Gravitational Waves

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Tue 16:00–17:30 ZARM, Room 1730 (Lectures)
Fr 14:15–15:00 ZARM, Room 1730 (Lectures)
Fri 15:00–15:45 ZARM, Room 1730 (Tutorials)

Complementary Reading

The following standard text-books contain useful chapters on gravitational waves:

S. Weinberg: "Gravitation and Cosmology" Wiley (1972)

C. Misner, K. Thorne, J. Wheeler: "Gravitation" Freeman (1973)

H. Stephani: "Relativity" Cambridge University Press (2004)

L. Ryder: "Introduction to General Relativity" Cambridge University Press (2009)

N. Straumann: "General Relativity" Springer (2012)

For regularly updated online reviews see the *Living Reviews on Relativity*, in particular

B. Sathyaprakash, B. Schutz: "Physics, astrophysics and cosmology with gravitational waves" `http://www.livingreviews.org/lrr-2009-2`

Contents:

1. Historic introduction

1915 A. Einstein establishes the field equation of general relativity

1916 A. Einstein demonstrates that the *linearised* vacuum field equation admits wavelike solutions which are rather similar to electromagnetic waves

1918 A. Einstein derives the *quadrupole formula* according to which gravitational waves are produced by a time-dependent mass quadrupole moment

1925 H. Brinkmann finds a class of *exact* wavelike solutions to the vacuum field equation, later called pp-waves ("plane-fronted waves with parallel rays") by J. Ehlers and W. Kundt

1936 A. Einstein submits, together with N. Rosen, a manuscript to *Physical Review* in which they claim that gravitational waves do not exist

1937 After receiving a critical referee report, A. Einstein withdraws the manuscript with the erroneous claim and publishes, together with N. Rosen, a strongly revised manuscript on wavelike solutions (*Einstein-Rosen waves*) in the *Journal of the Franklin Institute*

1957 F. Pirani gives an invariant (i.e., coordinate-independent) characterisation of gravitational radiation

1960 I. Robinson and A. Trautman discover a class of exact solutions to Einstein's vacuum field equation that describe outgoing gravitational radiation

1960 J. Weber starts his (unsuccessful) search for gravitational waves with the help of resonant bar detectors ("Weber cylinders")

1974 R. Hulse and J. Taylor (Nobel prize 1993) discover the binary pulsar PSR B1913+16 and interpret the energy loss of the system as an indirect proof of the existence of gravitational waves

2002 The first laser interferometric gravitational wave detectors go into operation (GEO66, LIGO, VIRGO,...)

2014 BICEP2 finds evidence for the existence of primordial gravitational waves in the cosmic background radiation

2. Brief review of general relativity

A general-relativistic spacetime is a pair (M, g) where:

M is a four-dimensional manifold; local coordinates will be denoted (x^0, x^1, x^2, x^3) and Einstein's summation convention will be used for greek indices $\mu, \nu, \sigma, \ldots = 0, 1, 2, 3$ and for latin indices $i, j, k, \ldots = 1, 2, 3$.

g is a Lorentzian metric on M, i.e. g is a covariant second-rank tensor field, $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, that is

(a) symmetric, $g_{\mu\nu} = g_{\nu\mu}$, and

(b) non-degenerate with Lorentzian signature, i.e., for any $p \in M$ there are coordinates defined near p such that $g|_p = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

We can, thus, introduce contravariant metric components by

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \, .$$

We use $g^{\mu\nu}$ and $g_{\sigma\tau}$ for raising and lowering indices, e.g.

$$g_{\rho\tau} A^\tau = A_\rho \, , \qquad B_{\mu\nu} g^{\nu\tau} = B_\mu{}^\tau \, .$$

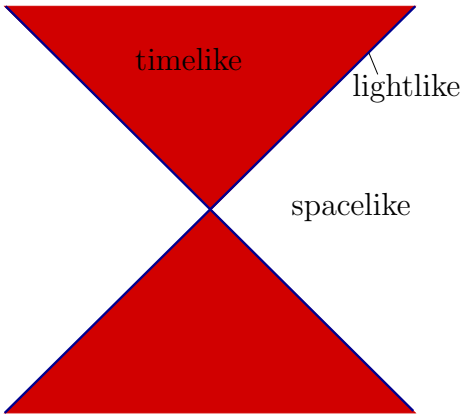
The metric contains all information about the spacetime geometry and thus about the gravitational field. In particular, the metric determines the following.

- The causal structure of spacetime:

  A curve $s \mapsto x(s) = \big(x^0(s), x^1(s), x^2(s), x^3(s)\big)$ is called

  $$\left.\begin{array}{l}\text{spacelike}\\ \text{lightlike}\\ \text{timelike}\end{array}\right\} \iff g_{\mu\nu}\big(x(s)\big)\dot{x}^\mu(s)\dot{x}^\nu(s) \left\{\begin{array}{l} > 0\\ = 0\\ < 0\end{array}\right.$$

  Timelike curves describe motion at subluminal speed and lightlike curves describe motion at the speed of light. Spacelike curves describe motion at superluminal speed which is forbidden for signals.

  For a timelike curve, we usually use proper time τ for the parameter which is defined by

  $$g_{\mu\nu}\big(x(\tau)\big)\dot{x}^\mu(\tau)\,\dot{x}^\mu(\tau) \;=\; -\,c^2 \, .$$

  A clock that shows proper time along its worldline is called a *standard clock*. All experiments to date are in agreement with the assumptions that atomic clocks are standard clocks.

  The motion of a material continuum, e.g. of a fluid, can be described by a vector field $U = U^\mu \partial_\mu$ with $g_{\mu\nu} U^\mu U^\nu = -c^2$. The integral curves of U are to be interpreted as the worldlines of the fluid elements parametrised by proper time.

- The geodesics:

  By definition, the geodesics are the solutions to the Euler-Lagrange equations

$$\frac{d}{ds}\frac{\partial \mathcal{L}(x,\dot{x})}{\partial \dot{x}^\mu} \;-\; \frac{\partial \mathcal{L}(x,\dot{x})}{\partial x^\mu} \;=\; 0$$

  of the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) \;=\; \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \;.$$

  These Euler-Lagrange equations take the form

$$\ddot{x}^\mu \;+\; \Gamma^\mu{}_{\nu\sigma}(x)\dot{x}^\nu\dot{x}^\sigma \;=\; 0$$

  where

$$\Gamma^\mu{}_{\nu\sigma} \;=\; \frac{1}{2}\, g^{\mu\tau}\left(\,\partial_\nu g_{\tau\sigma} \;+\; \partial_\sigma g_{\tau\nu} \;-\; \partial_\tau g_{\nu\sigma}\,\right)$$

  are the so-called Christoffel symbols.

  The Lagrangian $\mathcal{L}(x,\dot{x})$ is constant along a geodesic (see Worksheet 1), so we can speak of timelike, lightlike and spacelike geodesics. Timelike geodesics ($\mathcal{L} < 0$) are to be interpreted as the worldlines of freely falling particles, and lightlike geodesics ($\mathcal{L} = 0$) are to be interpreted as light rays.

  The Christoffel symbols define a *covariant derivative* that makes tensor fields into tensor fields, e.g.

$$\nabla_\nu U^\mu \;=\; \partial_\nu U^\mu \;+\; \Gamma^\mu{}_{\nu\tau}U^\tau \;,$$
$$\nabla_\nu A_\mu \;=\; \partial_\nu A_\mu \;-\; \Gamma^\rho{}_{\nu\mu}A_\rho \;.$$

  In Minkowski spacetime (i.e., in the "flat" spacetime of special relativity), we can choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ on the whole spacetime, where we have used the standard abbreviation $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$. In this coordinate system, the Christoffel symbols obviously vanish. Conversely, vanishing of the Christoffel symbols on an open neighbourhood implies that the $g_{\mu\nu}$ are constants; one can then perform a linear coordinate transformation such that $g_{\mu\nu} = \eta_{\mu\nu}$.

- The curvature:

  The Riemannian curvature tensor is defined, in coordinate notation, by

$$R^\tau{}_{\mu\nu\sigma} \;=\; \partial_\mu\Gamma^\tau{}_{\nu\sigma} \;-\; \partial_\nu\Gamma^\tau{}_{\mu\sigma} \;+\; \Gamma^\rho{}_{\nu\sigma}\Gamma^\tau{}_{\mu\rho} \;-\; \Gamma^\rho{}_{\mu\sigma}\Gamma^\tau{}_{\nu\rho}\,.$$

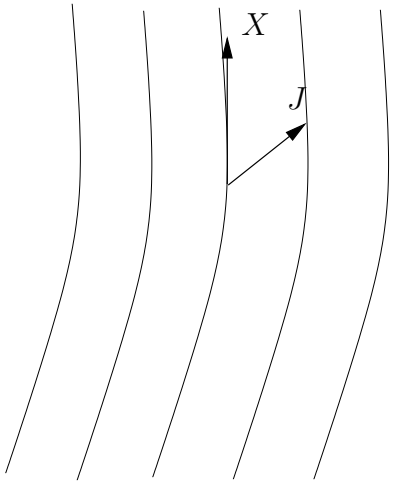
  This defines, indeed, a tensor field, i.e., if $R^\tau{}_{\mu\nu\sigma}$ vanishes in one coordinate system, then it vanishes in any coordinate system. The condition $R^\tau{}_{\mu\nu\sigma} = 0$ is true if and only if there is a local coordinate system, around any one point, such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\sigma} = 0$ on the domain of the coordinate system.

  The curvature tensor determines the relative motion of neighbouring geodesics: If $X = X^\mu\partial_\mu$ is a vector field whose integral curves are geodesics, and if $J = J^\nu\partial_\nu$ connects neighbouring integral curves of X (i.e., if the Lie bracket between X and J vanishes), then the *equation of geodesic deviation* or *Jacobi equation* holds:

$$\big(X^\mu \nabla_\mu\big)\big(X^\nu \nabla_\nu\big) J^\sigma \;=\; R^\sigma{}_{\mu\nu\rho} X^\mu J^\nu X^\rho \;.$$

If the integral curves of X are timelike, they can be interpreted as worldlines of freely falling particles. In this case the curvature term in the Jacobi equation gives the *tidal force* produced by the gravitational field.

If the integral curves of X are lightlike, they can be interpreted as light rays. In this case the curvature term in the Jacobi equation determines the influence of the gravitational field on the shapes of light bundles.



- Einstein's field equation:

  The fundamental equation that relates the spacetime metric (i.e., the gravitational field) to the distribution of energy is Einstein's field equation:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; \kappa\, T_{\mu\nu}$$

  where

  - $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ is the *Ricci tensor*;
  - $R = R_{\mu\nu} g^{\mu\nu}$ is the *Ricci scalar*;
  - $T_{\mu\nu}$ is the *energy-momentum tensor* which gives the energy density $T_{\mu\nu} U^\mu U^\nu$ for any observer field with 4-velocity U^μ normalised to $g_{\mu\nu} U^\mu U^\nu = -c^2$;
  - Λ is the cosmological constant;
  - κ is Einstein's gravitational constant which is related to Newton's gravitational constant G through $\kappa = 8\pi G/c^4$.

  Einstein's field equation can be justified in the following way: One looks for an equation of the form $(\mathcal{D}g)_{\mu\nu} = T_{\mu\nu}$ where $\mathcal{D}$ is a differential operator acting on the metric. One wants to have $\mathcal{D}g$ satisfying the following two properties:

  (A) $\mathcal{D}g$ contains partial derivatives of the metric up to second order.

  (B) $\nabla^\mu (\mathcal{D}g)_{\mu\nu} = 0$.

  Condition (A) is motivated by analogy to the Newtonian theory: The Poisson equation is a second-order differential equation for the Newtonian gravitational potential ϕ, and the metric is viewed as the general-relativistic analogue to ϕ . Condition (B) is motivated in the following way: For a closed system, in special relativity the energy-momentum tensor field satisfies the conservation law $\partial^\mu T_{\mu\nu} = 0$ in inertial coordinates. By the rule of minimal coupling, in general relativity the energy-momentum tensor field of a closed system should satisfy $\nabla^\mu T_{\mu\nu} = 0$. For consistency, the same property has to hold for the left-hand side of the desired equation.

D. Lovelock has shown in 1972 that these two conditions (A) and (B) are satisfied if and only if $\mathcal{D}g$ is of the form

$$(\mathcal{D}g)_{\mu\nu} = \frac{1}{\kappa}\left(R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu}\right)$$

with some constants Λ and κ, i.e., if and only if the desired equation has indeed the form of Einstein's field equation.

For vacuum ($T_{\mu\nu} = 0$), Einstein's field equation reads

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\ .$$

By contraction with $g^{\mu\nu}$ this implies $R = 4\Lambda$, so the vacuum field equation reduces to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Present-day cosmological observations suggest that we live in a universe with a positive cosmological constant whose value is $\Lambda \approx (10^{26}\mathrm{m})^{-2} \approx (10^{16}\mathrm{ly})^{-2}$. As the diameter of our galaxy is approximately 10^5ly, for any distance d within our galaxy the quantity $d^2\Lambda < 10^{-22}$ is negligibly small. As a consequence, the Λ term can be safely ignored for considerations inside our galaxy. Then the vacuum field equation takes the very compact form

$$R_{\mu\nu} = 0$$

which, however, is a complicated system of ten non-linear second-order partial differential equations for the ten independent components of the metric.

Gravitational waves travelling through empty space are wavelike solutions of the equation $R_{\mu\nu} = 0$.

3. Linearised field equation around flat spacetime

In 1916 Einstein predicted the existence of gravitational waves, based on his linearised vacuum field equation. In 1918 he derived his famous quadrupole formula which relates emitted gravitational waves to the quadrupole moment of the source. In Chapters 3, 4 and 5 we will review this early work on gravitational waves which is based on the *linearised* Einstein theory around flat spacetime. As a consequence, the results are true only for gravitational waves whose amplitudes are small. We will see that, to within this approximation, the theory of gravitational waves is very similar to the theory of electromagnetic waves.
We consider a spacetime metric $g_{\mu\nu}$ that takes, in appropriate coordinates, the form

$$g_{\mu\nu} \,=\, \eta_{\mu\nu} \,+\, h_{\mu\nu} \,.$$

Here $\eta_{\mu\nu}$ denotes the Minkowski metric, i.e., the spacetime metric of special relativity, in inertial coordinates,

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)\,,$$

and the $h_{\mu\nu}$ are assumed to be so small that all expressions can be linearised with respect to the $h_{\mu\nu}$ and their derivatives $\partial_\sigma h_{\mu\nu}$. In particular, it is our goal to linearise Einstein's field equation with respect to the $h_{\mu\nu}$ and their derivatives. This gives a valid approximation of Einstein's theory of gravity if the spacetime is very close to the spacetime of special relativity.

Our assumptions fix the coordinate system up to transformations of the form

$$x^\mu \mapsto \tilde{x}^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu + f^\mu(x) \qquad\qquad \text{(C)}$$

where $(\Lambda^\mu{}_\nu)$ is a Lorentz transformation, $\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta_{\mu\rho} = \eta_{\nu\sigma}$, and the f^μ are small of first order.

We agree that, in this chapter, greek indices are lowered and raised with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, respectively. Here $\eta^{\mu\nu}$ is defined by

$$\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma \,.$$

We write

$$h \,:=\, h_{\mu\nu}\,\eta^{\mu\nu} \,=\, h_\mu{}^\mu \,=\, h^\nu{}_\nu \,.$$

Then the inverse metric $g^{\nu\rho}$is of the form

$$g^{\nu\rho} \,=\, \eta^{\nu\rho} \,-\, h^{\nu\rho} \,.$$

Proof: $\big(\,\eta_{\mu\nu} \,+\, h_{\mu\nu}\,\big)\,\big(\,\eta^{\nu\rho} \,-\, h^{\nu\rho}\,\big) \,=\, \eta_{\mu\nu}\eta^{\nu\rho} \,+\, h_{\mu\nu}\eta^{\nu\rho} \,-\, \eta_{\mu\nu}h^{\nu\rho} \,+\, \ldots \,=\, \delta^\rho_\mu \,+\, h_\mu{}^\rho \,-\, h_\mu{}^\rho \,=\, \delta^\rho_\mu$, where the ellipses stand for a quadratic term that is to be neglected, according to our assumptions. □

We will now derive the linearised field equation. As a first step, we have to calculate the Christoffel symbols. We find

$$\Gamma^\rho{}_{\mu\nu} \,=\, \frac{1}{2}\,g^{\rho\sigma}\,\big(\,\partial_\mu g_{\sigma\nu} \,+\, \partial_\nu g_{\sigma\mu} \,-\, \partial_\sigma g_{\mu\nu}\,\big) \,=\, \frac{1}{2}\,\eta^{\rho\sigma}\,\big(\,\partial_\mu h_{\sigma\nu} \,+\, \partial_\nu h_{\sigma\mu} \,-\, \partial_\sigma h_{\mu\nu}\,\big) \,+\, \ldots$$

Thereupon, we can calculate the components of the Ricci tensor:.

$$R_{\mu\nu} \,=\, \partial_\mu \Gamma^\rho{}_{\rho\nu} \,-\, \partial_\rho \Gamma^\rho{}_{\mu\nu} \,+\, \ldots \,=$$

$$=\, \frac{1}{2}\,\eta^{\rho\sigma}\,\partial_\mu\,\big(\,\cancel{\partial_\rho h_{\sigma\nu}} \,+\, \partial_\nu h_{\sigma\rho} \,-\, \partial_\sigma h_{\rho\nu}\,\big) \,-\, \frac{1}{2}\,\eta^{\rho\sigma}\,\partial_\rho\,\big(\,\cancel{\partial_\mu h_{\sigma\nu}} \,+\, \partial_\nu h_{\sigma\mu} \,-\, \partial_\sigma h_{\mu\nu}\,\big) \,=$$

$$= \frac{1}{2}\big(\,\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu}\,\big)\;.$$

Here, $\Box$ denotes the wave operator (d'Alembert operator) that is formed with the Minkowski metric,

$$\Box \;=\; \eta^{\mu\nu}\partial_\mu\partial_\nu \;=\; \partial^\nu\partial_\nu\;.$$

From the last expression we can calculate the scalar curvature:

$$R \;=\; g^{\mu\nu}R_{\mu\nu} \;=\; \eta^{\mu\nu}R_{\mu\nu} \,+\, \ldots \;=\; \frac{1}{2}\,\eta^{\mu\nu}\,\big(\,\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu}\,\big)$$

$$= \frac{1}{2}\,\big(\,\Box h \,-\, \partial^\nu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial^\mu h_{\sigma\mu} \,+\, \Box h\,\big) \;=\; \Box h \,-\, \partial^\sigma\partial^\mu h_{\sigma\mu}\,.$$

Hence, the linearised version of Einstein's field equation (without a cosmological constant)

$$2\,R_{\mu\nu} \,-\, R\,g_{\mu\nu} \;=\; 2\,\kappa\,T_{\mu\nu}\;, \qquad \kappa = \frac{8\pi G}{c^4}$$

reads

$$\partial_\mu\partial_\nu h \,-\, \partial_\mu\partial^\rho h_{\rho\nu} \,-\, \partial^\sigma\partial_\nu h_{\sigma\mu} \,+\, \Box h_{\mu\nu} \,-\, \eta_{\mu\nu}\,\big(\,\Box h \,-\, \partial^\sigma\partial^\tau h_{\sigma\tau}\,\big) \;=\; 2\,\kappa\,T_{\mu\nu}\;. \quad (*)$$

This is a system of linear partial differential equations of second order for the $h_{\mu\nu}$. It can be rewritten in a more convenient form after substituting for $h_{\mu\nu}$ the quantity

$$\gamma_{\mu\nu} \;=\; h_{\mu\nu} \,-\, \frac{h}{2}\,\eta_{\mu\nu}\;.$$

As the relation between $h_{\mu\nu}$ and $\gamma_{\mu\nu}$ is linear, our assumptions are equivalent to saying that we linearise all equations with respect to the $\gamma_{\mu\nu}$ and the $\partial_\rho\gamma_{\mu\nu}$. In order to express the $h_{\mu\nu}$ in terms of the $\gamma_{\mu\nu}$, we calculate the trace,

$$\gamma \;:=\; \eta^{\mu\nu}\gamma_{\mu\nu} \;=\; h \,-\, \frac{1}{2}\,4\,h \;=\; -\,h\;,$$

$$h_{\mu\nu} \;=\; \gamma_{\mu\nu} \,-\, \frac{\gamma}{2}\,\eta_{\mu\nu}\;.$$

Upon inserting this expression into the linearised field equation $(*)$, we find

$$-\,\cancel{\partial_\mu\partial_\nu\gamma} \,-\, \partial_\mu\partial^\rho\gamma_{\rho\nu} \,+\, \frac{1}{2}\,\cancel{\eta_{\rho\nu}\partial_\mu\partial^\rho\gamma} \,-\, \partial^\sigma\partial_\nu\gamma_{\sigma\mu} \,+\, \frac{1}{2}\,\cancel{\eta_{\sigma\mu}\partial^\sigma\partial_\nu\gamma} \,+$$

$$+\,\Box\gamma_{\mu\nu} \,-\, \frac{1}{2}\,\cancel{\eta_{\mu\nu}\Box\gamma} \,-\, \eta_{\mu\nu}\,\big(\,-\,\cancel{\Box\gamma} \,-\, \partial^\sigma\partial^\tau\gamma_{\sigma\tau} \,+\, \frac{1}{2}\,\cancel{\eta_{\sigma\tau}\partial^\sigma\partial^\tau\gamma}\,\big) \;=\; 2\,\kappa\,T_{\mu\nu}\;,$$

$$\Box\gamma_{\mu\nu} \,-\, \partial_\mu\partial^\rho\gamma_{\rho\nu} \,-\, \partial_\nu\partial^\rho\gamma_{\rho\mu} \,+\, \eta_{\mu\nu}\,\partial^\sigma\partial^\tau\gamma_{\sigma\tau} \;=\; 2\,\kappa\,T_{\mu\nu}\;. \quad (**)$$

This equation can be simplified further by a coordinate transformation (C) with $a^\mu = 0$ and $\Lambda^\mu{}_\nu = \delta^\mu_\nu$,

$$x^\mu \;\mapsto\; x^\mu \,+\, f^\mu(x)$$

where the f^μ are of first order. For such a coordinate transformation, we have obviously

$$dx^\mu \;\mapsto\; dx^\mu \,+\, \partial_\rho f^\mu dx^\rho$$

and thus

$$\partial_\sigma \;\mapsto\; \partial_\sigma \;-\; \partial_\sigma f^\tau \partial_\tau \;.$$

Proof: $\big(\, dx^\mu \;+\; \partial_\rho f^\mu dx^\rho \,\big)\,\big(\, \partial_\sigma \;-\; \partial_\sigma f^\tau \partial_\tau \,\big) \;=\; dx^\mu(\partial_\sigma) \;+\; \partial_\rho f^\mu dx^\rho(\partial_\sigma) \;-\; \partial_\sigma f^\tau dx^\mu(\partial_\tau) + \ldots \;=$ $\delta^\sigma_\mu \;+\; \cancel{\partial_\rho f^\mu \delta^\rho_\sigma} \;-\; \cancel{\partial_\sigma f^\tau \delta^\mu_\tau}$. □

With the help of these equations, we can now calculate how the $g_{\mu\nu}$, the $h_{\mu\nu}$, and the $\gamma_{\mu\nu}$ behave under such a coordinate transformation:

$$g_{\mu\nu} \;=\; g\big(\partial_\mu, \partial_\nu\big) \;\mapsto\; g\big(\partial_\mu - \partial_\mu f^\tau \partial_\tau, \partial_\nu - \partial_\nu f^\sigma \partial_\sigma\big) \;=\; g_{\mu\nu} - \partial_\mu f^\tau g_{\tau\nu} - \partial_\nu f^\sigma g_{\mu\sigma} \;,$$

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \;\mapsto\; g_{\mu\nu} - \partial_\mu f^\tau g_{\tau\nu} - \partial_\nu f^\sigma g_{\mu\sigma} - \eta_{\mu\nu} \;=\; h_{\mu\nu} - \partial_\mu f^\tau \eta_{\tau\nu} - \partial_\nu f^\sigma \eta_{\mu\sigma} \;+\; \ldots$$

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \;\mapsto\; h_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu - \frac{1}{2}\eta_{\mu\nu}\big(h - 2\partial_\tau f^\tau\big) = \gamma_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu + \eta_{\mu\nu}\partial_\tau f^\tau \;.$$

For the divergence of $\gamma_{\mu\nu}$, which occurs three times in $(**)$, this gives the following transformation behaviour:

$$\partial^\mu \gamma_{\mu\nu} \;\mapsto\; \partial^\mu \gamma_{\mu\nu} - \partial^\mu \partial_\mu f_\nu - \cancel{\partial^\mu \partial_\nu f_\mu} + \cancel{\eta_{\mu\nu} \partial^\mu \partial_\tau f^\tau} = \partial^\mu \gamma_{\mu\nu} - \Box f_\nu \;.$$

This shows that, if it is possible to choose the f_ν such that

$$\Box f_\nu = \partial^\mu \gamma_{\mu\nu} \;,$$

then $\partial^\mu \gamma_{\mu\nu}$ is transformed to zero. Such a choice is, indeed, possible as the wave equation on Minkowski spacetime,

$$\Box f_\nu = \Phi_\nu \;,$$

has solutions for any Φ_ν. This is well-known from electrodynamics. (Particular solutions are the retarded potentials, see below.)

