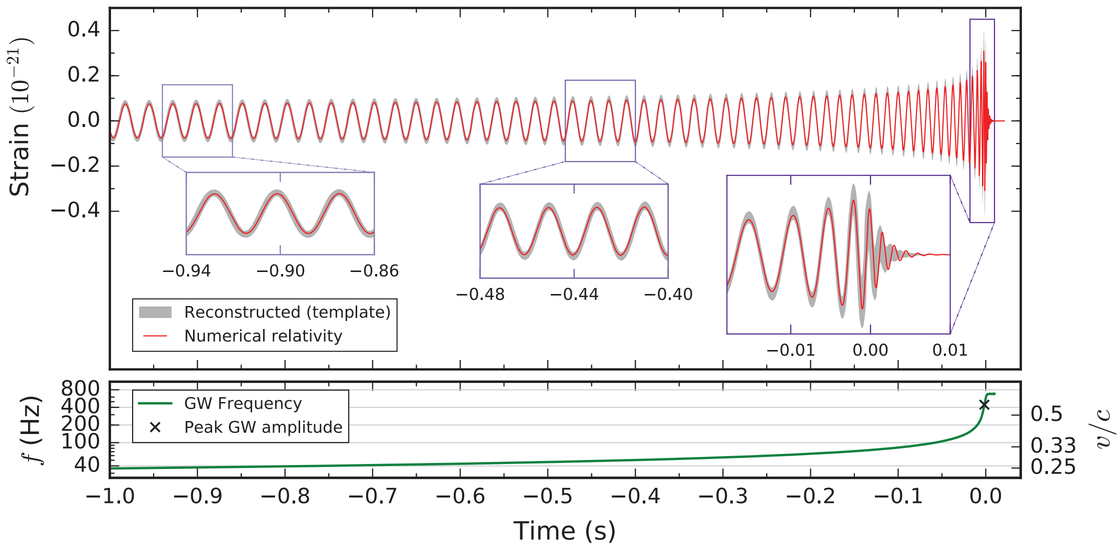
from B. Abbott et al., Phys. Rev. Lett. 116, 061102 (2016)

Here the (luminosity) distance has been calculated from Einstein's quadrupole formula: If the masses and the orbital elements have been determined by fitting the data to an appropriate template, the amplitude of the gravitational wave determines the distance.

From the table one reads that the sum of the two progenitor black holes exceeds the mass of the final black hole by about three solar masses: In other words, during the merger, whose essential phase lasted less than 20 ms, the energy equivalent of 3 solar masses was radiated away. This means that GW150914 was by far the most powerful event in our universe that has ever been observed. The radiated power (energy per time) is about 50 times the power that is emitted, in terms of electromagnetic radiation, by all visible stars in all galaxies

An intensive search has taken place for electromagnetic counter-parts of GW150914, i.e., one has been looking for electromagnetic signals (from the radio over the optical to the gamma ray regime) emitted at the same time from the same region. The only candidate was a very weak gamma-ray burst, observed by the Fermi Gamma-Ray Space Telescope within a millisecond of GW150914 from a location in the sky that is within the region from which GW150914 could have come. However, this signal was very weak, and it was not observed by the INTEGRAL and the AGILE instruments which were operational at the time; therefore, it is most likely that it was just some background and not a real gamma-ray burst. Actually, we wouldn't expect a black-hole merger to be accompanied by a strong electromagnetic signal. It is different for merging neutron stars: A neutron star has a surface, and if something hits this surface this is likely to produce an electromagnetic (X-ray) signal.

We now turn to the second gravitational wave signal GW151226 which is known as the "boxing day event". (In the UK and in the US 26 December is called the boxing day, referring to the gift boxes that are unpacked on this date.)



from B. Abbott et al., Phys. Rev. Lett. 116, 241103 (2016)

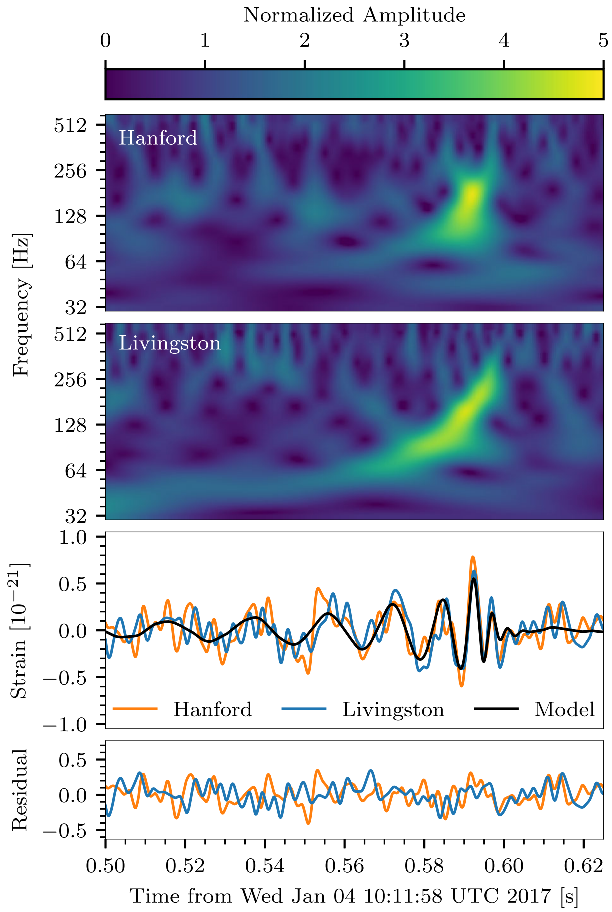
The picture above shows that, in comparison to GW150914, the amplitude was lower but the event lasted longer (about a second in comparison to 0.2 seconds). Whereas GW150914 could be followed for about 8 cycles before the merger, it was 55 cycles for the boxing day event. This already indicates that the masses involved were lower.

The table below shows the parameters that, according to comparison with the templates, fits the data best.

Primary black hole mass	$14.2^{+8.3}_{-3.7} M_\odot$
Secondary black hole mass	$7.5^{+2.3}_{-2.3} M_\odot$
Chirp mass	$8.9^{+0.3}_{-0.3} M_\odot$
Total black hole mass	$21.8^{+5.9}_{-1.7} M_\odot$
Final black hole mass	$20.8^{+6.1}_{-1.7} M_\odot$
Radiated gravitational-wave energy	$1.0^{+0.1}_{-0.2} M_\odot c^2$
Peak luminosity	$3.3^{+0.8}_{-1.6} \times 10^{56}$ erg/s
Final black hole spin	$0.74^{+0.06}_{-0.06}$
Luminosity distance	440^{+180}_{-190} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

from B. Abbott et al., Phys. Rev. Lett. 116, 241103 (2016)

The third event, GW170104, was announced on 1 June 2017 (one day before this lecture was delivered). The black hole masses are in between the ones from GW150914 and GW151226, and the distance is more than twice as much.



from B. Abbott et al., Phys. Rev. Lett. 118, 221101 (2017)

Primary black hole mass m_1	$31.2^{+8.4}_{-6.0} M_\odot$
Secondary black hole mass m_2	$19.4^{+5.3}_{-5.9} M_\odot$
Chirp mass $\mathcal{M}$	$21.1^{+2.4}_{-2.7} M_\odot$
Total mass M	$50.7^{+5.9}_{-5.0} M_\odot$
Final black hole mass M_f	$48.7^{+5.7}_{-4.6} M_\odot$
Radiated energy $E_{\rm rad}$	$2.0^{+0.6}_{-0.7} M_\odot c^2$
Peak luminosity $\ell_{\rm peak}$	$3.1^{+0.7}_{-1.3} \times 10^{56} \mathrm{erg\,s^{-1}}$
Effective inspiral spin parameter $\chi_{\rm eff}$	$-0.12^{+0.21}_{-0.30}$
Final black hole spin a_f	$0.64^{+0.09}_{-0.20}$
Luminosity distance D_L	880^{+450}_{-390} Mpc
Source redshift z	$0.18^{+0.08}_{-0.07}$

from B. Abbott et al., Phys. Rev. Lett. 118, 221101 (2017)

They use to say: Two observations may be a coincidence, but three observations are a pattern. We now clearly see the pattern, and it gave us a few surprises. To be sure, the fact that there are gravitational waves was no surprise at all. More or less everybody in the field was convinced that gravitational waves exist, at least since the observations of the Hulse-Taylor pulsar. The surprises are the following: Black holes with masses between 50 and 100 solar masses seem to be abundant, and mergers of such black holes (in galaxies that are a few hundred Megaparsecs away) seem to be more likely to observe than mergers of neutron stars (within our galaxy). Before the gravitational-wave observations we had good evidence for the existence of stellar black holes with masses between 1 solar mass and 25 solar masses and of supermassive black holes with masses of at least a few million solar masses. Stellar black holes in the range between 50 and 100 solar masses had not been expected. This could have important consequences. Even the debate of whether or not the mysterious "dark matter" could be made up of black holes has been revived, see S. Bird et al. ["Did LIGO detect dark matter?" Phys. Rev. Lett. 116, 201301 (2016)], co-authored by Nobel-prize laureate Adam Riess.

With Virgo, KAGRA and LIGO India coming online soon, we expect the observation of gravitational waves to become a matter of routine within a few years. This will give us a "second eye" (in addition to the observation of electromagnetic signals) to the universe. Clearly, this will allow us to study events in the universe about which we wouldn't have received any information with the help of ordinary (radio, optical, X-ray, ...) telescopes. It is this perspective of establishing a "gravitational wave astronomy" that gives the enormous importance to the LIGO discovery of gravitational waves.

6 Gravitational waves in the linearised theory around curved spacetime

In this section we consider gravitational waves that are small perturbations of a *curved* background. We will in particular treat the case that the background is the Schwarzschild spacetime. However, we begin by deriving the linearised field equation around an unspecified curved background

6.1 Linearisation of Einstein's field equation on a curved background

We assume that the metric is of the form

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$$

where $\overline{g}_{\mu\nu}$ is an arbitray Lorentzian (background) metric and the perturbation is assumed to be so small that all terms of second or higher order with respect to $h_{\mu\nu}$ or its derivatives can be neglected.

We want to work out the field equation (without a cosmological constant),

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \kappa T_{\mu\nu}\,,$$

in this linearised theory. As a first step, we have to calculate the Christoffel symbols.

For this entire chapter we agree to raise and to lower indices with the background metric. Then, to within our linear approximation, the inverse metric is of the form

$$g^{\nu\rho} = \overline{g}^{\nu\rho} - h^{\nu\rho}\,.$$

Proof: $g_{\mu\nu}\big(\overline{g}^{\nu\rho} - h^{\nu\rho}\big) = \big(\overline{g}_{\mu\nu} + h_{\mu\nu}\big)\big(\overline{g}^{\nu\rho} - h^{\nu\rho}\big) = \delta^\rho_\mu + h_\mu{}^\rho - h_\mu{}^\rho = \delta^\rho_\mu$.

Let $\overline{\Gamma}^\rho{}_{\mu\nu}$ denote the Christoffel symbols of the background metric and let $\overline{\nabla}$ be the covariant derivative with respect to the background metric. Then the Christoffel symbols of the perturbed spacetime are

$$\Gamma^\rho{}_{\mu\nu} \;=\; \frac{1}{2} g^{\rho\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big)$$

$$= \frac{1}{2}\big(\overline{g}^{\rho\sigma} - h^{\rho\sigma}\big)\Big(\partial_\mu\big(\overline{g}_{\sigma\nu} + h_{\sigma\nu}\big) + \partial_\nu\big(\overline{g}_{\sigma\mu} + h_{\sigma\mu}\big) - \partial_\sigma\big(\overline{g}_{\mu\nu} + h_{\mu\nu}\big)\Big)$$

$$= \frac{1}{2}\overline{g}^{\rho\sigma}\big(\partial_\mu\overline{g}_{\sigma\nu} + \partial_\nu\overline{g}_{\sigma\mu} - \partial_\sigma\overline{g}_{\mu\nu}\big) + \frac{1}{2}\overline{g}^{\rho\sigma}\big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\big) - \frac{1}{2}h^{\rho\sigma}\big(\partial_\mu\overline{g}_{\sigma\nu} + \partial_\nu\overline{g}_{\sigma\mu} - \partial_\sigma\overline{g}_{\mu\nu}\big)$$

$$= \overline{\Gamma}^\rho{}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\lambda}\big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\big) - \frac{1}{2}\overline{g}^{\rho\lambda}\overline{g}^{\sigma\tau}h_{\lambda\tau}\big(\partial_\mu\overline{g}_{\sigma\nu} + \partial_\nu\overline{g}_{\sigma\mu} - \partial_\sigma\overline{g}_{\mu\nu}\big)$$

$$= \overline{\Gamma}^\rho{}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\lambda}\Big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} - 2\overline{\Gamma}^\tau{}_{\mu\nu}h_{\lambda\tau}\Big)$$

$$= \overline{\Gamma}^\rho{}_{\mu\nu} + \frac{1}{2}\overline{g}^{\rho\lambda}\Big(\overline{\nabla}_\mu h_{\lambda\nu} + \cancel{\overline{\Gamma}^\tau{}_{\mu\lambda}h_{\tau\nu}} + \cancel{\overline{\Gamma}^\tau{}_{\mu\nu}h_{\lambda\tau}} + \overline{\nabla}_\nu h_{\lambda\mu} + \cancel{\overline{\Gamma}^\tau{}_{\nu\lambda}h_{\tau\mu}} + \cancel{\overline{\Gamma}^\tau{}_{\nu\mu}h_{\lambda\tau}}$$

$$-\overline{\nabla}_\lambda h_{\mu\nu} - \cancel{\overline{\Gamma}^\tau{}_{\lambda\mu}h_{\tau\nu}} - \cancel{\overline{\Gamma}^\tau{}_{\lambda\nu}h_{\mu\tau}} - \cancel{2\overline{\Gamma}^\tau{}_{\mu\nu}h_{\lambda\tau}}\Big) \;=\; \overline{\Gamma}^\rho{}_{\mu\nu} + \frac{1}{2}\Big(\overline{\nabla}_\mu h^\rho{}_\nu + \overline{\nabla}_\nu h^\rho{}_\mu - \overline{\nabla}^\rho h_{\mu\nu}\Big)$$

We write this result as

$$\Gamma^\rho{}_{\mu\nu} \;=\; \overline{\Gamma}^\rho{}_{\mu\nu} + \delta\Gamma^\rho{}_{\mu\nu}$$

where

$$\delta\Gamma^\rho{}_{\mu\nu} \;=\; \frac{1}{2}\Big(\overline{\nabla}_\mu h^\rho{}_\nu + \overline{\nabla}_\nu h^\rho{}_\mu - \overline{\nabla}^\rho h_{\mu\nu}\Big)$$

is a tensor field. (Recall that the *difference* of the Christoffel symbols of two connections is a tensor field.)

Next we calculate the curvature tensor.

$$R^\mu{}_{\nu\rho\sigma} \;=\; \partial_\nu\Gamma^\mu{}_{\rho\sigma} \;-\; \partial_\rho\Gamma^\mu{}_{\nu\sigma} \;+\; \Gamma^\mu{}_{\nu\kappa}\Gamma^\kappa{}_{\rho\sigma} \;-\; \Gamma^\mu{}_{\rho\kappa}\Gamma^\kappa{}_{\nu\sigma} \;=\; \partial_\nu\Big(\overline{\Gamma}^\mu{}_{\rho\sigma} + \delta\Gamma^\mu{}_{\rho\sigma}\Big) \;-\; \partial_\rho\Big(\overline{\Gamma}^\mu{}_{\nu\sigma} + \delta\Gamma^\mu{}_{\nu\sigma}\Big)$$

$$+\Big(\overline{\Gamma}^\mu{}_{\nu\kappa} + \delta\Gamma^\mu{}_{\nu\kappa}\Big)\Big(\overline{\Gamma}^\kappa{}_{\rho\sigma} + \delta\Gamma^\kappa{}_{\rho\sigma}\Big) \;-\; \Big(\overline{\Gamma}^\mu{}_{\rho\kappa} + \delta\Gamma^\mu{}_{\rho\kappa}\Big)\Big(\overline{\Gamma}^\kappa{}_{\nu\sigma} + \delta\Gamma^\kappa{}_{\nu\sigma}\Big)$$

$$= \overline{R}^\mu{}_{\nu\rho\sigma} \;+\; \partial_\nu\delta\Gamma^\mu{}_{\rho\sigma} \;-\; \partial_\rho\delta\Gamma^\mu{}_{\nu\sigma} \;+\; \overline{\Gamma}^\mu{}_{\nu\kappa}\delta\Gamma^\kappa{}_{\rho\sigma} \;+\; \overline{\Gamma}^\kappa{}_{\rho\sigma}\delta\Gamma^\mu{}_{\nu\kappa} \;-\; \overline{\Gamma}^\mu{}_{\rho\kappa}\delta\Gamma^\kappa{}_{\nu\sigma} \;-\; \overline{\Gamma}^\kappa{}_{\nu\sigma}\delta\Gamma^\mu{}_{\rho\kappa}\,.$$

This gives us the curvature tensor in the form

$$R^\mu{}_{\nu\rho\sigma} \;=\; \overline{R}^\mu{}_{\nu\rho\sigma} \;+\; \delta R^\mu{}_{\nu\rho\sigma}$$

where

$$\delta R^\mu{}_{\nu\rho\sigma} \;=\; \partial_\nu \delta\Gamma^\mu{}_{\rho\sigma} \;-\; \partial_\rho \delta\Gamma^\mu{}_{\nu\sigma} \;+\; \overline{\Gamma}^\mu{}_{\nu\kappa}\delta\Gamma^\kappa{}_{\rho\sigma} \;+\; \overline{\Gamma}^\kappa{}_{\rho\sigma}\delta\Gamma^\mu{}_{\nu\kappa} \;-\; \overline{\Gamma}^\mu{}_{\rho\kappa}\delta\Gamma^\kappa{}_{\nu\sigma} \;-\; \overline{\Gamma}^\kappa{}_{\nu\sigma}\delta\Gamma^\mu{}_{\rho\kappa}$$

$$= \;\overline{\nabla}_\nu \delta\Gamma^\mu{}_{\rho\sigma} \;-\; \overline{\Gamma}^\mu{}_{\nu\kappa}\delta\overline{\Gamma}^\kappa{}_{\rho\sigma} \;+\; \overline{\Gamma}^\kappa{}_{\nu\rho}\delta\overline{\Gamma}^\mu{}_{\kappa\sigma} \;+\; \overline{\Gamma}^\kappa{}_{\nu\sigma}\delta\overline{\Gamma}^\mu{}_{\kappa\rho}$$

$$-\,\overline{\nabla}_\rho \delta\Gamma^\mu{}_{\nu\sigma} \;+\; \overline{\Gamma}^\mu{}_{\rho\kappa}\delta\Gamma^\kappa{}_{\nu\sigma} \;-\; \overline{\Gamma}^\kappa{}_{\rho\nu}\delta\Gamma^\mu{}_{\kappa\sigma} \;-\; \overline{\Gamma}^\kappa{}_{\rho\sigma}\delta\Gamma^\mu{}_{\kappa\nu}$$

$$+\,\overline{\Gamma}^\mu{}_{\nu\kappa}\delta\Gamma^\kappa{}_{\rho\sigma} \;+\; \overline{\Gamma}^\kappa{}_{\rho\sigma}\delta\Gamma^\mu{}_{\nu\kappa} \;-\; \overline{\Gamma}^\mu{}_{\rho\kappa}\delta\Gamma^\kappa{}_{\nu\sigma} \;-\; \overline{\Gamma}^\kappa{}_{\nu\sigma}\delta\Gamma^\mu{}_{\rho\kappa}$$

$$= \;\overline{\nabla}_\nu \delta\Gamma^\mu{}_{\rho\sigma} - \overline{\nabla}_\rho \delta\Gamma^\mu{}_{\nu\sigma}\,.$$

With our earlier result for $\delta\Gamma^\mu{}_{\rho\sigma}$ this can be rewritten as

$$\delta R^\mu{}_{\nu\rho\sigma} \;=\; \frac{1}{2}\,\overline{\nabla}_\nu\Big(\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\sigma h^\mu{}_\rho - \overline{\nabla}^\mu h_{\rho\sigma}\Big) - \frac{1}{2}\,\overline{\nabla}_\rho\Big(\overline{\nabla}_\nu h^\mu{}_\sigma + \overline{\nabla}_\sigma h^\mu{}_\nu - \overline{\nabla}^\mu h_{\nu\sigma}\Big)$$

$$= \;\frac{1}{2}\left(\overline{\nabla}_\nu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\nu\overline{\nabla}_\sigma h^\mu{}_\rho - \overline{\nabla}_\nu\overline{\nabla}^\mu h_{\rho\sigma} - \overline{\nabla}_\rho\overline{\nabla}_\nu h^\mu{}_\sigma - \overline{\nabla}_\rho\overline{\nabla}_\sigma h^\mu{}_\nu + \overline{\nabla}_\rho\overline{\nabla}^\mu h_{\nu\sigma}\right).$$

Contraction gives the Ricci tensor

$$R_{\rho\sigma} \;=\; \overline{R}_{\rho\sigma} \;+\; \delta R_{\rho\sigma}$$

where

$$\delta R_{\rho\sigma} \;=\; \delta R^\mu{}_{\mu\rho\sigma} \;=\; \frac{1}{2}\left(\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho - \overline{\nabla}_\mu\overline{\nabla}^\mu h_{\rho\sigma} - \cancel{\overline{\nabla}_\rho\overline{\nabla}_\mu h^\mu{}_\sigma} - \overline{\nabla}_\rho\overline{\nabla}_\sigma h^\mu{}_\mu + \cancel{\overline{\nabla}_\rho\overline{\nabla}^\mu h_{\mu\sigma}}\right)$$

$$= \;\frac{1}{2}\left(\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho - \overline{\Box} h_{\rho\sigma} - \overline{\nabla}_\rho\overline{\nabla}_\sigma h\right).$$

Here we have introduced the trace of the perturbation,

$$h \;=\; h^\mu{}_\mu \;=\; \overline{g}^{\mu\nu} h_{\mu\nu}\,,$$

and the wave operator of the background metric,

$$\overline{\Box} \;=\; \overline{\nabla}_\mu\overline{\nabla}^\mu \;=\; \overline{g}^{\mu\nu}\overline{\nabla}_\mu\overline{\nabla}_\nu\,.$$

If we contract another time we get the Ricci scalar

$$R \;=\; \overline{R} \;+\; \delta R$$

with

$$\delta R \;=\; \delta R^\rho{}_\rho \;=\; \frac{1}{2}\left(\overline{\nabla}_\mu\overline{\nabla}_\rho h^{\mu\rho} + \overline{\nabla}_\mu\overline{\nabla}^\rho h^\mu{}_\rho - \overline{\Box} h_\rho{}^\rho - \overline{\nabla}_\rho\overline{\nabla}^\rho h\right) \;=\; \overline{\nabla}_\mu\overline{\nabla}_\rho h^{\mu\rho} - \overline{\Box} h\,.$$

Finally, the Einstein tensor $G_{\rho\sigma} = R_{\rho\sigma} - \dfrac{R}{2} g_{\rho\sigma}$ reads

$$G_{\rho\sigma} \;=\; \overline{G}_{\rho\sigma} \;+\; \delta G_{\rho\sigma}$$

where

$$\delta G_{\rho\sigma} \;=\; \delta\Big(R_{\rho\sigma} \;-\; \frac{1}{2}\,R\,g_{\rho\sigma}\Big)$$

$$=\; \delta R_{\rho\sigma} \;-\; \frac{1}{2}\,R\,\delta g_{\rho\sigma} - \frac{1}{2}\,\delta R\,g_{\rho\sigma} \;=\; \delta R_{\rho\sigma} \;-\; \frac{1}{2}\,\overline{R}\,h_{\rho\sigma} - \frac{1}{2}\,\delta R\,\overline{g}_{\rho\sigma}$$

$$=\; \frac{1}{2}\,\Big(\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho - \overline{\Box} h_{\rho\sigma} - \overline{\nabla}_\rho\overline{\nabla}_\sigma h - \overline{R}\,h_{\rho\sigma} - \overline{g}_{\rho\sigma}\big(\overline{\nabla}_\mu\overline{\nabla}_\nu h^{\mu\nu} - \overline{\Box} h\,.\big)\Big)\,.$$

Now we use that the commutator of covariant derivatives can be expressed in terms of the curvature tensor,

$$\big(\overline{\nabla}_\mu\overline{\nabla}_\rho - \overline{\nabla}_\rho\overline{\nabla}_\mu\big)h^\mu{}_\sigma \;=\; \overline{R}^\tau{}_{\mu\rho\sigma}h^\mu{}_\tau - \overline{R}^\mu{}_{\mu\rho\tau}h^\tau{}_\sigma\;.$$

We find

$$\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho \;=\; \overline{\nabla}_\rho\overline{\nabla}_\mu h^\mu{}_\sigma + \overline{R}^\tau{}_{\mu\rho\sigma}h^\mu{}_\tau - \overline{R}^\mu{}_{\mu\rho\tau}h^\tau{}_\sigma + \overline{\nabla}_\sigma\overline{\nabla}_\mu h^\mu{}_\rho + \overline{R}^\tau{}_{\mu\sigma\rho}h^\mu{}_\tau - \overline{R}^\mu{}_{\mu\sigma\tau}h^\tau{}_\rho\,.$$

This can be rewritten, with the first Bianchi identity

$$\overline{R}^\tau{}_{\mu\sigma\rho} + \overline{R}^\tau{}_{\rho\mu\sigma} + \overline{R}^\tau{}_{\sigma\rho\mu} \;=\; 0$$

as

$$\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho \;=\; \overline{\nabla}_\rho\overline{\nabla}_\mu h^\mu{}_\sigma + \overline{\nabla}_\sigma\overline{\nabla}_\mu h^\mu{}_\rho + \overline{R}^\tau{}_{\mu\rho\sigma}h^\mu{}_\tau - \overline{R}^\tau{}_{\rho\mu\sigma}h^\mu{}_\tau - \overline{R}^\tau{}_{\sigma\rho\mu}h^\mu{}_\tau - \overline{R}_{\rho\tau}h^\tau{}_\sigma - \overline{R}_{\sigma\tau}h^\tau{}_\rho$$

and, further, with the curvature identity

$$\overline{R}^\tau{}_{\rho\mu\sigma} \;=\; -\,\overline{R}^\tau{}_{\mu\rho\sigma}\,,$$

as

$$\overline{\nabla}_\mu\overline{\nabla}_\rho h^\mu{}_\sigma + \overline{\nabla}_\mu\overline{\nabla}_\sigma h^\mu{}_\rho \;=\; \overline{\nabla}_\rho\overline{\nabla}_\mu h^\mu{}_\sigma + \overline{\nabla}_\sigma\overline{\nabla}_\mu h^\mu{}_\rho + \;2\,\overline{R}^\tau{}_{\mu\rho\sigma}h^\mu{}_\tau - \underbrace{\overline{R}_{\tau\sigma\rho\mu}h^{\mu\tau}}_{=0} - \overline{R}_{\rho\tau}h^\tau{}_\sigma - \overline{R}_{\sigma\tau}h^\tau{}_\rho$$

where the underbraced term vanishes because of the curvature identity $\overline{R}_{\tau\sigma\rho\mu} = -\overline{R}_{\mu\sigma\rho\tau}$. Inserting this expression into our result for $\delta G_{\mu\nu}$ gives

$$2\,\delta G_{\rho\sigma} \;=\; \overline{\nabla}_\rho\overline{\nabla}_\mu h^\mu{}_\sigma + \overline{\nabla}_\sigma\overline{\nabla}_\mu h^\mu{}_\rho + \;2\,\overline{R}^\tau{}_{\mu\rho\sigma}h^\mu{}_\tau - \overline{R}_{\rho\tau}h^\tau{}_\sigma - \overline{R}_{\sigma\tau}h^\tau{}_\rho$$

$$-\overline{\Box} h_{\rho\sigma} - \overline{\nabla}_\rho\overline{\nabla}_\sigma h - \overline{R}\,h_{\rho\sigma} - \overline{g}_{\rho\sigma}\big(\overline{\nabla}_\mu\overline{\nabla}_\nu h^{\mu\nu} - \overline{\Box} h\big)\,.$$

This is the general expression for $\delta G_{\mu\nu}$ on an arbitrary background spacetime. It gives us the linearised field equation in the form

$$\overline{G}_{\rho\sigma} + \delta G_{\rho\sigma} = \kappa\,T_{\rho\sigma}\,.$$

From now on we specify to the case that the background spacetime satisfies the vacuum field equation (without a cosmological constant),

$$\overline{R}_{\mu\nu} = 0 \, .$$

Then

$$2\,\delta G_{\rho\sigma} \;=\; \overline{\nabla}_\rho \overline{\nabla}_\mu h^\mu{}_\sigma + \overline{\nabla}_\sigma \overline{\nabla}_\mu h^\mu{}_\rho + \,2\,\overline{R}^\tau{}_{\mu\rho\sigma} h^\mu{}_\tau - \overline{\Box} h_{\rho\sigma} - \overline{\nabla}_\rho \overline{\nabla}_\sigma h - \overline{g}_{\rho\sigma}\big(\overline{\nabla}_\mu \overline{\nabla}_\nu h^{\mu\nu} - \overline{\Box} h\big) \, .$$

This expression can be simplified if we use the gauge freedom. We follow our treatment in the case of a flat background (see p. 8) as closely as possible. We introduce

$$\gamma_{\mu\nu} \;=\; h_{\mu\nu} \;-\; \frac{h}{2}\,\overline{g}_{\mu\nu}$$

which implies

$$\gamma \;:=\; \overline{g}^{\mu\nu}\gamma_{\mu\nu} \;=\; h \;-\; \frac{1}{2}\,4\,h \;=\; -\,h \;,$$

hence

$$h_{\mu\nu} \;=\; \gamma_{\mu\nu} \;-\; \frac{\gamma}{2}\,\overline{g}_{\mu\nu} \; .$$

Then our expression for $\delta G_{\mu\nu}$ reads

$$\begin{aligned}
2\,\delta G_{\rho\sigma} \;=\;& \overline{\nabla}_\rho \overline{\nabla}_\mu \Big(\gamma^\mu{}_\sigma - \frac{\gamma}{2}\delta^\mu_\sigma\Big) + \overline{\nabla}_\sigma \overline{\nabla}_\mu \Big(\gamma^\mu{}_\rho - \frac{\gamma}{2}\delta^\mu_\rho\Big) + \,2\,\overline{R}^\tau{}_{\mu\rho\sigma}\Big(\gamma^\mu{}_\tau - \frac{\gamma}{2}\delta^\mu_\tau\Big) \\
& - \overline{\Box}\Big(\gamma_{\rho\sigma} - \frac{\gamma}{2}\overline{g}_{\rho\sigma}\Big) + \overline{\nabla}_\rho \overline{\nabla}_\sigma \gamma - \overline{g}_{\rho\sigma}\Big(\overline{\nabla}_\mu \overline{\nabla}_\nu \gamma^{\mu\nu} - \overline{\nabla}_\mu \overline{\nabla}_\nu \frac{\gamma}{2}\overline{g}^{\mu\nu} + \overline{\Box}\gamma\Big) \\
=\;& \overline{\nabla}_\rho \overline{\nabla}_\mu \gamma^\mu{}_\sigma - \cancel{\frac{1}{2}\overline{\nabla}_\rho \overline{\nabla}_\sigma \gamma} + \overline{\nabla}_\sigma \overline{\nabla}_\mu \gamma^\mu{}_\rho - \cancel{\frac{1}{2}\overline{\nabla}_\sigma \overline{\nabla}_\rho \gamma} + \,2\,\overline{R}^\tau{}_{\mu\rho\sigma}\gamma^\mu{}_\tau - \gamma\,\underbrace{\overline{R}^\mu{}_{\mu\rho\sigma}}_{=0} \\
& - \overline{\Box}\gamma_{\rho\sigma} + \cancel{\frac{1}{2}\overline{g}_{\rho\sigma}\overline{\Box}\gamma} + \cancel{\overline{\nabla}_\rho \overline{\nabla}_\sigma \gamma} - \overline{g}_{\rho\sigma}\overline{\nabla}_\mu \overline{\nabla}_\nu \gamma^{\mu\nu} + \cancel{\frac{1}{2}\overline{g}_{\rho\sigma}\overline{\Box}\gamma} - \cancel{\overline{g}_{\rho\sigma}\overline{\Box}\gamma} \\
=\;& -\overline{\Box}\gamma_{\rho\sigma} + \,2\,\overline{R}^\tau{}_{\mu\rho\sigma}\gamma^\mu{}_\tau + \overline{\nabla}_\rho \overline{\nabla}_\mu \gamma^\mu{}_\sigma + \overline{\nabla}_\sigma \overline{\nabla}_\mu \gamma^\mu{}_\rho - \overline{g}_{\rho\sigma}\overline{\nabla}_\mu \overline{\nabla}_\nu \gamma^{\mu\nu} \, .
\end{aligned}$$

The last three terms all involve the covariant divergence $\overline{\nabla}_\mu \gamma^{\mu\nu}$. We will now demonstrate that this can be transformed to zero by a gauge transformation, generalising the Hilbert gauge we already know from the linearisation around flat spacetime.

As before, by a gauge transformation we mean a coordinate transformation

$$x^\mu \mapsto x^\mu + f^\mu(x)$$

where $f^\mu(x)$ is small of first order, i.e., so small that only terms linear in f^μ and its derivatives have to be kept.

Then the metric transforms as

$$g_{\mu\nu}(x)dx^\mu dx^\nu \mapsto g_{\mu\nu}\big(x+f\big)\big(dx^\mu + \partial_\rho f^\mu(x)dx^\rho\big)\big(dx^\nu + \partial_\sigma f^\nu(x)dx^\sigma\big)$$

$$= g_{\mu\nu}(x)dx^\mu dx^\nu + \partial_\sigma g_{\mu\nu}(x)f^\sigma(x)dx^\mu dx^\nu + \partial_\rho f^\mu(x)g_{\mu\nu}(x)dx^\rho dx^\nu + \partial_\sigma f^\nu(x)g_{\mu\nu}(x)dx^\mu dx^\sigma + \ \ldots$$

$$= \Big(g_{\mu\nu}(x) + \partial_\sigma g_{\mu\nu}(x)f^\sigma(x) + \partial_\mu f^\rho(x)g_{\rho\nu}(x) + \partial_\nu f^\sigma(x)g_{\mu\sigma}(x)\Big)dx^\mu dx^\nu$$

i.e.

$$\overline{g}_{\mu\nu} + h_{\mu\nu} \mapsto \overline{g}_{\mu\nu} + h_{\mu\nu} + f^\sigma\partial_\sigma\overline{g}_{\mu\nu} + \overline{g}_{\rho\nu}\partial_\mu f^\rho + \overline{g}_{\mu\sigma}\partial_\nu f^\sigma + \ldots$$

hence

$$h_{\mu\nu} \ \mapsto \ h_{\mu\nu} + f^\sigma\partial_\sigma\overline{g}_{\mu\nu} + \overline{g}_{\rho\nu}\partial_\mu f^\rho + \overline{g}_{\mu\sigma}\partial_\nu f^\sigma + \ldots$$

$$= h_{\mu\nu} + f^\tau\partial_\tau\overline{g}_{\mu\nu} + \overline{g}_{\rho\nu}\big(\overline{\nabla}_\mu f^\rho - \overline{\Gamma}^\rho{}_{\mu\tau}f^\tau\big) + \overline{g}_{\mu\sigma}\big(\overline{\nabla}_\nu f^\sigma - \overline{\Gamma}^\sigma{}_{\nu\tau}f^\tau\big)$$

$$= h_{\mu\nu}+\overline{\nabla}_\mu f_\nu+\overline{\nabla}_\nu f_\mu+f^\tau\Big(\partial_\tau\overline{g}_{\mu\nu}-\frac{1}{2}\overline{g}_{\rho\nu}\overline{g}^{\rho\sigma}\big(\partial_\mu\overline{g}_{\sigma\tau}+\partial_\tau\overline{g}_{\sigma\mu}-\partial_\sigma\overline{g}_{\mu\tau}\big)-\frac{1}{2}\overline{g}_{\mu\sigma}\overline{g}^{\sigma\rho}\big(\partial_\nu\overline{g}_{\rho\tau}+\partial_\tau\overline{g}_{\rho\nu}-\partial_\rho\overline{g}_{\nu\tau}\big)\Big)$$

$$= h_{\mu\nu} + \overline{\nabla}_\mu f_\nu + \overline{\nabla}_\nu f_\mu + f^\tau\Big(\cancel{\partial_\tau\overline{g}_{\mu\nu}} - \frac{1}{2}\big(\cancel{\partial_\mu\overline{g}_{\nu\tau}} + \cancel{\partial_\tau\overline{g}_{\nu\mu}} - \cancel{\partial_\nu\overline{g}_{\mu\tau}}\big) - \frac{1}{2}\big(\cancel{\partial_\nu\overline{g}_{\mu\tau}} + \cancel{\partial_\tau\overline{g}_{\mu\nu}} - \cancel{\partial_\mu\overline{g}_{\nu\tau}}\big)\Big)$$

$$= h_{\mu\nu} + \overline{\nabla}_\mu f_\nu + \overline{\nabla}_\nu f_\mu\,.$$

This implies

$$h \ \mapsto \ h + 2\overline{\nabla}_\sigma f^\sigma$$

or equivalently

$$\gamma \ \mapsto \ \gamma - 2\overline{\nabla}_\sigma f^\sigma\,,$$

and

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\overline{g}_{\mu\nu} \ \mapsto \ h_{\mu\nu} + \overline{\nabla}_\mu f_\nu + \overline{\nabla}_\nu f_\mu - \frac{1}{2}\big(h + 2\overline{\nabla}_\sigma f^\sigma\big)\overline{g}_{\mu\nu} = \gamma_{\mu\nu} + \overline{\nabla}_\mu f_\nu + \overline{\nabla}_\nu f_\mu - \overline{\nabla}_\sigma f^\sigma\overline{g}_{\mu\nu}\,.$$

Hence, the covariant divergence of $\gamma_{\mu\nu}$ transforms according to

$$\overline{\nabla}^\mu\gamma_{\mu\nu} \mapsto \overline{\nabla}^\mu\gamma_{\mu\nu} + \overline{\nabla}^\mu\overline{\nabla}_\mu f_\nu + \overline{\nabla}^\mu\overline{\nabla}_\nu f_\mu - \overline{\nabla}^\mu\overline{\nabla}_\sigma f^\sigma\overline{g}_{\mu\nu}$$

$$= \overline{\nabla}^\mu\gamma_{\mu\nu} + \overline{\Box}f_\nu + \overline{\nabla}_\sigma\overline{\nabla}_\nu f^\sigma - \overline{\nabla}_\nu\overline{\nabla}_\sigma f^\sigma = \overline{\nabla}^\mu\gamma_{\mu\nu} + \overline{\Box}f_\nu + \underbrace{\overline{R}^\sigma{}_{\sigma\nu\tau}}_{=0}f^\tau\,.$$

If f^μ is chosen such that

$$\overline{\Box}f_\nu \ = \ -\overline{\nabla}^\mu\gamma_{\mu\nu}\,, \qquad \text{(WE)}$$

the covariant divergence of $\gamma_{\mu\nu}$ is transformed to zero. If initial data for f_ν and $\partial_0 f_\nu$ are given on a spacelike hypersurface $x^0 =$ constant, a unique solution to the wave equation (WE) exists on a neighbourhood of this hypersurface. If the spacetime is what is called "globally hyperbolic", we may choose this hypersurface such that the solution exists globally. The domain of outer communication of the Schwarzschild spacetime (i.e., the region outside the horizon) is globally hyperbolic. Details on the initial value problem for the wave equation on a curved background can be found in the book by R. Wald ["General Relativity" University of Chicago Press (1984)].

On a globally hyperbolic spacetime with vanishing Ricci tensor we can thus transform $\overline{\nabla}^\mu \gamma_{\mu\nu}$ to zero. This is called the (*generalised*) *Hilbert gauge*. Then the linearised field equation reads

$$-\frac{1}{2}\overline{\Box}\gamma_{\mu\nu} + \overline{R}^\tau{}_{\rho\mu\nu}\gamma_\tau{}^\rho \;=\; \kappa\, T_{\mu\nu}\,, \quad \overline{\nabla}^\mu \gamma_{\mu\nu} = 0\,.$$

This still leaves the freedom of making gauge transformations $x^\mu \to x^\mu + f^\mu(x)$ with $\overline{\Box} f^\mu = 0$.

6.2 "Geometric optics" of gravitational waves on a curved background

We will now discuss how gravitational waves propagate, on a curved background, in the high-frequency limit. This is the gravitational-wave analogue of geometric optics. In the last section we have seen that, in a gauge with $\overline{\nabla}^\mu h_{\mu\nu} = 0$, the linearised vacuum field equation reads

$$\overline{\Box}\gamma_{\mu\nu} - 2\overline{R}^\tau{}_{\rho\mu\nu}\gamma_\tau{}^\rho \;=\; 0\,.$$

On a flat background, we have found plane-wave solutions which are of the form

$$h_{\mu\nu}(x) = \mathrm{Re}\big\{ A_{\mu\nu} e^{k_\sigma x^\sigma} \big\}$$

with a constant complex amplitude $A_{\mu\nu}$ and a constant real wave covector k_μ. In the case of a curved background, exact solutions of this form do not exist. However, we may consider one-parameter families of metric perturbations that describe *approximate* plane-harmonic waves. We write this, with a real parameter α, as

$$\gamma_{\mu\nu}(x) = \mathrm{Re}\Big\{ \Big(a_{\mu\nu}(x) + \alpha\, b_{\mu\nu} + O(\alpha^2) \Big) e^{iS(x)/\alpha} \Big\}$$

where $a_{\mu\nu}(x)$ and $b_{\mu\nu}(x)$ are complex-valued functions of x and $S(x)$ is a real-valued function of x. For the interpretation of this ansatz, we expand the function S on a neighbourhood of a point x_0 in a Taylor series,

$$S(x) = S(x_0) + \overline{\nabla}_\mu S(x_0)\big(x^\mu - x_0^\mu\big) + \dots = S_0 + \overline{\nabla}_\mu S(x_0) x^\mu + \dots$$

where $S_0 = S(x_0) - \overline{\nabla}_\mu S(x_0) x_0^\mu$. We choose the neighbourhood so small that the higher-order terms, indicated by ellipses, may be neglected and that the leading-order amplitude satisfies $a_{\mu\nu}(x) \approx a_{\mu\nu}(x_0)$ on this neighbourhood. For α sufficiently small, our ansatz then gives indeed approximately a plane-harmonic wave with amplitude $a_{\mu\nu}(x_0) e^{iS_0/\alpha}$ and wave covector $k_\mu = \overline{\nabla}_\mu S(x_0)/\alpha$. Clearly, the smaller α the more maxima our wave has in the chosen neighbourhood.

As this construction can be done around *any* point, the function $k_\mu(x) = \overline{\nabla}_\mu S(x)/\alpha$ can be interpreted as the wave covector *field* of our approximate plane-harmonic wave. With respect to an observer with 4-velocity U^μ, we may interpret $-U^\mu \overline{\nabla}_\mu S/\alpha$ as the frequency and the part of $\overline{\nabla}_\mu S/\alpha$ that is perpendicular to U^μ as the spatial wave covector.
We will now assume that our approximate plane-harmomnic wave family satisfies the linearised vacuum field equation, for all α, and we will evaluate this condition in leading and next-to-leading order of $1/\alpha$. The Hilbert gauge condition requires

$$0 = \mathrm{Re}\Big\{ e^{iS/\alpha} \Big(\frac{i}{\alpha} \big(\overline{\nabla}^\mu S \, a_{\mu\nu} + O(\alpha) \big) \Big) \Big\} .$$

For non-constant S, this can be true only if

$$\overline{\nabla}^\mu S a_{\mu\nu} = 0 .$$

The field equation requires

$$0 = \overline{\nabla}^\sigma \mathrm{Re}\Big\{ e^{iS/\alpha} \Big(\frac{i}{\alpha} \overline{\nabla}_\sigma S \, a_{\mu\nu} + \overline{\nabla}_\sigma a_{\mu\nu} + i \overline{\nabla}_\sigma S \, b_{\mu\nu} + O(\alpha) \Big) \Big\} - \mathrm{Re}\Big\{ e^{iS/\alpha} \Big(2 \, \overline{R}^\tau{}_{\rho\mu\nu} \, a^\rho{}_\tau + O(\alpha) \Big) \Big\}$$

$$= \mathrm{Re}\Big\{ e^{iS/\alpha} \Big(-\frac{1}{\alpha^2} \overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma S \, a_{\mu\nu} + \frac{i}{\alpha} \overline{\nabla}_\sigma \overline{\nabla}^\sigma S \, a_{\mu\nu} + \frac{2\,i}{\alpha} \overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma a_{\mu\nu} - \frac{1}{\alpha} \overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma S \, b_{\mu\nu} + O(\alpha^0) \Big) \Big\} .$$

We find in leading order (α^{-2})

$$\overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma S \, a_{\mu\nu} = 0$$

and in next-to-leading order (α^{-1})

$$\overline{\nabla}_\sigma S \, \overline{\nabla}^\sigma a_{\mu\nu} + 2 \overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma a_{\mu\nu} + i \, \overline{\nabla}^\sigma S \, \overline{\nabla}_\sigma S \, b_{\mu\nu} = 0 .$$

As we want to have a wave with non-zero amplitude, the leading order gives us the condition

$$\overline{\nabla}^\mu S \, \overline{\nabla}_\mu S = 0 .$$

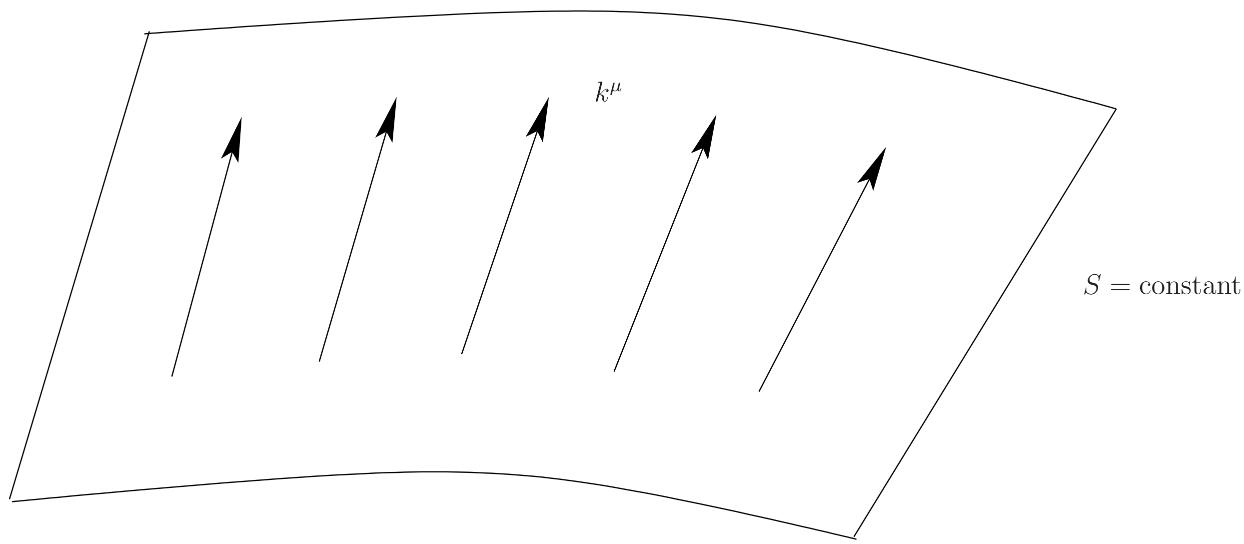
The integral curves of the vector field $k^\mu = \overline{\nabla}^\mu S$ are called the *rays* of the approximate plane-harmonic wave family. The latter condition says that the rays are lightlike geodesics of the background metric.

Proof: Clearly, the condition $0 = \overline{\nabla}^\mu S \, \overline{\nabla}_\mu S = k^\mu k_\mu = \overline{g}_{\mu\nu} k^\mu k^\nu$ says that the integral curves of k^μ are lightlike. We have to prove that they are geodesics, i.e., that $k^\mu \overline{\nabla}_\mu k^\sigma = 0$. To that end, we apply the operator $\overline{\nabla}_\rho$ to both sides of the equation $0 = \overline{g}_{\mu\nu} k^\mu k^\nu$. This results in

$$0 = 2 \overline{g}_{\mu\nu} k^\mu \overline{\nabla}_\rho k^\nu = 2 \, k^\mu \overline{\nabla}_\rho k_\mu = 2 \, k^\mu \overline{\nabla}_\rho \overline{\nabla}_\mu S = 2 \, k^\mu \overline{\nabla}_\mu \overline{\nabla}_\rho S = 2 \, k^\mu \overline{\nabla}_\mu k_\rho$$

where we have used the fact that covariant derivatives commute if they are applied to a scalar function. Multiplication with $\overline{g}^{\rho\sigma}/2$ yields indeed $0 = k^\mu \overline{\nabla}_\mu k^\sigma$. □

The fact that the rays are lightlike geodesics may be interpreted as saying that gravitational waves propagate at the speed of light. Note that the condition $k^\mu \overline{\nabla}_\mu S = 0$ means that the vector field k^μ is tangential to the surfaces $S =$ constant, see the picture on the next page: From ordinary geometry with a positive definite metric we are used to the fact that the gradient of a function S is transverse to the surfaces $S =$ constant; in the case of an indefinite metric, however, the gradient may be lightlike and then it is tangential to the surfaces $S =$ constant.



We now evaluate the next-to-leading order. With the result from the leading order this may be rewritten as

$$\overline{\nabla}^\sigma k_\sigma \, a_{\mu\nu} + 2k^\sigma \overline{\nabla}_\sigma a_{\mu\nu} = 0 \, .$$

This is a *transport law* for the amplitude $a_{\mu\nu}$ along the rays. We may write it in a more convenient form if we multiply $a_{\mu\nu}$ with a scalar function u,

$$k^\sigma \overline{\nabla}_\sigma \big(u \, a_{\mu\nu} \big) = a_{\mu\nu} k^\sigma \overline{\nabla}_\sigma u + u \, k^\sigma \overline{\nabla}_\sigma a_{\mu\nu} s = \Big(k^\sigma \overline{\nabla}_\sigma u - \, u \, \frac{1}{2} \overline{\nabla}_\sigma k^\sigma \Big) a_{\mu\nu} \, .$$

If we choose the function u such that it satisfies along each ray the first-order differential equation

$$k^\sigma \overline{\nabla}_\sigma u = \, u \, \frac{1}{2} \overline{\nabla}_\sigma k^\sigma$$

we have

$$k^\sigma \overline{\nabla}_\sigma \big(u \, a_{\mu\nu} \big) = 0 \, ,$$

i.e., $u \, a_{\mu\nu}$ is parallelly transported along each ray.

We summarise the findings of this section for gravitational waves that travel on a Ricci-flat background: The rays are lightlike geodesics of the background metric, i.e., the gravitational radiation travels at the speed of light. The amplitude is parallely transported along each ray if multiplied with an appropriate normalisation factor.

Note that we have assumed that the background geometry is kept fixed. This scheme is appropriate if the wavelength of the gravitational wave is much shorter than a typical length scale on which the background geometry changes. As an alternative to our scheme, one may assume that the background metric and, thus, the curvature tensor $\overline{R}^\tau{}_{\mu\rho\sigma}$ also depend on the parameter α. If the curvature tensor is proportional to $1/\alpha$, the result that the rays are lightlike geodesics is still true but the transport law for the amplitude is modified. If the curvature tensor is proportional to $1/\alpha^2$, the rays are no longer lightlike geodesics. If the wavelength of the gravitational wave is of the same order of magnitude as a typical curvature radius, or even longer, then our original scheme is not applicable.