We have thus shown that, by an appropriate coordinate transformation, we can put the linearised field equation $(**)$ into the following form:

$$\Box \gamma_{\mu\nu} \;=\; 2\,\kappa\, T_{\mu\nu} \;.$$

Now the $\gamma_{\mu\nu}$ have to satisfy the additional condition

$$\partial^\mu \gamma_{\mu\nu} \;=\; 0$$

which is known as the *Hilbert gauge*. (Some other authors call it the *Einstein gauge*, the *de Donder gauge*, or the *Fock gauge*.) The transformation of $\gamma_{\mu\nu}$ under a change of coordinates is analogous to a gauge transformation of the four-potential A_μ in electrodynamics. Even after imposing the Hilbert gauge condition, there is still the freedom to make coordinate transformations (C) with $\Box f^\mu = 0$. In particular, the theory is invariant under Lorentz transformations.

The linearised Einstein theory is a Lorentz invariant theory of the gravitational field on Minkowski spacetime. It is very similar to Maxwell's vacuum electrodynamics, which is a (linear) Lorentz invariant theory of electromagnetic fields on Minkowski spacetime. Of course, one has to keep in mind that the linearised Einstein theory is only an approximation; an *exact* Lorentz invariant theory of gravity on Minkowski spacetime cannot be formulated. Einstein and others tried this, without success, for several years before general relativity came into existence.

lin. Einstein theory	electrodynamics
$\gamma_{\mu\nu}$	A_μ
$T_{\mu\nu}$	J_μ
Hilbert gauge $\partial^\mu \gamma_{\mu\nu} = 0$	Lorenz gauge $\partial^\mu A_\mu = 0$
$\Box \gamma_{\mu\nu} = -2\kappa T_{\mu\nu}$	$\Box A_\mu = \mu_0^{-1} J_\mu$

The table illustrates the analogy. Here "electrodynamics" stands for "electrodynamics on Minkowski spacetime in vacuum, $G_{\mu\nu} = \mu_0^{-1} F_{\mu\nu}$".

4. Gravitational waves in the linearised theory around flat spacetime

In this section we consider the linearised vacuum field equation in the Hilbert gauge,

$$\Box \gamma_{\mu\nu} \,=\, 0 \,, \qquad \partial^\mu \gamma_{\mu\nu} \,=\, 0 \,.$$

In analogy to the electrodynamical theory, we can write the general solution as a superposition of plane harmonic waves. In our case, any such plane harmonic wave is of the form

$$\gamma_{\mu\nu}(x) \,=\, \mathrm{Re}\big\{ A_{\mu\nu} e^{i k_\rho x^\rho} \big\}$$

with a real wave covector k_ρ and a complex amplitude $A_{\mu\nu} = A_{\nu\mu}$.
Such a plane harmonic wave satisfies the linearised vacuum field equation if and only if

$$0 \,=\, \eta^{\sigma\tau} \partial_\sigma \partial_\tau \gamma_{\mu\nu}(x) \,=\, \mathrm{Re}\big\{ \eta^{\sigma\tau} A_{\mu\nu} i k_\sigma i k_\tau e^{i k_\rho x^\rho} \big\} \,.$$

This holds for all x, with $(A_{\mu\nu}) \neq (0)$, if and only if

$$\eta^{\sigma\tau} k_\sigma k_\tau \,=\, 0 \,.$$

In other words, (k_0, k_1, k_2, k_3) has to be a lightlike covector with respect to the Minkowski metric. This result can be interpreted as saying that, to within the linearised Einstein theory, gravitational waves propagate on Minkowski spacetime at the speed c, just as electromagnetic waves in vacuum.

Our plane harmonic wave satisfies the Hilbert gauge condition if and only if

$$0 \;=\; \eta^{\mu\tau}\partial_\tau \gamma_{\mu\nu}(x) \;=\; \mathrm{Re}\big\{\eta^{\mu\tau} A_{\mu\nu} i k_\tau e^{ik_\rho x^\rho}\big\}$$

which is true, for all $x = (x^0, x^1, x^2, x^3)$, if and only if

$$k^\mu A_{\mu\nu} \;=\; 0 \qquad \text{(H)}\,.$$

For a given k_μ, the Hilbert gauge condition restricts the possible values of the amplitude $A_{\mu\nu}$, i.e., it restricts the possible polarisation states of the gravitational wave. For electromagnetic waves, it is well known that there are two polarisation states ("left-handed and right-handed", or "linear in x-direction and linear in y-direction") from which all possible polarisation states can be formed by way of superposition. We will see that also for gravitational waves there are two independent polarisation states; however, they are of a different geometric nature which has its origin in the fact that $\gamma_{\mu\nu}$ has two indices while the electromagnetic four-potential A_μ has only one.

In order to find all possible polarisation states of a gravitational wave, we begin by counting the independent components of the amplitude: The $A_{\mu\nu}$ form a (4×4)-matrix which has 16 entries. As $A_{\mu\nu} = A_{\nu\mu}$, only 10 of them are independent; the Hilbert gauge condition (H) consists of 4 scalar equations, so one might think that there are 6 independent components and thus six independent polarisation states. This, however, is wrong. The reason is that we can impose additional conditions onto the amplitudes, even after the Hilbert gauge has been chosen: The Hilbert gauge condition is preserved if we make a coordinate transformation of the form

$$x^\mu \;\mapsto\; x^\mu \,+\, f^\mu(x) \quad \text{mit} \quad \Box f^\mu \;=\; 0\,.$$

We can use this freedom to impose additional conditions onto the amplitudes $A_{\mu\nu}$.

Claim: Assume we have a plane-harmonic-wave solution

$$\gamma_{\mu\nu}(x) \;=\; \mathrm{Re}\big\{A_{\mu\nu} e^{ik_\rho x^\rho}\big\}$$

of the linearised vacuum field equation in the Hilbert gauge. Let (u^μ) be a constant four-velocity vector, $\eta_{\mu\nu}u^\mu u^\nu \;=\; -\,c^2$. Then we can make a coordinate transformation such that the Hilbert gauge condition is preserved and such that

$$u^\mu A_{\mu\nu} \;=\; 0\,, \qquad \text{(T1)}$$

$$\eta^{\mu\nu} A_{\mu\nu} \;=\; 0\,, \qquad \text{(T2)}$$

in the new coordinates (TT *gauge*, *transverse-traceless gauge*).

Proof: We perform a coordinate transformation

$$x^\mu \;\mapsto\; x^\mu \,+\, f^\mu(x)\,, \qquad f^\mu(x) \;=\; \mathrm{Re}\big\{i\, C^\mu e^{ik_\rho x^\rho}\big\}$$

with the wave covector (k_ρ) from our plane-harmonic-wave solution and with some complex coefficients C^μ. Then we have $\Box f^\mu = 0$, i.e., the Hilbert gauge condition is satisfied in the new coordinates as well. We want to choose the C^μ such that in the new coordinates (T1) and (T2) hold true. As a first step, we calculate how the amplitudes $A_{\mu\nu}$ transform.

We start out from the transformation behaviour of the $\gamma_{\mu\nu}$ which was calculated above,

$$\gamma_{\mu\nu} \;\mapsto\; \gamma_{\mu\nu} - \partial_\mu f_\nu - \partial_\nu f_\mu + \eta_{\mu\nu}\partial_\rho f^\rho \;,$$

hence

$$\mathrm{Re}\big\{ A_{\mu\nu} e^{ik_\rho x^\rho} \big\} \;\mapsto\; \mathrm{Re}\big\{ \big(A_{\mu\nu} \,-\, i\,i\,k_\mu C_\nu \,-\, i\,i\,k_\nu C_\mu \,+\, \eta_{\mu\nu} i\,i\,k_\rho\, C^\rho \big)\, e^{ik_\rho x^\rho} \big\} \;,$$

$$A_{\mu\nu} \;\mapsto\; A_{\mu\nu} \,+\, k_\mu C_\nu \,+\, k_\nu C_\mu \,-\, \eta_{\mu\nu}\, k_\rho\, C^\rho \;.$$

We want to choose the C_μ such that the equations

$$0 \;=\; u^\mu \big(A_{\mu\nu} \,+\, k_\mu C_\nu \,+\, k_\nu C_\mu \,-\, \eta_{\mu\nu}\, k_\rho\, C^\rho \big) \;, \qquad \text{(T1)}$$

$$0 \;=\; \eta^{\mu\nu} \big(A_{\mu\nu} \,+\, k_\mu C_\nu \,+\, k_\nu C_\mu \,-\, \eta_{\mu\nu}\, k_\rho\, C^\rho \big) \;=\; \eta^{\mu\nu}\, A_{\mu\nu} \,-\, 2\, k_\rho\, C^\rho \qquad \text{(T2)}$$

hold. To demonstrate that such a choice is possible, we choose the coordinates such that

$$\big(u^\mu\big) \;=\; \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} \;.$$

This can be done by a Lorentz transformation which, as a linear coordinate transformation, preserves all the relevant properties of the coordinate system. Then the spatial part of the desired condition (T1) reads:

(T1) for $\nu = j$: $A_{0j} + k_0 C_j + k_j C_0 \;=\; 0 \quad\Longleftrightarrow\quad C_j \;=\; -\,k_0^{-1}(A_{0j} + k_j C_0)$.

These equations show that the C_j are determined by C_0. We will now check if the temporal part of (T1) gives any restriction on C_0.

(T1) for $\nu = 0$: $A_{00} \,+\, 2\, k_0\, C_0 \,+\, \eta^{\rho\sigma}\, k_\rho\, C_\sigma \;=\; 0 \quad\Longleftrightarrow$

$$A_{00} \,+\, \cancel{2}\, k_0\, C_0 \,-\, \cancel{k_0 C_0} \,+\, \eta^{ij}\, k_i\, C_j \;=\; 0 \quad\Longleftrightarrow$$

$$A_{00} \,+\, k_0\, C_0 \,-\, \eta^{ij}\, k_i\, k_0^{-1} \big(A_{0j} + k_j C_0 \big) \;=\; 0 \quad\Longleftrightarrow$$

$$A_{00} \,+\, k_0\, C_0 \,-\, \eta^{ij}\, k_i\, k_0^{-1}\, A_{0j} \,+\, \eta^{00}\, k_0\, \cancel{k_0}\, \cancel{k_0^{-1}}\, C_0 \;=\; 0 \quad\Longleftrightarrow$$

$$-\, k_0\, A_{00} \,+\, \eta^{ij}\, k_i\, A_{0j} \;=\; 0 \quad\Longleftrightarrow$$

$$\eta^{\mu\nu} k_\mu\, A_{0\nu} \;=\; 0 \;.$$

This is precisely the Hilbert gauge condition (H) that is satisfied by assumption. We have thus found that (T1) fixes the C_j in terms of C_0 but leaves C_0 arbitrary.
We now turn to the second desired condition (T2).

$$(\mathrm{T2}):\ \eta^{\mu\nu}\,A_{\mu\nu}\ +\ 2\,k_0\,C_0\ -\ 2\,\eta^{ij}\,k_i\,C_j\ =\ 0\quad\Longleftrightarrow$$

$$A_\mu{}^\mu\ +\ 2\,k_0\,C_0\ +\ 2\,\eta^{ij}\,k_i\,k_0^{-1}\left(A_{0j}+k_jC_0\right)\ =\ 0\quad\Longleftrightarrow$$

$$A_\mu{}^\mu\ +\ 2\,k_0\,C_0\ +\ 2\,\eta^{ij}\,k_i\,k_0^{-1}\,A_{0j}\ -\ 2\,\eta^{00}\,k_0\,\cancel{k_0^{-1}}\,\cancel{k_0}\,C_0\ =\ 0\quad\Longleftrightarrow$$

$$A_\mu{}^\mu\ +\ 4\,k_0\,C_0\ +\ 2\,\eta^{ij}\,k_i\,k_0^{-1}\,A_{0j}\ =\ 0\quad\Longleftrightarrow$$

$$C_0\ =\ \frac{-\,A_\mu{}^\mu\,k_0\ -\ 2\,\eta^{ij}\,k_i\,A_{0j}}{4\,k_0^2}\ .$$

If we choose C_0 according to this equation, and then the C_j as required above, (T1) and (T2) are indeed satisfied in the new coordinates. □

In the TT gauge we have $\gamma = 0$ and thus $h_{\mu\nu} = \gamma_{\mu\nu}$. As a consequence, the metric is of the form

$$g_{\mu\nu}\ =\ \eta_{\mu\nu}\ +\ \gamma_{\mu\nu}\ ,\qquad \gamma_{\mu\nu}\ =\ \mathrm{Re}\big\{A_{\mu\nu}e^{ik_\rho x^\rho}\big\}$$

and the amplitudes are restricted by the conditions

$$k^\mu A_{\mu\nu}\ =\ 0\ ,\qquad u^\mu A_{\mu\nu}\ =\ 0\ ,\qquad \eta^{\mu\nu}A_{\mu\nu}\ =\ 0\ .$$

If we choose the coordinates such that

$$\big(u^\mu\big)\ =\ \begin{pmatrix}c\\0\\0\\0\end{pmatrix}\ ,\qquad \big(k^\rho\big)\ =\ \begin{pmatrix}\omega/c\\0\\0\\\omega/c\end{pmatrix}$$

which can be achieved by a Lorentz transformation, the amplitudes $A_{\mu\nu}$ satisfy

$$(\mathrm{H})\qquad 0\ =\ k^\mu\,A_{\mu\nu}\ =\ \frac{\omega}{c}\,(A_{0\nu}+A_{3\nu})\ ,$$

$$(\mathrm{T1})\qquad 0\ =\ u^\mu\,A_{\mu\nu}\ =\ c\,A_{0\nu}\ ,$$

$$(\mathrm{T2})\qquad 0\ =\ \eta^{\mu\nu}\,A_{\mu\nu}\ =\ -\,A_{00}\ +\ A_{11}\ +\ A_{22}\ +\ A_{33}$$

in the TT gauge. In this representation, there are only two non-zero components of $A_{\mu\nu}$,

$$A_{11}\ =\ -\,A_{22}\ =:\ A_+\ =\ \big|A_+\big|\,e^{i\varphi}\ ,$$

$$A_{12}\ =\ A_{21}\ =:\ A_\times\ =\ \big|A_\times\big|\,e^{i\psi}\ .$$

The fact that only the 1- and the 2-components are non-zero demonstrates that gravitational waves are *transverse*. There are only two independent polarisation states, the *plus mode* (+) and the *cross mode* (×).

For the physical interpretation of these two modes we need the following result.

Claim: The x^0-lines, i.e. the worldlines $x^\mu(\tau)$ with $\dot{x}^\mu(\tau) = u^\mu$, are geodesics.

Proof: From $\dot{x}^\mu(\tau) = u^\mu$ we find $\ddot{x}^\mu(\tau) = 0$. The Christoffel symbols read

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}\, g^{\mu\tau}\left(\partial_\nu g_{\tau\sigma} + \partial_\sigma g_{\tau\nu} - \partial_\tau g_{\nu\sigma}\right) =$$

$$= \frac{1}{2}\,\eta^{\mu\tau}\left(\partial_\nu \gamma_{\tau\sigma} + \partial_\sigma \gamma_{\tau\nu} - \partial_\tau \gamma_{\nu\sigma}\right) =$$

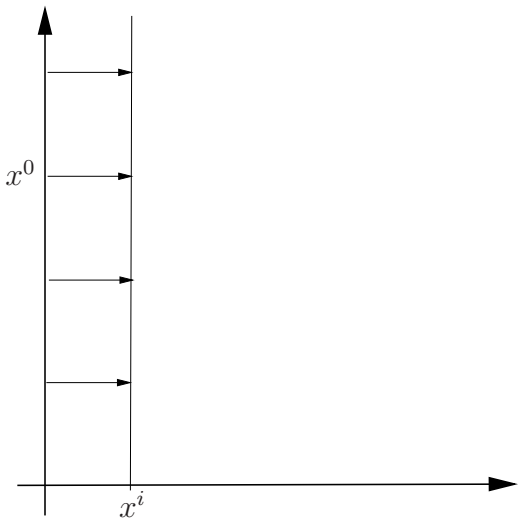
$$= \frac{1}{2}\,\eta^{\mu\tau}\,\mathrm{Re}\Big\{\left(i\,k_\nu\,A_{\tau\sigma} + i\,k_\sigma\,A_{\tau\nu} - i\,k_\tau\,A_{\nu\sigma}\right) e^{ik_\rho x^\rho}\Big\}\,.$$

Hence

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\,\dot{x}^\nu\,\dot{x}^\nu = 0 + \frac{1}{2}\,\eta^{\mu\tau}\,\mathrm{Re}\Big\{\left(i\,k_\nu\,\underbrace{A_{\tau\sigma}\,u^\sigma}_{=0}\,u^\nu + i\,k_\sigma\,\underbrace{A_{\tau\nu}\,u^\nu}_{=0}\,u^\sigma - i\,k_\tau\,\underbrace{A_{\nu\sigma}\,u^\sigma}_{=0}\,u^\nu\right) e^{ik_\rho x^\rho}\Big\} = 0\,.$$

□

In other words, the x^0-lines are the worldlines of freely falling particles. For any such particle the (x^1, x^2, x^3)-coordinates remain constant. This does, of course, *not* mean that the gravitational wave has no effect on freely falling particles. The distance, as it is measured with the metric, between neighbouring x^0-lines is not at all constant. We calculate the square of the distance between the x^0-line at the spatial origin $(0,0,0)$ and a neighbouring x^0-line at (x^1, x^2, x^3) for the case that the x^i are so small that the metric can be viewed as constant between 0 and x^i.



$$g_{ij}(x^0,0,0,0)\,(x^i-0)\,(x^j-0) =$$

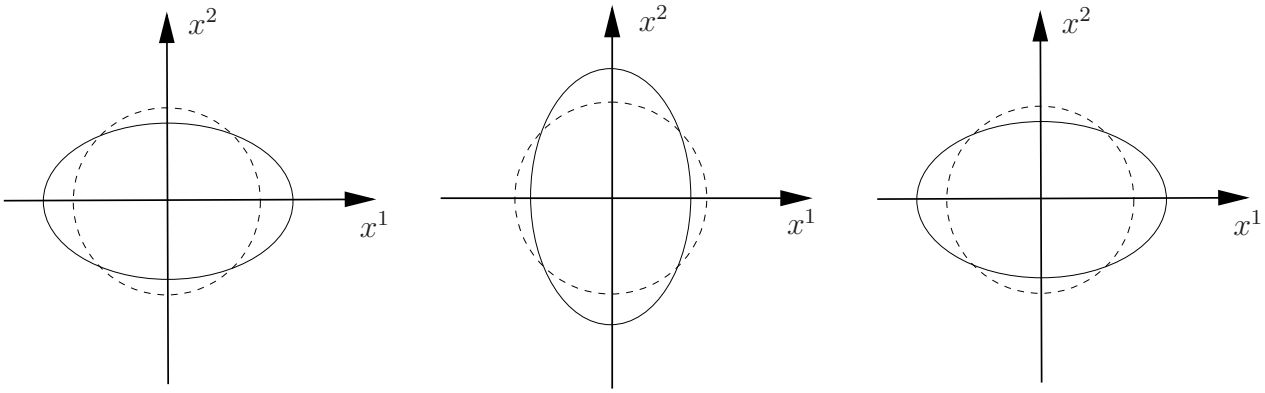
$$= \left(\eta_{ij} + \gamma_{ij}(x^0,0,0,0)\right) x^i\,x^j = \delta_{ij}\,x^i\,x^j + \mathrm{Re}\big\{A_{ij}x^i x^j e^{ik_0x^0}\big\} =$$

$$= \delta_{ij}\,x^i\,x^j + \mathrm{Re}\big\{A_+\big((x^1)^2-(x^2)^2\big)e^{-i\omega t}\big\} + \mathrm{Re}\big\{2\,A_\times\,x^1\,x^2 e^{-i\omega t}\big\} =$$

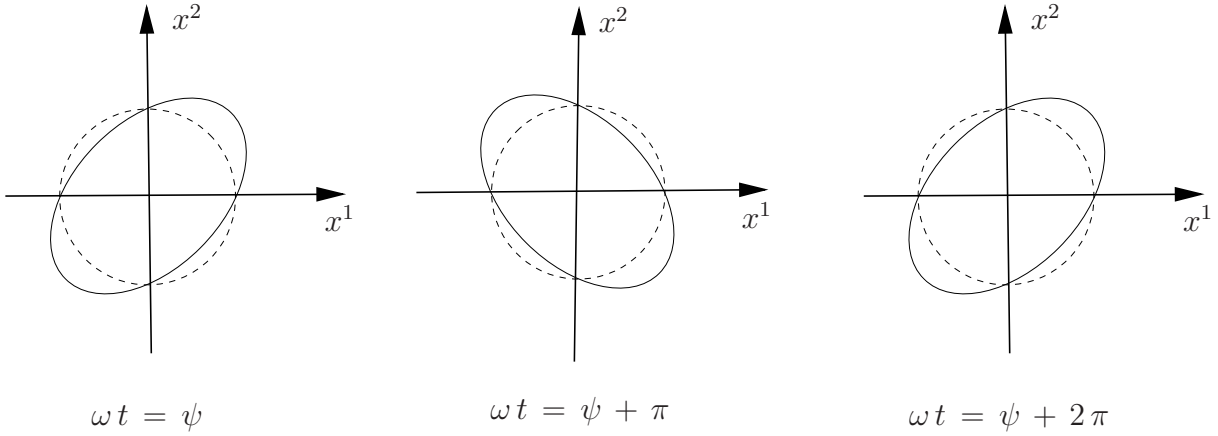
$$= \delta_{ij}\,x^i\,x^j + \big|A_+\big|\,\big((x^1)^2-(x^2)^2\big)\cos\big(\varphi-\omega t\big) + 2\,\big|A_\times\big|\,x^1\,x^2\cos\big(\psi-\omega t\big)\,.$$

The last equation demonstrates what happens to particles that are arranged on a small spherical shell and then released to free fall: Both the plus mode and the cross mode of the gravitational wave produce a time-periodic elliptic deformation in the plane perpendicular to the propagation direction. For the plus mode, the principal axes of the ellipse coincide with coordinate axes, for the cross mode they are rotated by 45 degrees. This explains the origin of the names "plus mode" and "cross mode".