6.3 Linearised field equation on Schwarzschild spacetime

In Section 6.1 we have derived the linearised vacuum field equation on a Ricci-flat curved background in the form

$$\overline{\Box} h_{\mu\nu} - 2\overline{R}^{\tau}{}_{\rho\mu\nu} h_{\tau}{}^{\rho} \; = \; 0$$

which is valid only in a gauge such that $\overline{\nabla}^{\mu} h_{\mu\nu} \; = \; 0$ and $h = 0$. As an alternative, we can write the linearised vacuum field equation in the form

$$0 \; = \; \delta R_{\nu\sigma} \; = \; \delta R^{\mu}{}_{\nu\mu\sigma} \; = \; \overline{\nabla}_{\nu} \delta \Gamma^{\mu}{}_{\mu\sigma} - \overline{\nabla}_{\mu} \delta \Gamma^{\mu}{}_{\nu\sigma}$$

which is true in *any* gauge, see p. 83. In the following we will use the latter form because it leaves us the freedom of making arbitrary gauge transformations.

It is our goal to evaluate this equation for the case that the background metric is the Schwarzschild spacetime,

$$\overline{g}_{\mu\nu} dx^{\mu} dx^{\nu} \; = \; - \Big(1 - \frac{r_S}{r} \Big) c^2 dt^2 \; + \; \frac{dr^2}{\Big(1 - \dfrac{r_S}{r} \Big)} \; + \; r^2 \big(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \big)$$

where r_S is parameter with the dimension of a length, called the Schwarzschild radius. Recall that the Schwarzschild metric describes the vacuum region outside a spherically symmetric gravitating body. The Schwarzschild radius is related to the mass M by the equation

$$r_S = \frac{2GM}{c^2} \, .$$

The central body may be a star, in which case the metric is valid in the domain $r_* < r < \infty$ where $r_* > r_S$ denotes the physical radius of the star, or it may be a black hole, in which case the metric is valid in the domain $0 < r < \infty$ with a (coordinate) singularity at $r = r_S$.

For later purpose, we list the non-vanishing Christoffel symbols of the Schwarzschild spacetime:

$$\overline{\Gamma}^t_{rt} \; = \; \overline{\Gamma}^t_{tr} \; = \; \frac{r_S}{2r^2 \Big(1 - \dfrac{r_S}{r} \Big)} \, ,$$

$$\overline{\Gamma}^r_{rr} = \frac{-\, r_S}{2r^2 \Big(1 - \dfrac{r_S}{r} \Big)} \, , \quad \overline{\Gamma}^r_{tt} = \frac{c^2 r_S}{2r^2} \Big(1 - \frac{r_S}{r} \Big) \, , \quad \overline{\Gamma}^r_{\varphi\varphi} = -r \Big(1 - \frac{r_S}{r} \Big) \sin^2\vartheta \, , \quad \overline{\Gamma}^r_{\vartheta\vartheta} \; = \; -\, r \, \Big(1 - \frac{r_S}{r} \Big)$$

$$\overline{\Gamma}^{\vartheta}_{r\vartheta} \; = \; \overline{\Gamma}^{\vartheta}_{\vartheta r} \; = \; \frac{1}{r} \, , \quad \overline{\Gamma}^{\vartheta}_{\varphi\varphi} \; = \; - \sin\vartheta \cos\vartheta$$

$$\overline{\Gamma}^{\varphi}_{\vartheta\varphi} \; = \; \overline{\Gamma}^{\varphi}_{\varphi\vartheta} \; = \; \cot\vartheta \, , \quad \overline{\Gamma}^{\varphi}_{r\varphi} \; = \; \overline{\Gamma}^{\varphi}_{\varphi r} \; = \; \frac{1}{r}$$

As the Schwarzschild spacetime is static and spherically symmetric, we can separate off the time part and the angle part so that, in the end, we are left with an ordinary differential equation for the radial part. The procedure is quite analogous to solving the Schrödinger equation with a time-independent spherically symmetric potential: One splits off the time part $\big(\sim e^{i\omega t} \big)$ and the angle part $\big(\sim Y_{\ell m}(\vartheta, \varphi) \big)$ and is then left with an ordinary differential equation for the radial part; in the case of the Coulomb potential, this radial differential equation has the Laguerre polynomials as the solutions.

In the case at hand, the situation is considerably more complicated than in the case of the Schrödinger equation as our unknown function is not a scalar field ψ but a tensor field $h_{\mu\nu}$. Therefore, we have to deal with (co)vectorial and tensorial spherical harmonics in addition to the ordinary (scalar) spherical harmonics $Y_{\ell m}(\vartheta, \varphi)$. Also, the gauge freedom has to be taken into account. We proceed in the following five steps.

Step 1: Expand $h_{\mu\nu}(t, r, \vartheta, \varphi)$ in terms of spherical harmonics.

Step 2: Decompose $h_{\mu\nu}(t, r, \vartheta, \varphi)$ into parts that are even and odd, respectively, with respect to parity transformations.

Step 3: Restrict to the odd parts. Simplify $h_{\mu\nu}(t, r, \vartheta, \varphi)$ with the help of an appropriate gauge transformation.

Step 4: Insert this simplified expression into the linearised vacuum field equation. After an appropriate substitution this results in a partial differential equation for a function that depends on t and r, known as the time-dependent *Regge-Wheeler equation*.

Step 5: Separate off the time part to get an ordinary differential equation for a funcion that depends on the radial variable r only, known as the time-independent Regge-Wheeler equation.

Here we restrict to perturbations that are odd with respect to parity transformations, following a pioneering paper by T. Regge and J. Wheeler ["Stability of a Schwarzschild singularity" Phys. Rev. 108, 1063 (1957)].

The treatment of even perturbations is considerably more difficult. We will not work this out here. In the end, also in this case one arrives at a Regge-Wheeler type equation for a radial function with an effective potential. It is called the *Zerilli equation* and was found only 13 years after the Regge-Wheeler paper by F. Zerilli ["Effective potential for even-parity Regge-Wheeler gravitational perturbation equations", Phys. Rev. Lett. 24, 737 (1970)].

We will now carry through our step-by-step procedure.

Step 1: As a preparation for expanding the metric perturbation into spherical harmonics, we write it in the form

$$h_{\mu\nu}(t, r, \vartheta, \varphi) dx^\mu dx^\nu = \underbrace{h_{AB}(t, r, \vartheta, \varphi)}_{\text{scalar}} dx^A dx^B$$

$$+ 2 \underbrace{h_{A\Sigma}(t, r, \vartheta, \varphi) dx^\Sigma}_{\text{covector}} dx^A + \underbrace{h_{\Sigma\Omega}(t, r, \vartheta, \varphi) dx^\Sigma dx^\Omega}_{\text{second-rank tensor}} .$$

Here and in the following, indices $A, B, C, \dots$ take values r and t while indices $\Sigma, \Omega, \Delta, \dots$ take values ϑ and φ. Recall that we write two covector fields without a symbol between them when we mean the symmetrised tensor product, $dx^A dx^\Sigma = \frac{1}{2}\big(dx^A \otimes dx^\Sigma + dx^\Sigma \otimes dx^A\big)$. We see that, with respect to the angular part, the perturbation splits into three scalar functions h_{AB}, two covector fields $h_{A\Sigma} dx^\Sigma$ and a symmetric second-rank tensor field $h_{\Sigma\Omega} dx^\Sigma dx^\Omega$. We do the expansion into spherical harmonics for these three cases separately.

Scalar part: For fixed (t, r), we have three scalar functions h_{tt}, $h_{rt} = h_{tr}$ and h_{rr} on the sphere. These can be expanded into the usual (scalar-valued) spherical harmonics. For non-negative m, they are defined as

$$Y_{\ell m}(\vartheta, \varphi) \;=\; C_{\ell m} P_{\ell m}\big(\cos\vartheta\big) e^{im\varphi}$$

where the $P_{\ell m}$ are the associated Legendre polynomials,

$$P_{\ell m}(x) \;=\; (-1)^m (1-x^2)^{m/2} \Big(\frac{d}{dx}\Big)^m P_\ell(x)\,,$$

the P_ℓ are the Legendre polynomials,

$$P_\ell(x) \;=\; \frac{1}{2^\ell \ell!}\Big(\frac{d}{dx}\Big)^\ell \big(x^2-1\big)^\ell\,,$$

and the $C_{\ell m}$ are normalisation factors,

$$C_{\ell m} \;=\; \sqrt{\frac{(2\ell+1)(l-m)!}{4\pi(\ell+m)!}}\,.$$

The definition is extended to the case of negative m by requiring

$$Y_{\ell(-m)}(\vartheta, \varphi) \;=\; (-1)^m \overline{Y_{\ell m}(\vartheta, \varphi)}$$

where overlining means complex conjugation. ℓ runs over all integers from 0 to ∞ and m runs, for fixed ℓ, over all integers from $-\ell$ to ℓ.

The case $\ell = 0$ (monopole perturbation) and the case $\ell = 1$ (dipole perturbation) are special and will, therefore, be excluded from the following discussion of scalar, covector and tensor perturbations. By the Jebsen-Birkhoff theorem, perturbations with $\ell = 0$ cannot do anything else but changing the mass parameter of the Schwarzschild black hole. With some more effort, it can be shown that the case $\ell = 1$ covers a shift of the origin and the introduction of a non-zero spin of the black hole; as we consider only linear perturbations, the resulting metric will be the Kerr metric linearised with respect to the spin parameter a. So one finds that the cases $\ell = 0$ and $\ell = 1$ result in a perturbed metric that is still stationary (time-independent), i.e, that it has nothing to do with gravitational waves. This is in parallel with our earlier observation, when we considered linear perturbations around flat spacetime, that the monopole term and the dipole term do not contribute to gravitational waves in the far field.

As we omit the monopole and the dipole term, the general form of a scalar perturbation is

$$h_{AB}(t, r, \vartheta, \varphi) \;=\; \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} u_{AB\ell m}(t, r) Y_{\ell m}(\vartheta, \varphi)\,,$$

so for each (ℓ, m) with $\ell \geq 2$ the scalar part of the perturbation is characterised by three functions $u_{tt\ell m}(t, r)$, $u_{tr\ell m}(t, r) = u_{rt\ell m}(t, r)$ and $u_{rr\ell m}(t, r)$.

Covector part: For fixed (t,r), we have two covector fields $h_{t\Sigma}dx^\Sigma$ and $h_{r\Sigma}dx^\Sigma$ on the sphere. The coefficients $h_{A\Sigma}$ are scalar-valued functions on the sphere, so one could expand them in terms of the scalar-valued spherical harmonics $Y_{\ell m}$. However, this would not be meaningful because the $h_{A\Sigma}$ are not *invariant* scalar functions; they change if a new coordinate basis is chosen on the sphere. To get an expansion that respects the invariance properties of the mathematical objects, one needs (co)vector-valued spherical harmonics. As the sphere is two-dimensional, we need two such basis covector fields for each (ℓ, m). Of course, there are different choices for such a basis. Here we choose the same basis as Regge and Wheeler: For the first basis covector field we choose the gradient of the scalar function $Y_{\ell m}$, which is indeed a non-zero covector field for all values of ℓ under consideration, $\ell \geq 2$,

$$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \overline{\nabla}_\Sigma Y_{\ell m}(\vartheta,\varphi)dx^\Sigma \,.$$

The second basis covector field is constructed orthogonal to the first one,

$$\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \varepsilon_{\Sigma\Omega}\overline{\nabla}^\Omega Y_{\ell m}(\vartheta,\varphi)dx^\Sigma \,,$$

where

$$\varepsilon_{\Sigma\Omega}dx^\Sigma \otimes dx^\Omega \;=\; r^2 \sin\vartheta \big(d\vartheta \otimes d\varphi - d\varphi \otimes d\vartheta \big)$$

is the Levi-Civita tensor field (volume form) on the sphere. The latter is uniquely characterised by the properties that it is anti-symmetric, $\varepsilon_{\Sigma\Omega} = -\varepsilon_{\Omega\Sigma}$, and that it evaluates to unity on an orthonormal basis,

$$\Big(\varepsilon_{\Sigma\Omega}dx^\Sigma \otimes dx^\Omega\Big)\Big(\frac{\partial_\vartheta}{\sqrt{\overline{g}_{\vartheta\vartheta}}}, \frac{\partial_\varphi}{\sqrt{\overline{g}_{\varphi\varphi}}}\Big) \;=\; \frac{\varepsilon_{\vartheta\varphi}}{\sqrt{\overline{g}_{\vartheta\vartheta}\,\overline{g}_{\varphi\varphi}}} \;=\; \frac{r^2\sin\vartheta}{\sqrt{r^2 r^2 \sin^2\vartheta}} \;=\; 1\,.$$

As the covariant derivative of a scalar function is the same as the partial derivative, the (co)vector-valued spherical harmonics can be rewritten as

$$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\vartheta \;+\; \partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\varphi \,,$$

$$\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \varepsilon_{\vartheta\varphi}\overline{g}^{\varphi\varphi}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;+\; \varepsilon_{\varphi\vartheta}\overline{g}^{\vartheta\vartheta}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi$$

$$= \cancel{r^2}\cancel{\sin\vartheta}\,\frac{1}{\cancel{r^2}\sin^{\cancel{2}}\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;-\; \cancel{r^2}\sin\vartheta\,\frac{1}{\cancel{r^2}}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi$$

$$= \frac{1}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;-\; \sin\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi \,.$$

If we expand the covector parts of the metric perturbation with respect to this basis,

$$h_{A\Sigma}dx^\Sigma \;=\; \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\Big(\hat{v}_{A\ell m}(t,r)\Psi_{\ell m\Sigma} + v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}\Big)dx^\Sigma \,,$$

it is characterised for each (ℓ,m) by four functions $\hat{v}_{t\ell m}(t,r)$, $\hat{v}_{r\ell m}(t,r)$, $v_{t\ell m}(t,r)$ and $v_{r\ell m}(t,r)$.

Tensor part: For fixed (t, r), we have a symmetric second-rank tensor field $h_{\Sigma\Omega} dx^\Sigma dx^\Omega$ on the sphere. On a two-dimensional space, a symmetric second-rank tensor has three independent components, so for each (ℓ, m) we need three linearly independent tensor-valued spherical harmonics for a basis. Again, we choose the basis in the same way as Regge and Wheeler. As the second covariant derivative of a scalar function gives a symmetric second-rank tensor field, we choose for the first basis tensor field

$$\Psi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \Big(\overline{\nabla}_\Sigma \overline{\nabla}_\Omega Y_{\ell m} \Big)(\vartheta,\varphi) dx^\Sigma dx^\Omega \, .$$

As the angular part of the metric, $\overline{g}_{\Sigma\Omega}$, is a symmetric second-rank tensor field proportional to r^2, we choose for the second basis tensor field

$$\Phi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \Big(Y_{\ell m} \, \frac{\overline{g}_{\Sigma\Omega}}{r^2} \Big)(\vartheta,\varphi) dx^\Sigma dx^\Omega \, .$$

The third basis tensor field we construct, as in the covector case, orthogonal to the $\Psi_{\ell m \Sigma\Omega}$,

$$\chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \frac{1}{2} \Big(\varepsilon_{\Sigma\Delta} \overline{\nabla}^\Delta \overline{\nabla}_\Omega Y_{\ell m} + \varepsilon_{\Omega\Delta} \overline{\nabla}^\Delta \overline{\nabla}_\Sigma Y_{\ell m} \Big)(\vartheta,\varphi) \, dx^\Sigma \, dx^\Omega \, .$$

It is not difficult to verify that, for all (ℓ, m) with $\ell \geq 2$, these three second-rank tensor fields are indeed linearly independent. With the help of the Christoffel symbols listed at the beginning of this section, we rewrite the tensor-valued spherical harmonics as

$$\Psi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \overline{\nabla}_\vartheta \overline{\nabla}_\vartheta Y_{\ell m}(\vartheta,\varphi) d\vartheta^2 + 2 \overline{\nabla}_\vartheta \overline{\nabla}_\varphi Y_{\ell m}(\vartheta,\varphi) d\vartheta d\varphi + \overline{\nabla}_\varphi \overline{\nabla}_\varphi Y_{\ell m}(\vartheta,\varphi) d\varphi^2$$

$$= \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) d\vartheta^2 + 2 \Big(\partial_\vartheta \partial_\varphi Y_{\ell m} - \overline{\Gamma}^\varphi{}_{\vartheta\varphi} \partial_\varphi Y_{\ell m} \Big)(\vartheta,\varphi) d\vartheta d\varphi + \Big(\partial_\varphi^2 Y_{\ell m} - \overline{\Gamma}^\vartheta{}_{\varphi\varphi} \partial_\vartheta Y_{\ell m} \Big)(\vartheta,\varphi) d\varphi^2$$

$$= \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) d\vartheta^2 + 2 \Big(\partial_\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta \, \partial_\varphi Y_{\ell m}(\vartheta,\varphi) \Big) d\vartheta d\varphi$$

$$+ \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta \cos\vartheta \, \partial_\vartheta Y_{\ell m}(\vartheta,\varphi) \Big) d\varphi^2 \, ,$$

$$\Phi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; Y_{\ell m}(\vartheta,\varphi) \Big(\frac{\overline{g}_{\vartheta\vartheta}}{r^2} d\vartheta^2 + \frac{\overline{g}_{\varphi\varphi}}{r^2} d\varphi^2 \Big) \;=\; Y_{\ell m}(\vartheta,\varphi) \big(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \big) \, ,$$

$$\chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \Big(\varepsilon_{\vartheta\varphi} \overline{g}^{\varphi\varphi} \overline{\nabla}_\varphi \overline{\nabla}_\vartheta Y_{\ell m} \Big)(\vartheta,\varphi) d\vartheta^2 + \Big(\varepsilon_{\vartheta\varphi} \overline{g}^{\varphi\varphi} \overline{\nabla}_\varphi \overline{\nabla}_\varphi Y_{\ell m} \Big)(\vartheta,\varphi) d\vartheta d\varphi$$

$$+ \Big(\varepsilon_{\varphi\vartheta} \overline{g}^{\vartheta\vartheta} \overline{\nabla}_\vartheta \overline{\nabla}_\vartheta Y_{\ell m} \Big)(\vartheta,\varphi) d\varphi d\vartheta + \Big(\varepsilon_{\varphi\vartheta} \overline{g}^{\vartheta\vartheta} \overline{\nabla}_\vartheta \overline{\nabla}_\varphi Y_{\ell m} \Big)(\vartheta,\varphi) d\varphi^2$$

$$= \frac{\cancel{r^2} \cancel{\sin\vartheta}}{\cancel{r^2} \sin^{\cancel{2}}\vartheta} \Big(\partial_\varphi \partial_\vartheta Y_{\ell m} - \overline{\Gamma}^\varphi{}_{\varphi\vartheta} \partial_\varphi Y_{\ell m} \Big)(\vartheta,\varphi) d\vartheta^2 + \frac{\cancel{r^2} \cancel{\sin\vartheta}}{\cancel{r^2} \sin^{\cancel{2}}\vartheta} \Big(\partial_\varphi^2 Y_{\ell m} - \overline{\Gamma}^\vartheta{}_{\varphi\varphi} \partial_\vartheta Y_{\ell m} \Big)(\vartheta,\varphi) d\vartheta d\varphi$$

$$- \frac{\cancel{r^2} \sin\vartheta}{\cancel{r^2}} \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) d\vartheta d\varphi - \frac{\cancel{r^2} \sin\vartheta}{\cancel{r^2}} \Big(\partial_\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \overline{\Gamma}^\varphi{}_{\vartheta\varphi} \partial_\varphi Y_{\ell m} \Big) d\varphi^2$$

$$= \frac{1}{\sin\vartheta} \Big(\partial_\varphi \partial_\vartheta Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta \, \partial_\varphi Y_{\ell m}(\vartheta,\varphi) \Big) \big(d\vartheta^2 - \sin^2\vartheta \, d\varphi^2 \big)$$

$$+ \frac{1}{\sin\vartheta} \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) - \sin^2\vartheta \, \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta \cos\vartheta \, \partial_\vartheta Y_{\ell m}(\vartheta,\varphi) \Big) d\vartheta d\varphi \, .$$

We expand the tensorial part of the metric perturbation in the form

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\;$$

$$\sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \Big(\hat{w}_{\ell m}(t,r) \Psi_{\ell m \Sigma\Omega}(\vartheta,\varphi) + \tilde{w}_{\ell m}(t,r) \Phi_{\ell m \Sigma\Omega}(\vartheta,\varphi) + w_{\ell m}(t,r) \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) \Big) dx^\Sigma dx^\Omega \, .$$

Step 2: We will now investigate the transformation behaviour of our various spherical harmonics with respect to parity transformations (i.e., reflections at the origin)

$$(\vartheta, \varphi) \mapsto (\pi - \vartheta, \varphi + \pi) .$$

Obviously, under such a transformation

$$\cos\vartheta \mapsto -\cos\vartheta , \quad \sin\vartheta \mapsto \sin\vartheta , \quad e^{im\varphi} \mapsto e^{im\varphi} e^{im\pi} = e^{im\varphi}\big(e^{i\pi}\big)^m = e^{im\varphi}(-1)^m ,$$

$$d\vartheta \mapsto -d\vartheta , \quad d\varphi \mapsto d\varphi , \quad \partial_\vartheta \mapsto -\partial_\vartheta , \quad \partial_\varphi \mapsto \partial_\varphi ,$$

We also need to know that

$$P_\ell(-x) = \frac{1}{2^\ell \ell!}\Big(-\frac{d}{dx}\Big)^\ell P_\ell\big(x^2-1\big)^\ell = (-1)^\ell P_\ell(x) ,$$

$$P_{\ell m}(-x) = (-1)^m (1-x^2)^{m/2}\Big(-\frac{d}{dx}\Big)^m P_\ell(-x) = (-1)^m(-1)^\ell P_{\ell m}(x) .$$

We introduce the following terminology. A function $F_{\ell m}(\vartheta, \varphi)$ is said to be

- *even* under parity transformations if $F_{\ell m}(\pi - \vartheta, \varphi + \pi) = (-1)^\ell F_{\ell m}(\vartheta, \varphi)$,
- *odd* under parity transformations if $F_{\ell m}(\pi - \vartheta, \varphi + \pi) = (-1)^{\ell+1} F_{\ell m}(\vartheta, \varphi)$.

Instead of even/odd, some authors say *polar/axial*, *electric/magnetic* or *poloidal/toroidal*.

With the help of the above transformation rules, we will now demonstrate that

$Y_{\ell m}(\vartheta, \varphi)$ is even,
$\Psi_{\ell m \Sigma}(\vartheta, \varphi) dx^\Sigma$ is even,
$\Phi_{\ell m \Sigma}(\vartheta, \varphi) dx^\Sigma$ is odd,
$\Psi_{\ell m \Sigma\Omega}(\vartheta, \varphi) dx^\Sigma dx^\Omega$ is even,
$\Phi_{\ell m \Sigma\Omega}(\vartheta, \varphi) dx^\Sigma dx^\Omega$ is even,
$\chi_{\ell m \Sigma\Omega}(\vartheta, \varphi) dx^\Sigma dx^\Omega$ is odd.

Proof: Under a parity transformation,

$$Y_{\ell m}(\vartheta, \varphi) = C_{\ell m} P_{\ell m}\big(\cos\vartheta\big) e^{im\varphi}$$

$$\mapsto C_{\ell m} P_{\ell m}\big(-\cos\vartheta\big) e^{im\varphi} e^{im\pi} = C_{\ell m}(-1)^m(-1)^\ell P_{\ell m}\big(\cos\vartheta\big) e^{im\varphi}(-1)^m = (-1)^\ell Y_{\ell m}(\vartheta, \varphi) ,$$

$$\Psi_{\ell m \Sigma}(\vartheta, \varphi) dx^\Sigma = \partial_\vartheta Y_{\ell m}(\vartheta, \varphi) d\vartheta + \partial_\varphi Y_{\ell m}(\vartheta, \varphi) d\varphi$$
$$\mapsto (-1)^\ell(-\partial_\vartheta) Y_{\ell m}(\vartheta, \varphi)(-d\vartheta) + (-1)^\ell \partial_\varphi Y_{\ell m}(\vartheta, \varphi) d\varphi = (-1)^\ell \Psi_{\ell m \Sigma}(\vartheta, \varphi) dx^\Sigma ,$$

$$\Phi_{\ell m \Sigma} dx^\Sigma = \frac{1}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta, \varphi) dx^\vartheta - \sin\vartheta\, \partial_\vartheta Y_{\ell m}(\vartheta, \varphi) dx^\varphi$$
$$\mapsto \frac{(-1)^\ell}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta, \varphi)(-dx^\vartheta) + \sin\vartheta\,(-\partial_\vartheta) Y_{\ell m}(\vartheta, \varphi) dx^\varphi = (-1)^{\ell+1}\Phi_{\ell m \Sigma} dx^\Sigma ,$$

$$\Psi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) d\vartheta^2$$

$$+\,2\Big(\partial_\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big) d\vartheta d\varphi + \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big) d\varphi^2$$

$$\mapsto\; (-1)^\ell \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) d\vartheta^2 + 2(-1)^\ell \Big(- \partial_\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi) + \cot\vartheta \partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)(-d\vartheta) d\varphi$$

$$+(-1)^\ell \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big) d\varphi^2 \;=\; (-1)^\ell \Psi_{\ell m\Sigma\Omega} dx^\Sigma dx^\Omega \,,$$

$$\Phi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; Y_{\ell m}(\vartheta,\varphi)\big(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2\big)$$

$$\mapsto\; (-1)^\ell Y_{\ell m}(\vartheta,\varphi)\big(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2\big) \;=\; (-1)^\ell \Phi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \,,$$

$$\chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \frac{1}{\sin\vartheta}\Big(\partial_\varphi \partial_\vartheta Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta\, \partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)\big(d\vartheta^2 - \sin^2\!\vartheta\, d\varphi^2\big)$$

$$+\frac{2}{\sin\vartheta}\Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) - \sin^2\!\vartheta\, \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\, \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big) d\vartheta d\varphi$$

$$\mapsto\; \frac{(-1)^\ell}{\sin\vartheta}\Big(- \partial_\varphi \partial_\vartheta Y_{\ell m}(\vartheta,\varphi) + \cot\vartheta\, \partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)\big(d\vartheta^2 - \sin^2\!\vartheta\, d\varphi^2\big)$$

$$+\frac{2(-1)^\ell}{\sin\vartheta}\Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) - \sin^2\!\vartheta\, \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\, \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)(-d\vartheta) d\varphi$$

$$=\; (-1)^{\ell+1} \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega$$

□

Step 3:

We restrict to odd metric perturbations,

$$h_{AB}(t,r,\vartheta,\varphi) \;=\; 0\,,$$

$$h_{A\Sigma}(t,r,\vartheta,\varphi) dx^\Sigma \;=\; \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} v_{A\ell m}(t,r) \Phi_{\ell m \Sigma} dx^\Sigma \,,$$

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi) dx^\Sigma dx^\Omega \;=\; \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} w_{\ell m}(t,r) \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \,.$$

We fix ℓ and m, i.e., we consider one partial wave,

$$h_{\mu\nu}(t,r,\vartheta,\varphi) dx^\mu dx^\nu \;=\; 2\, v_{A\ell m}(t,r) \Phi_{\ell m \Sigma}(\vartheta,\varphi) dx^A dx^\Sigma + w_{\ell m}(t,r) \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) dx^\Sigma dx^\Omega \,.$$

This partial wave is determined by three scalar functions, $v_{t\ell m}(t,r)$, $v_{r\ell m}(t,r)$ and $w_{\ell m}(t,r)$. We will use the gauge freedom for transforming $w_{\ell m}(t,r)$ to zero.

To that end, we recall that under a gauge transformation

$$x^\mu \mapsto x^\mu + f^\mu(x) = x^\mu + \overline{g}^{\mu\nu}(x) f_\nu(x)$$

the metric perturbation changes according to

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \overline{\nabla}_\mu f_\nu + \overline{\nabla}_\nu f_\mu \, .$$

We choose a gauge function of the form

$$f_A(t,r,\vartheta,\varphi) = 0 \, , \quad f_\Sigma(t,r,\vartheta,\varphi) = \Lambda_{\ell m}(t,r)\big(\varepsilon_\Sigma{}^\Omega \overline{\nabla}_\Omega Y_{\ell m}\big)(\vartheta,\varphi)$$

with a function $\Lambda_{\ell m}(t,r)$ to be determined. Note that such a gauge transformation depends on ℓ and m, i.e., it is done for the chosen partial wave. Our gauge transformation preserves the equation $h_{AB} = 0$, because with the Christoffel symbols listed at the beginning of this section we find

$$h_{AB} \mapsto h_{AB} + \overline{\nabla}_A f_B + \overline{\nabla}_B f_A = 0 + \partial_A f_B - \overline{\Gamma}^\mu{}_{AB} f_\mu + \partial_B f_A - \overline{\Gamma}^\mu{}_{BA} f_\mu \; = \; 0 - 2 \underbrace{\overline{\Gamma}^\Sigma{}_{AB}}_{=0} f_\Sigma \; = \; 0 \, .$$

The tensorial part transforms as

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi) \; = \; w_{\ell m}(t,r) \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) \; \mapsto \; \big(w_{\ell m}(t,r) + \Lambda_{\ell m}(t,r)\big) \chi_{\ell m \Sigma\Omega}(\vartheta,\varphi) \, .$$

Proof: We first observe that

$$\overline{\nabla}_\Sigma\big(\Lambda_{\ell m} \varepsilon_\Omega{}^\Delta\big) \; = \; \varepsilon_\Omega{}^\Delta \overline{\nabla}_\Sigma \Lambda_{\ell m} + \Lambda_{\ell m} \overline{\nabla}_\Sigma \varepsilon_\Omega{}^\Delta = \varepsilon_\Omega{}^\Delta \underbrace{\partial_\Sigma \Lambda_{\ell m}}_{=0} + \Lambda_{\ell m} \overline{g}^{\Delta\Phi} \underbrace{\overline{\nabla}_\Sigma \varepsilon_{\Omega\Phi}}_{=0} \; = \; 0 \, .$$

Here we have used the fact that the scalar function $\Lambda_{\ell m}$ is independent of ϑ and φ and that the Levi-Civita tensor is covariantly constant,

$$\overline{\nabla}_\Sigma \varepsilon_{\Omega\Delta} = 0 \, .$$

The latter can be proven in the following way.

$$\overline{\nabla}_\Sigma \varepsilon_{\Delta\Omega} = \partial_\Sigma \varepsilon_{\Delta\Omega} - \overline{\Gamma}^\Pi{}_{\Sigma\Delta} \varepsilon_{\Pi\Omega} - \overline{\Gamma}^\Pi{}_{\Sigma\Omega} \varepsilon_{\Delta\Pi}$$

demonstrates that the left-hand side is zero for $\Delta = \Omega$. This follows from the fact that then, because of the antisymmetry of $\varepsilon_{\Delta\Omega}$, the first term on the right-hand side vanishes and the other two compensate each other. Therefore, we only have to consider the case that $\Delta \neq \Omega$:

$$-\overline{\nabla}_\vartheta \varepsilon_{\varphi\vartheta} = \overline{\nabla}_\vartheta \varepsilon_{\vartheta\varphi} = \partial_\vartheta \varepsilon_{\vartheta\varphi} - \overline{\Gamma}^\vartheta{}_{\vartheta\vartheta} \varepsilon_{\vartheta\varphi} - \overline{\Gamma}^\varphi{}_{\vartheta\varphi} \varepsilon_{\vartheta\varphi}$$

$$= \partial_\vartheta\big(r^2 \sin\vartheta\big) - 0 - \cot\vartheta \, r^2 \sin\vartheta = r^2 \cos\vartheta - r^2 \cos\vartheta \; = \; 0 \, ,$$

$$-\overline{\nabla}_\varphi \varepsilon_{\varphi\vartheta} = \overline{\nabla}_\varphi \varepsilon_{\vartheta\varphi} = \partial_\varphi \varepsilon_{\vartheta\varphi} - \overline{\Gamma}^\vartheta{}_{\varphi\vartheta} \varepsilon_{\vartheta\varphi} - \overline{\Gamma}^\varphi{}_{\varphi\varphi} \varepsilon_{\vartheta\varphi} = 0 - 0 - 0 = 0 \, .$$

With this result at hand, it is now easy to complete the proof.

$$
\begin{aligned}
h_{\Sigma\Omega} \;\mapsto\; & h_{\Sigma\Omega} + \overline{\nabla}_\Sigma f_\Omega + \overline{\nabla}_\Omega f_\Sigma \\
= & \, w_{\ell m}\chi_{\ell m \Sigma\Omega} + \overline{\nabla}_\Sigma\big(\Lambda_{\ell m}\varepsilon_\Omega{}^\Delta\overline{\nabla}_\Delta Y_{\ell m}\big) + \overline{\nabla}_\Omega\big(\Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta\overline{\nabla}_\Delta Y_{\ell m}\big) \\
= & \, w_{\ell m}\Big(\varepsilon_{\Sigma\Delta}\overline{\nabla}^\Delta\overline{\nabla}_\Omega Y_{\ell m} + \varepsilon_{\Omega\Delta}\overline{\nabla}^\Delta\overline{\nabla}_\Sigma Y_{\ell m}\Big) + \Lambda_{\ell m}\varepsilon_\Omega{}^\Delta\overline{\nabla}_\Sigma\overline{\nabla}_\Delta Y_{\ell m} + \Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta\overline{\nabla}_\Omega\overline{\nabla}_\Delta Y_{\ell m} \\
= & \, w_{\ell m}\Big(\varepsilon_\Sigma{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Omega Y_{\ell m} + \varepsilon_\Omega{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Sigma Y_{\ell m}\Big) + \Lambda_{\ell m}\varepsilon_\Omega{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Sigma Y_{\ell m} + \Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Omega Y_{\ell m} \\
= & \, \big(w_{\ell m} + \Lambda_{\ell m}\big)\Big(\varepsilon_\Sigma{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Omega Y_{\ell m} + \varepsilon_\Omega{}^\Delta\overline{\nabla}_\Delta\overline{\nabla}_\Sigma Y_{\ell m}\Big)\,.
\end{aligned}
$$

□

If we choose $\Lambda_{\ell m}(t,r) = -w_{\ell m}(t,r)$, the tensorial part is transformed to zero and in the new gauge the metric perturbation is determined by just two scalar functions $v_{t\ell m}(t,r)$ and $v_{r\ell m}(t,r)$,

$$
h_{\mu\nu}(t,r,\vartheta,\varphi)dx^\mu dx^\nu \;=\; 2\,h_{A\Sigma}(t,r,\vartheta,\varphi)dx^A dx^\Sigma \;=\; 2\,v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^A dx^\Sigma\,.
$$

As the diagonal elements of $h_{\mu\nu}$ vanish, it is obvious that the condition of vanishing trace is satisfied

$$
\overline{g}^{\mu\nu}h_{\mu\nu} = 0\,.
$$

The generalised Hilbert gauge condition is, however, *not* satisfied,

$$
\overline{\nabla}^\mu h_{\mu\nu} \neq 0
$$

in general. Note that our choice of gauge, which is known as the *Regge-Wheeler gauge*, is done for a particular (ℓ, m).

Step 4:

Now comes the hard part of the construction. We plug our metric perturbation $h_{\mu\nu}$, whose only non-vanishing components are $h_{A\Sigma}(t,r,\vartheta,\varphi) = v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}(\vartheta,\varphi)$, into the linearised field equation. We use the latter in the gauge-independent form

$$
0 \;=\; \overline{\nabla}_\nu \delta\Gamma^\mu{}_{\mu\sigma} \;-\; \overline{\nabla}_\mu \delta\Gamma^\mu{}_{\nu\sigma}
$$

where

$$
2\delta\Gamma^\nu{}_{\rho\sigma} \;=\; \overline{g}^{\nu\lambda}\Big(\overline{\nabla}_\rho h_{\lambda\sigma} + \overline{\nabla}_\sigma h_{\lambda\rho} - \overline{\nabla}_\lambda h_{\rho\sigma}\Big)
$$

$$
= \overline{g}^{\nu\lambda}\Big(\partial_\rho h_{\lambda\sigma} - \cancel{\overline{\Gamma}^\tau{}_{\rho\lambda}h_{\tau\sigma}} - \overline{\Gamma}^\tau{}_{\rho\sigma}h_{\lambda\tau} + \partial_\sigma h_{\lambda\rho} - \cancel{\overline{\Gamma}^\tau{}_{\sigma\lambda}h_{\tau\rho}} - \overline{\Gamma}^\tau{}_{\sigma\rho}h_{\lambda\tau} - \partial_\lambda h_{\rho\sigma} + \cancel{\overline{\Gamma}^\tau{}_{\lambda\rho}h_{\tau\sigma}} + \cancel{\overline{\Gamma}^\tau{}_{\lambda\sigma}h_{\rho\tau}}\Big)
$$

$$
= \overline{g}^{\nu\lambda}\Big(\partial_\rho h_{\lambda\sigma} + \partial_\sigma h_{\lambda\rho} - \partial_\lambda h_{\rho\sigma} - 2\overline{\Gamma}^\tau{}_{\rho\sigma}h_{\lambda\tau}\Big)\,.
$$

We will show that, owing to our choice of gauge, $\delta\Gamma^\mu{}_{\mu\sigma} = 0$. We consider first the case $\sigma = A = t, r$, then the case $\sigma = \Sigma = \vartheta, \varphi$.

With the help of the Christoffel symbols listed at the beginning of this section we find

$$\delta\Gamma^{\mu}{}_{\mu A} \,=\, \overline{g}^{\mu\lambda}\Big(\cancel{\partial_\mu h_{\lambda A}} + \partial_A h_{\lambda\mu} - \cancel{\partial_\lambda h_{\mu A}} - 2\overline{\Gamma}^{\tau}{}_{\mu A} h_{\lambda\tau}\Big)$$

$$= \,\overline{g}^{CD}\underbrace{\partial_A h_{CD}}_{=0} + \overline{g}^{\Sigma\Omega}\underbrace{\partial_A h_{\Sigma\Omega}}_{=0} - 2\overline{g}^{CD}\underbrace{\overline{\Gamma}^{\Omega}{}_{CA}}_{=0} h_{D\Omega} - 2\overline{g}^{\Sigma\Omega}\underbrace{\overline{\Gamma}^{B}{}_{\Sigma A}}_{=0} h_{\Omega B} \,=\, 0\,,$$

$$\delta\Gamma^{\mu}{}_{\mu\Sigma} \,=\, \overline{g}^{\mu\lambda}\Big(\cancel{\partial_\mu h_{\lambda\Sigma}} + \partial_\Sigma h_{\lambda\mu} - \cancel{\partial_\lambda h_{\mu\Sigma}} - 2\overline{\Gamma}^{\tau}{}_{\mu\Sigma} h_{\lambda\tau}\Big)$$

$$= \,\overline{g}^{CD}\underbrace{\partial_\Sigma h_{CD}}_{=0} + \overline{g}^{\Omega\Delta}\underbrace{\partial_\Sigma h_{\Omega\Delta}}_{=0} - 2\overline{g}^{CD}\overline{\Gamma}^{\Delta}{}_{C\Sigma} h_{D\Delta} - 2\overline{g}^{\Omega\Delta}\overline{\Gamma}^{B}{}_{\Omega\Sigma} h_{\Delta B}$$

$$= \,-2\overline{g}^{CD}\frac{1}{r}\delta^r_C\delta^\Delta_\Sigma h_{D\Delta} + 2\overline{g}^{\Omega\Delta} r\big(1-\frac{r_S}{r}\big)\frac{\overline{g}_{\Omega\Sigma}}{r^2}\delta^B_r h_{\Delta B}$$

$$= \,-2\overline{g}^{rr}\frac{1}{r}h_{r\Sigma} + \frac{2}{r}\big(1-\frac{r_S}{r}\big)h_{r\Sigma} \,=\, 0\,.$$

This means that the linearised field equation simplifies to

$$0 \,=\, \overline{\nabla}_\mu \delta\Gamma^{\mu}{}_{\nu\sigma}\,.$$

As a preparation for working out the ten components of this tensor equation, we calculate the $\delta\Gamma^{\nu}{}_{\rho\sigma}$. In this calculation, we constantly use the Christoffel symbols of the Schwarzschild metric, and we will also need the eigenvalue equation of the angular momentum operator which reads

$$L^2 Y_{\ell m} = -\hbar^2(\sin\vartheta)^{-1}\Big(\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta Y_{\ell m}\big) + (\sin\vartheta)^{-1}\partial_\varphi^2 Y_{\ell m}\Big) = \hbar^2\ell(\ell+1)Y_{\ell m}$$

in standard quantum mechanics notation. We find

$$2\delta\Gamma^{C}{}_{AB} \,=\, \overline{g}^{CD}\Big(\underbrace{\partial_A h_{DB}}_{=0} + \underbrace{\partial_B h_{DA}}_{=0} - \underbrace{\partial_D h_{AB}}_{=0} - 2\underbrace{\overline{\Gamma}^{\Sigma}{}_{AB}}_{=0} h_{D\Sigma}\Big) \,=\, 0\,,$$

$$\begin{aligned} 2\delta\Gamma^{\Delta}{}_{AB} \,&=\, \overline{g}^{\Delta\Lambda}\Big(\partial_A h_{\Lambda B} + \partial_B h_{\Lambda A} - \underbrace{\partial_\Lambda h_{AB}}_{=0} - 2\overline{\Gamma}^{C}{}_{AB} h_{\Lambda C}\Big) \\ &=\, \big(\partial_A v_{B\ell m} + \partial_B v_{A\ell m} - 2\overline{\Gamma}^{C}{}_{AB} v_{C\ell m}\big)\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda} \\ &=\, \underbrace{\big(\partial_A v_{B\ell m} + \partial_B v_{A\ell m} - 2\overline{\Gamma}^{C}{}_{AB} v_{C\ell m}\big)}_{=:\,2\,q_{AB}}\overline{g}^{\Delta\Lambda}\varepsilon_{\Lambda\Omega}\overline{g}^{\Omega\Pi}\overline{\nabla}_\Pi Y_{\ell m}\,, \end{aligned}$$

$$\begin{aligned} 2\delta\Gamma^{C}{}_{A\Sigma} \,&=\, \overline{g}^{CD}\Big(\partial_A h_{D\Sigma} + \underbrace{\partial_\Sigma h_{DA}}_{=0} - \partial_D h_{A\Sigma} - 2\underbrace{\overline{\Gamma}^{\Delta}{}_{A\Sigma}}_{\frac{1}{r}\delta^r_A\delta^\Delta_\Sigma} h_{D\Delta}\Big) \\ &=\, \overline{g}^{CD}\underbrace{\big(\partial_A v_{D\ell m} - \partial_D v_{A\ell m} - \frac{2}{r}\delta^r_A v_{D\ell m}\big)}_{=:\,2\,p_{AD}}\Phi_{\ell m\Sigma}\,, \end{aligned}$$

$$2\delta\Gamma^{\Delta}{}_{A\Sigma} \;=\; \overline{g}^{\Delta\Lambda}\Big(\underbrace{\partial_A h_{\Lambda\Sigma}}_{=0} + \partial_\Sigma h_{\Lambda A} - \partial_\Lambda h_{A\Sigma} - 2\underbrace{\overline{\Gamma}^{C}{}_{A\Sigma}}_{=0} h_{\Lambda C}\Big) \;=\; v_{A\ell m}\overline{g}^{\Delta\Lambda}\big(\partial_\Sigma \Phi_{\ell m\Lambda} - \partial_\Lambda \Phi_{\ell m\Sigma}\big)$$

$$= \; v_{A\ell m}\overline{g}^{\Delta\Lambda}\big(\partial_\vartheta \Phi_{\ell m\varphi} - \partial_\varphi \Phi_{\ell m\vartheta}\big)\big(\delta^{\vartheta}_{\Sigma}\delta^{\varphi}_{\Lambda} - \delta^{\varphi}_{\Sigma}\delta^{\vartheta}_{\Lambda}\big)$$

$$= \; v_{A\ell m}\overline{g}^{\Delta\Lambda}\Big(\partial_\vartheta\big(\varepsilon_{\varphi\vartheta}\overline{g}^{\vartheta\vartheta}\partial_\vartheta Y_{\ell m}\big) - \partial_\varphi\big(\varepsilon_{\vartheta\varphi}\overline{g}^{\varphi\varphi}\partial_\varphi Y_{\ell m}\big)\Big)\frac{\varepsilon_{\Sigma\Lambda}}{r^2\sin\vartheta}$$

$$= \; v_{A\ell m}\overline{g}^{\Delta\Lambda}\Big(- \partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta Y_{\ell m}\big) - \partial_\varphi\big((\sin\vartheta)^{-1}\partial_\varphi Y_{\ell m}\big)\Big)\frac{\varepsilon_{\Sigma\Lambda}}{r^2\sin\vartheta}$$

$$= \; v_{A\ell m}\overline{g}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}\frac{1}{r^2}\ell(\ell+1)Y_{\ell m}\,,$$

$$2\delta\Gamma^{C}{}_{\Sigma\Omega} \;=\; \overline{g}^{CD}\Big(\partial_\Sigma h_{D\Omega} + \partial_\Omega h_{D\Sigma} - \underbrace{\partial_D h_{\Sigma\Omega}}_{=0} - 2\overline{\Gamma}^{\Delta}{}_{\Sigma\Omega}h_{D\Delta}\Big)$$

$$= \; \overline{g}^{CD}v_{D\ell m}\Big(\partial_\Sigma \Phi_{\ell m\Omega} + \partial_\Omega \Phi_{\ell m\Sigma} - 2\overline{\Gamma}^{\Delta}{}_{\Sigma\Omega}\Phi_{\ell m\Delta}\Big) \;=\; \overline{g}^{CD}v_{D\ell m}\Big(\overline{\nabla}_\Sigma \Phi_{\ell m\Omega} + \overline{\nabla}_\Omega \Phi_{\ell m\Sigma}\Big)$$

$$= \; 2\overline{g}^{CD}v_{D\ell m}\chi_{\ell m\Sigma\Omega}\,,$$

$$2\delta\Gamma^{\Delta}{}_{\Sigma\Omega} \;=\; \overline{g}^{\Delta\Lambda}\Big(\underbrace{\partial_\Sigma h_{\Lambda\Omega}}_{=0} + \underbrace{\partial_\Omega h_{\Lambda\Sigma}}_{=0} - \underbrace{\partial_\Lambda h_{\Sigma\Omega}}_{=0} - 2\overline{\Gamma}^{C}{}_{\Sigma\Omega}h_{\Lambda C}\Big) \;=\; -2\overline{g}^{\Delta\Lambda}\overline{\Gamma}^{C}{}_{\Sigma\Omega}\Phi_{\ell m\Lambda}v_{C\ell m}$$

$$= \; -2\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\Big(- v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\delta^{\vartheta}_{\Sigma}\delta^{\vartheta}_{\Omega} - v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\sin^2\vartheta\,\delta^{\varphi}_{\Sigma}\delta^{\varphi}_{\Omega}\Big)$$

$$= \; 2\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\frac{\overline{g}_{\Sigma\Omega}}{r^2} \;=\; 2\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}v_{r\ell m}\big(1-\frac{r_S}{r}\big)\frac{\overline{g}_{\Sigma\Omega}}{r}\,.$$

We are now ready to calculate the covariant derivatives

$$\overline{\nabla}_C\delta\Gamma^{C}{}_{\rho\sigma} = \partial_C\delta\Gamma^{C}{}_{\rho\sigma} + \overline{\Gamma}^{C}{}_{C\mu}\delta\Gamma^{\mu}{}_{\rho\sigma} - \overline{\Gamma}^{\mu}{}_{C\rho}\delta\Gamma^{C}{}_{\mu\sigma} - \overline{\Gamma}^{\mu}{}_{C\sigma}\delta\Gamma^{C}{}_{\rho\mu}$$

and

$$\overline{\nabla}_\Delta\delta\Gamma^{\Delta}{}_{\rho\sigma} = \partial_\Delta\delta\Gamma^{\Delta}{}_{\rho\sigma} + \overline{\Gamma}^{\Delta}{}_{\Delta\mu}\delta\Gamma^{\mu}{}_{\rho\sigma} - \overline{\Gamma}^{\mu}{}_{\Delta\rho}\delta\Gamma^{\Delta}{}_{\mu\sigma} - \overline{\Gamma}^{\mu}{}_{\Delta\sigma}\delta\Gamma^{\Delta}{}_{\rho\mu}$$

for all index combinations ρ,σ. On the right-hand sides we split the sum over $\mu = t, r, \vartheta, \varphi$ into two sums, over $A = t, r$ and $\Omega = \vartheta, \varphi$, and collect all non-zero terms. We find

$$\overline{\nabla}_C\delta\Gamma^{C}{}_{AB} = 0\,,$$

$$\overline{\nabla}_\Delta\delta\Gamma^{\Delta}{}_{AB} = \partial_\Delta\delta\Gamma^{\Delta}{}_{AB} + \overline{\Gamma}^{\Delta}{}_{\Delta\Omega}\delta\Gamma^{\Omega}{}_{AB} - \overline{\Gamma}^{\Omega}{}_{\Delta A}\delta\Gamma^{\Delta}{}_{\Omega B} - \overline{\Gamma}^{\Omega}{}_{\Delta B}\delta\Gamma^{\Delta}{}_{A\Omega}$$

$$= \partial_\Delta\big(\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}q_{AB}\big) + \overline{\Gamma}^{\Delta}{}_{\Delta\Omega}\overline{g}^{\Omega\Lambda}\Phi_{\ell m\Lambda}q_{AB} - \delta^{r}_{A}\frac{1}{r}\delta^{\Omega}_{\Delta}\delta\Gamma^{\Delta}{}_{\Omega B} - \delta^{r}_{B}\frac{1}{r}\delta^{\Omega}_{\Delta}\delta\Gamma^{\Delta}{}_{\Omega A}$$

$$= q_{AB}\overline{\nabla}_\Delta\big(\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\big) - \delta^{r}_{A}\frac{1}{r}\underbrace{\delta\Gamma^{\Omega}{}_{\Omega B}}_{=0} - \delta^{r}_{B}\tfrac{1}{r}\underbrace{\delta\Gamma^{\Omega}{}_{\Omega A}}_{=0}$$

$$= q_{AB}\overline{\nabla}_\Delta\big(\overline{g}^{\Delta\Lambda}\varepsilon_{\Lambda\Sigma}\overline{g}^{\Sigma\Pi}\overline{\nabla}_\Pi Y_{\ell m}\big) = q_{AB}\underbrace{\varepsilon_{\Lambda\Sigma}}_{=-\varepsilon_{\Sigma\Lambda}}\overline{g}^{\Delta\Lambda}\overline{g}^{\Sigma\Pi}\underbrace{\overline{\nabla}_\Delta\overline{\nabla}_\Pi Y_{\ell m}}_{=\overline{\nabla}_\Pi\overline{\nabla}_\Delta Y_{\ell m}} \;=\; 0\,,$$

$$\overline{\nabla}_C\delta\Gamma^C{}_{A\Sigma} = \partial_C\delta\Gamma^C{}_{A\Sigma} - \overline{\Gamma}^B{}_{CA}\delta\Gamma^C{}_{B\Sigma} - \overline{\Gamma}^\Omega{}_{C\Sigma}\delta\Gamma^C{}_{A\Omega} = \partial_C\big(\overline{g}^{CD}p_{AD}\Phi_{\ell m\Sigma}\big)$$