Plus mode ($A_+ \neq 0\,,\ A_\times = 0$):



Cross mode ($A_+ = 0\,,\ A_\times \neq 0$):



The motion of test paticles under the influence of a gravitational wave can also be derived as a solution of a differential equation. We will derive this differential equation, which is a specification to the situation at hand of the Jacobi equation, in Worksheet 2. It will show that the "driving force" that generates the change of distances between neighbouring free particles is the curvature tensor.

We have found, as our main result, that a gravitational wave produces a change of the distances between freely falling particles in the plane perpendicular to the propagation direction. There are two types of gravitational wave detectors that try to measure this effect :

- **Resonant bar detectors:** The first gravitational wave detectors of this type were developed by J. Weber in the 1960s. They were aluminium cylinders of about 1.5 m length. A gravitational wave of an appropriate frequency would excite a resonant oscillation of such a cylinder. With the uprise of laser interferometric gravitational wave detectors, the bar detectors have lost their relevance. However, some of them are still used.

- **Laser interferometric gravitational wave detectors:** They are Michelson interferometers with an effective arm length of a few hundred meters at least. An incoming gravitational wave would influence the distance between the mirrors and also the path of the light beam inside the interferometer. Both effects produce a change in the interference pattern. Several such detectors are in operation since the early 2000s, e.g. GEO600 (near Hannover, the effective arm length is 600 meters) and LIGO (USA, three interferometers at two sites, the effective arm length is 2 kilometers and 4 kilometers, respectively). A space-borne interferometer (originally called LISA and planned with 5 million kilometers arm length) might be launched around 2034.

We will discuss both types of gravitational wave detectors in greater detail below.

As an aside, we mention that the wave equation for $\gamma_{\mu\nu}$ and its solutions in the TT gauge are a possible starting point for quantising the gravitational field. The resulting quanta, called *gravitons*, have spin 2. This is related to the transformation behaviour of the solutions in the TT gauge under spatial rotations about the propagation direction. The latter will be calculated in Worksheet 3.

5. Generation of gravitational waves

We will now discuss what sort of sources would produce a gravitational wave. We will see that, in the far-field approximation, the gravitational wave field is determined by the second time-derivative of the quadrupole moment of the source. In other words, gravitational radiation predominantly is quadrupole radiation. By contrast, it is well known that electromagnetic radiation predominantly is dipole radiation.

5.1 The far-field approximation of a gravitational wave

We consider the linearised field equation in the Hilbert gauge,

$$\Box\gamma^{\mu\nu} \,=\, 2\,\kappa\,T^{\mu\nu}\,, \qquad \partial_\mu\gamma^{\mu\nu} \,=\, 0\,.$$

For given $T_{\mu\nu}$, the general solution to this inhomogeneous wave equation is the general solution to the homogeneous wave equation (superposition of plane harmonic waves) plus a particular solution to the inhomogeneous equation. Such a particular solution can be written down immediately by analogy with the retarded potentials from electrodynamics:

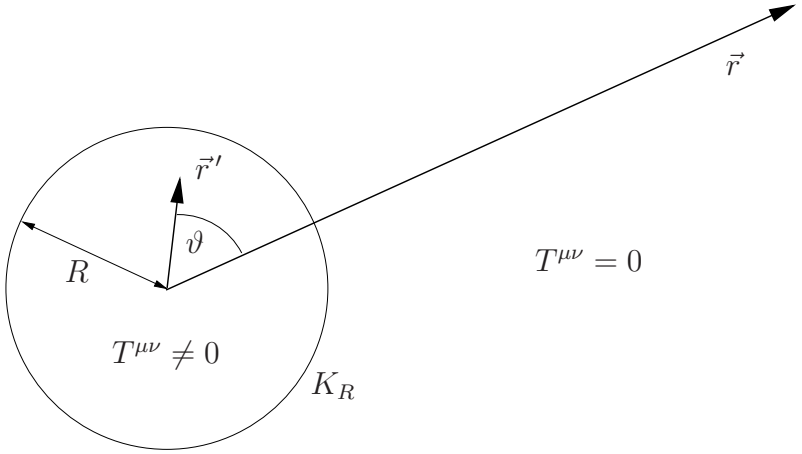
$$\gamma^{\mu\nu}\big(ct,\vec{r}\,\big) \,=\, \frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3} \frac{2\,\kappa\,T^{\mu\nu}\big(\,c\,t\,-\,|\vec{r}\,'-\vec{r}\,|\,,\,\vec{r}\,'\big)\,d^3\vec{r}\,'}{|\vec{r}\,'-\vec{r}\,|}\,. \qquad \text{(RP)}$$

Here and in the following we write

$$x^0 \,=\, c\,t\,, \qquad (x^1,x^2,x^3) \,=\, \vec{r}\,, \qquad r \,=\, |\vec{r}\,|$$

and $d^3\vec{r}\,'$ is the volume element with respect to the primed coordinates, $d^3\vec{r}\,' \,=\, dx'^1\,dx'^2\,dx'^3$. By differentiating twice one easily verifies that the $\gamma^{\mu\nu}$ from (RP) satisfy, indeed, the equation $\Box\gamma^{\mu\nu} \,=\, 2\,\kappa\,T^{\mu\nu}$.

The general solution to the inhomogeneous wave equation is given by adding an arbitrary superposition of plane-harmonic waves that satisfy the homogeneous equation, see Chapter 4. If there are no waves coming in from infinity, (RP) alone gives the physically correct solution. We will now discuss this solution far away from the sources. To that end, we assume that $T^{\mu\nu}$ is different from zero only in a compact region of space. We can then surround this region by a sphere K_R of radius R around the origin, such that $T^{\mu\nu} = 0$ outside of K_R and on the boundary. We are interested in the field $\gamma^{\mu\nu}$ at a point $\vec{r}$ with $|\vec{r}| \gg R$.



Then

$$|\vec{r}\,' - \vec{r}| \,=\, \sqrt{\big(\vec{r}\,' - \vec{r}\big)\cdot\big(\vec{r}\,' - \vec{r}\big)} \;=\; \sqrt{\vec{r}\,'\cdot\vec{r}\,' \,+\, \vec{r}\cdot\vec{r} \,-\, 2\,\vec{r}\,'\cdot\vec{r}} \;=$$

$$= \sqrt{r'^2 \,+\, r^2 \,-\, 2\,r'\,r\cos\vartheta} \;=\; r\,\sqrt{1 \,-\, 2\,\frac{r'}{r}\cos\vartheta \,+\, \frac{r'^2}{r^2}} \;=\; r\,\big(\,1 \,+\, O(r'/r)\,\big)\;.$$

Inserting the result into (RP) yields

$$\gamma^{\mu\nu}\big(ct,\vec{r}\,\big) \,=\, \frac{\kappa}{2\,\pi}\int\limits_{\mathbb{R}^3} \frac{T^{\mu\nu}\big(\,c\,t \,-\, r\,\big(1 + O(r'/r)\big)\,,\,\vec{r}\,'\,\big)\,d^3\vec{r}\,'}{r\,\big(1 + O(r'/r)\big)}\,.$$

If $r \gg R$, the $O(r'/r)$-terms can be neglected, as $r' \leq R$ on the whole domain of integration. This is known as the *far-field approximation*,

$$\gamma^{\mu\nu}\big(ct,\vec{r}\,\big) \,=\, \frac{\kappa}{2\,\pi\,r}\int\limits_{\mathbb{R}^3} T^{\mu\nu}\big(\,c\,t \,-\, r\,,\,\vec{r}\,'\,\big)\,d^3\vec{r}\,'\,. \qquad (FF)$$

In this approximation, the $\gamma^{\mu\nu}$ depend on $\vec{r}$ only in terms of its modulus $r = |\vec{r}|$, i.e., the wave fronts are spheres, $r =$ constant. As the radii of these spheres are large, they can be approximated as planes on a sufficiently small neighbourhood of any point $\vec{r}$. This means that, on any such neighbourhood, our gravitational wave resembles a plane wave of the type we have studied in Chapter 4.

We will now investigate which properties of the source determine the γ^{ij} in the far-field approximation. To that end we introduce the multipole moments of the source. They are defined in analogy to electrodynamics, with the charge density replaced by the energy density $T_{00} = -T_0{}^0 = T^{00}$.

$$M(t) \;=\; \int\limits_{K_R} T^{00}\big(ct,\vec{r}\,\big)\, d^3\vec{r} \qquad \text{(monopole moment)}\,,$$

$$D^k(t) \;=\; \int\limits_{K_R} T^{00}\big(ct,\vec{r}\,\big)\, x^k\, d^3\vec{r} \qquad \text{(dipole moment)}\,,$$

$$Q^{k\ell}(t) \;=\; \int\limits_{K_R} T^{00}\big(ct,\vec{r}\,\big)\, x^k\, x^\ell\, d^3\vec{r} \qquad \text{(quadrupole moment)}\,,$$

$$\ldots$$

Note that each quadrupole moment is determined by its trace-free part and the quadrupole moments of lower order. For this reason, some authors define the multipole moments as the trace-free parts of our moments.

We calculate the first and second time derivative of the quadrupole moments. To that end, we need to know that, because of the Hilbert gauge condition,

$$\partial_\mu T^{\mu\nu} \;=\; \frac{1}{2\,\kappa}\,\partial_\mu \Box\, \gamma^{\mu\nu} \;=\; \frac{1}{2\,\kappa}\,\Box\,\partial_\mu \gamma^{\mu\nu} \;=\; 0\,.$$

We find

$$\frac{d}{dt}Q^{k\ell}(t) \;=\; \int\limits_{K_R} c\,\partial_0 T^{00}(ct,\vec{r}\,)\, x^k\, x^\ell\, d^3\vec{r} \;=\; -\,c \int\limits_{K_R} \partial_i T^{i0}(ct,\vec{r}\,)\, x^k\, x^\ell\, d^3\vec{r} \;=$$

$$=\; -\,c \int\limits_{K_R} \Big(\partial_i\big(T^{i0}(ct,\vec{r}\,)\, x^k\, x^\ell\big) \;-\; T^{i0}(ct,\vec{r}\,)\,\delta_i^k\, x^\ell \;-\; T^{i0}(ct,\vec{r}\,)\, x^k\,\delta_i^\ell \Big)\, d^3\vec{r}\,.$$

The first integral can be rewritten, with the Gauss theorem, as a surface integral over the boundary ∂K_R of K_R,

$$\int\limits_{K_R} \partial_i\big(T^{i0}(ct,\vec{r}\,)\, x^k\, x^\ell\big) d^3\vec{r} \;=\; \int\limits_{\partial K_R} T^{i0}(ct,\vec{r}\,)\, x^k\, x^\ell df_i$$

where df_i is the surface element on ∂K_R. As the sphere K_R surrounds all sources, $T^{\mu\nu}$ is equal to zero on ∂K_R, so the last integral vanishes. Hence

$$\frac{d}{dt}Q^{k\ell}(t) \;=\; c \int\limits_{K_R} \big(T^{k0}(ct,\vec{r}\,)\, x^\ell \;+\; T^{\ell 0}(ct,\vec{r}\,)\, x^k\,\big)\, d^3\vec{r}\,.$$

Analogously we calculate the second derivative.

$$\frac{d^2}{dt^2}Q^{k\ell}(t) \,=\, c^2 \int\limits_{K_R} \Big(\partial_0 T^{k0}(ct,\vec{r}\,)\, x^\ell \,+\, \partial_0 T^{\ell 0}(ct,\vec{r}\,)\, x^k \Big)\, d^3\vec{r} \,=$$

$$= \, c^2 \int\limits_{K_R} \Big(\,-\, \partial_i T^{ki}(ct,\vec{r}\,)\, x^\ell \,-\, \partial_i T^{\ell i}(ct,\vec{r}\,)\, x^k \Big)\, d^3\vec{r} \,=$$

$$= \, c^2 \int\limits_{K_R} \Big(\,-\, \partial_i \big(T^{ki}(ct,\vec{r}\,)\, x^\ell\big) \,+\, T^{ki}(ct,\vec{r}\,)\, \delta_i^\ell \,-\, \partial_i \big(T^{\ell i}(ct,\vec{r}\,)\, x^k\,\big) \,+\, T^{\ell i}(ct,\vec{r}\,)\, \delta_i^k \Big)\, d^3\vec{r} \,=$$

$$= \, 0 \,+\, c^2 \int\limits_{K_R} T^{k\ell}(ct,\vec{r}\,)\, d^3\vec{r} \,-\, 0 \,+\, c^2 \int\limits_{K_R} T^{\ell k}(ct,\vec{r}\,)\, d^3\vec{r}\,.$$

As $T^{k\ell} = T^{\ell k}$, this can be rewritten as

$$\frac{d^2}{dt^2}Q^{k\ell}(t) \,==\, 2\,c^2 \int\limits_{K_R} T^{k\ell}(ct,\vec{r}\,')\, d^3\vec{r}\,'$$

where we have renamed the integration variable. Upon inserting this result into (FF) we find that, in the far-field approximation,

$$\gamma^{k\ell}(ct,\vec{r}\,) \,=\, \frac{\kappa}{2\,\pi\, r} \int\limits_{\mathbb{R}^3} T^{k\ell}\Big(c\,t \,-\, r, \vec{r}\,' \Big)\, d^3\vec{r}\,' \,=\, \frac{\kappa}{2\,\pi\, r}\, \frac{1}{2\,c^2}\, \frac{d^2 Q^{k\ell}}{dt^2}\Big(t \,-\, \frac{r}{c} \Big)\,.$$

If Einstein's gravitational constant is expressed with the help of Newton's gravitational constant, $\kappa = 4\pi G/c^4$, the result reads

$$\gamma^{k\ell}(ct,\vec{r}\,) \,=\, \frac{G}{c^6\, r}\, \frac{d^2 Q^{k\ell}}{dt^2}\Big(t \,-\, \frac{r}{c} \Big)\,.$$

The only property of the source that a gravitational wave detector can measure far away from the source is, thus, the second time derivative of the quadrupole moment at a retarded time. In this sense, gravitational radiation is quadrupole radiation, while electromagnetic radiation is dipole radiation. Roughly speaking, the difference has its origin in the fact that $\gamma^{\mu\nu}$ and $T^{\mu\nu}$ have two indices while the analogous quantities A^μ and J^μ in electrodynamics have only one index.

A time-dependent monopole moment (e.g. a pulsating spherically symmetric star) does not produce gravitational radiation. This is a consequence of Birkhoff's theorem acoording to which the spacetime outside of a spherically symmetric source is always the static Schwarzschild solution. We have now seen that, moreover, a time-dependent dipole moment does not produce any gravitational radiation in the far-field approximation. We need a time-dependent quadrupole moment. As an exmaple, we may think of a periodically squashed ball. Also, two masses orbiting their barycentre have a time-dependent quadrupole moment.

Note that, according to our results on the preceding page, only the *spatial* components $\gamma^{k\ell}$ are given by the second time derivative of the quadrupole moment. What about the time-time and the time-space components?

Claim: For a source $T^{\mu\nu}$ that is confined to a finite sphere for all times, in the far-field the components γ^{i0} vanish and the component γ^{00} is time-independent and falls off like r^{-1}.

Proof: See Worksheet 4.

For this reason, the components $\gamma^{0\mu}$ give no contribution to the radiation field in the far zone. In the next section we calculate the loss of energy of a system that emits gravitational waves.

5.2 Energy and momentum of a gravitational wave

The question of how to assign energy and momentum to a gravitational wave is conceptually subtle. According to general relativity, the gravitational field is *not* to be considered as a field on the spacetime, it is coded in the geometry of the spacetime itself. The energy-momentum tensor on the right-hand side of Einstein's field equation comprises everything *with the exception of the gravitational field*. An energy-momentum tensor of the gravitational field is not defined and cannot be defined. This is in correspondence with the equivalence principle according to which the gravitational field (coded in the Christoffel symbols which act as the "guiding field" for test particles and light) can be transformed to zero in any one point. As a non-zero tensor is non-zero in *any* coordinates, this is a clear indication that something like an energy-momentum tensor of the gravitational field cannot exist.

However, a (non-tensorial) quantity that describes energy and momentum of a gravitational field can be defined with respect to a background metric. We assume that we have a spacetime metric which takes, in the chosen coordinates the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \; .$$

For the time being, we do *not* assume that the $h_{\mu\nu}$ are small. The coordinates are then fixed up to Lorentz transformations

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \; , \qquad \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta_{\mu\nu} = \eta_{\sigma\tau} \; . \qquad \text{(LT)}$$

The Ricci tensor of the metric g is then of the form

$$R_{\mu\nu}(h) = R^{(1)}_{\mu\nu}(h) + R^{(2)}_{\mu\nu}(h) + \ldots$$

where $R^{(n)}_{\mu\nu}(h)$ comprises all terms of n^{th} order in the $h_{\rho\sigma}$ and their first and second derivatives. Similarly, the Einstein tensor is of the form

$$G_{\mu\nu}(h) \; = \; R_{\mu\nu}(h) \; - \; \frac{1}{2}\, g_{\mu\nu}\, g^{\rho\sigma} R_{\rho\sigma}(h) \; = \; G^{(1)}_{\mu\nu}(h) + G^{(2)}_{\mu\nu}(h) + \ldots$$

where

$$G^{(1)}_{\mu\nu}(h) = R^{(1)}_{\mu\nu}(h) - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} R^{(1)}_{\rho\sigma}(h) \, ,$$

$$G^{(2)}_{\mu\nu}(h) = R^{(2)}_{\mu\nu}(h) - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}(h) - \frac{1}{2}h_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma}(h) + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma}(h)\,,$$

and so on. Here we have used that $g^{\rho\sigma} = \eta^{\rho\sigma} - h^{\rho\sigma} + \dots$

We assume that our metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies Einstein's field equation, with a source term $T_{\mu\nu}$, *exactly*,

$$G_{\mu\nu}(h) = \kappa T_{\mu\nu}\,.$$

We rewrite this equation by keeping only the first-order terms on the left-hand side and shifting all higher-order terms to the right-hand side,

$$G^{(1)}_{\mu\nu}(h) = \kappa\big(T_{\mu\nu} + t_{\mu\nu}\big)\,, \qquad \text{(FEB)}$$

$$t_{\mu\nu} \;=\; -\frac{1}{\kappa}\left(G_{\mu\nu}(h) - G^{(1)}_{\mu\nu}(h)\right) \;=\; -\frac{1}{\kappa}\left(G^{(2)}_{\mu\nu}(h) \;+\; \dots\;\right)$$

According to (FEB), h satisfies the *linearised* field equation with a source term $T_{\mu\nu} + t_{\mu\nu}$. Of course, this is still the same Einstein equation which is non-linear. We have just renamed the non-linear terms into $t_{\mu\nu}$ and re-interpreted them as additional sources.

The $t_{\mu\nu}$ are not the components of a tensor; it is easy to check that the $G^{(1)}_{\mu\nu}$ and hence the $t_{\mu\nu}$ transform as tensor components under Lorentz transformations (LT), but not under arbitrary coordinate changes. $t_{\mu\nu}$ is called the *energy-momentum pseudotensor* of the gravitational field.

The following observation is crucial.

Claim: The combined source term $T_{\mu\nu} = t_{\mu\nu}$ satisfies the continuity equation

$$\partial^{\mu}\big(T_{\mu\nu} + t_{\mu\nu}\big)\,. \qquad \text{(CL)}$$

Proof: See Worksheet 4.

This conservation law is not a covariant equation. It holds only in the special coordinates in which our background metric has components $\eta_{\mu\nu}$. However, it really gives rise to a conservation law in integral form if integrated over a spacetime region ("the change of the energy content inside a spatial volume equals the energy flux over the boundary"). By contrast, the covariant divergence law $\nabla^{\mu}T_{\mu\nu}$, which is satisfied by our true matter source, is only a conservation law in "infinitesimally small regions"; it does not give rise to a conservation law in integral form.

If our matter source loses energy, exactly the same amount of energy must be carried away in the form of gravitational waves according to the conservation law (CL). This is the way in which the observations of the Hulse-Taylor pulsar are interpreted (which will be discussed in detail below): One observes that the system loses energy and one concludes that this energy is caried away in the form of gravitational waves.

Linearising the field equation with respect to the $h_{\mu\nu}$ and their derivatives is tantamount to setting $t_{\mu\nu}$ equal to zero. In this approximation, $T_{\mu\nu}$ alone satisfies the conservation law (CL). We have to go at least to the second order if we want to have a non-trivial $t_{\mu\nu}$. In the second-order theory, we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots$$

where $h^{(1)}_{\mu\nu}$ is a solution to the linearised field equation. The $h^{(1)}_{\mu\nu}$ are small of first order while the $h^{(2)}_{\mu\nu}$ are small of second order. In other words, terms linear in the $h^{(2)}_{\mu\nu}$ are treated at the same footing as terms quadratic in the $h^{(1)}_{\mu\nu}$.

Expanding both sides of (FEB) up to second order results in

$$G^{(1)}_{\mu\nu}(h^{(1)} + h^{(2)}) \;=\; \kappa\,\big(T_{\mu\nu} + t_{\mu\nu}\big)\,,$$

$$t_{\mu\nu} \;=\; -\,\frac{1}{\kappa}\,G^{(2)}_{\mu\nu}(h^{(1)})\,.$$

In other words, we get the energy-momentum pseudotensor of a gravitational field in its lowest non-trivial approximation if we insert the corresponding solution to the linearised field equation $h^{(1)}_{\rho\sigma}$ into $G^{(2)}_{\mu\nu}$. We will now carry out this calculation for a plane-harmonic wave of the kind we have considered in Chapter 4. On the basis of this result, we will then determine the power that is radiated away from a source that is confined to a finite sphere, as we have considered in Section 5.1.