$$-\overline{\Gamma}^B{}_{CA}\overline{g}^{CD}p_{BD}\Phi_{\ell m\Sigma} - \delta^r_C\frac{1}{r}\delta^\Omega_\Sigma\overline{g}^{CD}p_{AD}\Phi_{\ell m\Omega} = \Phi_{\ell m\Sigma}\Big(\partial_C\big(\overline{g}^{CD}p_{AD}\big) - \overline{\Gamma}^B{}_{CA}\overline{g}^{CD}p_{BD} - \frac{1}{r}\overline{g}^{rr}p_{Ar}\Big)\,,$$

$$\overline{\nabla}_\Delta\delta\Gamma^\Delta{}_{A\Sigma} = \underbrace{\partial_\Delta\delta\Gamma^\Delta{}_{A\Sigma} + \overline{\Gamma}^\Delta{}_{\Delta\Omega}\delta\Gamma^\Omega{}_{A\Sigma} - \overline{\Gamma}^\Omega{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{A\Omega}}_{=\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\overline{\nabla}_\Delta\left(\overline{g}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}Y_{\ell m}\right)} - \overline{\Gamma}^\Omega{}_{\Delta A}\delta\Gamma^\Delta{}_{\Omega\Sigma} + \overline{\Gamma}^\Delta{}_{\Delta C}\delta\Gamma^C{}_{A\Sigma}$$

$$-\overline{\Gamma}^C{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{AC} = \ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\overline{g}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}\overline{\nabla}_\Delta Y_{\ell m} - \delta^r_A\frac{1}{r}\delta^\Omega_\Delta\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\overline{g}_{\Omega\Sigma}\frac{v_{r\ell m}}{r}\big(1-\frac{r_S}{r}\big)$$

$$+\delta^r_C\frac{2}{r}\overline{g}^{CD}p_{AD}\Phi_{\ell m\Sigma} + \delta^C_r\frac{1}{r}\big(1-\frac{r_S}{r}\big)\overline{g}_{\Delta\Sigma}\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}q_{AC}$$

$$= \ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\Phi_{\ell m\Sigma} - \delta^r_A\Phi_{\ell m\Sigma}\frac{v_{r\ell m}}{r^2}\big(1-\frac{r_S}{r}\big) + \frac{2}{r}\overline{g}^{rr}p_{Ar}\Phi_{\ell m\Sigma} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)\Phi_{\ell m\Sigma}q_{Ar}$$

$$= \Phi_{\ell m\Sigma}\Big(\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2} - \delta^r_A\frac{v_{r\ell m}}{r^2}\big(1-\frac{r_S}{r}\big) + \frac{2}{r}\overline{g}^{rr}p_{Ar} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{Ar}\Big)\,,$$

$$\overline{\nabla}_C\delta\Gamma^C{}_{\Sigma\Omega} = \partial_C\delta\Gamma^C{}_{\Sigma\Omega} - \overline{\Gamma}^\Delta{}_{C\Sigma}\delta\Gamma^C{}_{\Delta\Omega} - \overline{\Gamma}^\Delta{}_{C\Omega}\delta\Gamma^C{}_{\Sigma\Delta}$$

$$= \frac{1}{r}\delta^\Delta_\Omega\overline{g}^{CD}v_{D\ell m}\chi_{\ell m\Sigma\Delta} = \chi_{\ell m\Sigma\Omega}\Big(\partial_C\big(\overline{g}^{CD}v_{D\ell m}\big) - \frac{2}{r}g^{rr}v_{r\ell m}\Big)\,,$$

$$\overline{\nabla}_\Delta\delta\Gamma^\Delta{}_{\Sigma\Omega} = \partial_\Delta\delta\Gamma^\Delta{}_{\Sigma\Omega} + \overline{\Gamma}^\Delta{}_{\Delta\Pi}\delta\Gamma^\Pi{}_{\Sigma\Omega} - \overline{\Gamma}^\Pi{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{\Pi\Omega} - \overline{\Gamma}^\Pi{}_{\Delta\Omega}\delta\Gamma^\Delta{}_{\Sigma\Pi} + \overline{\Gamma}^\Delta{}_{\Delta C}\delta\Gamma^C{}_{\Sigma\Omega}$$

$$= \frac{v_{r\ell m}}{r}\big(1-\frac{r_S}{r}\big)\underbrace{\overline{\nabla}_\Delta\big(\overline{g}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\overline{g}_{\Sigma\Omega}\big)}_{=0} + \delta^r_C\frac{2}{r}\overline{g}^{CD}v_{D\ell m}\chi_{\ell m\Sigma\Omega} = \frac{2}{r}\overline{g}^{rr}v_{r\ell m}\chi_{\ell m\Sigma\Omega}\,.$$

Hence

$$\overline{\nabla}_\mu\delta\overline{\Gamma}^\mu{}_{AB} = 0\,,$$

$$\overline{\nabla}_\mu\delta\overline{\Gamma}^\mu{}_{A\Sigma} = \Phi_{\ell m\Sigma}\Big(\partial_C\big(\overline{g}^{CD}p_{AD}\big) - \overline{\Gamma}^B{}_{CA}\overline{g}^{CD}p_{BD} - \cancel{\frac{1}{r}\overline{g}^{rr}p_{Ar}}$$

$$+\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2} - \delta^r_A\frac{v_r}{r^2}\big(1-\frac{r_S}{r}\big) + \frac{\cancel{2}}{r}\overline{g}^{rr}p_{Ar} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{Ar}\Big)\,,$$

$$\overline{\nabla}_\mu\delta\overline{\Gamma}^\mu{}_{\Sigma\Omega} = \chi_{\ell m\Sigma\Omega}\partial_C\big(\overline{g}^{CD}v_{D\ell m}\big)\,.$$

We see that the (AB) components of the linearised field equation are identically satisfied.

We now turn to the $(t\Sigma)$ and $(r\Sigma)$ components.

$$0 = \partial_r\big(\overline{g}^{rr}p_{tr}\big) - \overline{\Gamma}^t{}_{rt}\overline{g}^{rr}p_{tr} - \overline{\Gamma}^r{}_{tt}\overline{g}^{tt}p_{rt} + \ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} + \frac{1}{r}\overline{g}^{rr}p_{tr} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{tr}$$

$$= \frac{1}{2}\partial_r\Big(\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big)\Big) - \frac{r_S}{4r^2}\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big) + \frac{r_S}{4r^2}\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)$$

$$+\ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} + \frac{1}{2r}\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} - \cancel{\partial_r v_{t\ell m}}\big) + \frac{1}{2r}\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} + \cancel{\partial_r v_{t\ell m}} - 2\overline{\Gamma}^t{}_{tr}v_{t\ell m}\big)$$

$$= \frac{1}{2}\big(1-\frac{r_S}{r}\big)\partial_r\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big) + \big(1-\frac{r_S}{r}\big)\frac{\partial_t v_{r\ell m}}{r} + \ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} - \frac{r_S v_{t\ell m}}{r^3}\,, \qquad \text{(F1)}$$

$$0 = \partial_r\big(\overline{g}^{rr}p_{rr}\big) + \partial_t\big(\overline{g}^{tt}p_{rt}\big) - \overline{\Gamma}^r{}_{rr}\overline{g}^{rr}p_{rr} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2}\big(1-\frac{r_S}{r}\big) + \frac{1}{r}\overline{g}^{rr}p_{rr} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{rr}$$

$$= \cancel{-\partial_r\Big(\big(1-\frac{r_S}{r}\big)\frac{v_{r\ell m}}{r}\Big)} - \frac{\partial_t\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)}{2\,c^2\big(1-\frac{r}{r_S}\big)} - \frac{r_S v_{r\ell m}}{2\,r^3} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2}\big(1-\cancel{\frac{r_S}{r}}\big)$$

$$\cancel{-\big(1-\frac{r_S}{r}\big)\frac{v_{r\ell m}}{r^2}} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)\big(\cancel{\partial_r v_{r\ell m}} - \overline{\Gamma}^r{}_{rr}v_{r\ell m}\big)$$

$$= -\frac{\partial_t\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)}{2\,c^2\big(1-\frac{r}{r_S}\big)} - \cancel{\frac{r_S v_{r\ell m}}{2\,r^3}} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2} + \cancel{\frac{r_S v_{r\ell m}}{2\,r^3}}$$

$$= -\frac{\partial_t\partial_r v_{t\ell m} - \partial_t^2 v_{r\ell m} - \frac{2}{r}\partial_t v_{t\ell m}}{2\,c^2\big(1-\frac{r}{r_S}\big)} + \frac{\big(\ell(\ell+1)-2\big)v_{r\ell m}}{2\,r^2}\,. \qquad \text{(F2)}$$

The $(\Sigma\Omega)$ component of the linearised field equation gives one equation,

$$0 = \partial_r\big(\overline{g}^{rr}v_{r\ell m}\big) + \partial_t\big(\overline{g}^{tt}v_{t\ell m}\big) = \partial_r\Big(\big(1-\frac{r_S}{r}\big)v_{r\ell m}\Big) - \frac{\partial_t v_{t\ell m}}{c^2\big(1-\frac{r_S}{r}\big)}\,. \qquad \text{(F3)}$$

The field equations (F1), (F2) and (F3) can be decoupled. To that end we replace $v_{r\ell m}$ with

$$Q_{\ell m} = \big(1-\frac{r}{r_S}\big)\frac{v_{r\ell m}}{r}$$

which allows to rewrite (F3) as

$$\frac{\partial_t v_{t\ell m}}{c^2} = \big(1-\frac{r_S}{r}\big)\partial_r\big(r\,Q_{\ell m}\big) = \big(1-\frac{r_S}{r}\big)Q_{\ell m} + \big(1-\frac{r_S}{r}\big)\,r\,\partial_r Q_{\ell m}\,. \qquad \text{(F3')}$$

Differentiation with respect to r yields

$$\frac{\partial_r\partial_t v_{t\ell m}}{c^2} = \frac{r_S}{r^2}Q_{\ell m} + 2\,\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m} + r\,\partial_r\Big(\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m}\Big)\,. \qquad \text{(F3'')}$$

With the help of (F3') and (F3"), (F2) can be rewritten as an equation for $Q_{\ell m}$ alone,

$$0 = \frac{1}{r}(1 - \frac{r_S}{r})\Big(\frac{\partial_t \partial_r v_{t\ell m}}{c^2} - \frac{\partial_t^2 v_{r\ell m}}{c^2} - \frac{2\partial_t v_{t\ell m}}{r\, c^2}\Big) - (1 - \frac{r_S}{r})^2 \frac{\big(\ell(\ell+1) - 2\big) v_{r\ell m}}{r^3}$$

$$= \frac{1}{r}(1 - \frac{r_S}{r})\Big\{\frac{r_S}{r^2} Q_{\ell m} + \cancel{2\,(1 - \frac{r_S}{r}) \partial_r Q_{\ell m}} + r\,\partial_r\Big((1 - \frac{r_S}{r})\partial_r Q_{\ell m}\Big)\Big\}$$

$$-\frac{\partial_t^2 Q_{\ell m}}{c^2} - \frac{2}{r^2}\,(1 - \frac{r_S}{r})^2 \Big(Q_{\ell m} + \cancel{r\,\partial_r Q_{\ell m}}\Big) - (1 - \frac{r_S}{r})\frac{\big(\ell(\ell+1) - 2\big) Q_{\ell m}}{r^2}$$

$$= (1 - \frac{r_S}{r})\partial_r\Big((1 - \frac{r_S}{r})\partial_r Q_{\ell m}\Big) - \frac{\partial_t^2 Q_{\ell m}}{c^2} - \frac{1}{r^2}\,(1 - \frac{r_S}{r})\Big(\big(\ell(\ell+1) - \frac{3 r_S}{r}\big)\Big) Q_{\ell m}\,.$$

If we introduce Wheeler's *tortoise coordinate*

$$r_* = r + r_S \ln\Big(\frac{r}{r_S} - 1\Big)\,, \qquad \partial_{r_*} = \Big(1 - \frac{r_S}{r}\Big)\partial_r \partial_{r_*} = \Big(1 - \frac{r_S}{r}\Big)\partial_r\,,$$

which shifts the horizon to $r_* = -\infty$, we have derived the standard form of the time-dependent *Regge-Wheeler equation*

$$\partial_{r_*}^2 Q_{\ell m} - \frac{1}{c^2}\partial_t^2 Q_{\ell m} - V_\ell(r_*) Q_{\ell m} = 0.$$

Here the *Regge-Wheeler potential* $V_\ell(r_*)$ is given implicitly by

$$V_\ell(r_*) = \frac{1}{r^2}(1 - \frac{r_S}{r})\Big(\ell(\ell+1) - \frac{3 r_S}{r}\Big)\,.$$

Note that the potential depends on ℓ but not on m. This means that we could drop the index m on $Q_{\ell m}$, i.e., $Q_{\ell m'} = Q_{\ell m} =: Q_\ell$. (Actually, this could have been anticipated because of the spherical symmetry.) This implies that $v_{r\ell m}$ and $v_{t\ell m}$ can be replaced with $v_{r\ell}$ and $v_{t\ell}$, respectively.

Step 5: Finally, we separate the time coordinate with the help of the ansatz

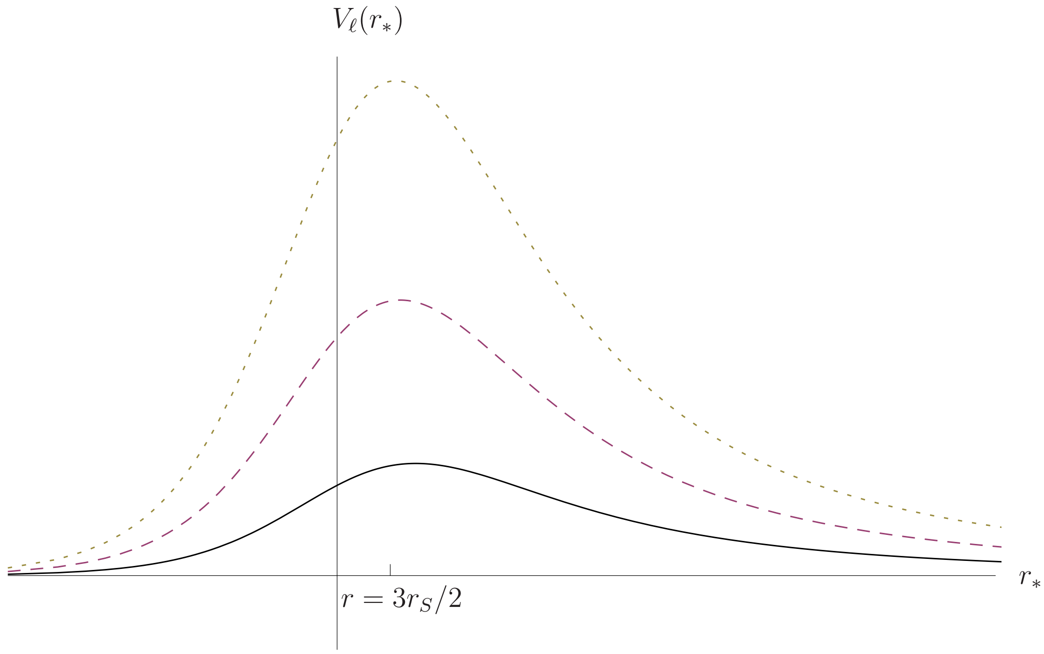
$$Q_\ell(t, r_*) = e^{-i\omega t} Z_\ell(r_*)\,.$$

Inserting this expression into the time-dependent Regge-Wheeler equation yields

$$\frac{d^2 Z_\ell}{dr_*^2} e^{-i\omega t} + \frac{\omega^2}{c^2} Z_\ell e^{-i\omega t} - V_\ell(r_*) Z_\ell e^{-i\omega t} = 0,\,,$$

$$-\frac{d^2 Z_\ell}{dr_*^2} + V_\ell(r_*) Z_\ell = \frac{\omega^2}{c^2} Z_\ell\,.$$

This is the time-independent *Regge-Wheeler equation.* It is very similar to the radial part of the time-independent Schrödinger equation with a spherically symmetric potential. There are some differences, however. (i) The frequency occurs quadratic, rather than linear, because the Regge-Wheeler equation is of *second* order in time. (ii) The radius coordinate r_* ranges from $-\infty$ to ∞, rather than from 0 to ∞. (iii) We have to impose the condition on our complex function Z_ℓ that the corresponding metric perturbations $h_{\mu\nu}$ are real. (iv) In contrast to the wave function in quantum mechanics, there is no physical reason why Z_ℓ should have to satisfy a square-integrability condition; instead, one has to impose physically motivated boundary conditions.



The Regge-Wheeler potential describes a potential *barrier* rather than a potential *well*, see the picture. Correspondingly, we do not expect any bound states to exist. As we have omitted monopole perturbations ($\ell = 0$) and dipole perturbations ($\ell = 1$), which cannot describe gravitational waves, we plot the potential for $\ell = 2$ (solid), $\ell = 3$ (dashed) and $\ell = 4$ (dotted). The maximum of the potential is near the light sphere at $r = 3r_S/2$. In the limit $\ell \to \infty$ it approaches this value.

From any solution $Q_\ell(t, r_*) = Z_\ell(r_*)e^{-i\omega t}$ of the Regge-Wheeler equation we can construct the metric perturbations $v_{r\ell}$ and $v_{t\ell}$ in the following way. $v_{r\ell}$ is given directly as

$$v_{r\ell} = Q_{\ell m} r\big(1 - \frac{r_S}{r}\big)^{-1}$$

and $v_{t\ell}$ follows if we plug the ansatz $v_{t\ell}(t, r_*) = U_\ell e^{-i\omega t}$ into (F3'),

$$\frac{1}{c^2}\partial_t v_{t\ell} = \big(1 - \frac{r_S}{r}\big)Q_\ell + r\partial_{r_*}Q_\ell \,,$$

$$-\frac{i\omega}{c^2}U_\ell = \big(1 - \frac{r_S}{r}\big)Z_\ell + r\frac{dZ_\ell}{dr_*} \,.$$

It can be shown that then the field equation (F1), which has not been used so far, is automatically satisfied.

From time-harmonic solutions to the Regge-Wheeler equation we can construct more general odd linear perturbations of the Schwarzschild metric by forming superpositions of solutions with different ω and different (ℓ, m). Note that such a superposition is indeed possible, although we have chosen a gauge (i.e., a coordinate system) that depends on (ℓ, m); the reason is that in the representation $h_{\mu\nu}(x)dx^\mu dx^\nu$ of the metric perturbation the $x = (x^0, x^1, x^2, x^3)$ denote the *unperturbed* Schwarzschild coodinates. (As the $h_{\mu\nu}$ are small of first order, the difference between their values at the perturbed coordinates and at the unperturbed coordinates is of second order and hence to be neglected.) Therefore the $h_{\mu\nu}$ can be superimposed, even if we use different *perturbed* coordinates for different (ℓ, m).

The general solution to the Regge-Wheeler equation cannot be written in terms of elementary functions. However, by an appropriate substitution it can be transformed into a *c*onfluent Heun equation, a differential equation that was studied in the 19th century by German mathematician Karl Heun. The general solution of this equation can be expressed in terms of the so-called HeunC function which is implemented, e.g., in Maple.

In Worksheet 8 we will demonstrate that the Regge-Wheeler potential falls off exponentially for $r_* \to -\infty$. This implies that, near the horizon, the Regge-Wheeler equation can be approximated by

$$-\frac{d}{dr_*^2} Z_\ell(r_*) \approx \frac{\omega^2}{c^2} Z_\ell(r_*)$$

which is solved by $e^{\pm i\omega r_*/c}$. Therefore there must be two solutions of the Regge-Wheeler equation that behave asymptotically as

$$Z_\ell^I(r_*) \sim e^{-i\omega r_*/c} \quad \text{for } r_* \to -\infty \,,$$

$$Z_\ell^O(r_*) \sim e^{i\omega r_*/c} \quad \text{for } r_* \to -\infty \,.$$

These two solutions are called the IN mode and the OUT mode, respectively. As they are linearly independent, and as the set of solutions to a *linear* second-order differential equation is a two-dimensional complex vector space, any solution to the Regge-Wheeler equation must be a linear combination of the IN mode and the OUT mode with constant complex coefficients. We have characterised here the IN mode and the OUT mode by their asymptotic behaviour only; actually, they can be written in terms of infinite power series that converge on a certain neighbourhood of the horizon (but not at the horizon).

In Worksheet 8 we will also show that the Regge-Wheeler potential falls off like r_*^{-2} for $r_* \to \infty$, As a consequence, the approximation

$$-\frac{d}{dr_*^2} Z_\ell(r_*) \approx \frac{\omega^2}{c^2} Z_\ell(r_*)$$

is valid near infinity as well. This gives rise to two other solutions of the Regge-Wheeler equation that behave asymptotically as

$$Z_\ell^D(r_*) \sim e^{-i\omega r_*/c} \quad \text{for } r_* \to \infty \,,$$

$$Z_\ell^U(r_*) \sim e^{i\omega r_*/c} \quad \text{for } r_* \to \infty \,.$$

These two solutions are called the DOWN mode and the UP mode, respectively. Again, they are linearly independent, so they may be used as a basis for the set of all solutions. The DOWN mode and the UP mode may also be written in terms of power series that converge on an interval up to (but not including) infinity; in the following we will only need the asymptotic behaviour of these modes.

Two interesting types of problems are related with the Regge-Wheeler equation (and, analogously, with the Zerilli equation for even modes). Firstly, one can study quasi-normal modes, and secondly one can study scattering problems.

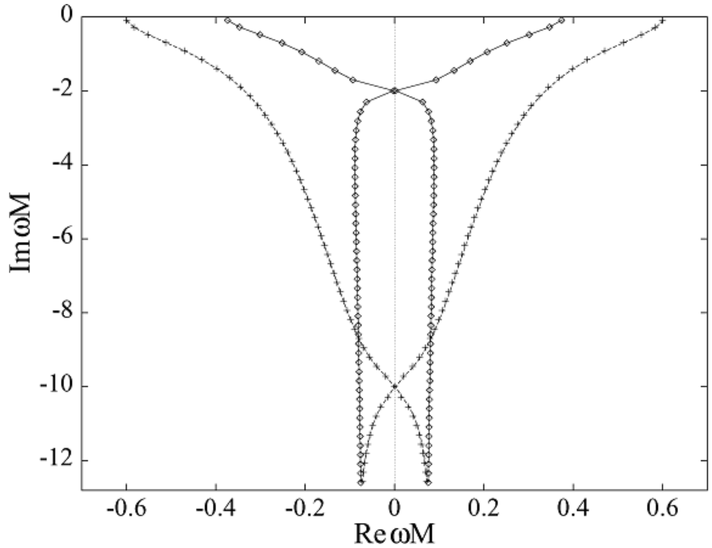
A quasi-normal mode is a solution which is a pure IN mode and, at the same time, a pure UP mode. This means that nothing comes out of the black hole and nothing comes in from infinity. We will demonstrate in Worksheet 8 that such a solution does not exist for real frequencies ω and also not for frequencies with a non-zero real part and a positive imaginary part. In the first case $Q_\ell(t, r_*) = Z_\ell(r_*)e^{-i\omega t}$ would describe a stationary oscillation and in the second case a solution that grows exponentially in time. Quasi-normal modes exist only with strictly negative imaginary part of the frequency, so they decay exponentially in time. There is a discrete set of complex frequencies for such quasi-normal modes which can be determined numerically or with analytical approximation methods. The following table shows the complex frequencies for the first four quasi-normal modes for $\ell = 2, 3, 4$, taken from a paper by E. Leaver ["An analytic representation for the quasi-normal modes of Kerr black holes", Proc. R. Soc. London, Ser. A, 402, 285298, (1985)].

n	$\ell = 2$	$\ell = 3$	$\ell = 4$
0	0.37367 -0.08896 i	0.59944 -0.09270 i	0.80918 -0.09416 i
1	0.34671 -0.27391 i	0.58264 -0.28130 i	0.79663 -0.28443 i
2	0.30105 -0.47828 i	0.55168 -0.47909 i	0.77271 -0.47991 i
3	0.25150 -0.70514 i	0.51196 -0.69034 i	0.73984 -0.68392 i

TABLE I: *The first four QNM frequencies (ωM) of the Schwarzschild black hole for $\ell = 2, 3,$ and 4[3].*

Natural units are chosen; for conversion into Hz one has to multiply with $2\pi \times 5142\,\text{Hz} \times M_\odot/M$. We see that for fixed ℓ the real part of the frequency is *maximal* for the fundamental mode ($n = 0$). This is in contrast to normal modes where the frequency of the fundamental mode is minimal. The fact that all quasinormal modes have frequencies with a strictly negative imaginary part implies that a Schwarzschild black hole is stable against this type of linear perturbations. (There are various ways of how to define stability. The notion considered here is known as *modal stability.*) The damping time (i.e., the inverse of the imaginary part of ω) is surprisingly small: From the table we read that, for a black hole with a few solar masses, the frequency is in the order of kilohertz and the damping time is in the order of milliseconds! For a supermassive black hole with some million Solar masses the damping time is in the order of hours.

The picture below shows the frequencies of the quasinormal modes of a Schwarzschild black hole in the complex plane.

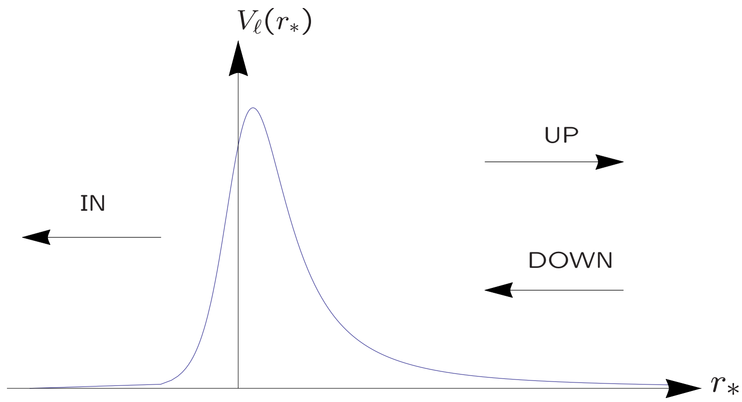


This diagram, taken from N. Andersson and S. Linnaeus ["Quasinormal modes of a Schwarzschild black hole: Improved phase-integral treatment", Phys. Rev. D 46, 4179, (1992)], displays the values for the quasi-normal modes with $\ell = 2$ as diamonds and with $\ell = 3$ as crosses. A similar diagram can also be produced with the even (Zerilli) quasi-normal modes. One finds that they lie along the same curves but at different values.

Quasi-normal modes have also been calculated for charged black holes (i.e., for the Reissner-Nordstöm metric), and, with much greater difficulty, for rotating black holes (i.e., for the Kerr metric). The differences could be used, in principle, for distinguishing different types of black holes by the gravitational radiation they emit when they are perturbed.

We now turn to the scattering problem. To that end we consider solutions $Z_\ell(r_*)$ to the time-independent Regge-Wheeler equation with *real* frequency $\omega \neq 0$, i.e., such that $Q_\ell(t, r_*) = Z_\ell(r_*)e^{-i\omega t}$ is neither decaying nor increasing but oscillatory in time, and we assume that nothing comes out of the horizon. The latter assumption means that we consider a solution that is a pure IN mode; as the DOWN mode and the UP mode are linearly independent for every $\omega \neq 0$, such a solution must be a linear combination of the DOWN mode and the UP mode.

The picture on the next page illustrates this situation: The DOWN mode describes what we send in; part of it is reflected by the potential barrier as an UP mode, and part of it goes to the horizon as an IN mode. From elementary physical intuition one would take it for granted that what is reflected is always less than what is sent in. We will demonstrate that this is, indeed, true for scattering by a Schwarzschild black hole. However, at the end of this section we will comment on other situations (i.e., other types of black holes) where this is not true; one then speaks of *superradiance*.



As a preparation, we first observe that the IN mode must be a linear combination of the UP mode and the DOWN mode because the latter are linearly independent,

$$Z_\ell^I(r_*) = A_\ell Z_\ell^D(r_*) + B_\ell Z_\ell^U(r_*)$$

with complex coefficients A_ℓ and B_ℓ. As for real ω the UP mode is the complex conjugate of the DOWN mode, this may be rewritten as

$$Z_\ell^I(r_*) = A_\ell Z_\ell^D(r_*) + B_\ell \overline{Z}_\ell^D(r_*)$$

where overlining means complex conjugation. Moreover, for real ω the OUT mode is the complex conjugate of the IN mode, hence

$$Z_\ell^O(r_*) = \overline{A}_\ell \overline{Z}_\ell^D(r_*) + \overline{B}_\ell Z_\ell^D(r_*)\,.$$

There is no simple way of calculating the coefficients A_ℓ and B_ℓ (which, of course, also depend on ω). However, it is easy to derive a relation that their absolute squares has to satisfy:

Claim: $\left|A_\ell\right|^2 - \left|B_\ell\right|^2 = 1$.

Proof: Z_ℓ^I and Z_ℓ^O are solutions of the same second-order linear differential equation. This implies that their Wronskian

$$W(Z_\ell^I, Z_\ell^O) = \det \begin{pmatrix} Z_\ell^I & Z_\ell^O \\ \dfrac{dZ_\ell^I}{dr_*} & \dfrac{dZ_\ell^O}{dr_*} \end{pmatrix}$$

is constant. This is a standard result from the theory of linear differential equations which, in the case at hand, is demonstrated by the following elementary calculation.

$$\frac{d}{dr_*} W(Z_\ell^I, Z_\ell^O)(r_*) = \frac{d}{dr_*}\Big(Z_\ell^I(r_*) \frac{dZ_\ell^O(r_*)}{dr_*} - Z_\ell^O(r_*)\frac{dZ_\ell^I(r_*)}{dr_*}\Big)$$

$$= Z_\ell^I(r_*)\frac{d^2 Z_\ell^O(r_*)}{dr_*^2} - Z_\ell^O(r_*)\frac{d^2 Z_\ell^I(r_*)}{dr_*^2}$$

$$= Z_\ell^I(r_*)\Big(V_\ell(r_*) Z_\ell^O(r_*) + \frac{\omega^2}{c^2} Z_\ell^O(r_*)\Big) - Z_\ell^O(r_*)\Big(V_\ell(r_*) Z_\ell^I(r_*) + \frac{\omega^2}{c^2} Z_\ell^I(r_*)\Big) = 0\,.$$

As a consequence,

$$0 = \lim_{r_*\to-\infty}\Big(Z^I_\ell(r_*)\frac{dZ^O_\ell(r_*)}{dr_*} - Z^O_\ell(r_*)\frac{dZ^I_\ell(r_*)}{dr_*}\Big)$$

$$-\lim_{r_*\to\infty}\Big(Z^I_\ell(r_*)\frac{dZ^O_\ell(r_*)}{dr_*}-Z^O_\ell(r_*)\frac{dZ^I_\ell(r_*)}{dr_*}\Big) = \lim_{r_*\to-\infty}\Big(Z^I_\ell(r_*)\frac{d\overline{Z}^I_\ell(r_*)}{dr_*}-\overline{Z}^I_\ell(r_*)\frac{dZ^I_\ell(r_*)}{dr_*}\Big)$$

$$-\lim_{r_*\to\infty}\Big(\big[A_\ell Z^D_\ell(r_*) + B_\ell\overline{Z}^D_\ell(r_*)\big]\frac{d}{dr_*}\big[\overline{A}_\ell\overline{Z}^D_\ell(r_*) + \overline{B}_\ell Z^D_\ell(r_*)\big]\Big)$$

$$+\lim_{r_*\to\infty}\Big(\big[\overline{A}_\ell\overline{Z}^D_\ell(r_*) + \overline{B}_\ell Z^D_\ell(r_*)\big]\frac{d}{dr_*}\big[A_\ell Z^D_\ell(r_*) + B_\ell\overline{Z}^D_\ell(r_*)\big]\Big)$$

$$= \lim_{r_*\to-\infty}\Big(e^{-i\omega r_*/c}\frac{de^{i\omega r_*/c}}{dr_*} - e^{i\omega r_*/c}\frac{de^{-i\omega r_*/c}}{dr_*}\Big)$$

$$-\lim_{r_*\to\infty}\Big(\big[A_\ell e^{-i\omega r_*/c} + B_\ell e^{i\omega r_*/c}\big]\frac{d}{dr_*}\big[\overline{A}_\ell e^{i\omega r_*/c} + \overline{B}_\ell e^{-i\omega r_*/c}\big]$$

$$+\lim_{r_*\to\infty}\Big(\big[\overline{A}_\ell e^{i\omega r_*/c} + \overline{B}_\ell e^{-i\omega r_*/c}\big]\frac{d}{dr_*}\big[A_\ell e^{-i\omega r_*/c} + B_\ell e^{i\omega r_*/c}\big]$$

$$= \lim_{r_*\to-\infty}\Big(e^{-i\omega r_*/c}e^{i\omega r_*/c}\frac{i\omega}{c} + e^{i\omega r_*/c}e^{-i\omega r_*/c}\frac{i\omega}{c}\Big)$$

$$-\lim_{r_*\to\infty}\Big(\big[A_\ell e^{-i\omega r_*/c} + B_\ell e^{i\omega r_*/c}\big]\big[\overline{A}_\ell e^{i\omega r_*/c}\frac{i\omega}{c} - \overline{B}_\ell e^{-i\omega r_*/c}\frac{i\omega}{c}\big]\Big)$$

$$+\lim_{r_*\to\infty}\Big(\big[\overline{A}_\ell e^{i\omega r_*/c} + \overline{B}_\ell e^{-i\omega r_*/c}\big]\big[-A_\ell e^{-i\omega r_*/c}\frac{i\omega}{c} + B_\ell e^{i\omega r_*/c}\frac{i\omega}{c}\big]\Big)$$

$$= \lim_{r_*\to-\infty}\Big(\frac{2i\omega}{c}\Big) - \lim_{r_*\to\infty}\Big(\big|A_\ell\big|^2\frac{2i\omega}{c} - \big|B_\ell\big|^2\frac{2i\omega}{c}\Big) = \frac{2i\omega}{c}\Big(1-\big|A_\ell\big|^2+\big|B_\ell\big|^2\Big)\,.$$

□

After these preparations we are now ready to discuss the scattering problem. We consider a solution to the Regge-Wheeler equation with real $\omega \neq 0$ that is a pure IN mode and, thus, a linear combination of an UP and a DOWN mode,

$$Z_\ell(r_*) = C^I_\ell Z^I_\ell(r_*) = C^D_\ell Z^D_\ell(r_*) + C^U_\ell Z^U_\ell(r_*)$$

with complex coefficients C^I_ℓ, C^D_ℓ and C^U_ℓ. As

$$Z^I_\ell(r_*) = A_\ell Z^D_\ell(r_*) + B_\ell Z^U_\ell(r_*)\,,$$

comparing coefficients yields

$$C^I_\ell A_\ell = C^D_\ell\,,\quad C^I_\ell B_\ell = C^U_\ell\,.$$

One calls

$$\mathcal{T}_\ell = \frac{\left|C_\ell^I\right|^2}{\left|C_\ell^D\right|^2}$$

the *transmission coefficient* and

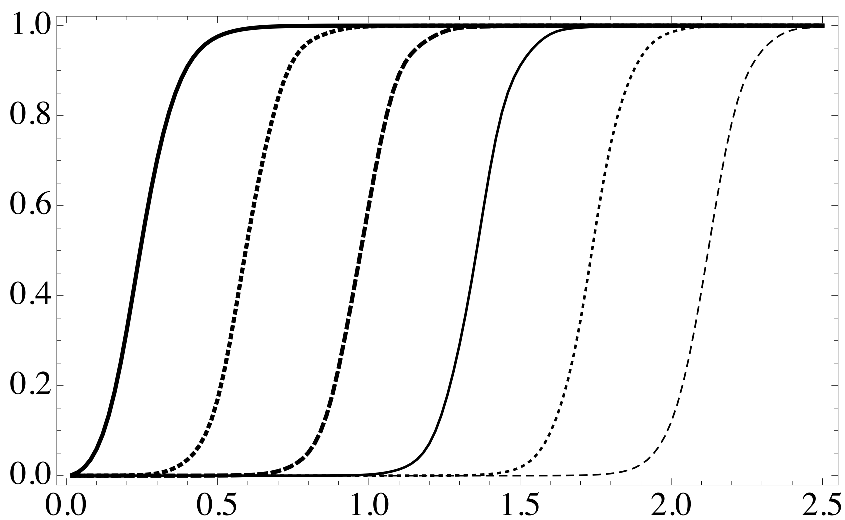
$$\mathcal{R}_\ell = \frac{\left|C_\ell^U\right|^2}{\left|C_\ell^D\right|^2}$$

the *reflection coefficient*. We now find

$$\mathcal{T}_\ell + \mathcal{R}_\ell = \frac{\left|C_\ell^I\right|^2}{\left|C_\ell^D\right|^2} + \frac{\left|C_\ell^U\right|^2}{\left|C_\ell^D\right|^2}$$

$$= \frac{1}{\left|A_\ell\right|^2} + \frac{\left|B_\ell\right|^2}{\left|A_\ell\right|^2} = \frac{1 + \left|B_\ell\right|^2}{\left|A_\ell\right|^2} = \frac{\left|A_\ell\right|^2}{\left|A_\ell\right|^2} = 1\,.$$

As by definition $\mathcal{T}_\ell \geq 0$ and $\mathcal{R}_\ell \geq 0$, this demonstrates that $\mathcal{T}_\ell \leq 1$ and $\mathcal{R}_\ell \leq 1$. The latter inequality proves that there is no superradiance: The reflected intensity cannot be bigger than the intensity that is sent in.



While it was rather easy to demonstrate this inequality, calculating the precise dependence of $\mathcal{T}_\ell$ and $\mathcal{R}_\ell$ on ω is a difficult task. It can be done only numerically or semi-analytically. In the diagram, $\mathcal{T}_\ell$ is plotted for different values of ℓ against the frequency ω; the latter is given in units of c/r_S. We see that $\mathcal{T}_\ell$ tends to zero for $\omega \to 0$ and to 1 for $\omega \to \infty$. However, the $\mathcal{T}_\ell$ turn out to be strictly smaller than 1 for all real ω. As an ideal black body (i.e., a body that totally absorbs all the radiation that is sent in from infinity) would be characterised by $\mathcal{T}_\ell = 1$, it is also common to call the $\mathcal{T}_\ell$ the *greybody factors*.

We have discussed the quasi-normal modes and the scattering problem here for perturbations of the gravitational field, i.e., for gravitational waves. Instead of the linearised Einstein field equation, one may also study the Klein-Gordon equation or the Maxwell equations on the Schwarzschild background. This gives rise to very similar Regge-Wheeler-type equations and to quasi-normal modes and scattering of scalar waves or electromagnetic waves. The fact that there is no superradiance is common for all types of waves on the Schwarzschild background.

By contrast, scattering by a rotating black hole (mathematically modelled by the Kerr metric) does show superradiance. This gives rise to the *black hole bomb*: If a Kerr black hole is enclosed by mirrors, radiation that is sent towards the horizon is back-scattered with an amplification factor bigger than 1, so after repeated reflection at the mirror the amplitude of the radiation becomes unboundedly big. The energy needed for this process is extracted from the black hole by spinning it down.

Superradiance also occurs with charged black holes (mathematically modelled by the Reissner-Nordström metric). In this case one needs a charged field (e.g. a complex Klein-Gordon field) and the needed energy is extracted from the black hole by reducing its charge.

7 Exact wave solutions of Einstein's field equation

Up to now we have treated gravitational waves as perturbations of a background spacetime that are so small that all equations can be linearised with respect to them. This is a viable theory for explaining any observations that are expected for the foreseeable future. Nonetheless it is helpful, and even necessary for a full understanding, to study gravitational waves at the level of the *full nonlinear Einstein equation*. It could well be that some of the observations made in the linear theory, e.g. about the polarisation states or about the multipole characters of gravitational waves, are just an artefact of the linearisation. In this chapter we are going to discuss some classes of *exact* wave solutions to Einstein's vacuum equations.

7.1 Brinkmann solutions (pp waves)

We begin with a class of exact plane-wave solutions which is known as the Brinkmann metrics or as pp waves. For introducing them we first consider Minkowski spacetime in double null coordinates (x^1, x^2, u, v), where the u lines are the straight lightlike lines in negative x^3 direction and the v lines are the straight lightlike lines in positive x^3 direction. We then modify the spacetime in such a way that it is no longer flat but that the v lines remain lightlike, geodesic and orthogonal to planes. The idea is that the v lines can then be interpreted as the "rays" of a gravitational wave with planar wave surfaces, if the vacuum Einstein equation is satisfied.

From Minkowski spacetime in standard coordinates, $g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^0)^2$, we transform to "double-null coordinates", $(x^1, x^2, x^3, x^0) \mapsto (x^1, x^2, u, v)$, defined by

$$x^0 = \frac{1}{\sqrt{2}}(v+u)\,, \qquad x^3 = \frac{1}{\sqrt{2}}(v-u)\,.$$

Then

$$(dx^0)^2 - (dx^3)^2 = \frac{1}{2}\big(dv + du\big)^2 - \frac{1}{2}\big(dv - du\big)^2 =$$

$$= \frac{1}{2}\big(\cancel{du^2} + 2\,dv\,du + \cancel{dv^2}\big) - \frac{1}{2}\big(\cancel{du^2} - 2\,dv\,du + \cancel{dv^2}\big) = 2\,dv\,du\,,$$

hence the Minkowski metric reads

$$g = (dx^1)^2 + (dx^2)^2 - 2\,dv\,du\,.$$

We now add a term that makes the spacetime dynamic (time-dependent), but in such a way that ∂_v remains lightlike,

$$g = (dx^1)^2 + (dx^2)^2 - 2\,dv\,du + H(x^1, x^2, u)\,du^2 \qquad \text{(B)}$$

with some function $H(x^1, x^2, u)$. The dependence of H on u (i.e., on $x^0 - x^3$) makes the metric time-dependent. The vector field ∂_u is no longer lightlike, but the vector field ∂_v still is. We will show in Worksheet 8 that, in addition, ∂_v is absolutely parallel, i.e., covariantly constant in *any* direction, hence in particular geodesic. This allows to interpret the v lines as the propagation direction of a wave that travels at the speed of light. Each x^1-x^2-surface (i.e., each surface $\{u = \text{constant}, v = \text{constant}\}$) is a Euclidean plane perpendicular to the propagation direction of the wave.

Below we will calculate the Christoffel symbols of the metric (B) from which one can easily determine the Ricci tensor. One finds that the vacuum Einstein equation $R_{\mu\nu} = 0$ holds if and only if H satisfies the Laplace equation with respect to the variables x^1 and x^2.

$$\delta^{AB}\partial_A\partial_B H = 0\,. \qquad \text{(T)}$$

Here an in the following, capital indices $A, B, \dots$ takes values 1 and 2. If the condition (T) is satisfied, the metric (B) can be interpreted as a (pure) gravitational wave. For the case that (T) is not satisfied, one finds that the energy-momentum tensor has the form of that of an electromagnetic field; the metric can then be interpreted as a combination of a gravitational wave and an electromagnetic wave.

Metrics of the form (B) made their first appearence in a purely mathematical paper by H. Brinkmann ["Einstein spaces which are mapped conformally on each other" Math. Annalen 94, 119 (1925)]. The coordinates (x^1, x^2, u, v) are known as *Brinkmann coordinates.* A. Peres ["Some gravitational waves " Phys. Rev. Lett. 3, 571 (1959)] rediscovered these metrics with condition (T) and interpreted them as gravitational waves. The same solutions to Einstein's vacuum field equation were studied by J. Ehlers and W. Kundt ["Exact solutions of the gravitational field equations" in L. Witten (ed.) "Gravitation: an introduction to current research" Wiley, New York (1962) p.49] who called them *plane-fronted waves with parallel rays* or *pp-waves* for short. Obviously, "plane-fronted" refers to the (x^1, x^2)-surfaces and "parallel rays" refers to the v-lines.

We will now write down the geodesic equation for the metric (B) which will give us the Christoffel symbols. As usual, the most convenient way is to start from the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) \;=\; \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu} \;=\; \frac{1}{2}\Big((\dot{x}^1)^2 + (\dot{x}^2)^2 - 2\,\dot{u}\,\dot{v} \,+\, H(x^1,x^2,u)\,\dot{u}^2 \Big)$$

where an overdot means derivative with respect to an affine parameter s. From this we get the geodesics as the solutions to the Euler-Lagrange equation

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}\big(x,\dot{x}\big)}{\partial \dot{x}^{\mu}} \right) \,-\, \frac{\partial \mathcal{L}\big(x,\dot{x}\big)}{\partial x^{\mu}} \;=\; 0\,.$$

Doing this for $x^\mu = v$, u and x^A yields

$$-\,\ddot{u} \;=\; 0\,, \qquad -\ddot{v} + \frac{d\big(H\,\dot{u}\big)}{ds} \,-\, \frac{1}{2}\,\partial_u H\,\dot{u}^2 \;=\; 0\,, \qquad \ddot{x}^A \,-\, \frac{1}{2}\,\delta^{AB}\partial_B H\,\dot{u}^2 \;=\; 0\,,$$

hence

$$\ddot{u} \;=\; 0\,, \qquad \ddot{v} \,-\, \frac{1}{2}\,\partial_u H\,\dot{u}^2 \,-\, \partial_A H\,\dot{u}\,\dot{x}^A \;=\; 0\,, \qquad \ddot{x}^A \,-\, \frac{1}{2}\,\delta^{AB}\partial_B H\,\dot{u}^2 \;=\; 0\,.$$

From these equations we read that the only non-vanishing Christoffel symbols are

$$\Gamma^{v}{}_{uu} \;=\; -\,\frac{1}{2}\,\partial_u H \qquad \Gamma^{v}{}_{uA} \;=\; -\,\frac{1}{2}\,\partial_A H \qquad \Gamma^{A}{}_{uu} \;=\; -\,\delta^{AB}\,\partial_B H\,.$$

From the Christoffel symbols one can calculate the Ricci tensor

$$R_{\mu\sigma} = \partial_\tau \Gamma^{\tau}{}_{\mu\sigma} - \partial_\mu \Gamma^{\tau}{}_{\tau\sigma} + \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\tau}{}_{\tau\rho} - \Gamma^{\rho}{}_{\tau\sigma}\Gamma^{\tau}{}_{\mu\rho}\,.$$

One finds that the only non-vanishing component of the Ricci tensor is

$$R_{uu} \;=\; -\,\frac{1}{2}\,\delta^{AB}\partial_A\partial_B H$$

so that indeed the vacuum field equation $R_{\mu\nu} = 0$ is equivalent to the Laplace equation (T), as was already anticipated above.

In the following we will consider those pp-waves for which the function H is a quadratic form in the variables x^1 and x^2,

$$H(x^1,x^2,u) \;=\; h_{AB}(u)x^A x^B$$

with a symmetric (2×2) matrix $\big(h_{AB}(u)\big)$. If the vacuum field equation (T) is satisfied, i.e., if the matrix $\big(h_{AB}(u)\big)$ is trace-free,

$$\delta^{AB}h_{AB}(u) = 0\,,$$

these special pp-waves are called *plane gravitational waves*; otherwise they describe a coupled system of a plane gravitational wave and a plane electromagnetic wave. Both cases were first studied by O. Baldwin and G. Jeffery ["The relativity theory of plane waves", Proc. Roy. Soc. London A 111, 95 (1926)] who did not know about Brinkmann's earlier work on the larger class of what we now call pp-waves.

The condition of vanishing trace means that for a plane gravitational wave the matrix $h_{AB}(u)$ can be written as

$$\big(h_{AB}(u)\big) \;=\; \begin{pmatrix} f_+(u) & f_\times(u) \\ f_\times(u) & -\,f_+(u) \end{pmatrix} .$$

The *profile functions* $f_+(u)$ and $f_\times(u)$ determine the shape of the gravitational wave. The fact that (within the class of metrics considered) two scalar functions are necessary to determine the wave can be interpreted by saying that "a gravitational wave has two polarisation states". This is in perfect agreement with what we have found for plane harmonic waves in the linearised theory about Minkowski spacetime, where we also had two polarisation states, the *plus-mode* and the *cross-mode*).

For a plane gravitational wave the geodesic equation specifies to

$$\ddot{u} \;=\; 0\,,$$

$$\ddot{v} \;=\; \frac{1}{2}\Big(f_+'(u)\big((x^1)^2-(x^2)^2\big)\big)+2\,f_\times'(u)x^1x^2\Big)\,\dot{u}^2$$
$$+\Big(f_+(u)\big(x^1\dot{x}^1-x^2\dot{x}^2\big)+f_\times(u)\big(x^1\dot{x}^2+x^2\dot{x}^1\big)\Big)\dot{u}\,,$$

$$\begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} f_+(u) & f_\times(u) \\ f_\times(u) & -f_+(u) \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} .$$

We see that there are geodesics that are completely contained in a lightlike hypersurface $u =$ constant. For them we have $u(s) = u_0$, $\dot{u}(s) = 0$ and $\ddot{u}(s) = 0$, so the u component of the geodesic equation is satisfied. The other components read

$$\ddot{v}(s) = 0\,,$$

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} f_+(u_0) & f_\times(u_0) \\ f_\times(u_0) & -f_+(u_0) \end{pmatrix}\begin{pmatrix} x^1(s) \\ x^2(s) \end{pmatrix} ,$$

which can be integrated easily.

For all the other geodesics we have $\dot{u}(s) \neq 0$. Then the u component of the geodesic equation, $\ddot{u} = 0$, says that u can be used as the affine parameter. (Recall that the affine parametrisation along a geodesic is unique only up to a transformation of the form $s \mapsto as + b$ with a non-zero constant a.) With $u(s) = s$, the other components of the geodesic equation read

$$\ddot{v}(s) \;=\; \frac{1}{2}\Big(f_+'(s)\big((x^1(s))^2-(x^2(s))^2\big)\big)+2\,f_\times'(s)x^1(s)x^2(s)\Big)$$
$$+\Big(f_+(s)\big(x^1(s)\dot{x}^1(s)-x^2(s)\dot{x}^2(s)\big)+f_\times(s)\big(x^1(s)\dot{x}^2(s)+x^2(s)\dot{x}^1(s)\big),\,,$$

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} f_+(s) & f_\times(s) \\ f_\times(s) & -f_+(s) \end{pmatrix}\begin{pmatrix} x^1(s) \\ x^2(s) \end{pmatrix} .$$

We see that the (x^1, x^2) equation decouples. After having solved this equation, $v(s)$ is determined by a straight-forward integration. Therefore we concentrate on the matrix differential equation for x^1 and x^2. This equation gives the motion of the geodesics in the (x^1, x^2) plane, i.e., in the plane orthogonal to the propagation direction of the wave. For the plus-mode, $f_\times = 0$, we have

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} = \frac{f_+(s)}{2} \begin{pmatrix} x^1(s) \\ -x^2(s) \end{pmatrix} .$$

At points where f_+ is positive, there is focussing in the x^1 direction and defocussing in the x^2 direction; at points where f_+ is negative, it is vice versa.

To discuss the cross mode, we may rotate the coordinates by 45^o,

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} , \qquad \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} .$$

Then

$$\begin{pmatrix} \ddot{y}^1 \\ \ddot{y}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{f_\times(s)}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$= \frac{f_\times(s)}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{f_\times(s)}{4} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{f_\times(s)}{2} \begin{pmatrix} y^1 \\ -y^2 \end{pmatrix} ,$$

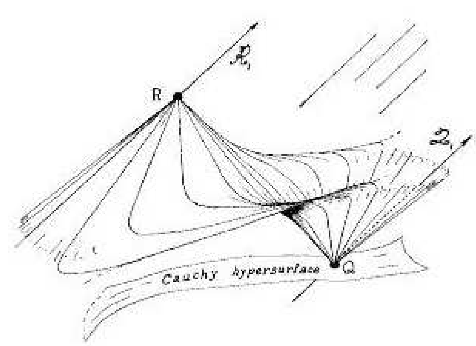
so we have the same focussing and defocussing properties as for the plus-mode, just rotated by 45^o.