As before, we raise and lower indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. We need to calculate $R^{(2)}_{\mu\nu}(h)$ which is a bit tedious. We begin with the Christoffel symbols

$$\Gamma^{\rho}{}_{\mu\nu} \;=\; \frac{1}{2}\,g^{\rho\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big) \;=\; \frac{1}{2}\,\big(\eta^{\rho\sigma} - h^{\rho\sigma} + \dots\big)\big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\big)\,.$$

The Ricci tensor is

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\mu\Gamma^{\rho}{}_{\rho\nu} - \partial_\rho\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\rho\nu} - \Gamma^{\rho}{}_{\sigma\rho}\Gamma^{\sigma}{}_{\mu\nu}\,,$$

hence

$$\begin{aligned} R^{(2)}_{\mu\nu} \;=\; &-\frac{1}{2}\,\partial_\mu\big(h^{\rho\sigma}(\cancel{\partial_\rho h_{\sigma\nu}} + \partial_\nu h_{\sigma\rho} - \cancel{\partial_\sigma h_{\rho\nu}})\big) \;+\; \frac{1}{2}\,\partial_\rho\big(h^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})\big) \\ &+\frac{1}{4}\,\eta^{\rho\tau}\big(\partial_\sigma h_{\tau\mu} + \partial_\mu h_{\tau\sigma} - \partial_\tau h_{\sigma\mu}\big)\eta^{\sigma\lambda}\big(\partial_\rho h_{\lambda\nu} + \partial_\nu h_{\lambda\rho} - \partial_\lambda h_{\rho\nu}\big) \\ &-\frac{1}{4}\,\eta^{\rho\tau}\big(\partial_\sigma h_{\tau\rho} + \cancel{\partial_\rho h_{\tau\sigma}} - \cancel{\partial_\tau h_{\sigma\rho}}\big)\eta^{\sigma\lambda}\big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\big) \end{aligned}$$

We want to calculate the time-space components

$$t_{0j} = -\,\frac{1}{\kappa}\,G^{(2)}_{0j}(h^{(1)})$$

for a plane-harmonic wave in the TT gauge,

$$h^{(1)}_{\mu\nu}(x) = \gamma_{\mu\nu}(x) = \mathrm{Re}\big\{A_{\mu\nu}e^{ik_\rho x^\rho}\big\}$$

where $k_\mu k^\mu = 0$, $\gamma_{\mu\nu}(x)k^\nu = 0$, $\gamma_\mu{}^\mu(x) = 0$, $\gamma_{0\nu}(x) = 0$. Up to a factor $-c$, the time-space components t_{0j} give the energy current density s_j with respect to an observer whose four-velocity u^ρ is tangent to the x^0 lines,

$$s_j = -u^\rho t_{\rho j} = -u^0 t_{0j} = -c t_{oj}\,.$$

We find

$$\kappa\,t_{0j} \;=\; R^{(2)}_{0j}(h^{(1)}) \;+\; 0 \;+\; 0 \;+\; 0 \;=\; \frac{1}{2}\,\partial_0\big(\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho}\big) \;-\; \frac{1}{2}\,\partial_\rho\big(\gamma^{\rho\sigma}\partial_0 h_{\sigma j}\big)$$

$$
\begin{aligned}
&-\frac{1}{4}\,\eta^{\rho\tau}\partial_0\gamma_{\tau\sigma}\eta^{\sigma\lambda}\big(\cancel{\partial_\rho\gamma_{\lambda j}}+\partial_j\gamma_{\lambda\rho}-\cancel{\partial_\lambda\gamma_{\rho j}}\big)+\frac{1}{4}\underbrace{\eta^{\rho\tau}\partial_\sigma\gamma_{\tau\rho}}_{=\partial_\sigma\gamma_\tau{}^\tau=0}\eta^{\sigma\lambda}\partial_0\gamma_{\lambda j}\\
&=\frac{1}{2}\,\partial_0\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho}+\frac{1}{2}\,\gamma^{\rho\sigma}\partial_0\partial_j\gamma_{\sigma\rho}\;-\;0\;-\frac{1}{2}\underbrace{\gamma^{\rho\sigma}\partial_\rho\partial_0\gamma_{\sigma j}}_{\sim\gamma^{\rho\sigma}k_\rho=0}-\frac{1}{4}\,\partial_0\gamma^{\rho\lambda}\partial_j\gamma_{\lambda\rho}\\
&=\frac{1}{4}\,\partial_0\gamma^{\rho\sigma}\partial_j\gamma_{\sigma\rho}+\frac{1}{2}\,\gamma^{\rho\sigma}\partial_0\partial_j\gamma_{\sigma\rho}\;=\;\frac{1}{4}\,\partial_0\gamma^{k\ell}\partial_j\gamma_{k\ell}+\frac{1}{2}\,\gamma^{k\ell}\partial_0\partial_j\gamma_{k\ell}\\
&=\frac{1}{4}\,\mathrm{Re}\big\{A^{k\ell}ik_0e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{A_{k\ell}ik_je^{ik_\rho x^\rho}\big\}\;+\;\frac{1}{2}\,\mathrm{Re}\big\{A^{k\ell}e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{-A_{k\ell}k_0k_je^{ik_\rho x^\rho}\big\}\,.
\end{aligned}
$$

We introduce the covector

$$
n_j=\frac{k_j}{k_0}
$$

which is parallel to the spatial wave covektor k_j and normalised, because

$$
n_jn^j\;=\;\frac{k_jk^j}{k_0^2}\;=\;\frac{k_\rho k^\rho-k_0k^0}{k_0^2}\;=\;\frac{0\;+\;k_0^2}{k_0^2}\;=\;1\,.
$$

Hence

$$
\begin{aligned}
\kappa\,t_{0j}\;&=\;\frac{k_0k_j}{4}\,\mathrm{Re}\big\{A^{k\ell}\,i\,e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{A_{k\ell}\,i\,e^{ik_\rho x^\rho}\big\}\;-\;\frac{k_0k_j}{2}\,\mathrm{Re}\big\{A^{k\ell}\,e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{A_{k\ell}\,e^{ik_\rho x^\rho}\big\}\\
&=\;\frac{k_0^2n_j}{16}\big(A^{k\ell}\,i\,e^{ik_\rho x^\rho}-\overline{A^{k\ell}}\,i\,e^{-ik_\rho x^\rho}\big)\big(A_{k\ell}\,i\,e^{ik_\rho x^\rho}-\overline{A^{k\ell}}\,i\,e^{-ik_\rho x^\rho}\big)\\
&\quad-\frac{k_0^2n_j}{8}\,\big(A_{k\ell}\,e^{ik_\rho x^\rho}+\overline{A^{k\ell}}\,e^{-ik_\rho x^\rho}\big)\big(A^{k\ell}\,e^{ik_\rho x^\rho}+\overline{A^{k\ell}}\,e^{-ik_\rho x^\rho}\big)\\
&=\;-\frac{3k_0^2n_j}{16}\Big(A^{k\ell}A_{k\ell}\,e^{2ik_\rho x^\rho}\;+\;\overline{A^{k\ell}A_{k\ell}}\,e^{-2ik_\rho x^\rho}\Big)\;-\;\frac{k_0^2n_j}{8}A^{k\ell}\,\overline{A_{k\ell}}\\
&=\;-\frac{3k_0^2n_j}{8}\Big(\mathrm{Re}\big\{A^{k\ell}A_{k\ell}\big\}\cos\big(2k_\rho x^\rho\big)\;-\;\mathrm{Im}\big\{A^{k\ell}A_{k\ell}\big\}\sin\big(2k_\rho x^\rho\big)\Big)\;-\;\frac{k_0^2n_j}{8}A^{k\ell}\,\overline{A_{k\ell}}
\end{aligned}
$$

where an overbar means complex conjugation. The first two terms, which are proportional to $\cos(2k_\rho x^\rho)=\cos(2k_ix^i-2\omega t)$ and $\sin(2k_\rho x^\rho)=\sin(2k_ix^i-2\omega t)$, respectively, vary periodically with time around zero. If we consider the time-average, denoted by $\langle\cdot\rangle$, they drop out. The time-averaged energy current density of a plane-harmonic gravitational wave in the TT gauge is

$$
\langle s_j\rangle=-\,c\,\langle t_{0j}\rangle\;=\;\frac{c\,k_0^2n_j}{8\kappa}A^{k\ell}\,\overline{A_{k\ell}}\,.
$$

This expression can be rewritten as

$$
\langle s_j\rangle\;=\;\frac{c\,n_j}{4\,\kappa}\,\big\langle\partial_0\gamma^{k\ell}\partial_0\gamma_{k\ell}\big\rangle\qquad\qquad\text{(EC)}
$$

as follows from comparison with

$$
\begin{aligned}
\big\langle\partial_0\gamma^{k\ell}\partial_0\gamma_{k\ell}\big\rangle\;&=\;\big\langle\mathrm{Re}\big\{A^{k\ell}\,i\,k_0\,e^{ik_\rho x^\rho}\big\}\mathrm{Re}\big\{A_{k\ell}\,i\,k_0\,e^{ik_\sigma x^\sigma}\big\}\big\rangle\\
&=\;\frac{k_0^2}{4}\Big\langle\big(A^{k\ell}\,i\,e^{ik_\rho x^\rho}-\overline{A^{k\ell}}\,i\,e^{-ik_\rho x^\rho}\big)\big(A_{k\ell}\,i\,e^{ik_\sigma x^\sigma}-\overline{A_{k\ell}}\,i\,e^{-ik_\sigma x^\sigma}\big)\Big\rangle\;=\;\frac{k_0^2}{2}A^{k\ell}\,\overline{A_{k\ell}}\,.
\end{aligned}
$$

We now turn back to the situation of an energy-momentum tensor field which has support inside a sphere of radius R, for all time, see picture on p.17. We know from Section 5.1 that the solution to the linearised field equation satisfies, in the far zone,

$$\gamma^{k\ell}(ct, \vec{r}\,) \;=\; \frac{\kappa}{4\,\pi\, r\, c^2}\, \frac{d^2 Q^{k\ell}}{dt^2}\Big(\, t \,-\, \frac{r}{c}\,\Big)\,.$$

This $\gamma^{k\ell}$, which satisfies the Hilbert gauge condition but not in general the TT gauge condition, can be viewed, in a sufficiently small neighbourhood of any one point in the far zone, as a superposition of plane-harmonic waves propagating in the direction n^j, where n^j is the unit vector in the radial direction. However, we cannot apply (EC) directly for calculating the time-averaged energy current of this gravitational wave, because (EC) holds only for a plane-harmonic wave in the TT gauge. We have to project onto the transverse-traceless part of $\gamma^{k\ell}$ first.

Projecting onto the transverse part means projecting onto the orthocomplement of n_j, i.e. applying the projection operator

$$P_i{}^j = \delta_i^j - n_i n^j$$

which satisfies the projection property

$$P_i{}^j P_j{}^k = \big(\delta_i^j - n_i n^j\big)\big(\delta_j^k - n_j n^k\big) = \delta_i^k - n_i n^k - \cancel{n_i n^k} + \cancel{n_i n^k} = P_i^k$$

and the symmetry property $P^{rs} = P^{sr}$. After applying this projection operator we have to subtract the trace to get the transverse-traceless part of $\gamma_{k\ell}$,

$$\gamma^{TT}_{k\ell} = P_k{}^i P_\ell{}^j \gamma_{ij} - \frac{1}{2} P_{k\ell} P^{rs} P_r{}^i P_s{}^j \gamma_{ij} = P_k{}^i P_\ell{}^j \gamma_{ij} - \frac{1}{2} P_{k\ell} P^{ij} \gamma_{ij}\,.$$

For applying (EC) to our gravitational field in the far zone we need to calculate

$$\partial_0 \gamma^{TTk\ell} \partial_0 \gamma^{TT}_{k\ell} \;=\; \partial_0\big(P^{km} P^{\ell n} \gamma_{mn} - \frac{1}{2} P^{k\ell} P^{mn} \gamma_{mn}\big)\partial_0\big(P_k{}^r P_\ell{}^s \gamma_{rs} - \frac{1}{2} P_{k\ell} P^{rs} \gamma_{rs}\big)$$

$$= \big(P^{km} P^{\ell n} P_k{}^r P_\ell{}^s - \frac{1}{2} P^{km} P^{\ell n} P_{k\ell} P^{rs} - \frac{1}{2} P^{k\ell} P^{mn} P_k{}^r P_\ell{}^s + \frac{1}{4} P^{k\ell} P^{mn} P_{k\ell} P^{rs}\big)\partial_0\gamma_{mn}\partial_0\gamma_{rs}$$

$$= \big(P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs} - \frac{1}{2} P^{rs} P^{mn} + \frac{1}{4} P_k{}^k P^{mn} P^{rs}\big)\partial_0\gamma_{mn}\partial_0\gamma_{rs}$$

$$= \big(P^{mr} P^{ns} - P^{mn} P^{rs} + \frac{1}{4}\underbrace{(\delta_k^k - n_k n^k)}_{=2} P^{mn} P^{rs}\big)\partial_0\gamma_{mn}\partial_0\gamma_{rs} \;=\; \big(P^{mr} P^{ns} - \frac{1}{2} P^{mn} P^{rs}\big)\partial_0\gamma_{mn}\partial_0\gamma_{rs}$$

$$= \big((\delta^{mr} - n^m n^r)(\delta^{ns} - n^n n^s) - \frac{1}{2}(\delta^{mn} - n^m n^n)(\delta^{rs} - n^r n^s)\big)\partial_0\gamma_{mn}\partial_0\gamma_{rs}$$

$$= \Big(\delta^{mr}\delta^{ns} - \frac{\delta^{mn}\delta^{rs}}{2} - 2\delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2}\Big)\partial_0\gamma_{mn}\partial_0\gamma_{rs}$$

Note that n^j is the unit vector in radial direction, so it depends on $\vec{r}$ but not on t. Time-averaging over an appropriate interval yields

$$\big\langle \partial_0 \gamma^{TTk\ell} \partial_0 \gamma^{TT}_{k\ell} \big\rangle \;=\; \Big(\delta^{mr}\delta^{ns} - \frac{\delta^{mn}\delta^{rs}}{2} - 2\delta^{mr} n^n n^s + \delta^{mn} n^r n^s + \frac{n^m n^n n^r n^s}{2}\Big)\,\big\langle \partial_0\gamma_{mn}\partial_0\gamma_{rs}\big\rangle\ .$$

This gives us the time-averaged energy current density

$$\langle s_j\rangle\,(ct,\,\vec{r}\,)\;=\;\frac{c\,n_j}{4\kappa}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{2}-2\delta^{mr}n^n n^s+\delta^{mn}n^r n^s+\frac{n^m n^n n^r n^s}{2}\Big)\,\langle\partial_0\gamma_{mn}\partial_0\gamma_{rs}\rangle\,(ct,\,\vec{r}\,)\,.$$

With

$$\partial_0\gamma_{k\ell}(ct,\,\vec{r}\,)\;=\;\partial_0\Big\{\frac{\kappa}{4\,\pi\,r\,c^2}\,\frac{d^2Q_{k\ell}}{dt^2}\Big(\,t\,-\,\frac{r}{c}\,\Big)\Big\}\;=\;\frac{\kappa}{4\,\pi\,r\,c^3}\,\frac{d^3Q_{k\ell}}{dt^3}\Big(\,t\,-\,\frac{r}{c}\,\Big)$$

we find

$$\langle s_j\rangle\,(ct,\,\vec{r}\,)$$

$$=\;\frac{\kappa n_j}{64\pi^2r^2c^5}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{2}-2\delta^{mr}n^n n^s+\delta^{mn}n^r n^s+\frac{n^m n^n n^r n^s}{2}\Big)\,\Big\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\Big\rangle\,\Big(t-\frac{r}{c}\Big)\,.$$

This equation holds at any point $\vec{r}$ in the far zone where n^j denotes the unit vector in radial direction at this point. We can integrate this equation over a sphere of radius $r(\gg R)$ to get the radiated power (energy per time) that passes through this sphere

$$P(t,r)=\int\limits_0^{2\pi}\int\limits_0^{\pi}\langle s_j\rangle\,(ct,\,\vec{r}\,)r^2n^j\sin\vartheta\,d\vartheta\,d\varphi\;=$$

$$\int\limits_0^{2\pi}\int\limits_0^{\pi}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{2}-2\delta^{mr}n^n n^s+\delta^{mn}n^r n^s+\frac{n^m n^n n^r n^s}{2}\Big)\frac{\kappa n_j\cancel{r^2}n^j\sin\vartheta\,d\vartheta\,d\varphi}{64\pi^2\cancel{r^2}c^5}\,\Big\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\Big\rangle\,\Big(t-\frac{r}{c}\Big)\,.$$

Claim: $\int\limits_0^{2\pi}\int\limits_0^{\pi}n^k n^\ell\sin\vartheta\,d\vartheta\,d\varphi=\frac{4\pi}{3}\delta^{k\ell}$ and $\int\limits_0^{2\pi}\int\limits_0^{\pi}n^k n^\ell n^r n^s\sin\vartheta\,d\vartheta\,d\varphi=\frac{4\pi}{15}(\delta^{k\ell}\delta^{rs}+\delta^{kr}\delta^{\ell s}+\delta^{ks}\delta^{r\ell})$.

Proof: We calculate for all $(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$,

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\xi_i\xi_j n^i n^j\sin\vartheta\,d\varphi\,d\vartheta\;=\;\int_0^{\pi}\int_0^{2\pi}\big(\xi_1\sin\vartheta\cos\varphi+\xi_2\sin\vartheta\sin\varphi+\xi_3\cos\vartheta\big)^2\sin\vartheta\,d\varphi\,d\vartheta$$

$$=\;\xi_1^2\int\limits_0^{\pi}\sin^3\vartheta\,d\vartheta\underbrace{\int\limits_0^{2\pi}\cos^2\varphi\,d\varphi}_{=\,\pi}+\xi_2^2\int\limits_0^{\pi}\sin^3\vartheta\,d\vartheta\underbrace{\int\limits_0^{2\pi}\sin^2\varphi\,d\varphi}_{=\,\pi}+\xi_3^2\int\limits_0^{\pi}\cos^2\vartheta\sin\vartheta\,d\vartheta\underbrace{\int\limits_0^{2\pi}d\varphi}_{=\,2\,\pi}\,.$$

For all other terms the φ integration gives zero. Hence

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\xi_i\xi_j n^i n^j d\varphi\,d\vartheta\;=\;\big(\pi\,\xi_1^2+\pi\,\xi_2^2\big)\underbrace{\int\limits_0^{\pi}\sin^3\vartheta\,d\vartheta}_{=\,4/3}\;+\;2\,\pi\,\xi_3^2\underbrace{\int\limits_0^{\pi}\cos^2\vartheta\sin\vartheta\,d\vartheta}_{=\,2/3}$$

$$=\;\frac{4\pi}{3}\big(\xi_1^2+\xi_2^2+\xi_3^2\big)\;=\;\frac{4\pi}{3}\delta^{ij}\xi_i\xi_j\,.$$

Similarly,

$$\int_0^\pi\int_0^{2\pi}\xi_i\xi_j\xi_k\xi_\ell n^i n^j n^k n^\ell \sin\vartheta\,d\varphi\,d\vartheta \;=\; \int_0^\pi\int_0^{2\pi}\big(\xi_1\sin\vartheta\cos\varphi+\xi_2\sin\vartheta\sin\varphi+\xi_3\cos\vartheta\big)^4\sin\vartheta\,d\varphi\,d\vartheta$$

$$= \;\xi_1^4\int_0^\pi \sin^5\vartheta\,d\vartheta\underbrace{\int_0^{2\pi}\cos^4\varphi\,d\varphi}_{=3\,\pi/4}+\xi_2^4\int_0^\pi \sin^5\vartheta\,d\vartheta\underbrace{\int_0^{2\pi}\sin^4\varphi\,d\varphi}_{=3\,\pi/4}+6\xi_1^2\xi_2^2\int_0^\pi \sin^5\vartheta\underbrace{\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\,d\varphi}_{=\pi/4}$$

$$+6\xi_1^2\xi_3^2\int_0^\pi \cos^2\vartheta\sin^3\vartheta\,d\vartheta\underbrace{\int_0^{2\pi}\cos^2\varphi\,d\varphi}_{=\pi}+6\xi_2^2\xi_3^2\int_0^\pi \cos^2\vartheta\sin^3\vartheta\,d\vartheta\underbrace{\int_0^{2\pi}\sin^2\varphi\,d\varphi}_{=\pi}+\xi_3^4\int_0^\pi \cos^4\vartheta\sin\vartheta\,d\vartheta\underbrace{\int_0^{2\pi}d\varphi}_{=2\,\pi}$$

$$= \;\frac{3\pi}{4}\big(\xi_1^4+\xi_2^4+2\xi_1^2\xi_2^2\big)\underbrace{\int_0^\pi \sin^5\vartheta\,d\vartheta}_{=16/15}\;+\;6\pi(\xi_1^2+\xi_2^2)\xi_3^2\underbrace{\int_0^\pi \cos^2\vartheta\sin^3\vartheta\,d\vartheta}_{=4/15}+2\,\pi\,\xi_3^4\underbrace{\int_0^\pi \cos^4\vartheta\sin\vartheta\,d\vartheta}_{=2/5}$$

$$= \;\frac{4\pi}{5}\big(\xi_1^2+\xi_2^2+\xi_3^2\big)^2 \;=\; \frac{4\pi}{5}\delta^{ij}\delta^{k\ell}\xi_i\xi_j\xi_k\xi_\ell\,.$$

Symmetrisation of the coefficients gives the desired result. □

Hence

$$P(t,r) =$$

$$\frac{\kappa}{16\pi c^5}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{2}-\frac{2\delta^{mr}\delta^{ns}}{3}+\frac{\delta^{mn}\delta^{rs}}{3}+\frac{\delta^{mn}\delta^{rs}+\delta^{mr}\delta^{ns}+\delta^{ms}\delta^{rn}}{30}\Big)\left\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\right\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \;\frac{\kappa}{16\pi c^5}\Big(\frac{2\delta^{mr}\delta^{ns}}{5}-\frac{2\delta^{mn}\delta^{rs}}{15}\Big)\left\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\right\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \;\frac{\kappa}{40\pi c^5}\Big(\delta^{mr}\delta^{ns}-\frac{\delta^{mn}\delta^{rs}}{3}\Big)\left\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q_{rs}}{dt^3}\right\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \;\frac{\kappa}{40\pi c^5}\left\langle\frac{d^3Q_{mn}}{dt^3}\,\frac{d^3Q^{mn}}{dt^3}-\frac{1}{3}\frac{d^3Q_m{}^m}{dt^3}\,\frac{d^3Q_r{}^r}{dt^3}\right\rangle\Big(t-\frac{r}{c}\Big)\,.$$

This can be rewritten more conveniently if we introduce the *reduced quadrupole moment* $\mathbb{Q}_{k\ell}$ which is defined as the trace-free part of $Q_{k\ell}$,

$$\mathbb{Q}_{k\ell} = Q_{k\ell}-\frac{1}{3}\delta_{k\ell}Q_j{}^j\,.$$

Then the energy flux through the sphere of radius r reads

$$P(t,r) \;=\; \frac{\kappa}{40\pi c^5}\left\langle\frac{d^3}{dt^3}\Big(\mathbb{Q}_{mn}+\frac{1}{3}\delta_{mn}Q_k{}^k\Big)\,\frac{d^3}{dt^3}\Big(\mathbb{Q}^{mn}+\frac{1}{3}\delta^{mn}Q_\ell{}^\ell\Big)-\frac{1}{3}\frac{d^3Q_m{}^m}{dt^3}\,\frac{d^3Q_r{}^r}{dt^3}\right\rangle\Big(t-\frac{r}{c}\Big)$$

$$= \frac{\kappa}{40\pi c^5}\left\langle \frac{d^3\mathbb{Q}_{mn}}{dt^3}\frac{d^3\mathbb{Q}^{mn}}{dt^3} + 0 + 0 + \frac{1}{\cancel{9}}\cancel{3}\frac{d^3Q_k{}^k}{dt^3}\frac{d^3\cancel{Q_\ell{}^\ell}}{dt^3} - \frac{1}{\cancel{3}}\frac{d^3Q_k{}^k}{dt^3}\frac{d^3\cancel{Q_\ell{}^\ell}}{dt^3}\right\rangle\left(t-\frac{r}{c}\right)$$

which eventually gives us Einstein's *quadrupole formula*

$$P(t,r) \;=\; \frac{\kappa}{40\pi c^5}\left\langle \frac{d^3\mathbb{Q}_{mn}}{dt^3}\frac{d^3\mathbb{Q}^{mn}}{dt^3}\right\rangle\left(t-\frac{r}{c}\right).$$

This formula allows us to calculate the power that is radiated away by a time-dependent matter source. If we want to apply this formula, we need to know the reduced quadrupole moment of the source. Note that this is the *energy* quadrupole moment,

$$\mathbb{Q}_{k\ell} = Q_{k\ell} - \frac{1}{3}\delta_{k\ell}Q_j{}^j\,,$$

$$Q_{k\ell}(t) = \int_{\mathbb{R}^3} T_{00}(ct,\,\vec{r})x_k x_\ell d^3\vec{r}$$

which requires to know the energy density T_{00}. The latter contains the whole energy content of the source which is difficult to determine. For slowly moving bodies the biggest contribution to the energy density comes from the mass density $\mu(ct,\,\vec{r})$. As long as the source involves only motions that are slow in comparison to the speed of light, we can write

$$T_{00}(ct,\,\vec{r}) \approx c^2\mu(ct,\,\vec{r})\,.$$

as a valid approximation. We can then replace the reduced energy quadrupole moment $\mathbb{Q}_{k\ell}$ by the reduced mass quadrupole moment

$$\mathbb{I}_{k\ell} = I_{k\ell} - \frac{1}{3}\delta_{k\ell}I_j{}^j\,,$$

$$I_{k\ell}(t) = \int_{\mathbb{R}^3} \mu(ct,\,\vec{r})x_k x_\ell d^3\vec{r}$$

With the aproximation

$$\mathbb{Q}_{k\ell} \approx c^2\,\mathbb{I}_{k\ell}$$

Einstein's quadrupole formula reads

$$P(t,r) \;=\; \frac{\kappa}{40\,\pi\,c}\left\langle \frac{d^3\mathbb{I}_{mn}}{dt^3}\frac{d^3\mathbb{I}^{mn}}{dt^3}\right\rangle\left(t-\frac{r}{c}\right)$$

or, with $\kappa = 8\pi G/c^4$,

$$P(t,r) \;=\; \frac{G}{5\,c^5}\left\langle \frac{d^3\mathbb{I}_{mn}}{dt^3}\frac{d^3\mathbb{I}^{mn}}{dt^3}\right\rangle\left(t-\frac{r}{c}\right).$$

This is the form in which the formula is usually applied. In this version, the quadrupole formula involves the following approximations.