This consideration holds for timelike, lightlike and spacelike geodesics. For timelike geodesics it gives the motion of freely falling test particles, in analogy to what we have discussed in the linearised theory. We see that the plus-mode and the cross-mode have the same physical interpretation for the exact plane gravitational waves, but now x^1 and x^2 may be arbitrarily large. To make the analogy with our treatment of the linearised theory perfect, we may Fourier-expand the matrix-valued function $h_{AB}(u)$ (i.e., the profile functions $f_+(u)$ and $f_\times(u)$). Then we get exactly the same expression for each Fourier mode

$$h_{AB}(u) = \mathrm{Re}\big\{ h^0_{AB} e^{-i\omega u/c} \big\}$$

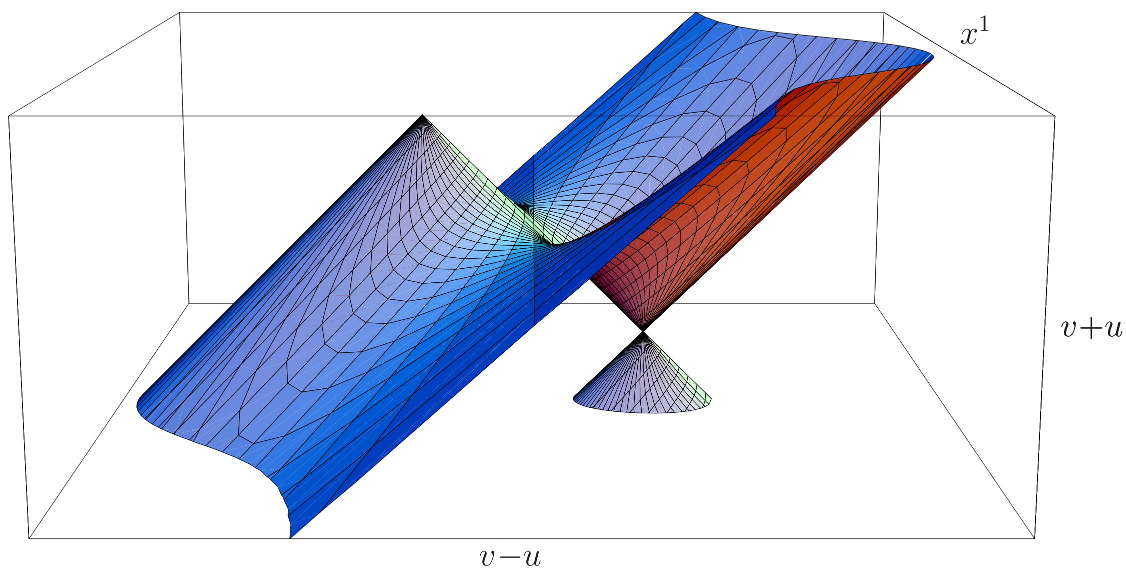
as we had in the linearised theory.

We now turn to the lightlike geodesics. The picture on the right shows the past light-cone of an event R, in a famous hand-drawing by Roger Penrose [" A remarkable property of plane waves in general relativity" Rev. Modern Phys. 37, 215 (1965)]. One sees that, with the exception of the v-line through R (which is a straight line), all light rays that are issuing from the event R into the past are refocussed into another event Q.

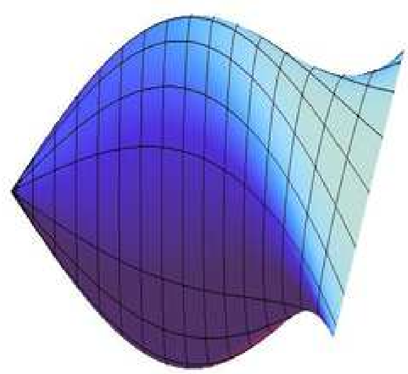


Actually, taking the fourth dimension into account which is missing in the picture, a pure gravitational wave refocusses light rays into a line ("astigmatic focussing"). A combined gravitational and electromagnetic wave can refocus light rays into a point ("anastigmatic focussing"). The picture also indicates that a plane-wave spacetime cannot admit a Cauchy hypersurfaces, i.e., a hypersurface that intersects any causal curve exactly once: Such a hypersurface would have to intersect the v-line through R. But then some of the other past-oriented lightlike geodesics from R to Q have to be intersected twice.

The following picture of the light cone was produced with Mathematica. The profile functions were chosen as $f_\times(u) = 0$ and $f_+(u) = k^2\chi(u)$, where k is a non-zero constant and $\chi(u)$ is the characteristic function of a finite interval (i.e., the gravitational wave is “sandwiched“ between two flat spacetime regions, bounded by hypersurfaces u = constant). The x^2 dimension is omitted. The similarity with the Penrose drawing is striking.



The picture on the right gives a purely spatial view of the light-cone above. Now both spatial dimensions x^1 and x^2 are shown and the temporal dimension, $u + v$, is omitted. One clearly sees the astigmatic focussing: There is focussing in one spatial dimension and defocussing in the other spatial dimension, so that the lightlike geodesics are refocussed in a line.



7.2 Beck-Einstein-Rosen solutions

In this section we want to discuss a class of excact wave-like solutions to Einstein's vacuum equation with cylindrical symmetry. These solutions are usually called *Einstein-Rosen waves* although they were found by Austrian physicist Guido Beck already 12 years before Einstein and Rosen [G. Beck: "Zur Theorie binärer Gravitationsfelder" Zeitschr. f. Physik 33, 713 (1925)].

Beck started out from known results on axisymmetric and static metrics which had been found by H. Weyl in 1917. A metric is axisymmetric and static if it can be written in cylindrical polar coordinates (t, ρ, φ, z) such that the $g_{\mu\nu}$ are independent of t, independent of φ, and invariant under a transformation $\varphi \mapsto -\varphi$. Such metrics describe the gravitational fields of time-independent non-rotating bodies with axial symmetry. (If the invariance under the transformation $\varphi \mapsto -\varphi$ is dropped one speaks of axisymmetric stationary metrics; then rotating bodies are included.) Beck took the known axisymmetric and static metrics and performed the formal substitution $t \mapsto iz$, $z \mapsto it$. Then the metric is still axisymmetric but, instead of being time-independent, it is now invariant under translations in z-direction. In this way one gets time-dependent metrics (waves) with cylindrical symmetry.

The ansatz for the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\gamma-2\psi}\big(d\rho^2 - c^2dt^2\big) + e^{-2\psi}W^2d\varphi^2 + e^{2\psi}dz^2$$

where γ, ψ and W are functions of t and ρ. This is precisely the same ansatz, with the above-mentioned substitution, as it is used for the axisymmetric and static metrics; in the latter context, one speaks of *Weyl canonical coordinates*. This is the most general form of a cylindrically symmetric metric apart from the fact that we have assumed invariance under $\varphi \mapsto -\varphi$ (in analogy to the axisymmetric static case). Note that the ansatz of the metric in the (t, ρ) plane being proportional to $\big(d\rho^2 - c^2dt^2\big)$ is no restriction as every two-dimensional metric is conformal to the flat metric.

To find vacuum solutions with the prescribed symmetry we have to calculate the Ricci tensor. As usual, the easiest way to find the Christoffel symbols is by starting from the Lagrangian for the geodesics,

$$\mathcal{L}(x,\dot{x}) \;=\; \frac{1}{2}\Big(e^{2\gamma-2\psi}\big(\dot{\rho}^2 - c^2\dot{t}^2\big) + e^{-2\psi}W^2\dot{\varphi}^2 + e^{2\psi}\dot{z}^2\Big)$$

where the overdot means differentiation with respect to an affine parameter s. The Euler-Lagrange equations give the four components of the geodesic equation. After some elementary algebra they take the following form.

$$\ddot{z} + 2\,\partial_\rho\psi\,\dot{\rho}\,\dot{z} + 2\,\partial_t\psi\,\dot{t}\,\dot{z} = 0\,,$$

$$\ddot{\varphi} + 2\Big(\frac{\partial_\rho W}{W} - \partial_\rho\psi\Big)\dot{\rho}\,\dot{\varphi} + 2\Big(\frac{\partial_t W}{W} - \partial_t\psi\Big)\dot{t}\,\dot{\varphi} = 0\,,$$

$$\begin{aligned}\ddot{\rho} + \big(\partial_\rho\gamma - \partial_\rho\psi\big)\dot{\rho}^2 - 2\big(\partial_t\gamma - \partial_t\psi\big)\dot{\rho}\,\dot{t} + \big(\partial_\rho\gamma - \partial_\rho\psi\big)c^2\dot{t}^2 \\ -e^{-2\gamma}W^2\Big(\frac{\partial_\rho W}{W} - \partial_\rho\psi\Big)\dot{\varphi}^2 - e^{-2\gamma+4\psi}\partial_\rho\psi\,\dot{z}^2 = 0\,,\end{aligned}$$

$$\begin{aligned}\ddot{t} + 2\big(\partial_\rho\gamma - \partial_\rho\psi\big)\dot{\rho}\,\dot{t} + \big(\partial_t\gamma - \partial_t\psi\big)\dot{t}^2 + \big(\partial_t\gamma - \partial_t\psi\big)\frac{1}{c^2}\dot{\rho}^2 \\ +e^{-2\gamma}\frac{W^2}{c^2}\Big(\frac{\partial_t W}{W} - \partial_t\psi\Big)\dot{\varphi}^2 + e^{-2\gamma+4\psi}\partial_t\psi\,\frac{1}{c^2}\,\dot{z}^2 = 0\,.\end{aligned}$$

From these equations we can read the Christoffel symbols and, thereupon, calculate the Ricci tensor. We find

$$R_{zz} = e^{-2\gamma+4\psi}\Big(\partial_\rho^2\psi - \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho W}{W}\partial_\rho\psi - \frac{\partial_t W}{c^2 W}\partial_t\psi\Big)\,,$$

$$R_{\varphi\varphi} = W^2 e^{-2\gamma}\Big(- \partial_\rho^2\psi + \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho^2 W}{W} - \frac{\partial_t^2 W}{c^2 W} - \frac{\partial_\rho W}{W}\partial_\rho\psi + \frac{\partial_t W}{c^2 W}\partial_t\psi\Big)\,,$$

$$R_{tt} = \partial_t^2\gamma - c^2\partial_\rho^2\gamma - \partial_t^2\psi + c^2\partial_\rho^2\psi + \frac{\partial_t^2 W}{W} - c^2\frac{\partial_\rho W}{W}\big(\partial_\rho\gamma - \partial_\rho\psi\big)$$
$$-\frac{\partial_t W}{W}\big(\partial_t\gamma + \partial_t\psi\big) + 2\big(\partial_t\psi\big)^2\,,$$

$$R_{\rho\rho} = \partial_\rho^2\gamma - \frac{1}{c^2}\partial_t^2\gamma - \partial_\rho^2\psi + \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho^2 W}{W} - \frac{\partial_\rho W}{W}\big(\partial_\rho\gamma + \partial_\rho\psi\big)$$
$$-\frac{\partial_t W}{c^2 W}\big(\partial_t\gamma - \partial_t\psi\big) + 2\big(\partial_\rho\psi\big)^2\,,$$

$$R_{\rho t} = R_{t\rho} = \frac{\partial_\rho\partial_t W}{W} - \frac{\partial_\rho W}{W}\partial_t\gamma - \frac{\partial_t W}{W}\partial_\rho\gamma + 2\partial_\rho\psi\,\partial_t\psi\,,$$

The other components of the Ricci tensor are zero. This reduces the vacuum field equation to five scalar equations. The first two equations, $R_{zz} = 0$ and $R_{\varphi\varphi} = 0$, are equivalent to the two equations

$$\partial_\rho^2 W - \frac{1}{c^2}\partial_t^2 W = 0\,, \qquad \text{(B1)}$$

$$\partial_\rho^2\psi - \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho W}{W}\partial_\rho\psi - \frac{\partial_t W}{c^2 W}\partial_t\psi = 0\,. \qquad \text{(B2)}$$

Similarly, the equations $R_{\rho\rho} = 0$ and $R_{tt} = 0$ are equivalent to the two equations

$$\frac{\partial_t^2 W}{2c^2 W} + \frac{\partial_\rho^2 W}{2W} - \frac{\partial_\rho W}{W}\partial_\rho\gamma - \frac{\partial_t W}{c^2 W}\partial_t\gamma + \frac{1}{c^2}\big(\partial_t\psi\big)^2 + \big(\partial_\rho\psi\big)^2 = 0\,, \qquad \text{(B3)}$$

$$\partial_\rho^2\gamma - \frac{1}{c^2}\partial_t^2\gamma - \frac{1}{c^2}\big(\partial_t\psi\big)^2 + \big(\partial_\rho\psi\big)^2 = 0\,. \qquad \text{(B4)}$$

The last component requires

$$\frac{\partial_\rho\partial_t W}{W} - \frac{\partial_\rho W}{W}\partial_t\gamma - \frac{\partial_t W}{W}\partial_\rho\gamma + 2\partial_\rho\psi\,\partial_t\psi = 0\,. \qquad \text{(B5)}$$

We will solve these equations for two cases.

Case A : $\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2 > 0$

This condition, which says that the gradient of the function W is spacelike, guarantees, in particular, that this gradient has no zeros. We can, therefore, use

$$\tilde{\rho} = W(t,\rho)$$

as a new coordinate. We use this freedom for performing a coordinate transformation $(t,\rho)\mapsto\big(\tilde{t},\tilde{\rho}\big)$ such that

$$d\tilde{t} = \partial_\rho W\, dt + \frac{\partial_t W}{c^2} d\rho\,, \qquad d\tilde{\rho} = \partial_\rho W\, d\rho + \partial_t W\, dt\,.$$

The second equation is just the differential version of the definition of $\tilde{\rho}$. We have to check if the first equation defines, indeed, a function $\tilde{t}$. The equations

$$\partial_t \tilde{t} = \partial_\rho W\,, \qquad \partial_\rho \tilde{t} = \frac{\partial_t W}{c^2}$$

can be satisfied only if the integrability condition

$$\partial_\rho^2 W = \frac{\partial_t^2 W}{c^2}$$

is satisfied. This, however, is guaranteed by the field equation, see (B1). Our new coordinates satisfy

$$\begin{aligned} d\tilde{\rho}^2 - c^2 d\tilde{t}^2 &= \Big(\partial_\rho W\, d\rho + \partial_t W\, dt\Big)^2 - c^2\Big(\partial_\rho W\, dt + \frac{\partial_t W}{c^2} d\rho\Big)^2 \\ &= \Big(\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2\Big)\big(d\rho^2 - c^2 dt^2\big)\,. \end{aligned}$$

Note that the factor on the right-hand side is positive by assumption. Therefore, we can replace the function γ by a new function $\tilde{\gamma}$, defined by

$$e^{2\tilde{\gamma}} \;=\; \frac{e^{2\gamma}}{\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2}\,.$$

Then the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\tilde{\gamma}-2\psi}\big(d\tilde{\rho}^2 - c^2 d\tilde{t}^2\big) + e^{-2\psi}\tilde{\rho}^2 d\varphi^2 + e^{2\psi}dz^2\,.$$

In the following we drop the tildas. Now we have to evaluate our field equations (B1) to (B5) with $W(t,\rho)=\rho$. (B1) is automatically satisfied. (B2) becomes

$$\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi - \frac{1}{c^2}\partial_t^2\psi = 0\,. \qquad \text{(B2')}$$

(B3) and (B5) can be solved for the partial derivatives of γ,

$$\partial_\rho\gamma = \rho\Big(\big(\partial_\rho\psi\big)^2 + \frac{1}{c^2}\big(\partial_t\psi\big)^2\Big)\,, \qquad \text{(B3')}$$

$$\partial_t\gamma \;=\; 2\,\rho\,\partial_\rho\psi\,\partial_t\psi\,. \qquad \text{(B5')}$$

(B4) is then automatically satisfied. Note that (B2') is a differential equation for ψ alone. We can solve this equation with a standard separation ansatz. After splitting off the time part we are left with the radial part of the Laplace equation in cylindrical coordinates which is the well-known Bessel equation. Therefore the general solution to (B2') is

$$\psi(t,\rho) \;=\; A\,J_0(\omega\rho)\cos(\omega t) + B\,Y_0(\omega\rho)\sin(\omega t)$$

where J_0 and Y_0 are the Bessel functions of first and second kind, respectively. While J_0 is regular everywhere, Y_0 goes to $-\infty$ for $\rho \to 0$. If we want to have a solution that is regular on the axis we have to choose $B = 0$. Having solved (B2'), we can determine γ from (B3') and (B5'). It is obvious that the solution γ is unique up to an additive constant. Existence of the solution is less trivial. We have to check if the integrability condition is satisfied:

$$\partial_t\Big\{\rho\Big(\big(\partial_\rho\psi\big)^2 + \frac{1}{c^2}\big(\partial_t\psi\big)^2\Big)\Big\} \;\stackrel{?}{=}\; \partial_\rho\big\{2\,\rho\,\partial_\rho\psi\,\partial_t\psi\big\}\,,$$

$$\rho\Big(\cancel{2\partial_\rho\psi\,\partial_t\partial_\rho\psi} + \frac{2}{c^2}\partial_t\psi\partial_t^2\psi\Big) \;\stackrel{?}{=}\; 2\,\partial_\rho\psi\,\partial_t\psi + 2\,\rho\,\partial_\rho^2\psi\,\partial_t\psi + \cancel{2\,\rho\,\partial_\rho\psi\,\partial_\rho\partial_t\psi}\,,$$

$$0 \;\stackrel{!}{=}\; 2\,\rho\,\partial_t\psi\Big(\frac{1}{\rho}\partial_\rho\psi\, + \partial_\rho^2\psi\, - \frac{1}{c^2}\partial_t^2\psi\Big)\,.$$

We see that the integrability condition of (B3') and (B5') is just the equation (B2'). This guarantees that to every solution of (B2') we find a corresponding γ such that all components of the vacuum field equation are satisfied.

This class of solutions describes gravitational waves with cylindrical symmetry. For $B = 0$ they are well-defined, as source-free vacuum solutions, on all of $\mathbb{R}^4$. There is a coordinate singularity on the axis, as always when using cylindrical polar coordinates, but no curvature singularity. This class of vacuum solutions was (re-)discovered by A. Einstein and N. Rosen ["On gravitational waves" J. Franklin Inst. 223, 43 (1937)]. In an earlier version of this paper, Einstein and Rosen had interpreted the coordinate we called φ as a non-periodic, Cartesian-like coordinate and, correspondingly, the waves as planar rather than as cylindrical. The (coordinate) singularity at $\rho = 0$ gave them the impression that this solution is unphysical and they even concluded from this observation that gravitational waves do not exist in the full non-linear theory. Einstein and Rosen submitted their paper with this (completely false) conclusion to Physical Review. The Editor sent the article to a referee (which had never been happened to Einstein before) who pointed out that the conclusion was erroneous and that, actually, the solutions are cylindrical. Einstein was so angry about the fact that his article had been sent for refereeing that he withdraw the paper and decided never again to publish in Physical Review. After H. P. Robertson (who, as we know now, was the referee) explained to him his error, Einstein wrote a completely new version of the article (N. Rosen had left for Russia by that time) which was then published in the Journal of the Franklin Institute. The cylindrical solutions presented in this paper are now known as Einstein-Rosen waves although they had already been found by Beck 12 years earlier.

Case B : $\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2 = 0\,,\quad \partial_\rho W \neq 0$

This condition says that the gradient of the function W is lightlike and non-zero. Then we have

$$\partial_\rho W = \pm\frac{1}{c}\partial_t W\,.$$

Here and in the following, either the upper sign or the lower sign holds. The components (B3) and (B5) of the field equations read

$$\frac{\partial_\rho^2 W}{W} - \frac{\partial_\rho W}{W}\partial_\rho \gamma \mp \frac{\partial_\rho W}{cW}\partial_t \gamma + \frac{1}{c^2}\big(\partial_t \psi\big)^2 + \big(\partial_\rho \psi\big)^2 = 0\,,$$

$$\frac{\partial_\rho^2 W}{W} \mp \frac{\partial_\rho W}{cW}\partial_t \gamma - \frac{\partial_\rho W}{W}\partial_\rho \gamma \pm \frac{2}{c}\partial_\rho \psi \partial_t \psi = 0\,.$$

Subtracting the second equation from the first yields

$$\partial_\rho \psi = \pm \frac{1}{c}\partial_t \psi\,,$$

i.e.,

$$\psi(t,\rho) = f(ct \pm \rho)\,.$$

Upon inserting this result into (B4), and using that $\partial_\rho W$ has no zeros, we find

$$\partial_\rho^2 \gamma - \frac{1}{c^2}\partial_t^2 \gamma = 0\,,$$

i.e.,

$$\gamma(t,\rho) = p(ct \pm \rho) + q(ct \mp \rho)\,.$$

With these results our metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = -\Big(e^{2p-2f}(cdt \pm d\rho)\Big)\Big(e^{2q}(cdt \mp d\rho)\Big) + e^{-2f}W^2 d\varphi^2 + e^{2f}dz^2\,.$$

We replace t and ρ by new coordinates $\big(\tilde{u},\tilde{v}\big)$ such that

$$d\tilde{u} = \frac{1}{\sqrt{2}}\,e^{2p-2f}(cdt \pm d\rho)\,, \qquad d\tilde{v} = \frac{1}{\sqrt{2}}\,e^{2q}(cdt \mp d\rho)\,.$$

This is possible as the integrability conditions are obviously satisfied. Then the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = 2d\tilde{u}d\tilde{v} + C_{11}(\tilde{u})d\varphi^2 + C_{22}(\tilde{u})dz^2$$

where we have used that f and W depend on $t \pm \rho$ only which, in turn, can be expressed in terms of $\tilde{u}$ alone. Metrics of the form

$$g_{\mu\nu}dx^\mu dx^\nu = -2d\tilde{u}d\tilde{v} + C_{AB}(\tilde{u})d\tilde{x}^A d\tilde{X}^B$$

are known as *Rosen waves*. They were discussed in a paper by N. Rosen which he wrote after he had left Princeton for the Soviet Union [N. Rosen: "Plane polarized waves in the general theory of relativity". Phys. Z. Soviet Union 12, 366 (1937)]. With our metric ansatz we have found only those Rosen waves for which the matrix $C_{AB}(\tilde{u})$ is diagonal; one gets the general class if one drops the assumption of the φ lines being orthogonal to the z lines.

The Rosen waves are actually locally isometric to the plane waves we have studied in the preceding section in the Brinkmann coordinates. We demonstrate this for the case that the matrix C_{AB} is diagonal.

We start out from the metric in Rosen coordinates with

$$\big(C_{AB}\big) = \begin{pmatrix} e_1(\tilde{u}^2)^2 & 0 \\ 0 & e_2(\tilde{u}^2)^2 \end{pmatrix} .$$

We express the Rosen coordinates $\big(\tilde{u}, \tilde{v}, \tilde{x}^1, \tilde{x}^2\big)$ in terms of new coordinates (which will turn out to be the Brinkmann coordinates) (u, v, x^1, x^2) by

$$\tilde{u} = u\,, \qquad \tilde{v} = v - \frac{1}{2}\Big(\frac{\dot{e}_1(u)}{e_1(u)}(x^1)^2 + \frac{\dot{e}_2(u)}{e_2(u)}(x^2)^2\Big)\,,$$

$$\tilde{x}^1 = \frac{x^1}{e_1(u)}\,, \qquad \tilde{x}^2 = \frac{x^2}{e_2(u)}\,.$$

Then

$$g_{\mu\nu}dx^\mu dx^\nu = -2d\tilde{u}d\tilde{v} + e_1^2\big(d\tilde{x}^1\big)^2 + e_2^2\big(d\tilde{x}^2\big)^2$$

$$= -2du\Big\{dv - \frac{1}{2}\frac{\dot{e}_1}{e_1}2x^1dx^1 - \frac{1}{2}\frac{\dot{e}_2}{e_2}2x^2dx^2 - \frac{1}{2}\Big(\frac{\dot{e}_1}{e_1}\Big)^{\cdot}(x^1)^2du + \frac{1}{2}\Big(\frac{\dot{e}_2}{e_2}\Big)^{\cdot}(x^2)^2du\Big\}$$

$$+e_1^2\Big(\frac{dx^1}{e_1} - \frac{\dot{e}_1}{e_1^2}x^1du\Big)^2 + e_2^2\Big(\frac{dx^2}{e_2} - \frac{\dot{e}_2}{e_2^2}x^2du\Big)^2$$

$$= -2dudv + dudx^1\Big\{2\cancel{\frac{\dot{e}_1}{e_1}x^1} - 2\cancel{\frac{\dot{e}_1}{e_1}x^1}\Big) + dudx^2\Big(2\cancel{\frac{\dot{e}_2}{e_2}x^2} - 2\cancel{\frac{\dot{e}_2}{e_2}x^1}\Big\}$$

$$+du^2\Big\{\Big(\frac{\ddot{e}_1}{e_1} - \cancel{\frac{\dot{e}_1^2}{e_1^2}}\Big)(x_1)^2 + \Big(\frac{\ddot{e}_1}{e_2} - \cancel{\frac{\dot{e}_2^2}{e_2^2}}\Big)(x_2)^2 + \cancel{\frac{\dot{e}_1^2}{e_1^2}(x^1)^2} + \cancel{\frac{\dot{e}_2^2}{e_2^2}(x^2)^2}\Big) + (dx^1)^2 + (dx^2)^2\,.$$

This is precisely the form of a plane wave in Brinkmann coordinates,

$$g_{\mu\nu}dx^\mu dx^\nu = -2dudv + h_{AB}(u)x^Ax^Bdu^2 + (dx^1)^2 + (dx^2)^2\,,$$

with

$$\big(h_{AB}(u)\big) = \begin{pmatrix} \dfrac{\ddot{e}_1(u)}{e_1} & 0 \\ 0 & \dfrac{\ddot{e}_2(u)}{e_2} \end{pmatrix} .$$

The vacuum field equation requires

$$\frac{\ddot{e}_1(u)}{e_1} + \frac{\ddot{e}_2(u)}{e_2} = 0\,.$$

The other cases, where the gradient of W is timelike, or where it changes its causal character from point to point, will not be treated here. The latter case is of relevance for *colliding* waves.