- The energy-momentum pseudotensor was calculated only to within second order (which is the lowest non-trivial order); the solution to the field equation that is needed to calculate this order is a first-order solution, i.e., a solution to the linearised field equation.

- The formula holds in the far zone, i.e., it gives the energy flux per time through a sphere of radius r which is big in comparison to the radius R of the sphere to which the matter source $T_{\mu\nu}$ is confined.
- The formula is based on the assumption that all motions inside the source are slow in comparison to the speed of light.

In addition, the formula involves a time-averaging over an interval that covers the periods of all Fourier components that contribute to the gravitational wave.

5.3 Gravitational waves from a binary source

We consider a binary system consisting of two masses M_1 and M_2 moving around each other. It is sufficient to describe the system at the Newtonian level, that is, to describe the motion of the two masses in terms of their Keplerian orbits. (We make later remarks about possible effects which are neglected herewith.)

The relative distance between the two masses is denoted by r. From the Kepler problem we know that

$$\mathsf{r}(\varphi) = \frac{R_0}{1 + e\cos\varphi}\,, \qquad \text{(K)}$$

where R_0 is the semi-latus rectum and e is the eccentricity of the orbit. The semi-major axis a and the semi-minor axis b of the orbit are given by

$$\begin{aligned} 2a &= \mathsf{r}(0) + \mathsf{r}(\pi) = \frac{R_0}{1+e} + \frac{R_0}{1-e} = 2\frac{R_0}{1-e^2} \\ b &= \frac{R_0}{\sqrt{1-e^2}}\,. \end{aligned}$$

We first calculate the mass quadrupole tensor in the co-rotating system, that is, in the body-fixed coordinate system. Then we transform into the non-rotating observer system. At last, the third time-derivative of this mass quadrupole tensor has to be inserted into the radiation formula. From that we can calculate the change of the orbital parameters of the system due to the loss of energy.

For the body-fixed coordinate system we choose as the origin the centre of mass. Then the distances of the two masses from the centre of mass are given by

$$\mathsf{r}_1 = -\frac{M_2}{M_1+M_2}\mathsf{r}\,, \qquad \mathsf{r}_2 = \frac{M_1}{M_1+M_2}\mathsf{r}\,.$$

Consistency requires $\mathsf{r} = -\mathsf{r}_1 + \mathsf{r}_2$ which is fulfilled. We call the direction of the line between the masses the 1-direction. The 2-direction is in the orbital plane and the 3-direction is orthogonal to the orbital plane.

Then the mass quadrupole tensor is diagonal, with

$$I_1 = M_1\mathsf{r}_1^2 + M_2\mathsf{r}_2^2 = M_1\left(-\frac{M_2}{M_1+M_2}\mathsf{r}\right)^2 + M_2\left(\frac{M_1}{M_1+M_2}\mathsf{r}\right)^2 = \frac{M_1M_2}{M_1+M_2}\mathsf{r}^2\,.$$

The other components vanish. Thus, the mass quadrupole tensor is

$$I' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The traceless version of this mass quadrupole tensor is

$$\mathbb{I}' = I' - \frac{1}{3}\,\mathrm{tr} I'\,\mathbf{1} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2I_1 & 0 & 0 \\ 0 & -I_1 & 0 \\ 0 & 0 & -I_1 \end{pmatrix} .$$

This mass quadrupole tensor, given in the body-fixed coordinate system, has to be transformed to the non-rotating observer system. This has to be done by means of a rotation matrix for a rotation around the 3-axis,

$$\alpha = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In matrix form the mass quadrupole tensor is then given by

$$\begin{aligned}
\mathbb{I} &= \alpha\,\mathbb{I}'\,\alpha^T \\
&= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 2I_1 & 0 & 0 \\ 0 & -I_1 & 0 \\ 0 & 0 & -I_1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2I_1\cos\varphi & 2I_1\sin\varphi & 0 \\ I_1\sin\varphi & -I_1\cos\varphi & 0 \\ 0 & 0 & -I_1 \end{pmatrix} \\
&= \frac{I_1}{3}\begin{pmatrix} 2\cos^2\varphi - \sin^2\varphi & 3\cos\varphi\sin\varphi & 0 \\ 3\cos\varphi\sin\varphi & 2\sin^2\varphi - \cos^2\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \frac{I_1}{3}\begin{pmatrix} 3\cos^2\varphi - 1 & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -3\cos^2\varphi + 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \frac{I_1}{3}\begin{pmatrix} \frac{3}{2}(1-\cos(2\varphi)) - 1 & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -\frac{3}{2}(1-\cos(2\varphi)) + 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \frac{I_1}{3}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{I_1}{2}\begin{pmatrix} -\cos(2\varphi) & \sin(2\varphi) & 0 \\ \sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} ,
\end{aligned}$$

which consists of a constant part and a part depending on the angle of rotation φ.

Instead of the angle of rotation φ one can introduce the mean anomaly $M = \varphi - e\sin\varphi$ which increases uniformly with time. The time derivative of φ can be expressed through the constant angular momentum $L = \frac{M_1M_2}{M_1+M_2}\mathsf{r}^2\dot{\varphi}$, that is, $\dot{\varphi} = \frac{(M_1+M_2)}{M_1M_2}\frac{L}{\mathsf{r}^2}$, where r is given by (K). Differentiating $\mathbb{I}$ three times requires some calculation, see Worksheet 5. In order to shorten

the calculations and to highlight the relevant effect of inspiralling we restrict here to a circular orbit, that is, to $e = 0$. In this case we have a constant r. From Kepler's third law

$$4\pi^2 \frac{\mathsf{r}^3}{T^2} = G(M_1 + M_2)$$

we obtain the angular frequency of the rotation of the binary system

$$\Omega = \frac{2\pi}{T} = \sqrt{\frac{G(M_1 + M_2)}{\mathsf{r}^3}}\,,$$

so that

$$\varphi = \Omega t\,.$$

We now calculate the power loss with the help of the quadrupole formula

$$P(t, r) = \frac{G}{5c^5} \left\langle \mathrm{tr}\left(\dddot{\mathbb{I}} \cdot \dddot{\mathbb{I}} \right) \right\rangle \left(t - \tfrac{r}{c}\right)\,.$$

Recall that this formula gives the power that passes through a (big) sphere of radius r at time t. (Don't confuse $r = |\,\vec{r}\,|$, the radius coordinate of the observer, with r, the distance of the two masses.) In the case at hand, $\langle\,\cdot\,\rangle$ denotes averaging over a period $2\pi/\Omega$.
We find

$$\begin{aligned}
\dot{\mathbb{I}} &= I_1\Omega \begin{pmatrix} \sin(2\varphi) & \cos(2\varphi) & 0 \\ \cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\ddot{\mathbb{I}} &= 2I_1\Omega^2 \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) & 0 \\ -\sin(2\varphi) & -\cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\dddot{\mathbb{I}} &= 4I_1\Omega^3 \begin{pmatrix} -\sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}\left(\dddot{\mathbb{I}} \cdot \dddot{\mathbb{I}} \right) &= (4I_1\Omega^3)^2 \mathrm{tr} \left(\begin{pmatrix} -\sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\
&= (4I_1\Omega^3)^2 \mathrm{tr} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 2(4I_1\Omega^3)^2\,.
\end{aligned}$$

With this we obtain

$$P = \frac{G}{5c^5} 2(4I_1\Omega^3)^2 = \frac{G}{5c^5} 32 \frac{M_1^2 M_2^2}{(M_1 + M_2)^2} \mathsf{r}^4 \frac{G^3 (M_1 + M_2)^3}{\mathsf{r}^9} = \frac{32G^4}{5c^5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{\mathsf{r}^5}\,.$$

P is the power radiated away by the binary system. Therefore, the binary system loses energy. During this process r decreases slowly with time. We have to insert r at the retarded time, $\mathsf{r}(t - r/c)$, to get the power that passes through a sphere of (big) radius r at time t.
The energy in a binary system is given by

$$E = \frac{1}{2}\frac{M_1 M_2}{(M_1 + M_2)}v^2 - \frac{GM_1M_2}{\mathsf{r}}\,.$$

For the circular orbit we obtain

$$\begin{aligned} E &= \frac{1}{2}\frac{M_1 M_2}{(M_1 + M_2)}(\Omega\mathsf{r})^2 - \frac{GM_1M_2}{\mathsf{r}} \\ &= \frac{1}{2}\frac{M_1 M_2}{(M_1 + M_2)}\frac{G(M_1 + M_2)}{\mathsf{r}^3}\mathsf{r}^2 - \frac{GM_1M_2}{\mathsf{r}} \\ &= -\frac{1}{2}M_1M_2\frac{G}{\mathsf{r}} \end{aligned}$$

so that the loss in energy can be described as

$$P = -\frac{dE}{dt} = -\frac{GM_1M_2}{2\mathsf{r}^2}\frac{d\mathsf{r}}{dt}\,.$$

With the power P calculated above we obtain

$$\frac{d\mathsf{r}}{dt} = -\frac{64G^3}{5c^5}\frac{M_1M_2(M_1 + M_2)}{\mathsf{r}^3}\,.$$

We can rewrite this as

$$-A = \text{const} = \mathsf{r}^3\dot{\mathsf{r}} = \frac{1}{4}\frac{d}{dt}\mathsf{r}^4\,.$$

Therefore

$$\mathsf{r}^4 = A_0 - 4At\,.$$

A_0 is the fourth power of the initial radius r_0 at $t = 0$. Then

$$\mathsf{r}(t) = \left(\mathsf{r}_0^4 - 4At\right)^{\frac{1}{4}} = \mathsf{r}_0\left(1 - \frac{4A}{\mathsf{r}_0^4}t\right)^{\frac{1}{4}} =: \mathsf{r}_0\left(1 - \frac{t}{t_{\text{spiral}}}\right)^{\frac{1}{4}}\,,$$

where

$$t_{\text{spiral}} := \frac{\mathsf{r}_0^4}{4A} = \frac{5c^5}{256G^3}\frac{\mathsf{r}_0^4}{M_1M_2(M_1 + M_2)}$$

is the time the system needs to inspiral completely.

It is clear that for a weaker gravitational coupling, that is, for a smaller G, the time for the inspiralling becomes larger. It is also interesting to see the velocity of light in the numerator. This means that for larger c, i.e., for approaching the nonrelativistic regime, also the inspiral time becomes larger. Therefore gravitational radiation is not only a gravitational but also a relativistic effect. For typical values like Solar masses and distances of several Solar radii the inspiral time is of the order of billions of years.

In addition to the approximations on which the quadrupole formula is based, we have made a number of additional approximations. We add now some remarks on the limitations of our model.

1. We assumed a circular orbit. The calculation with eccentric orbits is more involved, see Worksheet 5.

2. We assumed the stars to be point particles. Stars are extended bodies and will be deformed when they approach each other. So, during the last phase of the inspiralling this would lead to modifications of our result due to a shift of the centre of gravity and due to additional couplings of the star with the gravity gradient.

3. Due to gravitomagnetic effects also the spin of stars will lead to modifications of the inspiralling.

We recall that the quadrupole formula is valid only to within the linearised theory of gravity and that it gives the power that passes through a sphere in the far zone. Using the mass quadrupole tensor instead of the energy quadrupole tensor is justified unless the motion of the source is highly relativistic. During the last phase of the inspiralling such highly relativistic effects may contribute, in particular for merging black holes.

In Worksheet 5 we generalise the results of this section to the case of non-circular orbits. In particluar, we will derive formulas for the time-dependence of the semi-major axis a and of the period T. We will find that

$$\dot{a} = \frac{2a^2\dot{E}}{GM_1M_2} = \frac{-64M_1M_2G^3(M_1+M_2)}{5c^5a^3(1-e^2)^{7/2}}\Big(1 + \frac{73e^2}{24} + \frac{37e^4}{96}\Big) ,$$

$$\frac{\dot{T}}{T} = \frac{-96M_1M_2G^3(M_1+M_2)}{5c^5a^4(1-e^2)^{7/2}}\Big(1 + \frac{73e^2}{24} + \frac{37e^4}{96}\Big) .$$

As the masses are in the numerator and the semi-major axis is in the denominator, the latter even with a power of 4, we see that a measurable effect can be expected only for compact binaries, i.e., not for main sequence stars or planets, but for neutron stars or black holes. In the next section we will discuss how the predictions of general relativity have been verified with binary pulsars, i.e., binaries where at least one partner is a rotating neutron star.

5.4 Indirect evidence for gravitational waves from binary pulsars

Before coming to binary pulsars, we will briefly recall what pulsars are and how they were discovered.

Pulsars were discovered in 1967 by Jocelyn Bell, who is shown in Fig. 5.1, then a PhD student in the group of Antony Hewish at Cambridge University. Fig. 5.2 shows Hewish in front of the do-it-yourself radio telescope with which the discovery was made.

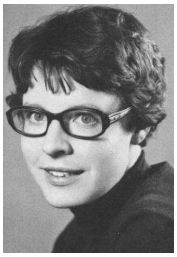


Fig. 5.1: from cwp.library.ucla.edu/

Fig. 5.2: from www.mrao.cam.ac.uk/

After having constructed the radio telescope, together with other students, with her own hands, Jocelyn Bell concentrated in her PhD work on the search for quasars with the scintillation method. The observation was often affected by interferences caused by terrestrial sources such as cars. On 6 August 1967 Jocelyn Bell observed some "scruff", as she later put it, that appeared to be different from these usual interferences, see Fig. 5.3. She discussed the observation with her supervisor. After having verified that the source remained fixed with respect to the stars it seemed certain that it was an astronomical object. Hewish and Bell decided to look at it more closely.

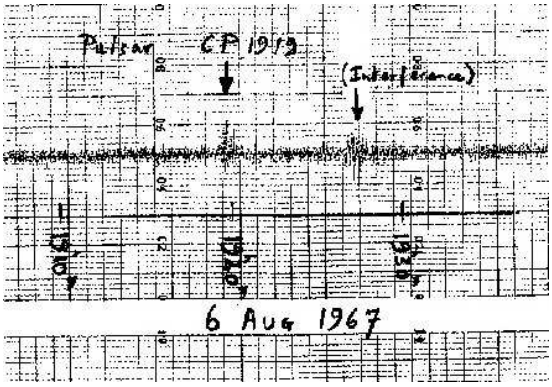


Fig. 5.3: from pulsar.ca.astro.it/

On 28 November 1967 Jocelyn Bell observed the mysterious object with a higher time resolution, see Fig. 5.4. It showed highly regularly pulses with a period of 1.337 seconds. It was seriously discussed in the group whether the signal could come from an alien civilisation, and it was only half-jocular that the object was initially called LGM-1, with LGM standing for *Little Green Men*. Later, the object was given the systematic name PSR B1919+21. Here PSR stands for *Pulsating Source of Radio emission*, which was soon abbreviated as *pulsar* and the numbers give the celestial coordinates of the source, a point in the constellation Vulpecula: $19^{\mathrm{h}}19^{\mathrm{m}}$ is the right ascension and $+21^{\mathrm{o}}$ is the declination; the letter B is added for coordinates refering to the epoch 1950 while a letter J is added for the epoch 2000.

Within a few weeks the Cambridge group found three more similar objects. In early 1968, they published their observations, see A. Hewish, J. Bell, J. Pilkington, P. Scott and R. Collins ["Observation of a rapidly pulsating radio source" Nature 217, 709 (1968)].

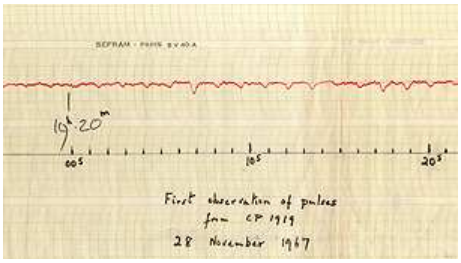


Fig. 5.4: from www.bbc.co.uk/

Passionate discussions started about the nature of the radiation. A majority, including Hewish, first thought that it might come from radial oscillations of a white dwarf. However, it turned out that not even a white dwarf, let alone a main sequence star, could perform oscillations with such a high frequency. After about a year, it was the prevailing opinion that the radiation comes from a rotating neutron star. Thomas Gold was the first to suggest such a model in early 1968 [T. Gold: "Rotating neutron stars as the Origin of the pulsating radio sources" Nature 218, 731-732 (1968)], which was initially ridiculed by many astrophysicists. The idea was that the neutron star has a magnetic field that is not aligned with the rotation axis. Radiation is emitted in a cone around the magnetic field axis, and this cone rotates like the beacon of a lighthouse. The observer registers a pulse whenever the cone hits the Earth. Neutron stars had been introduced, as a theoretical possibility, in 1934 by Walter Baade and Fritz Zwicky, but up to the discovery of pulsars there was no indication that they actually exist in Nature. An animation of the lighthouse model can be found in Section 2.1 of D. Lorimer ["Binary and Millisecond Pulsars", Living Rev. Relativity 11, (2008), http://www.livingreviews.org/lrr-2008-8].

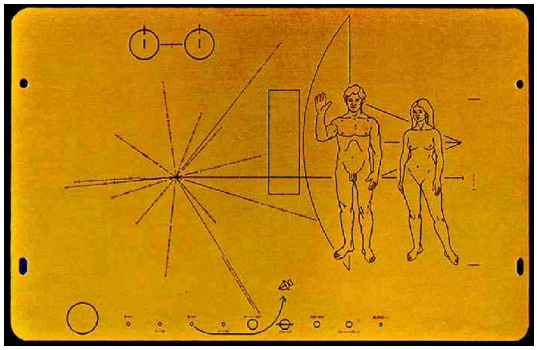


Fig. 5.5: from en.wikipedia.org

Within a few years after the discovery of PSR B1919+21, several dozens of pulsars were found. 14 of them are shown in the plaques that are on board the spacecraft Pioneer 10 and 11. They were launched in 1972 and 1973 and are the first spacecraft to leave the Solar system. The positions of the pulsars are shown, relative to the Earth, in the diagram in the left part of the plaque, see Fig. 5.5. This should tell an extraterrestrial civilisation where the spacecraft came from, in case that Pioneer 10 or 11 is intercepted by them.

The best known example of a pulsar is the neutron star at the centre of the Crab Nebula. It is the remnant of a supernova that was observed from the Earth in 1054. It is also visible in the optical and X-ray parts of the spectrum. There are also some pulsars that emit gamma rays, e.g. the Vela pulsar.

In 1974 Hewish received the Nobel Prize for the discovery of pulsars. Some people thought that it would have been fair if Jocelyn Bell had shared the prize. By now more than 2000 radio pulsars are known. Most of them are within our galaxy, but there are also a few in the Magellanic Clouds. The periods vary from a few milliseconds to about 10 seconds.

Fig. 5.6: from th.physik.uni-frankfurt.de/

After these remarks on pulsars in general, we turn now to binary pulsars. This term refers to binary systems in which at least one partner is a pulsar. About 10 % of all known pulsars have a companion. The first binary pulsar, PSR B1913+16, was discovered in 1974. It was again the work of a PhD student and a supervisor, this time Russell Hulse and Joseph Taylor from Cornell University who are shown in Fig. 5.6. In contrast to the earlier story, both were awarded the Nobel Prize in 1993.

Fig. 5.7: from en.wikipedia.org

The discovery was made with the 305-meter Arecibo radio telescope, see Fig. 5.7. The object is a pulsar with a period of 59 milliseconds. Evidence for the existence of a companion, which is dark and mute, came from the fact that the arrival time of the pulses varied periodically. Obviously, the pulsar is moving towards us, then away from us, then again towards us, and so on.

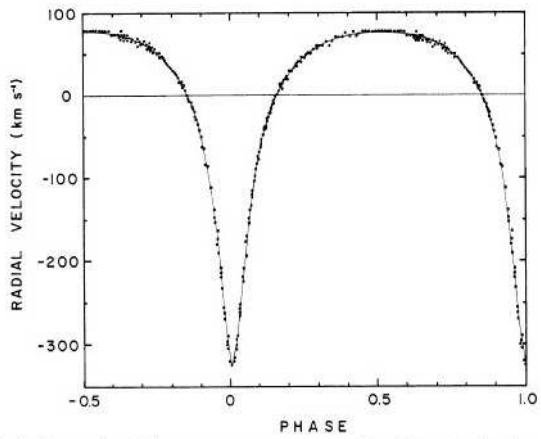


FIG. 1.—Velocity curve for the binary pulsar. Points represent measurements of the pulsar period distributed over parts of 10 different orbital periods. The curve corresponds to equations (1)–(4), with parameters from table 2.

Fig. 5.8: from Hulse and Taylor, loc. cit.

The plot of the radial velocity in Fig. 5.8 is taken from the original paper by R.Hulse and J. Taylor ["Discovery of a pulsar in a binary system" Astrophys. J. 195, L51 (1975)]. After correcting for the motion of the Earth, for dispersion in the intergalactic medium and for other effects, Hulse and Taylor fitted the observed time dependence of the radial velocity to a Kepler orbit. There is a certain degeneracy, i.e., not all orbital elements can be uniquely determined, but the following parameters of the system were found. The numbers are taken, again, from the paper by Hulse and Taylor.

T	7.75 hours
e	0.16
$a_1 \sin i$	$1.0\, R_\odot$
$\dfrac{(M_2 \sin i)^3}{(M_1+M_2)^2}$	$0.13\, M_\odot$

Here an index 1 stands for the pulsar and an index 2 stands for the invisible companion. i is the inclination angle. From the Newtonian analysis one cannot determine the individual masses M_1 and M_2. However, this is possible with the help of relativistic corrections, using the post-Newtonian (PN) approximation. Roughly speaking, this is an expansion in powers of v/c. If relativistic effects are taken into account, in particular the transverse Doppler effect and the gravitational Doppler effect, the individual masses and all orbital parameters can be determined. The method, which was worked out by V. Brumberg, Y. Zeldovich, I. Novikov and N. Shakura ["Determination of the component masses and inclination of a binary system containing a pulsar from relativistic effects", Sov. Astr. Lett. 1, 2 (1975)], is sketched in Straumann's book. One finds

M_1	$1.44\, M_\odot$
M_2	$1.39\, M_\odot$
i	45^o
periastron shift	$4.2\,^o$/yr

At periastron, the separation of the two stars is only 1.1 Solar radii, at apastron it is 4.5 Solar radii. The companion is thought to be a neutron star as well. We do not know the radii of the two stars precisely, but typically neutron stars have radii in the order of 10 kilometers.

Already in the original Hulse-Taylor paper it is remarked that the system should be a highly promising candidate for testing general relativity. In the above-mentioned paper by Brumberg et al. it was noted that it could provide indirect evidence for the existence of gravitational waves: With the masses and the orbital elements known, one could check if the period T depends on time according to the formula derived from general relativity.

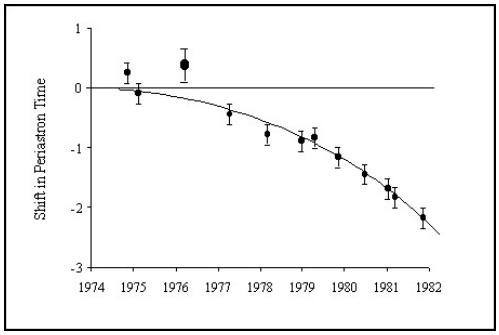


Fig. 5.9: From Taylor and Weisberg, loc. cit.

Such a dependence of T on time was reported by J. Taylor, L. Fowler and P. McCulloch ["Measurements of general relativistic effects in the binary pulsar PSR 1913+16" Nature 277, 437 (1979)] and confirmed, on the basis of more data, by J. Taylor and J. Weisberg ["A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16" Astrophys. J. 253, 908 (1982)]. The plot in Fig. 5.9 is taken from the latter paper. It clearly shows the decrease of the orbital period. The solid line gives the prediction according to general relativity, on the basis of the determined orbital parameters. In the course of time, the agreement between observation and theory became even more impressive, see Fig. 5.10.

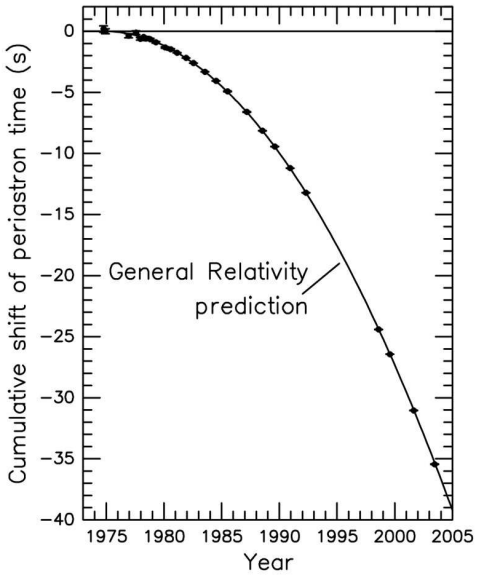


Fig. 5.10: from www.ast.cam.ac.uk

After the discovery of the Hulse-Taylor pulsar, several other binary pulsars were detected. They are used on a regular basis for testing general relativity and alternative theories of gravity. Up to now, general relativity has passed all tests with flying colours, whereas severe restrictions have been found for many alternative theories.

In addition to the Hulse-Taylor pulsar, there are some other binary pulsars that deserve special attention.

- In 2003, Marta Burgay et al. found a *double pulsar*, PSR J0737-3039A and PSR J0737-3039B, i.e., a binary system in which both stars are pulsars. This allows for even more precise tests of general relativity. Pulsar A has a period of 23 Milliseconds, pulsar B of 2.8 seconds. The masses are $M_A = 1.34 M_\odot$ and $M_B = 1.25 M_\odot$. The period is only 2.4 hours. Correspondingly, the separation of the two stars is even smaller than for PSR B1913+16 and its companion; the whole system would fit within the Sun. As the orbital plane is seen almost edge-on, there are eclipses. The apparent irregularity of the eclipses caused a puzzle for a while.

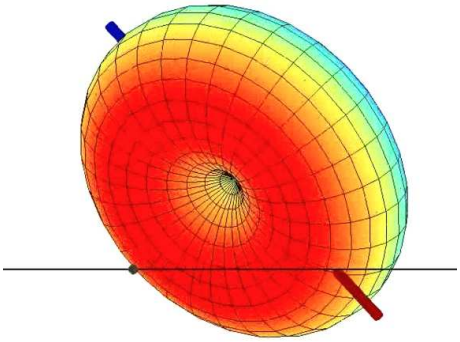


Fig. 5.11: From Breton et al., loc. cit.

A model that could solve this puzzle was brought forward by R. Breton, V. Kaspi, M. McLaughlin, M. Lyutikov, M. Kramer, I. Stairs, S. Ransom, R. Ferdman, F. Camilo and A. Possenti ["The double pulsar eclipses. I. Phenomenology and multi-frequency analysis" Astrophys. J. 747, 89, (2012)]. According to this model, one of the pulsars is surrounded by a doughnut-shaped magnetosphere which, in the course of its rotation, sometimes eclipses the other pulsar. The picture in Fig. 5.11 is taken from the paper by Breton et al.

- In 2013, a magnetar (i.e., a neutron star with a very strong magnetic field) was found at an angular distance of only 3 arcseconds from the centre of our galaxy, PSR J1745-2900. Of course, in terms of the Schwarzschild radius of the supermassive black hole at the centre of our galaxy, this is still a fairly large distance; the Schwarzschild radius corresponds to about 10 microarcseconds. Therefore, there is not a strong gravitational coupling of this magnetar to the centre. Finding a pulsar that is in a close orbit around a black hole is considered as the Holy Grail of pulsar research.

- In 2014, a ternary pulsar was discovered, PSR J0337+1715. Both companions are white dwarfs. Already in the late 1990s a pulsar in a triple system had been found, but the separations were quite large with orbital periods of several decades. The newly found system is much closer so that it is a much more promising candidate for additional tests of general relativity.

6. Gravitational wave detectors

In the preceding chapter we have discussed the generation of gravitational waves. As the most important results, we have found that, to within certain approximations, the gravitational field in the far zone is determined by the second time-derivative of the (mass) quadrupole tensor of the source and that the radiated power is determined by the third time-derivative of this quadrupole tensor. In this chapter we will now introduce various types of gravitational wave detectors that have been conceived and we will discuss what are the chances that actually gravitational waves will be observed with them.

6.1 Resonant bar detectors

Resonant bar detectors are vibrating systems in which a gravitational wave would excite a resonant oscillation. The idea was brought forward in 1960 by Joseph Weber ["Detection and generation of gravitational waves", Phys. Rev. 117, 306 (1960)]. A few years later, the first resonant bar detectors constructed by Weber went into operation. Some more sophisticated resonant bar detectors are still in use.

To explain the basic idea, we begin by considering the simplest vibrating system that can be used as a gravitational wave detector, namely two masses connected by a spring. This simple example is also treated in the first part of Weber's 1960 paper and it is dicussed in fairly great detail in the book by Misner, Thorne and Wheeler.

We have to recall some of our earlier results. In Worksheet 2 we derived a differential equation for the motion of freely falling particles under the influence of a gravitational wave,

$$\frac{d^2y^\ell(t)}{c^2dt^2} = R^\ell{}_{0k0}\big(ct,\vec{0}\,\big)y^k(t)\;, \qquad \text{(J)}$$

where the curvature tensor can be expressed as

$$R^\ell{}_{0k0}\big(ct,\vec{0}\,\big) = \frac{1}{2}\partial_0^2\gamma^\ell{}_k\big(ct,\vec{0}\,\big)\;.$$

Here $\gamma^\ell{}_k$ is a plane-harmonic gravitational wave in the TT gauge, with the four-velocity u^μ of the chosen observer tangent to the x^0-lines. The coordinates y^k are chosen such that the freely falling particle at $y^k(t)$ has distance $\sqrt{y^k(t)y_k(t)}$ from the freely falling particle at the origin. The differential equation is linearised with respect to $y^k(t)$, i.e., it is valid only as long as this quantity is sufficiently small.

(J) is a version of the Jacobi equation (or equation of geodesic deviation). If looked at with Newtonian eyes, the right-hand side of (J) is to be interpreted as the gravitational force. The solutions to (J) give, for $\gamma^\ell{}_k$ either a plus mode or a cross mode, the familiar patterns from p.15.

We will now consider a particle with mass m that is acted on by an additional (i.e., non-gravitational) force $f^\ell(t)$. Then we have to replace (J) with the equation of motion

$$\frac{d^2y^\ell(t)}{dt^2} = c^2R^\ell{}_{0k0}\big(ct,\vec{0}\,\big)y^k(t) + \frac{1}{m}f^\ell(t)\;.$$

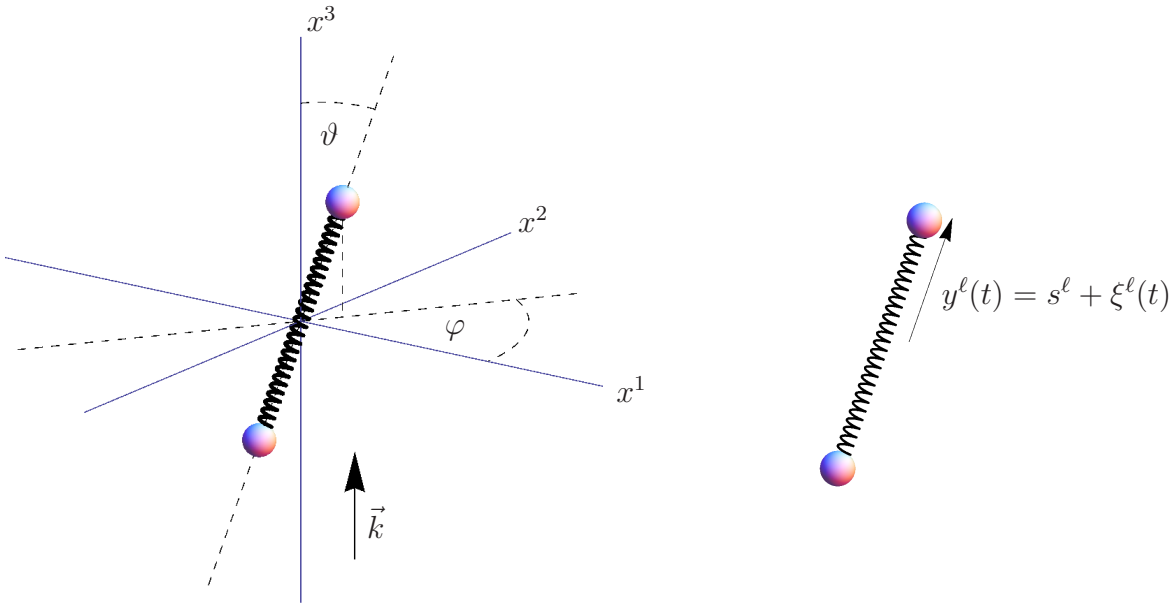


Fig. 6.1: gravitational wave propagating in x^3 direction excites oscillations in a spring system

For a system of two masses with $m_1 = m_2 = m$ connected by a spring, the position $y^\ell(t)$ of mass m_1 satisfies this equation with

$$y^\ell(t) = s^\ell + \xi^\ell(t)\,, \qquad f^\ell(t) = -k\,\xi^\ell(t) - \gamma\frac{d\xi^\ell(t)}{dt}\,,$$

see Fig. 6.1. Here s^ℓ gives the position of m_1 in the equilibrium state, $-k\,\xi^\ell(t)$ is the restoring force with a spring constant k, and $-\gamma d\xi^\ell(t)/dt$ is the damping force with a damping constant γ. The equation of motion reads

$$\frac{d^2\xi^\ell(t)}{dt^2} = c^2 R^\ell{}_{0k0}\big(ct,\vec{0}\,\big)\big(s^k + \xi^k(t)\big) - \frac{k}{m}\xi^\ell(t) - \frac{\gamma}{m}\frac{d\xi^\ell(t)}{dt}\,.$$

If the elongation of the spring from the equilibrium state is small, we can neglect $\xi^k(t)$ in comparison to s^k, i.e.

$$\frac{d^2\xi^\ell(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi^\ell(t)}{dt} + \frac{k}{m}\xi^\ell(t) = c^2 R^\ell{}_{0k0}\big(ct,\vec{0}\,\big)s^k\,.$$

As given above, we can express the curvature tensor by the second derivative of the $\gamma^\ell{}_k$. With

$$\gamma^\ell{}_k\big(ct,\vec{r}\,\big) = \mathrm{Re}\big\{A^\ell{}_k \mathsf{e}^{i(\vec{k}\cdot\vec{r}-\omega t)}\big\}\,,$$

$$\partial_0^2\gamma^\ell{}_k\big(ct,\vec{r}\,\big) = \frac{1}{c^2}\mathrm{Re}\big\{-\omega^2 A^\ell{}_k \mathsf{e}^{i(\vec{k}\cdot\vec{r}-\omega t)}\big\}\,,$$

$$c^2 R^\ell{}_{0k0}\big(ct,\vec{0}\,\big) = \frac{1}{2}\partial_0^2\gamma^\ell{}_k\big(ct,\vec{0}\,\big) = -\frac{\omega^2}{2}\mathrm{Re}\big\{A^\ell{}_k \mathsf{e}^{-i\omega t}\big\}\,.$$

If we assume that the masses at the ends of the spring can be displaced only in the longitudinal direction of the spring, we have

$$\xi^\ell(t) = \xi(t)\frac{s^\ell}{s}\,,$$

where, according to Fig. 6.1,

$$\big(s^\ell\big) = s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\,, \qquad \big(\xi^\ell(t)\big) = \xi(t)\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\,.$$

Then the equation of motion reads

$$\frac{s^\ell}{s}\Big(\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t)\Big) = -\frac{\omega^2}{2}\mathrm{Re}\big\{A^\ell{}_k s^k \mathsf{e}^{-i\omega t}\big\}\,,$$

or, after multiplication with s_ℓ/s,

$$\frac{\cancel{s^\ell s_\ell}}{\cancel{s^2}}\Big(\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t)\Big) = -\frac{\omega^2}{2}\mathrm{Re}\big\{A^\ell{}_k s^k \frac{s_\ell}{s}\mathsf{e}^{-i\omega t}\big\}\,.$$

We evaluate the right-hand side for a pure plus mode. The gravitational wave is assumed to propagate in the x^3 direction, as indicated in Fig. 6.1 by the wave vector $\vec{k}$. We find

$$\big(A^\ell{}_k\big) = \begin{pmatrix}A_+ & 0 & 0\\ 0 & -A_+ & 0\\ 0 & 0 & 0\end{pmatrix}\,,$$

$$s_\ell A^\ell{}_k s^k = s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}A_+ & 0 & 0\\ 0 & -A_+ & 0\\ 0 & 0 & 0\end{pmatrix} s\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

$$= s^2\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}A_+\cos\varphi\sin\vartheta\\ -A_+\sin\varphi\sin\vartheta\\ 0\end{pmatrix} = s^2A_+\big(\cos^2\varphi - \sin^2\varphi\big)\sin^2\vartheta = s^2A_+\cos(2\varphi)\sin^2\vartheta\,.$$

This gives us the equation of motion in its final form,

$$\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t) = -\frac{s\omega^2}{2}\cos(2\varphi)\sin^2\vartheta\,\mathrm{Re}\big\{A_+\mathsf{e}^{-i\omega t}\big\}\,,$$

which is the equation of a one-dimensional damped harmonic oscillator with a driving force.

Solving this equation is an elementary text-book matter. The general solution to the inhomogeneous ODE is the general solution of the homogeneous ODE plus a particular solution to the inhomogeneous ODE.

To solve the homogeneous ODE,

$$\frac{d^2\xi(t)}{dt^2} + \frac{\gamma}{m}\frac{d\xi(t)}{dt} + \frac{k}{m}\xi(t) = 0\,,$$

we insert the ansatz

$$\xi(t) = C\,\mathsf{e}^{\kappa t}\,.$$

This gives us

$$C\,\mathsf{e}^{\kappa t}\Big(\kappa^2+\frac{\gamma\,\kappa}{m}+\frac{k\,\kappa}{m}\Big)=0\,,$$

hence

$$\kappa_{1/2}=-\frac{\gamma}{2m}\pm\sqrt{\frac{\gamma^2}{4m^2}-\frac{k}{m}}\,.$$

As long as the damping is undercritical,

$$0<\frac{k}{m}-\frac{\gamma^2}{4m^2}=:\omega_0^2\,,$$

we have

$$\kappa_{1/2}=-\frac{\gamma}{2m}\pm i\,\omega_0\,,$$

and the general solution to the homogeneous equation is

$$\xi_{\rm hom}=C_1\mathsf{e}^{\kappa_1 t}+C_2\mathsf{e}^{\kappa_2 t}=\mathsf{e}^{-\gamma t/(2m)}\big(C_1e^{i\omega_0 t}+C_2e^{-i\omega_0 t}\big)\,.$$

C_1 and C_2 are determined by initial conditions. Whatever the initial conditions are, the solution dies down in the course of time.

We have now to find one particular solution to the inhomogeneous equation

$$\frac{d^2\xi(t)}{dt^2}+\frac{\gamma}{m}\frac{d\xi(t)}{dt}+\frac{k}{m}\xi(t)=-\frac{s\,\omega^2}{2}\cos(2\varphi)\sin^2\vartheta\,\mathrm{Re}\big\{A_+\,\mathsf{e}^{-i\omega t}\big\}\,.$$

With the ansatz

$$\xi(t)=\mathrm{Re}\big\{a\,\mathsf{e}^{-i\omega t}\big\}$$

we get

$$\mathrm{Re}\Big\{a\,\mathsf{e}^{-i\omega t}\Big(-\omega^2-\frac{i\omega\gamma}{m}+\frac{k}{m}\Big)\Big\}=-\frac{s\,\omega^2}{2}\cos(2\varphi)\sin^2\vartheta\,\mathrm{Re}\big\{A_+\,\mathsf{e}^{-i\omega t}\big\}\,,$$

$$\mathrm{Re}\Big\{\mathsf{e}^{-i\omega t}\left(a\Big(\omega^2-\frac{k}{m}+\frac{i\,\omega\,\gamma}{m}\Big)-\frac{s\,\omega^2 A_+}{2}\cos(2\varphi)\sin^2\vartheta\right)\Big\}=0\,,$$

$$a=\frac{s\,\omega^2 A_+\cos(2\varphi)\sin^2\vartheta}{\omega^2-\dfrac{k}{m}+\dfrac{i\,\omega\,\gamma}{m}}\,.$$

Therefore, if we wait until the solution to the homogeneous equation has died down, the oscillation of our spring system driven by the gravitational wave is given by

$$\xi(t)=\mathrm{Re}\big\{a\,\mathsf{e}^{-i\omega t}\big\}=s\,\omega^2\cos(2\varphi)\sin^2\vartheta\,\mathrm{Re}\Big\{\frac{A_+\,\mathsf{e}^{-i\omega t}}{\omega^2-\dfrac{k}{m}+\dfrac{i\,\omega\,\gamma}{m}}\Big\}\,.$$

The amplitude

$$\big|a\big|=\frac{s\,\big|A_+\big|\,\big|\cos(2\varphi)\big|\sin^2\vartheta\,\omega^2}{\Big|\omega^2-\dfrac{k}{m}+\dfrac{i\,\omega\,\gamma}{m}\Big|}=\frac{s\,\big|A_+\big|\,\big|\cos(2\varphi)\big|\sin^2\vartheta\,\omega^2}{\sqrt{\Big(\omega^2-\dfrac{k}{m}\Big)^2+\dfrac{\omega^2\,\gamma^2}{m^2}}}$$

takes, as a function of ω, its maximum at the *resonance frequency*

$$\omega_{\rm res}=\frac{k}{m}\Big(\frac{k}{m}-\frac{\gamma^2}{4m^2}\Big)^{-1/2}\,.$$

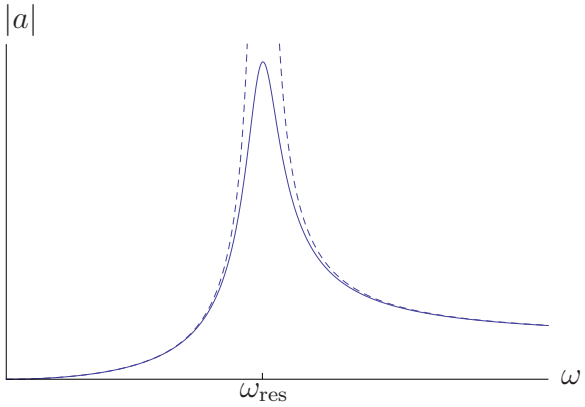


Fig. 6.2: dependence of the amplitude on the frequency of the gravitational wave

In the case of vanishing damping, $\gamma = 0$, the amplitude is even infinite at $\omega_{\text{res}} = \sqrt{k/m}$, see dashed curve in Fig. 6.2.

The optimal orientation of the spring is transverse to the direction of the incoming gravitational wave, $\sin^2\vartheta = 1$. In the case of longitudinal orientation, $\sin^2\vartheta = 0$, the amplitude is zero. With respect to the φ dependence, which gives the orientation in the plane perpendicular to the propagation direction of the wave, there is not only a 2π periodicity but even a π periodicity. This reflects the fact that the (linearised) gravitational field has spin 2, recall Worksheet 3.

We have used the spring system to explain the basic idea of how to use vibrating systems for detecting gravitational waves. The resonant bar detectors which were built by Weber and others are based on the same idea. However, instead of masses connected by a spring one uses elastic solids, traditionally with a cylindrical shape.

In this case, $y^\ell(t)$ denotes the position vector of an arbitray mass element of the solid with respect to a body-fixed reference point. Again, we write $y^\ell(t) = s^\ell + \xi^\ell(t)$ where s^ℓ gives the position in equilibrium. One introduces a second rank tensor $\varepsilon^\ell{}_k(t)$ by the equation $\xi^\ell(t) = \varepsilon^\ell{}_k(t)s^k$. The antisymmetric part of $\varepsilon_{\ell k}(t)$ describes a rotation of the mass element, while the symmetric part describes expansion and shear. The symmetric part of $\varepsilon_{\ell k}(t)$ is known as the *strain tensor*. By assuming again a linear restoring force (i.e., Hooke's law now in the version of continuum mechanics) and a linear damping, one gets a differential equation for the strain tensor which is very similar to the damped oscillator equation for the spring system. As a consequence, a cylinder that is positioned transverse to a plane-harmonic gravitational wave undergoes periodic deformations as shown in Fig. 6.3.

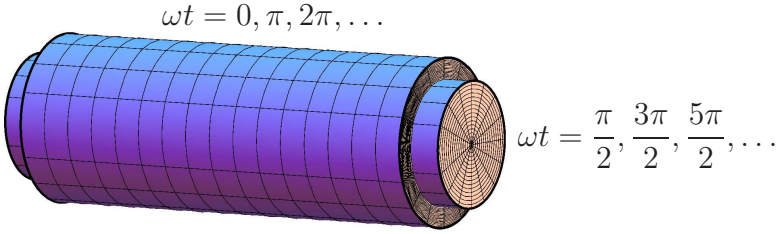


Fig. 6.3: oscillating Weber cylinder

Weber's first gravitational wave detector was an aluminium cylinder with a weight of 1.5 tons (150 centimeters long, 60 centimeters in diameter). The fundamental resonance frequency was at about 1660 Hertz. Quartz crystals glued to the surface were used for measuring the

deformations; as quartz crystals are piezoelectric, they transform a strain into a voltage which can be measured, see Fig. 6.4.

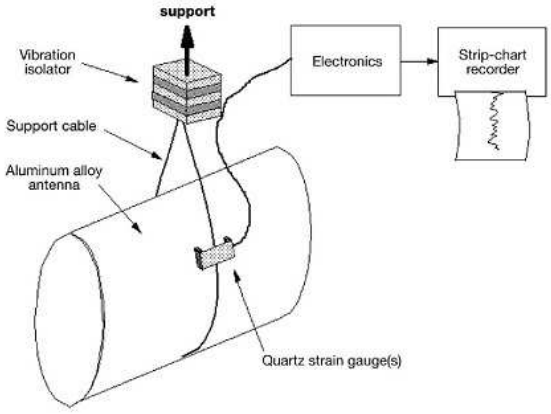


Fig. 6.4: from Levine, loc. cit.

The early attempts to measure gravitational waves with the help of resonant bar detectors are described in detail by J. Levine ["Early gravity-wave detection experiments", Phys. perspect. 6, 42 (2004)]. Fig. 6.5 shows Joe Weber working on a resonant bar detector.

Fig. 6.5: from physics.aps.org/

Weber operated his resonant bar detectors in pairs, searching for coincidences. In the beginning, he had two detectors on the campus of Maryland University, then he moved one of them to Chicago. There was even an attempt to station a (smaller) bar detector on the Moon with the Apollo 17 mission but the instrument malfunctioned. Weber claimed that he had found significant statistical evidence for coincident events which he thought to be gravitational wave signals. Nowadays there is agreement that his detectors were too crude to measure gravitational waves.

Joseph Weber died in the year 2000. By that time, attempts to detect gravitational waves had shifted to interferometric methods, see next section. However, there are still a few resonant bar detectors in operation. Fig. 6.6 shows the AURIGA instrument, near Pisa in Italy, which is a resonant bar detector of the traditional cylindrical shape.