7.3 Robinson-Trautman solutions

While plane waves are associated with bounded sources only approximately, at a large distance from the sources, and cylindrical waves are not associated with bounded sources at all, we will finally study a class of solutions that do give a valid description of gravitational radiation from bounded sources. It was constructed by I. Robinson and A. Trautman ["Some spherical gravitational waves in general relativity" Proc. Roy. Soc. London A 265, 463 (1962)] in analogy to the Liénard-Wiechert field from electrodynamics. The latter is the electromagnetic field of an accelerated point charge in Minkowski spacetime. The radiation field propagates along the lightlike geodesics (i.e., lightlike straight lines) that issue from the worldline of the point charge into the future. These lightlike geodesics, which generate the future light-cones from the events of the worldline of the charge, are hypersurface-orthogonal, shear-free and expanding. The basic idea of Robinson and Trautman was to construct vacuum solutions to Einstein's field equation which admit a family of lightlike geodesics with the same properties. One could then interpret these lightlike geodesics as the rays of gravitational radiation.

We begin by writing down the general form of a spacetime that is foliated into lightlike hypersurfaces. These hypersurfaces, which generalise the light-cones in Minkowski spacetime, can be written as hypersurfaces $\sigma =$ constant where σ is a scalar function with a lightlike gradient,

$$g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 0\,.$$

We define a vector field $K^\mu\partial_\mu$ by

$$K^\mu = g^{\mu\nu}\partial_\nu\sigma\,.$$

Clearly, this vector field is lightlike,

$$g_{\mu\nu}K^\mu K^\nu = g_{\mu\nu}g^{\mu\rho}\partial_\rho\sigma g^{\nu\lambda}\partial_\lambda\sigma = g^{\rho\lambda}\partial_\rho\sigma\partial_\lambda\sigma = 0\,,$$

and geodesic,

$$0 = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \quad\Longrightarrow\quad 0 = \nabla_\lambda\big(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\big) = 2g^{\mu\nu}\partial_\mu\nabla_\lambda\partial_\nu\sigma$$

$$= 2K^\nu\big(\partial_\lambda\partial_\nu\sigma - \Gamma^\tau{}_{\lambda\nu}\sigma\big) = 2K^\nu\big(\partial_\nu\partial_\lambda\sigma - \Gamma^\tau{}_{\nu\lambda}\sigma\big) = 2K^\nu\nabla_\nu\partial_\lambda\sigma = 2K^\nu\nabla_\nu K_\lambda\,.$$

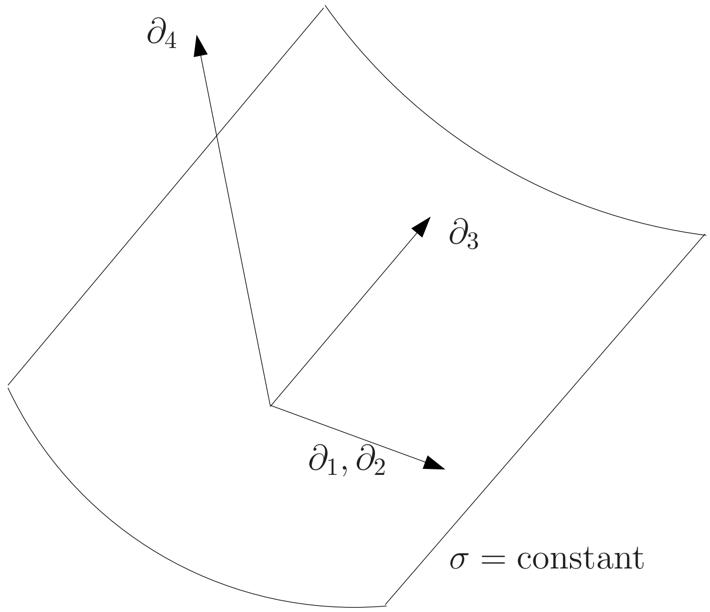
Note that the vector field $K^\mu\partial_\mu$ is tangent to the hypersurfaces $\sigma =$ constant and at the same time orthogonal to them.

We can choose coordinates $x^1 = \xi$, $x^2 = \eta$, $x^3 = \rho$ and $x^4 = \sigma$ in such a way that

$$\frac{\partial}{\partial\rho} = \partial_3 = K^\mu\partial_\mu\,,$$

see picture on the next page.

This can be achieved by assigning the value $\rho = \rho_0$ to a hypersurface that is transverse to the hypersurfaces $\sigma =$ constant and dragging it along with the flow of $K^\mu\partial_\mu$ to get the other hypersurfaces $\rho =$ constant; the coordinates ξ and η have to be chosen transverse to σ, but arbitrarily otherwise, on the initial hypersurface $\rho = \rho_0$ and are then again fixed by dragging them along with the flow of $K^\mu\partial_\mu$.



By construction,

$$g^{14} = g^{41} = g^{\mu\nu}\delta^4_\mu\delta^1_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\xi = K^\nu\partial_\nu\xi = \frac{\partial\xi}{\partial\rho} = 0\,,$$

$$g^{24} = g^{42} = g^{\mu\nu}\delta^4_\mu\delta^2_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\eta = K^\nu\partial_\nu\eta = \frac{\partial\eta}{\partial\rho} = 0\,,$$

$$g^{34} = g^{43} = g^{\mu\nu}\delta^4_\mu\delta^3_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\rho = K^\nu\partial_\nu\rho = \frac{\partial\rho}{\partial\rho} = 1\,,$$

$$g^{44} = g^{\mu\nu}\delta^4_\mu\delta^4_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 0\,.$$

This demonstrates that the contravariant components of the metric can be written as

$$\left(g^{\mu\nu}\right) = \begin{pmatrix} P^2\gamma^{11} & P^2\gamma^{12} & a & 0 \\ P^2\gamma^{12} & P^2\gamma^{22} & b & 0 \\ a & b & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{with} \qquad \det\begin{pmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{12} & \gamma^{22} \end{pmatrix} = 1\,.$$

Here we have used that the two-surfaces parametrised by ξ and η are spacelike, so the determinant of $\left(g^{AB}\right)$ must be positive. (As before, capital indices $A, B, \dots$ take values 1 and 2.) From the minors of the matrix $(g^{\mu\nu})$ we read that $g_{31} = g_{32} = g_{33} = 0$, hence

$$\delta^B_A = g_{A\mu}g^{\mu B} = g_{AC}g^{CB} = g_{AC}P^2\gamma^{CB} \quad \Longrightarrow \quad (\gamma^{-1})_{AC} = P^2 g_{AC}\,.$$

We will now add the condition that $K^\mu\partial_\mu$ should be shear-free and expanding. Twist, expansion and shear of the lightlike vector field $K^\mu\partial_\mu$ are defined as

twist : $\quad \Omega_{AB} = \dfrac{1}{2}\big(\nabla_A K_B - \nabla_B K_A\big)\,,$

expansion : $\quad \Theta = \nabla_A K^A\,,$

shear : $\quad \Sigma_{AB} = \dfrac{1}{2}\big(\nabla_A K_B + \nabla_B K_A\big) - \dfrac{\Theta}{2} g_{AB}\,.$

In the case at hand,

$$\nabla_\mu K_\nu = \nabla_\mu \partial_\nu \sigma = \partial_\mu \partial_\nu \sigma - \Gamma^\lambda{}_{\mu\nu}\partial_\lambda \sigma = 0 - \frac{1}{2} g^{\lambda\tau}\big(\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}\big)\delta^4_\lambda$$

$$= -\frac{1}{2}\underbrace{g^{43}}_{=1}\big(\partial_\mu \underbrace{g_{3\nu}}_{=0} + \partial_\nu \underbrace{g_{3\mu}}_{=0} - \partial_3 g_{\mu\nu}\big) = \frac{1}{2}\partial_3 g_{\mu\nu}\,.$$

Hence, the twist vanishes, $\Omega_{AB} = 0$. (Quite generally, the property of being hypersurface-orthogonal is equivalent to being twist-free.) To calculate the expansion, we observe that the Jacobi formula

$$\partial_3\big(\det(\boldsymbol{\gamma})\big) = \mathrm{trace}\big(\boldsymbol{\gamma}^{-1}\partial_3\boldsymbol{\gamma}\big)$$

applied to the matrix $\boldsymbol{\gamma} = (\gamma^{AB})$ results in

$$0 \,=\, \big(\gamma^{-1}\big)_{AB}\partial_3\gamma^{AB}\,,$$

hence

$$\Theta = g^{AB}\nabla_A K_B = \frac{1}{2} g^{AB}\partial_3 g_{AB} = -\frac{1}{2} g_{AB}\partial_3 g^{AB} = -\frac{1}{2}P^{-2}(\gamma^{-1})_{AB}\partial_3(P^2\gamma^{AB})$$

$$= -\frac{1}{2}P^{-2}(\gamma^{-1})_{AB}\gamma^{AB} 2P\partial_3 P = -2P^{-1}\partial_3 P\,.$$

Finally, we find the shear as

$$\Sigma_{AB} = \frac{1}{2}\partial_3 g_{AB} + P^{-1}\partial_3 P g_{AB} = \frac{1}{2}\partial_3\big(P^{-2}(\gamma^{-1})_{AB}\big) + P^{-3}(\gamma^{-1})_{AB}\partial_3 P$$

$$= \frac{1}{2}P^{-2}\partial_3\big((\gamma^{-1})_{AB}\big)\,.$$

We assume that the shear vanishes, i.e., that $\partial_3\big((\gamma^{-1})_{AB}\big) = 0$. This condition is equivalent to $\partial_3\gamma^{AB} = 0$. If we choose the coordinates ξ and η such that $\gamma^{AB} = \delta^{AB}$ on the initial hypersurface $\rho = \rho_0$, this condition will hold everywhere, so the contravariant components of the metric simplify to

$$\big(g^{\mu\nu}\big) = \begin{pmatrix} P^2 & 0 & a & 0 \\ 0 & P^2 & b & 0 \\ a & b & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix can be easily inverted,

$$\big(g_{\mu\nu}\big) = \begin{pmatrix} P^{-2} & 0 & 0 & -P^{-2}a \\ 0 & P^{-2} & 0 & -P^{-2}b \\ 0 & 0 & 0 & 1 \\ -P^{-2}a & -P^{-2}b & 1 & -c + P^{-2}a^2 + P^{-2}b^2 \end{pmatrix},$$

so the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = P^{-2}\Big(\big(d\xi - a\,d\sigma\big)^2 + \big(d\eta - b\,d\sigma\big)^2\Big) + 2\,d\rho\,d\sigma - c\,d\sigma^2\,.$$

This is the general form of a metric that admits a hypersurface-orthogonal, shear-free geodesic lightlike vector field.

Finally, we add the conditions that the expansion is non-zero and that the vacuum field equation $R_{\mu\nu} = 0$ holds. We begin with the 33-component of the field equation.

$$0 = R_{33} = R^\tau{}_{\mu\tau\nu} K^\mu K^\nu = K^\mu \big(\nabla_\mu \nabla_\tau K^\tau - \nabla_\tau \nabla_\mu K^\tau\big)$$

$$= K^\mu \nabla_\mu \nabla_\tau K^\tau - \nabla_\tau \big(\underbrace{K^\mu \nabla_\mu K^\tau}_{=0}\big) + \nabla_\mu K^\mu \nabla_\tau K^\tau$$

$$= \partial_3 \big(\nabla_A K^A + 0\big) + \nabla_A K^B \nabla_B K^A + 0 = \partial_3 \Theta + \frac{1}{4} g^{BC} g^{AD} \partial_3 g_{AC} \partial_3 g_{BD}$$

$$= \partial_3 \Theta + \frac{P^4}{4} \delta^{BC} \delta^{AD} \partial_3 \big(P^{-2} \delta_{AC}\big) \partial_3 \big(P^{-2} \delta_{BD}\big)$$

$$= \partial_3 \Theta + \frac{P^4}{4} \big(\underbrace{-2P^{-3} \partial_3 P}_{=P^{-2}\Theta}\big)^2 \underbrace{\delta^B_A \delta^A_B}_{=2} = \frac{\partial \Theta}{\partial \rho} + \frac{\Theta^2}{2}\,.$$

Quite generally, evaluating the expression $R_{\mu\nu} K^\mu K^\nu$ results in a differential equation for the expansion Θ along the integral curves of $K^\mu \partial_\mu$ which is known as the *Raychudhuri equation*. In the case at hand, assuming $R_{33} = 0$, it simply reads

$$\frac{\partial \Theta}{\partial \rho} = \frac{\Theta^2}{2}\,.$$

Now we use our assumption that $\Theta \neq 0$. Then we can integrate the Raychudhuri equation,

$$-\frac{2}{\Theta^2} \frac{\partial \Theta}{\partial \rho} = 1 \qquad \Longrightarrow \qquad 2 \frac{\partial \Theta^{-1}}{\partial \rho} = 1 \qquad \Longrightarrow \qquad \frac{2}{\Theta} = \rho + f(\xi, \eta, \sigma)\,.$$

As ρ was introduced by assigning a value ρ_0 to an arbitrary hypersurface transverse to the lightlike hypersurfaces $\sigma = $ constant, we are free to make a coordinate transformation $\rho \mapsto \rho - f(\xi, \eta, \sigma)$. Then we have

$$\rho = \frac{2}{\Theta} = -P \left(\frac{\partial P}{\partial \rho}\right)^{-1} \qquad \Longrightarrow \qquad \frac{\partial P}{\partial \rho} = -\frac{P}{\rho}$$

$$\Longrightarrow \qquad \frac{\partial \big(\rho\, P\big)}{\partial \rho} = P + \rho \frac{\partial P}{\partial \rho} = P - \rho \frac{P}{\rho} = 0\,.$$

So our assumption that (at least the 33-component of) the vacuum field equation holds and that $\Theta \neq 0$ has led to the conclusion that

$$p := \rho\, P \quad \text{satisfies} \quad \frac{\partial p}{\partial \rho} = 0\,.$$

So we can write the metric as

$$g_{\mu\nu} dx^\mu dx^\nu = \frac{\rho^2}{p^2} \Big(\big(d\xi - a\, d\sigma\big)^2 + \big(d\eta - b\, d\sigma\big)^2 \Big) + 2\, d\rho\, d\sigma - c\, d\sigma^2 \qquad \text{with} \quad \partial_3 p = 0\,.$$

For evaluating the vacuum field equation, we now have to calculate the other components of the Ricci tensor for this metric. This is straight-forward but rather tedious. Mathematica gives the following results.

$$R_{13} = \frac{\partial_3(\rho^4 \partial_3 a)}{2p^2\rho^2}\,,$$

$$R_{23} = \frac{\partial_3(\rho^4 \partial_3 b)}{2p^2\rho^2}\,,$$

$$R_{11} - R_{22} = \frac{\rho}{2p^4}\Big(\rho^2(\partial_3 a)^2 - \rho^2(\partial_3 b)^2 - 2p^2\big(2\partial_2 b + \rho\partial_2\partial_3 b - 2\partial_1 a - \rho\partial_1\partial_3 a\big)\Big)\,,$$

$$R_2 = \frac{\rho}{2p^4}\Big(\rho^2\partial_3 a\,\partial_3 b + p^2\big(2\partial_2 a + \rho\partial_2\partial_3 a + 2\partial_1 b + \rho\partial_1\partial_3 b\big)\Big)\,.$$

The first two and the last two equations can be combined in complex form, respectively, if we introduce the complex function $z := a + ib$,

$$R_{13} + iR_{23} = \frac{\partial_3(\rho^4\partial_3 z)}{2p^2\rho^2}\,,$$

$$R_{11} - R_{22} + 2iR_{12} = \frac{\rho}{2p^4}\Big(\rho^2(\partial_3 z)^2 + 2p^2\big(\partial_1 + i\partial_2\big)\big(2z + \rho\partial_3 z\big)\Big)\,.$$

The vacuum field equation requires $R_{13} + iR_{23} = 0$, hence

$$\rho^4\partial_3 z = v \qquad \Longrightarrow \qquad z = u - \frac{v}{3\rho^3} \qquad \text{with} \quad \partial_3 u = \partial_3 v = 0\,.$$

Inserting this result into the equation $R_{11} - R_{22} + 2iR_{12} = 0$ yields

$$0 = v^2 + 2p^2\big(\partial_1 + i\partial_2\big)\big(2u\rho^6 + \frac{v}{3}\rho^3\big)\,.$$

By comparing equal powers of ρ we find

$$v = 0 \quad \text{and} \quad (\partial_1 + i\partial_2)u = 0\,,$$

i.e., the function $z = a + ib = u$ is independent of $\rho = x^3$ and analytic in the complex variable $\xi + i\eta = x^1 + ix^2$,

$$\partial_3 z = 0\,, \quad (\partial_1 + i\partial_2)z = 0\,.$$

The second condition means that, if real and imaginary parts are written separately, the Cauchy-Riemann equations

$$\frac{\partial a}{\partial \xi} = \frac{\partial b}{\partial \eta}\,, \qquad \frac{\partial a}{\partial \eta} = -\frac{\partial b}{\partial \xi}$$

hold.

On the basis of these observations we will now show that a and b can be transformed to zero. To that end we perform a coordinate transformation of the form

$$\xi = \alpha(\tilde{\xi}, \tilde{\eta}, \tilde{\sigma})\,, \quad \eta = \beta(\tilde{\xi}, \tilde{\eta}, \tilde{\sigma})\,, \quad \sigma = \gamma(\tilde{\sigma})\,, \quad \rho = \frac{\tilde{\rho}}{\gamma'(\tilde{\sigma})}\,.$$

Here $\alpha + i\beta$ is an analytic function of $\tilde{\xi} + i\tilde{\eta}$, i.e.,

$$\frac{\partial\alpha}{\partial\tilde{\xi}} = \frac{\partial\beta}{\partial\tilde{\eta}}\,, \qquad \frac{\partial\alpha}{\partial\tilde{\eta}} = -\frac{\partial\beta}{\partial\tilde{\xi}}\,.$$

We choose α and β such that

$$a\gamma'(\tilde{\sigma}) = \frac{\partial\alpha}{\partial\tilde{\sigma}}\,, \qquad b\gamma'(\tilde{\sigma}) = \frac{\partial\beta}{\partial\tilde{\sigma}}\,.$$

Such a choice is possible because a and b are independent of ρ and satisfy the Cauchy-Riemann equations, which guarantees that the necessary integrability conditions are satisfied,

$$\frac{\partial}{\partial\tilde{\xi}}\left(a\,\gamma'(\tilde{\sigma})\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial a}{\partial\xi}\frac{\partial\alpha}{\partial\tilde{\xi}} + \frac{\partial a}{\partial\eta}\frac{\partial\beta}{\partial\tilde{\xi}}\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial b}{\partial\eta}\frac{\partial\beta}{\partial\tilde{\eta}} + \frac{\partial b}{\partial\xi}\frac{\partial\alpha}{\partial\tilde{\eta}}\right) = \frac{\partial}{\partial\tilde{\eta}}\left(b\,\gamma'(\tilde{\sigma})\right),$$

$$\frac{\partial}{\partial\tilde{\eta}}\left(a\,\gamma'(\tilde{\sigma})\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial a}{\partial\xi}\frac{\partial\alpha}{\partial\tilde{\eta}} + \frac{\partial a}{\partial\eta}\frac{\partial\beta}{\partial\tilde{\eta}}\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial b}{\partial\eta}\frac{\partial\beta}{\partial\tilde{\xi}} + \frac{\partial b}{\partial\xi}\frac{\partial\alpha}{\partial\tilde{\xi}}\right) = \frac{\partial}{\partial\tilde{\xi}}\left(b\,\gamma'(\tilde{\sigma})\right).$$

Under such a transformation the form of the metric is preserved, with

$$\frac{1}{\tilde{p}^2} = \frac{1}{p^2}\left(\left(\frac{\partial\alpha}{\partial\xi}\right)^2 + \left(\frac{\partial\beta}{\partial\xi}\right)^2\right) = \frac{1}{p^{-2}}\left(\left(\frac{\partial\alpha}{\partial\eta}\right)^2 + \left(\frac{\partial\beta}{\partial\eta}\right)^2\right)$$

$$\tilde{a} = 0\,, \qquad \tilde{b} = 0\,, \qquad \tilde{c} = c\gamma'(\tilde{\sigma})^2\,.$$

If we perform such a coordinate transformation, and then drop the tildas, the metric takes the simple form

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho^2}{p^2}\left(d\xi^2 + d\eta^2\right) + 2\,d\rho\,d\sigma - c\,d\sigma^2 \tag{RT}$$

with functions $p(\xi,\eta,\sigma)$ and $c(\xi,\eta,\rho,\sigma)$.

For this metric we now calculate the remaining components of the Ricci tensor. Again with Mathematica, we find

$$R_{11} + R_{22} = \frac{2}{p}\Big(\partial_3(\rho c) - 4\rho\,\partial_4 \ln p - \tilde{\Delta}\ln p\Big)$$

where we introduced the modified Laplace operator

$$\tilde{\Delta} = p^2\left(\partial_1^2 + \partial_2^2\right).$$

Integration of the equation $R_{11} + R_{22} = 0$ yields

$$c = 2\rho\partial_4 \ln p + \tilde{\Delta}\ln p - \frac{2m}{\rho} \quad \text{with} \quad \partial_3 m = 0\,. \qquad (*)$$

With this input we find that $R_{34} = 0$ is satisfied while

$$R_{14} = \frac{\partial_1 m}{\rho^2}\,, \qquad R_{24} = \frac{\partial_2 m}{\rho^2}\,.$$

Hence, the equations $R_{14} = 0$ and $R_{24} = 0$ require m to be a function of σ only. Finally, the remaining component of the Ricci tensor is

$$R_{44} = \frac{1}{2\rho^2}\Big(\tilde{\Delta}^2 \ln p + 12m\partial_4 \ln p - 4\frac{dm}{d\sigma}\Big) .$$

The condition $R_{44} = 0$ gives a fourth-order differential equation for p which is known as the *Robinson-Trautman equation*,

$$\tilde{\Delta}^2 \ln p + 12m\partial_4 \ln p - 4\frac{dm}{d\sigma} = 0 .$$

We can now summarise the procedure of how to construct a Robinson-Trautman vacuum solution. We choose a function $m(\sigma)$. With this function, we have to find a solution p to the Robinson-Trautman equation. With this p and the chosen m, we define a function c via $(*)$. Then the metric (RT) is a solution to Einstein's vacuum equation with the integral curves of $\partial/\partial\rho$ being a twist-free, shear-free, geodesic lightlike congruence with non-zero expansion.

Clearly, the Schwarzschild solution must be included. To verify this, choose for m a positive constant. Then the Robinson-Trautman equation is solved by

$$p = 1 + \frac{1}{4}\big(\xi^2 + \eta^2\big) ,$$

because

$$\partial_4 \ln p = 0 , \qquad \tilde{\Delta} \ln p = 1 .$$

In this case the function c reads

$$c = \tilde{\Delta} \ln p - \frac{2m}{\rho} = 1 - \frac{2m}{\rho} .$$

We express the coordinates ξ and η in terms of new coordinates ϑ and φ via

$$\xi + i\eta = \; 2\tan\tfrac{\vartheta}{2}\, e^{i\varphi}$$

which is the stereographic projection mapping from a sphere to a plane. Then

$$d\xi^2 + d\eta^2 = \frac{4\sin^2\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}\, d\varphi^2 + \, d\vartheta^2}{\cos^4\frac{\vartheta}{2}} = \frac{\sin^2\vartheta\, d\varphi^2 + d\vartheta^2}{\cos^4\frac{\vartheta}{2}} ,$$

$$p = 1 + \frac{1}{4}\big(\xi^2 + \eta^2\big) = 1 + \tan^2\tfrac{\vartheta}{2} = \frac{1}{\cos^2\frac{\vartheta}{2}} ,$$

so the metric reads

$$g_{\mu\nu} dx^\mu dx^\nu = \rho^2\big(\sin^2\vartheta\, d\varphi^2 + d\vartheta^2\big) + 2d\rho d\sigma - \Big(1 - \frac{2m}{\rho}\Big) d\sigma^2 .$$

If we rename $(\rho, \sigma) \mapsto (r, \pm c\tilde{t})$ we recognise the Schwarzschild metric in ingoing and outgoing Eddington-Finkelstein coordinates, respectively.

The Robinson-Trautman class of solutions also contains the socalled C-metric which describes a *uniformly accelerated* black hole. It can be viewed as the gravitational analogue of the *Born-Schott electromagnetic field* produced by a uniformly accelerated charge. Just as an accelerated charge produces *stationary* electromagnetic radiation, the C-metric describes *stationary* gravitational radiation.

Other Robinson-Trautman solutions describe *non-stationary* gravitational radiation produced by bounded sources. At the level of exact solutions to Einstein's field equation, the Robinson-Trautman metrics are the most realistic models of gravitational radiation we have. As they do not include any (over-idealised) symmetry assumptions, their variety is much richer than that of the Brinkmann or Beck-Einstein-Rosen solutions. For a detailed discussion of Robinson-Trautman metrics, including the C-metric, see J. Griffiths and J. Podolský: "Exact Space-Times in General Relativity" Cambridge University Press, 2009.

There are several generalisations of the Robinson-Trautman solutions. In particular, the condition of the rays being hypersurface-orthogonal (twist-free) has been dropped. This is important to include *rotating* sources. A twisting null congruence can be rather complicated. In Roger Penrose's *twistor* formalism any twistor is associated with a certain twisting, shear-free, geodesic null congruence, called a "Robinson congruence", on (complexified, compactified) Minkowski spacetime. The picture below is a hand-drawing by Roger Penrose. It shows a time-slice of a Robinson congruence.