Fig. 6.6: from www.auriga.lnl.infn.it/

Fig. 6.7 shows the MiniGRAIL instrument at the Kamerlingh Onnes Institute in Leiden, Netherlands. It has a spherical shape, so it can detect gravitational waves from all spatial directions. There is a similar instrument, named after the late physicist Mario Schenberg, in Brazil.

Fig. 6.7: from www.minigrail.nl/

While in the beginning Weber did his observations at room temperature, all modern resonant bar detectors are operated at a temperature of a few millikelvins to reduce thermal noise. They can detect waves only in a narrow frequency band around the resonance frequency which is above or slightly below 1 kHz. Gravitational waves from the Hulse-Taylor pulsar, e.g., cannot be detected with these instruments; this is not only because their sensitivity is too low but also because they cannot detect gravitational waves of such a low frequency. (The revolution period in the Hulse-Taylor binary is 7.75 hours, i.e. the frequency of the resulting gravitational waves is $2\Omega = 4\pi/T \approx 0.00045$ Hz.) Spinning bumpy neutron stars could produce gravitational waves with a frequency close to 1 kHz, but their amplitude would probably be too low for being detected with resonant bar detectors. Therefore, the search with such instruments concentrates on burst sources.

6.2 Interferometric gravitational wave detectors

With the help of a Michelson interferometer, tiny distance changes can be measured. The idea to use this well-known fact for the detection of gravitational waves came up in the early 1960s. The first published paper on the subject was by M. Gerstenshtein and V. Pustovoit ["On the detection of low-frequency gravitational waves"(in Russian), Sov. Phys. JETP 16, 433 (1962)]. The idea was strongly supported by V. Braginsky. However, concrete plans to build such gravitational wave detectors came up only in the 1970s. J. Forward actually built a small model detector in the mid-1970s. The construction of big instruments (LIGO, Geo600, VIRGO etc., see below) started in the 1990s. Many people were instrumental, among them R. Weiss and R. Drever (who were awarded the Einstein prize in 2007) and K. Thorne (who was awarded the Einstein medal in 2009).

For understanding the basic idea of how an interferometric gravitational wave detector works, we have to recall what a Michelsopn interferometer is, see Fig. 6.8.

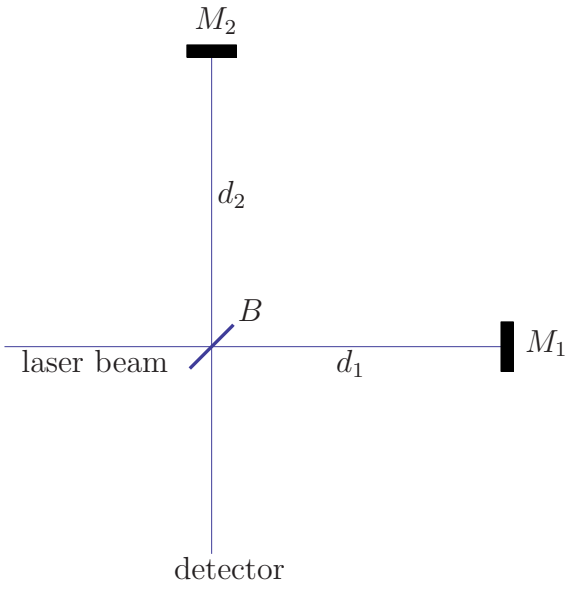


Fig. 6.8: Michelson interferometer

A laser beam is sent through the beam splitter B. One beam is reflected at mirror M_1, the other one at mirror M_2. When arriving at the detector the two beams have a phase difference that can be observed in terms of an interference pattern. If the instrument is operated in vacuo, the phase difference is

$$\Delta\phi = \frac{2\pi}{\lambda}\, 2\,(d_1 - d_2)$$

where λ is the wave length of the laser. As sophisticated Michelson interferometers can measure phase differences down to 10^{-5}, this is a method to detect changes in the distance $d_1 - d_2$ that are considerably smaller than the wavelength λ. If the Michelson interferometer is operated with visible light, the latter is about 600 nanometers.

To use this device as a gravitational wave detector, we think of the beam splitter B, the mirror M_1 and the mirror M_2 as being suspended with the help of files in such a way that they can move freely in the plane of the interferometer. For their motion in this plane, we can thus use

the equation of motion for freely falling particles. Under the influence of a gravitational wave whose propagation direction is orthogonal to the plane of the interferometer, they will move according to the patterns of p.15. Here we should identify the beam splitter with the particle at the centre of the coordinate system, and the mirrors M_1 and M_2 with particles on the x^1 axis and on the x^2 axis, respectively. For determining the time-dependence of the distances d_1 and d_2, and thus of the phase difference, we use our results from Worksheet 2. The distance from the origin of a particle with coordinates x^i was given as

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \gamma^k{}_j \delta_{k\ell} x^j x^\ell = \delta_{k\ell} x^k x^\ell + \mathrm{Re}\big\{ A^k{}_j \mathsf{e}^{-i\omega t} \big\} \delta_{k\ell} x^j x^\ell \; .$$

If we consider, for simplicity, a pure plus mode, this simplifies to

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \mathrm{Re}\big\{ A_+ \big((x^1)^2 - (x^2)^2 \big) \mathsf{e}^{-i\omega t} \big\}$$

and, with $A_+ = \big|A_+\big| \mathsf{e}^{i\varphi}$, to

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \big|A_+\big| \, \big((x^1)^2 - (x^2)^2 \big) \cos\big(\omega t - \varphi\big) \; .$$

For the mirror M_1, we have $x^2 = x^3 = 0$, hence

$$d_1(t)^2 = (x^1)^2 + \big|A_+\big| \, (x^1)^2 \cos\big(\omega t - \varphi\big)$$

and for the mirror M_2, we have $x^1 = x^3 = 0$, hence

$$d_2(t)^2 = (x^2)^2 - \big|A_+\big| \, (x^2)^2 \cos\big(\omega t - \varphi\big) \; .$$

If we assume that in the unperturbed state both arms have the same length d_0, we can write

$$d_1(t)^2 = d_0^2 \Big(1 + \big|A_+\big| \cos\big(\omega t - \varphi\big) \Big) \; ,$$

$$d_2(t)^2 = d_0^2 \Big(1 - \big|A_+\big| \cos\big(\omega t - \varphi\big) \Big) \; .$$

As a consequence, the phase difference reads

$$\Delta\phi(t) = \frac{4\pi}{\lambda} \Big(d_1(t) - d_2(t) \Big) = \frac{4\pi}{\lambda} \, d_0 \, \left(\sqrt{1 + \big|A_+\big| \cos\big(\omega t - \varphi\big)} - \sqrt{1 - \big|A_+\big| \cos\big(\omega t - \varphi\big)} \right)$$

which, according to our general agreement to linearise all expressions with respect to the gravitational wave, simplifies to

$$\begin{aligned} \Delta\phi(t) &= \frac{4\pi}{\lambda} \, d_0 \, \Big(1 + \frac{1}{2} \big|A_+\big| \cos\big(\omega t - \varphi\big) - 1 + \frac{1}{2} \big|A_+\big| \cos\big(\omega t - \varphi\big) + \dots \Big) \\ &= \frac{4\pi}{\lambda} \, d_0 \, \big|A_+\big| \cos\big(\omega t - \varphi\big) \; . \end{aligned}$$

Clearly, the phase difference is proportional to the amplitude $|A_+|$ of the incoming gravitational wave. It is also proportional to the armlength d_0 of the interferometer. This is the reason why gravitational wave detectors need a long armlength, several hundred meters at least. As always with Michelson interferometers, the phase difference is proportional to the inverse of the wave length λ of the laser. λ is not to be confused with the wave length of the gravitational wave. The frequency ω of the gravitational wave enters into the formula for the phase shift only insofar as it gives the periodicity with which the interference pattern changes. In contrast to the resonant bar detectors, interferometric detectors are not restricted to a narrow frequency band. The observable frequency ω is mainly limited by seismic noise which, for ground-based interferometric detectors, will render gravitational wave signals of less than 1 Hz practically unobservable.

In addition to the noise produced by seismic vibrations and by a (time-dependent) gradient of the gravitational acceleration, resulting from the fact that the Earth is not a perfect homogeneous sphere, there are several other sources of noise. Thermal noise has the effect that interferometric gravitational wave detectors have to be cooled down if they are to operate below $\approx 30\,\mathrm{Hz}$. The existing detectors (TAMA300, GEO600, LIGO, VIRGO, see below) operate at room temperature, but the next generation of detectors will use cryogenic techniques to reach lower frequencies. At the upper end of the frequency band, quantum noise plays a major role. The elementary theoretical explanation of how an interferometer works is based on a classical wave theory of light. If it is taken into account that, actually, light consists of quantum particles (photons), deviations from the classical interference patterns occur. Roughly speaking, the mirrors in the interferometer are hit not by a classical wave but rather by a stream of photons, similar to a stream of pellets from a shot gun. The resulting deviations from the classical interference pattern are known as *shot noise*. These deviations are small if the laser beam consists of many photons, i.e., if the laser power is high. Noise resulting from the quantum nature of light restricts the existing interferometric wave detectors to frequencies below $\approx 10\,\mathrm{kHz}$.

We now give a brief overview on the existing and planned interferometric gravitational wave detectors. The first small model detector of this type was built by J. Forward in Malibu, USA, in 1970. This was followed by a number of similar detectors at a laboratory scale, too small to actually detect gravitational waves but useful for testing the technology, e.g. in Garching, Germany, and in Glasgow, UK. In the mid-nineties the construction of detectors with an armlength of at least a few hundred meters began. In chronological order of the date when they became operational, these are the following.

TAMA300: This is a detector of 300 m arm length, located at the Mitaka Campus in Tokyo, Japan. It became operational in 1999. As a comparatively small instrument its main purpose was to develop advanced technologies to be used in bigger detectors.

GEO600: This is a German-British project, originally planned to be realised near Munich. Finally, the detector was built near Ruthe near Sarstedt near Hannover in the middle of nowhere in Northern Germany. It became operational in 2001.

Fig. 6.9: from http://www.questhannover.de

The design is quite inconspicuous. In Fig. 6.9. we see the two vacuum tubes around the two arms of the interferometer, each of which has a length of 600 m. The two tubes meet at the main building.

Fig. 6.10: from http://www.2physics.com

The main building hosts the laser, the beam splitter and several additional mirrors, e.g. for power recyling and for mode cleaning, each in a vacuum container. In Fig. 6.10 we can see these vacuum containers from the outside. Fig. 6.11 gives an inside view of the container that houses the beam splitter.

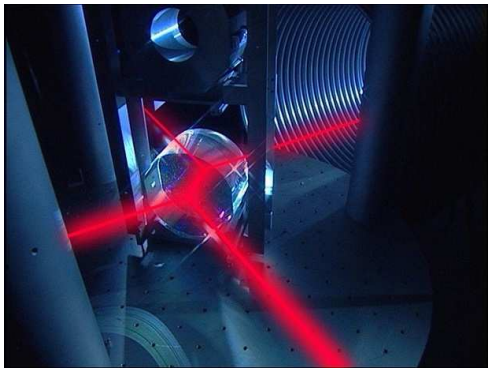


Fig. 6.11: from http://u-182-ls004.am10.uni-tuebingen.de

GEO600 is sensitive in the frequency band between 50 Hz and 1.5 kHz. It is operated with an Nd:Yag laser with an output power of 10 W at a wavelength of $\lambda = 1064\,\text{nm}$. With the help of power recycling, the laser power that is actually circulating in the interferometer is much bigger, namely ≈ 10 kW. Since 2011 GEO600 uses a second laser that produces squeezed light for reducing quantum noise. This laser is seen in the foreground of Fig. 6.10. Squeezed light is light in a state that minimises Heisenberg's uncertainty relation in such a way that the uncertainty in space is very small while the uncertainty in Fourier space is correspondingly big. The reduction of quantum noise is achieved by feeding this squeezed light into the interferometer (from below in Fig. 6.8), in addition to the light from the main laser (which comes from the left in Fig. 6.8). GEO600 is sensitive enough to detect length changes $d_1 - d_2$ in the order of $10^{-18}\,\text{m}$. Recall for the sake of comparison that the diameter of a proton is about $10^{-15}\,\text{m}$. In contrast to other existing interferometric gravitational wave detectors, GEO600 has no Fabry-Perrot cavities in the arms. While LIGO and VIRGO are out of operation at present, undergoing upgrades to Advanced LIGO and Advanced VIRGO respectively, GEO600 is operational. It will be upgraded, afterwards, to GEO-HF.

LIGO: There are two LIGO sites, one in Hanford, Washington, USA, and one in Livingston, Louisiana, USA. At each site there is an interferometer with 4 km arm length. At the Hanford site there is a second interferometer with 2 km arm length in the same vacuum tube.

Fig. 6.12: from http://www.mpa-garching.mpg.de

LIGO went operational in 2002. The Livingston site, situated in the swamps of Louisiana, is shown in Fig. 6.12. The vacuum tubes of 4 km length and 1.2 m diameter are the biggest existing ultra-high vacua. LIGO operates in the range between 30 Hz and 7 kHz. Having two smilar instruments working in parallel allows searching for coincident events. The data of LIGO and GEO600 are pooled and analysed jointly. The data analysis team is known as the LIGO Scientific Collaboration (LSC). Amateurs are included in the data analysis. Within the Einstein@home project, everybody is invited to provide his or her computer for analysing scientic data. Einstein@home was already very succesful in analysing data from the radio telescopes at Arecibo and at Green Banks; more than a dozen new pulsars were found by amateurs. Until now there was no spectacular success in the search for gravitational wave signals in the LIGO/GEO600 data, but maybe we just have to wait for a nearby burst source. At present, LIGO is not operational. After being upgraded, it will be back under the name of Adanced LIGO. Also, a third detector of the Advanced-LIGO type is going to be built in India.

VIRGO: This is an Italian-French gravitational wave detector at Cascina near Pisa in Italy that became operational in 2007. The geometrical arm length is 3 km, but by folding the laser beams the effective arm length can be extended up to 100 km. VIRGO is located within the site of the European Gravitational Observatory (EGO), see Fig. 6.13. It is operated at frequencies between 10 Hz and 10 kHz. At present, VIRGO is shut down. Similar to LIGO, it will return after being upgraded under the name of Advanced VIRGO.

Fig. 6.13: from http://www.ego-gw.it

There are plans for some other ground-based interferometric gravitational wave detectors.

KAGRA: The original name of this Japanese project was LCGT (Large Scale Cryogenic Gravitational Wave Telescope). As suggested by the C in the name, it is a detector that will use cryogenic materials such that it can be operated at low temperatures. The instrument is to be built in tunnels of the Kamioka mine, with an arm length of 3 km. It is planned to become operational in 2018.

Einstein Telescope: This is a joint project of eight European institutions, including the Albert Einstein Institute in Hannover, Germany. At the moment it is unclear if, when and where the project will be realised.

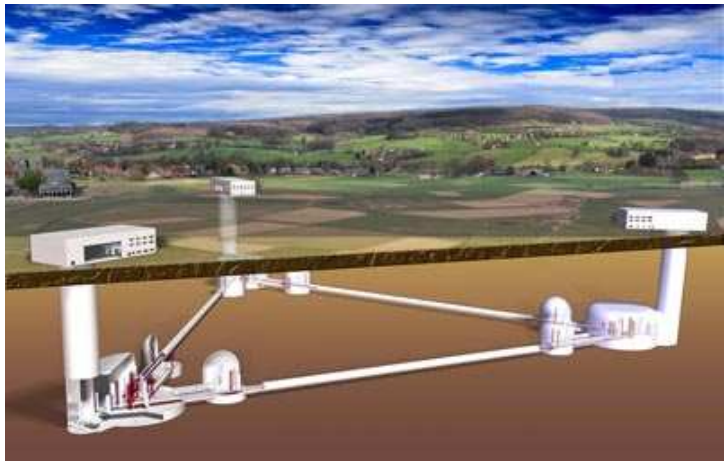


Fig. 6.14: from http://physicsworld.com

Similarly to KAGRA, it will be an underground detector (at a depth of 100−200 m), see Fig. 6.14, and it will use cryogenic materials for low thermal noise.

The sensitivity of existing and planned ground-based interferometric gravitational wave detectors is shown in Fig. 6.15. The resonant bar detector AURIGA is included for the sake of comparison. The picture is taken from S. Hild, Class. Quantum Grav. 29, 124006 (2012).

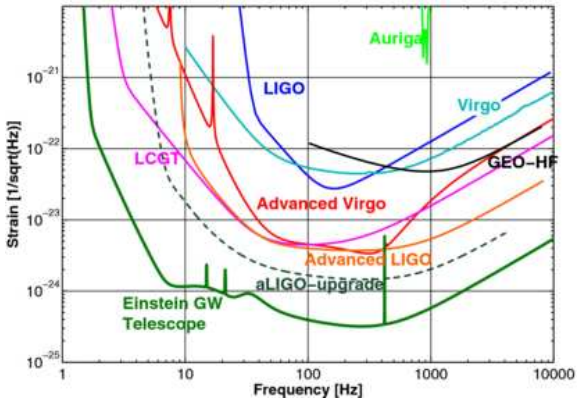


Fig. 6.15: from S. Hild, loc. cit.

As mentioned above, ground-based interferometers are limited to frequencies above 1 Hz, because of seismic noise. Therefore, e.g. gravitatonal waves emitted by the Hulse-Taylor pulsar (with a frequency of less than $\approx 10^{-4}$ Hz) or by similar binary pulsars are outside of the range of such detectors. There are plans for space-based interferometric gravitational wave detectors that could overcome this limit. They include the following.

eLISA: This is a long-standing project, designed already in the 1990s, for a space-based interferometric detector. The original name of the project was LISA (Laser Interferometer Space Antenna), and it was planned as a joint project of NASA and ESA.

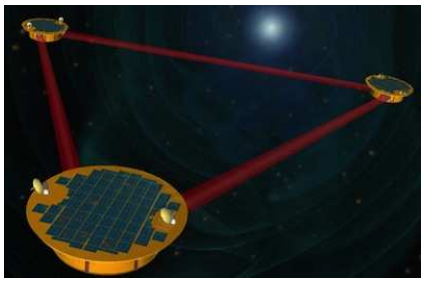


Fig. 6.16: from http://lisa.nasa.gov

In this original version, LISA should consist of three satellites, see Fig. 6.16, arranged in an equilateral triangle with a side length of 5 million kilometers. (That's about 12 times the separation of the Earth and the Moon.) This triangular array was supposed to fly along the orbit of the Earth around the Sun, trailing the Earth by 20 degrees. The inclination of the plane of the triangle with respect to the ecliptic was planned to be 60 degress. Each of the three satellites was to host two laser sources and two test masses, so that from each satellite a laser beam could be sent to a test mass in either of the two others. As it is impossible to receive a reflected laser beam with a measurable intensity over a distance of 5 million kilometers, it was planned that each satellite should host two transponders which would send back, after receiving a laser beam from a partner satellite, coherently a laser beam with the same frequency. In 2011, NASA stopped funding for LISA. Since then, it is a European-only project. Under the name NGO (New Gravitational wave Observatory) it entered into ESA's L1 mission selection, together with two competitors: The Jupiter Icy Moon Explorer (JUICE) and the X-ray observatory ATHENA. The winner was JUICE. NGO was re-designed and was elected as an L3 mission under the name eLISA (evolved LISA).

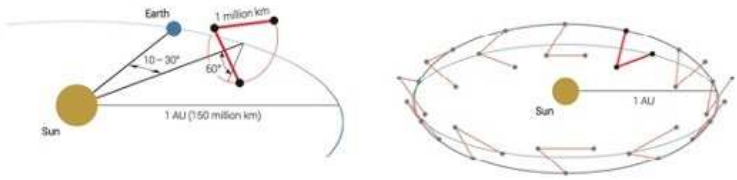


Fig. 6.17: http://www.dlr.de

A tentative launch date for eLISA is 2034. It is now planned as a system of a mother spacecraft with two daughter spacecraft. The mother emits laser beams that are sent back from transponders on board the daughters. There is no laser beam between the two daughters. The separation between the spacecraft has been down-sized to 1 million kilometers, see Fig. 6.17. eLISA would be sensitive in the range between 0.1 mHz und 1 Hz where ground-based detectors cannot operate.

As a preparation for the (e)LISA mission, a spacecraft called LISA Pathfinder has been constructed, see Fig. 6.17. It is waiting for being launched in 2015. It houses laser and test masses at a separation of $\approx 40\,\mathrm{cm}$ in one spacecraft. The main purpose of the project is to test the technology for eLISA (drag-free control, transponders for laser beams, etc.) under space conditions. A model of LISA Pathfinder was on display at the ILA in Berlin, May 2014. There is also a plan to use the LISA Pathfinder spacecraft, after its main mission is completed, for a test of MOND (MOdified Newtonian Dynamics), an alternative theory of gravity.

Fig. 6.18: from http://news.softpedia.com

DECIGO: The acronym stands for DECI-Hertz Interferometer Gravitational wave Observatory. It is a proposed Japanese space-based instrument. The name refers to the fact that this detector is planned to operate in the frequency range between 0.1 Hz and 10 Hz (a decihertz). At present, it is unclear if and when this project will be actually realised.

BBO: The Big Bang Observatory is a far-future project that has been suggested by physicists from the USA. As the name suggests, its main goal is the detection of gravitational waves that came into existence shortly after the big bang. The proposed instrument consists of 12 spacecraft, arranged into 4 LISA-type triangular patterns. It is written in the stars if BBO wil ever fly.

6.3 Doppler tracking of satellites

Resonant bar detectors and interferometric detectors are instruments that are constructed for the sole purpose of detecting gravitational waves. In particular the advanced interferometric detectors are rather expensive instruments. In this subsection we discuss a method of searching for gravitational waves that is much cheaper because it is not necessary to build new equipment. The method uses spacecraft which have been launched for some other purpose, in particular spacecraft investigating the outer parts of our Solar system like Voyager, Pioneer 10, Pioneer 11 and Cassini.

The path of such a spacecraft is routinely monitored with the help of Doppler tracking. The idea is to search in the Doppler tracking data for signatures of gravitational waves. Doppler tracking works in the following way. From the Earth, a radio wave signal is sent to the spacecraft which is highly monochromatic with a stable frequency ν_{em}. On board the spacecraft, a transponder

receives the signal with a frequency $\nu_{\rm rec}$ and coherently sends back to the Earth a signal with the same frequency $\nu_{\rm rec}$. The station on the Earth measures the freqency $\nu'_{\rm em}$ with which the signal arrives. The frequency ratio $\nu'_{\rm em}/\nu_{\rm em}$ is different from one for the following reasons. First, the motion of the spacecraft relative to the Earth causes a Doppler shift which is well understood on the basis of special relativity. Second, the gravitational field of the Sun produces a gravitational frequency shift which is also well understood. If we assume that a gravitational wave is sweeping over our Solar system, this would produce an additional frequency shift.

The idea of using Doppler tracking data for detecting gravitational waves came up in the early 1970s. The mathematical formalism was worked out by F. Estabrook and H. Wahlquist [Gen. Rel. Grav. 6, 439 (1975)]. A comprehensive overview of the method can be found in the Living Review by J. Armstrong ["Low-frequency gravitational wave searches using spacecraft Doppler tracking", Living Rev. Relativity 9, (2006), `http://www.livingreviews.org/lrr-2006-1`.]

In the following we calculate the effect of a gravitational wave on Doppler tracking data under highly idealised assumptions. We ignore the motion of the spacecraft relative to the Earth and the effect of the gravitational field of the Sun, i.e., we only calculate the effect of the gravitational wave that comes on top of the well-understood Doppler shift and the gravitational frequency shift produced by the Sun. Also, we ignore the influence of the interplanetary medium on radio waves.

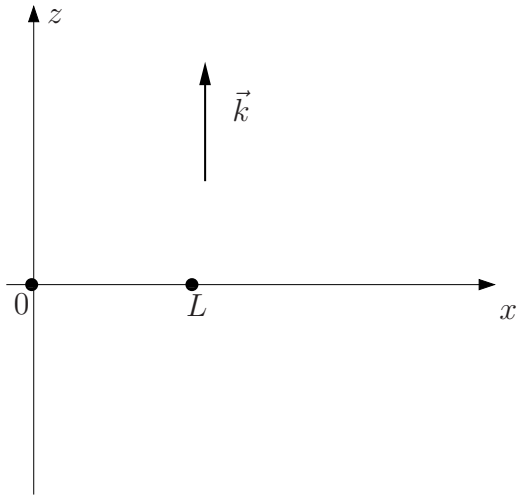


Fig. 6.19: Earth and spacecraft at rest in Minkowski background

The Earth and the spacecraft are assumed at rest in the Minkowski background at $(ct, 0, 0, 0)$ and $(ct, L, 0, 0)$, respectively, see Fig. 6.19. We treat the gravitational wave as plane-harmonic which is reasonable for a periodic source that is sufficiently far away. We work in the TT gauge and we restrict, for simplicity, to a pure plus mode that propagates in the z direction, see Fig. 6.19. This is the geometry which gives the maximal effect. The metric reads

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$$

with

$$\gamma_{\mu\nu}(ct, x, y, z) = \mathrm{Re}\big\{A_{\mu\nu} e^{i(kz-\omega t)}\big\}\,, \qquad \big(A_{\mu\nu}\big) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & 0 & 0 \\ 0 & 0 & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,.$$

Without loss of generality, we assume A_+ to be real. (This can be achieved by shifting the zero point on the time axis. Of course, if we had a superposition of different modes, we could make the amplitude real only for one of them.) Then

$$g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + dx^2 + dy^2 + dz^2 + A_+\cos(kz - \omega t)\left(dx^2 - dy^2\right) .$$

This demonstrates that for the emitter (Earth) and the receiver (satellite), whose worldlines are t lines, the time coordinate t coincides with proper time. Therefore, we can calculate the measured frequency shift in terms of t.

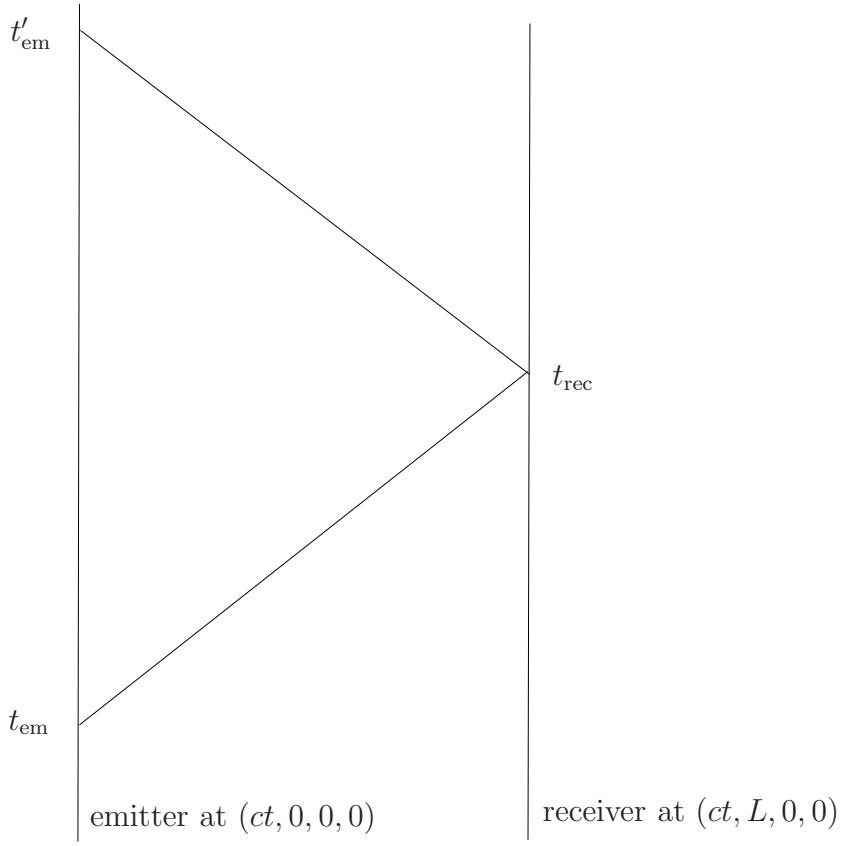


Fig. 6.20: worldine diagram of the emitter-receiver system

We assume that the emitter sends a light ray at $t_{\rm em}$ which is reflected at $t_{\rm rec}$ by the receiver and arrives back at $t'_{\rm em}$, see the worldline diagram in Fig. 6.20. As the light rays are travelling along the x axis, they satisfy

$$0 = -c^2dt^2 + dx^2 + A_+\cos(\omega t)dx^2 .$$

On the first leg, x is increasing, hence

$$dx \;=\; \frac{+\,c\,dt}{\sqrt{1 + A_+\cos(\omega t)}} .$$

In the linearised theory, to which all our calculations are restricted, this can be simplified to

$$\frac{dx}{c} \;=\; \Big(1 - \frac{A_+}{2}\cos(\omega t) + \ldots\Big)\,dt .$$

Integration over the first leg gives

$$\frac{L}{c} \;=\; \int_{t_{\rm em}}^{t_{\rm rec}} \Big(1 - \frac{A_+}{2}\cos(\omega t)\Big)\, dt\;. \qquad \text{(F1)}$$

For later purpose, we observe that this implies

$$T \;:=\; \frac{L}{c} \;=\; t_{\rm rec} - t_{\rm em} \;+\; O(A_+)\,, \qquad \text{(F2)}$$

Equation (F1) determines $t_{\rm rec}$ as a function of $t_{\rm em}$. Differentiating both sides of (F1) with respect to $t_{\rm em}$ results in

$$0 \;=\; \frac{dt_{\rm rec}}{dt_{\rm em}}\Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big)\Big) \;-\; \Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm em}\big)\Big)\;.$$

Linearisation with respect to A_+ yields

$$\frac{dt_{\rm rec}}{dt_{\rm em}} \;=\; \Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big)\Big)^{-1}\Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm em}\big)\Big) \;=\; 1 + \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big) - \frac{A_+}{2}\cos\big(\omega t_{\rm em}\big) + \ldots$$

and, with the help of (F2),

$$\frac{dt_{\rm rec}}{dt_{\rm em}} \;=\; 1 + \frac{A_+}{2}\cos\big(\omega(t_{\rm em}+T)\big) - \frac{A_+}{2}\cos\big(\omega t_{\rm em}\big) + \ldots$$

On the second leg, x is decreasing, hence

$$dx \;=\; \frac{-\,c\,dt}{\sqrt{1 + A_+\cos(\omega t)}} \;=\; -\,\Big(1 - \frac{A_+}{2}\cos(\omega t) + \ldots\Big)\, dt\;.$$

Integration over the second leg gives

$$-\,\frac{L}{c} \;=\; -\int_{t_{\rm rec}}^{t'_{\rm em}} \Big(1 - \frac{A_+}{2}\cos(\omega t)\Big)\, dt \qquad \text{(S1)}$$

which implies

$$T \;=\; \frac{L}{c} \;=\; t'_{\rm em} - t_{\rm rec} \;+\; O(A_+)\,. \qquad \text{(S2)}$$

Differentiating both sides of (S1) with respect to $t_{\rm em}$ results in

$$0 \;=\; \frac{dt'_{\rm em}}{dt_{\rm rec}}\Big(1 - \frac{A_+}{2}\cos\big(\omega t'_{\rm em}\big)\Big) \;-\; \Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big)\Big)\;.$$

Linearising this expression gives us

$$\begin{aligned}\frac{dt'_{\rm em}}{dt_{\rm rec}} \;&=\; \Big(1 - \frac{A_+}{2}\cos\big(\omega t'_{\rm em}\big)\Big)^{-1}\Big(1 - \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big)\Big) \\ &=\; 1 + \frac{A_+}{2}\cos\big(\omega t'_{\rm em}\big) - \frac{A_+}{2}\cos\big(\omega t_{\rm rec}\big) + \ldots\end{aligned}$$

and, with the help of (S2) and (S1),

$$\frac{dt'_{\rm em}}{dt_{\rm rec}} \;=\; 1 + \frac{A_+}{2}\cos\bigl(\omega(t_{\rm rec}+T)\bigr) - \frac{A_+}{2}\cos\bigl(\omega t_{\rm rec}\bigr) + \ldots$$

$$= 1 + \frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+2T)\bigr) - \frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+T)\bigr) + \ldots$$

The observed redshift is

$$z \;=\; \frac{\lambda'_{\rm em}-\lambda_{\rm em}}{\lambda_{\rm em}} \;=\; \frac{\nu_{\rm em}}{\nu'_{\rm em}} \;-\; 1 \;=\; \frac{dt'_{\rm em}}{dt_{\rm em}} \;-\; 1 \;=\; \frac{dt'_{\rm em}}{dt_{\rm rec}}\,\frac{dt_{\rm rec}}{dt_{\rm em}} \;-\; 1$$

$$= \Bigl(1+\frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+2T)\bigr)-\frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+T)\bigr)\Bigr)\Bigl(1+\frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+T)\bigr)-\frac{A_+}{2}\cos\bigl(\omega t_{\rm em}\bigr)\Bigr) - 1$$

$$= \cancel{1} + \frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+2T)\bigr) - \cancel{\frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+T)\bigr)} + \cancel{\frac{A_+}{2}\cos\bigl(\omega(t_{\rm em}+T)\bigr)} - \frac{A_+}{2}\cos\bigl(\omega t_{\rm em}\bigr)\Bigr) - \cancel{1}$$

$$= \frac{A_+}{2}\Bigl(\cos\bigl(\omega t_{\rm em}\bigr)\cos\bigl(2\omega T\bigr) - \sin\bigl(\omega t_{\rm em}\bigr)\sin\bigl(2\omega T\bigr) - \cos\bigl(\omega t_{\rm em}\bigr)\Bigr)$$

$$= \frac{A_+}{2}\Bigl(\cos\bigl(\omega t_{\rm em}\bigr)\bigl\{\cos^2\bigl(\omega T\bigr) - \sin^2\bigl(\omega T\bigr) - 1\bigr\} - \sin\bigl(\omega t_{\rm em}\bigr)\,2\sin\bigl(\omega T\bigr)\cos\bigl(\omega T\bigr)\Bigr)$$

$$= \frac{A_+}{2}\Bigl(-\,2\cos\bigl(\omega t_{\rm em}\bigr)\sin^2\bigl(\omega T\bigr) - \sin\bigl(\omega t_{\rm em}\bigr)\,2\sin\bigl(\omega T\bigr)\cos\bigl(\omega T\bigr)\Bigr)$$

$$= -\,A_+\sin\bigl(\omega T\bigr)\Bigl(\cos\bigl(\omega t_{\rm em}\bigr)\sin\bigl(\omega T\bigr) + \sin\bigl(\omega t_{\rm em}\bigr)\cos\bigl(\omega T\bigr)\Bigr) \;=\; A_+\sin\bigl(\omega T\bigr)\sin\bigl(\omega t_{\rm em}+\omega T\bigr)$$

This demonstrates that the redshift z oscillates, as a function of $t_{\rm em}$, with the frequency ω and the amplitude

$$a \;=\; \bigl|A_+\sin\bigl(\omega T\bigr)\bigr| \;\le\; \bigl|A_+\bigr|\,.$$

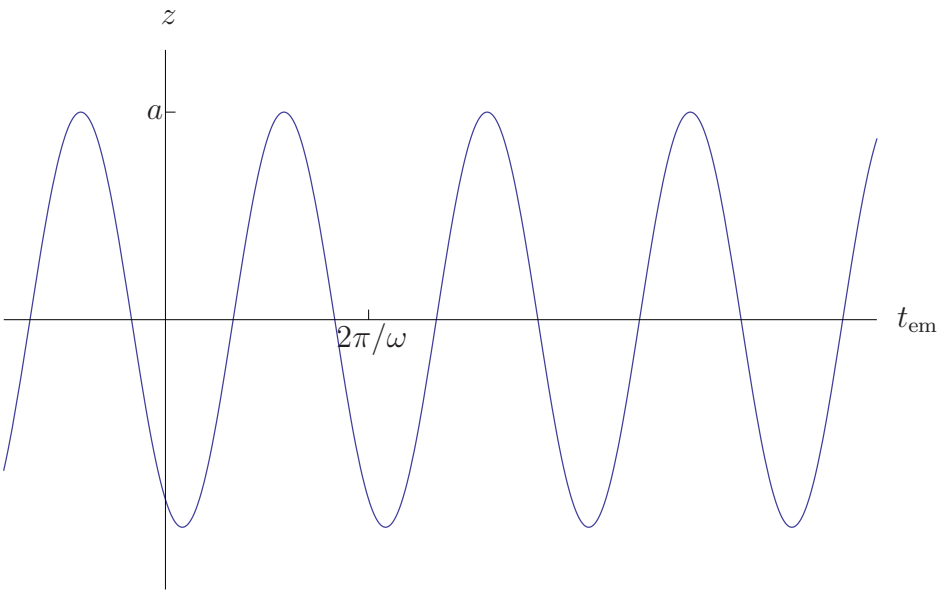


Fig. 6.22: Plot of redshift against emission time

Searches for gravitational waves with the help of Doppler tracking have been carried through, e.g., with Voyager 1 and 2, with Pioneer 10 and 11, and in particular with the Cassini spacecraft that was launched in 1997 and reached Saturn in 2004. From Fig. 6.23 we can read the frequencies and the amplitudes of gravitational waves that could have been detected by Doppler tracking of the Cassini spacecraft. We see that this method is sensitive only for low frequencies, around 10^{-3} Hz, and for amplitudes of about 10^{-16}.

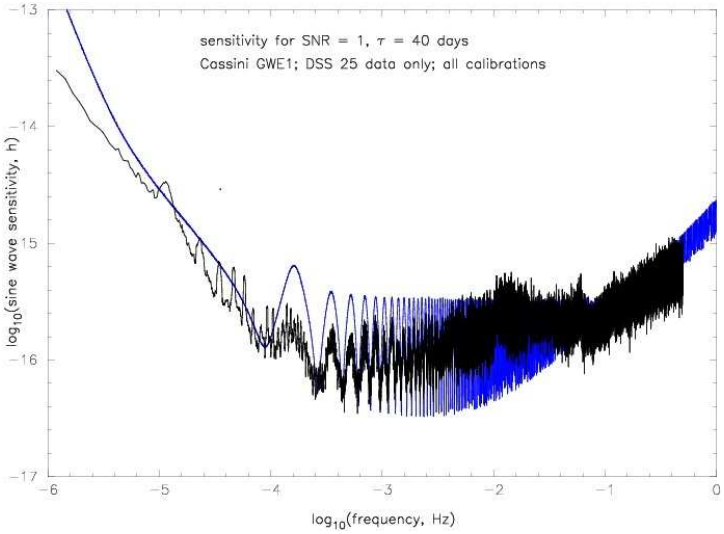


Fig. 6.23: from http://www.livingreviews.org/lrr-2006-1

The radio links with spacecraft in the outer region of our Solar system are established with a system of radio telscopes that is known as the Deep Space Network (DSN). It comprises sites in the USA, in Spain and in Australia such that at any time of the day at least one of the stations can communicate with the spacecraft. Fig. 6.24 shows one of the 34-meter telescopes of the DSN, at Goldstone in the Mojave Desert in the USA.

Fig. 6.24: from http://www.livingreviews.org/lrr-2006-1

6.4 Pulsar timing arrays

The basic idea of using pulsar timing arrays for detecting gravitational waves is similar to that of using Doppler tracking. Again, the change of the distance betwen two test masses is measured with the help of its effect onto the frequency of a radio signal. In the case of Doppler tracking, the two test masses are the Earth and a spacecraft; in the case of pulsar timing observations they are a (millisecond) pulsar and the Earth. A major difference is in the fact that pulsar timing is a one-way method (there are signals from the pulsar to the Earth, but no return signals) while Doppler tracking of spacecraft is a two-way method.

A pulsar emits radio pulses at a rate that is highly stable. For millisecond pulsars, the stability of the pulse frequency is comparable to the stablity of the best clocks we have. This, however does not mean that the pulses arrive with a constant frequency here on Earth. Changes in the times of arrival are caused e.g. by the relative motion of the pulsar and the Earth, by the influence of the gravitatonal field of the Sun and of other masses the signal might pass, and by the interstellar medium. All these known influences are taken into account in the socalled timing formulas used by radio astronomers for evaluating their observations. Remaining differences between theory and observation are known as *timing residuals*. A gravitational wave should produce such residuals.

The idea of searching for signatures from gravitational waves in the timing residuals of pulsars was brought forward by M. Sazhin ["Opportunities for detecting ultralong gravitational waves" Astron. Zh. 55, 65 (1978)] and further developed by S. Detweiler ["Pulsar timing measurements and the search for gravitational waves" Astrophys. J. 234, 1100 (1979)]. For a recent review on the planned International Pulsar Timing Array see G. Hobbs et al. ["The International Pulsar Timing Array project: using pulsars as a gravitational wave detector" Class. Quantum Grav. 27, 084013 (2010)].

Here we will give a calculation under highly idealised assumptions, just to outline the basic idea. We treat the pulsar and the Earth as at rest in a Minkowski background, and we ignore the influence of the interstellar medium. The gravitational wave is considered as a perturbation of the Minkowski background within the linearised theory,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,,$$

where the perturbation is assumed to satisfy the following two properties: (i) components with a time index vanish, $h_{0\mu} = 0$, and (ii) the worldlines of constant spatial coordinates, i.e. the curves $\big(x^\mu(t)\big) = (ct, \vec{r}_0)$ with a constant $\vec{r}_0$, are geodesics. These two assumptions are satisfied, in particular, if $h_{\mu\nu}$ is an arbitrary superposition of gravitational waves in the TT gauge.

By assumption, the worldlines of the pulsar and of the Earth are both t lines. From the form of the metric,

$$g_{\mu\nu}(x)dx^\mu dx^\nu = -c^2dt^2 + \delta_{ij}dx^i dx^j + h_{ij}(x)x^i dx^j\,,$$

we read that along these worldlines the time coordinate t coincides with proper time. Therefore, we can identify frequencies with respect to the time coordinate t with frequencies with respect to proper time of the pulsar or of the Earth. We assume that the pulsar emits signals at a fixed frequency ν_P. They will arrive at the Earth with a frequency $\nu_E(t_E)$ that depends on the time of arrival, t_E. It is our goal to determine this function $\nu_E(t_E)$.

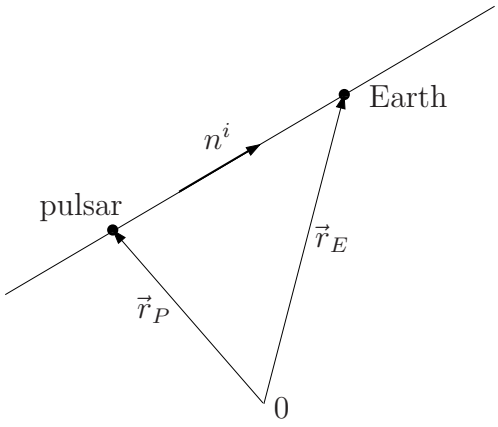


Fig. 6.25: Position of the pulsar and of the Earth

Along a light ray from the pulsar to the Earth, we must have

$$0 = -c^2 dt^2 + \delta_{ij} dx^i dx^j + h_{ij}(x) dx^i dx^j$$

and thus

$$c^2 \left(\frac{dt}{d\ell}\right)^2 \;=\; 1 \;+\; h_{ij}(x) \frac{dx^i}{d\ell} \frac{dx^j}{d\ell}$$

where ℓ denotes arclength with respect to the flat background metric, i.e.

$$d\ell^2 \;=\; \delta_{ij} dx^i dx^j \;.$$

Without a gravitational wave, the light ray moves on a straight line from the pulsar to the Earth, i.e., $dx^i/d\ell$ is a constant unit vecor n^i. With a gravitational wave, we have

$$\frac{dx^i}{d\ell} \;=\; n^i \;+\; O(h)$$

and thus

$$c\,\frac{dt}{d\ell} \;=\; \sqrt{1 + h_{ij}(x) n^i n^j + O(h^2)}\,,$$

$$d\ell \;=\; c\left(1 + h_{ij}(x) n^i n^j + O(h^2)\right)^{-1/2} dt \;=\; c\Big(1 - \frac{1}{2} h_{ij}(x) n^i n^j + \,\ldots\Big) dt$$

where the ellipses indicate terms of quadratic and higher order that will be neglected in the following. Integration over the path of the light ray, from its emission time t_P to the arrival time t_E, yields

$$\int_{t_P}^{t_E} \left(1 - \frac{1}{2} h_{ij}(x) n^i n^j\right) dt \;=\; \frac{L}{c}$$

where L is the distance from the pulsar to the Earth measured in the flat background. This equation gives t_E as a function of t_P. Differentiation with respect to t_P yields

$$\frac{dt_E}{dt_P}\Big(1 - \frac{1}{2} h_{ij}(ct_E, \vec{r}_E) n^i n^j\Big) - \Big(1 - \frac{1}{2} h_{k\ell}(ct_P, \vec{r}_P) n^k n^\ell\Big) \;=\; 0\;,,$$

$$\frac{dt_P}{dt_E} \;=\; \frac{\Big(1 - \frac{1}{2} h_{k\ell}(ct_E, \vec{r}_E) n^k n^\ell\Big)}{\Big(1 - \frac{1}{2} h_{ij}(ct_P, \vec{r}_P) n^i n^j\Big)} \;=\; 1 - \frac{1}{2} h_{k\ell}(ct_E, \vec{r}_E) n^k n^\ell \;+\; \frac{1}{2} h_{ij}(ct_P, \vec{r}_P) n^i n^j \;+\; \ldots\,,$$

$$\frac{dt_P}{dt_E} \;=\; 1 \;+\; \frac{n^i n^j}{2}\Big(h_{ij}(ct_P,\,\vec{r}_P) \;-\; h_{ij}(ct_E,\,\vec{r}_E)\Big)\,.$$

We have thus found that the pulses, which are emitted with a constant frequency ν_P, arrive with a frequency $\nu_E(t_E)$ given by

$$\frac{\nu_E(t_E)-\nu_P}{\nu_P} \;=\; \frac{dt_P}{dt_E} - 1 \;=\; \frac{n^i n^j}{2}\Big(h_{ij}(ct_P,\,\vec{r}_P) \;-\; h_{ij}(ct_E,\,\vec{r}_E)\Big)\,.$$

We see that the frequency shift depends on the projection onto n^i of the wave amplitude h_{ij} at the pulsar and at the Earth. If there is a single wave in TT gauge propagating in the x^3 direction, n^i must have a non-vanishing component in the $x^1 - x^2$ plane to give a non-zero frequency shift. If the same gravitational wave is observed with at least three pulsars, the propagation direction of the wave can be deduced.

Pulsar timing arrays are used for many applications; the search for gravitational waves is only one of them. Three pulsar timing arrays have been established which routinely observe the times of arrivals of many pulsars:

- Parkes Pulsar Timing Array (PPTA): This uses the Parkes Telescope in Australia and takes data since 2005.
- European Pulsar Timing Array (EPTA): This uses data from five radio telescopes in Europe, namely Effelsberg, Jodrell Bank, Westerbork, Nançay, and a new one in Sardinia.
- North American Nanohertz Observatory for Gravitational Waves (NanoGRAV): This is an Americal pulsar timing array using data from Arecibo and Green Bank.

It is planned to join them into an International Pulsar Timing Array (IPTA), see Hobbs et al., loc. cit. Great progress with pulsar timing data is expected from the Square Kilometer Array (SKA), an array of radio telescopes in the Southern hemisphere with an effective aperture of a square kilometer that is planned to be operational around 2020.

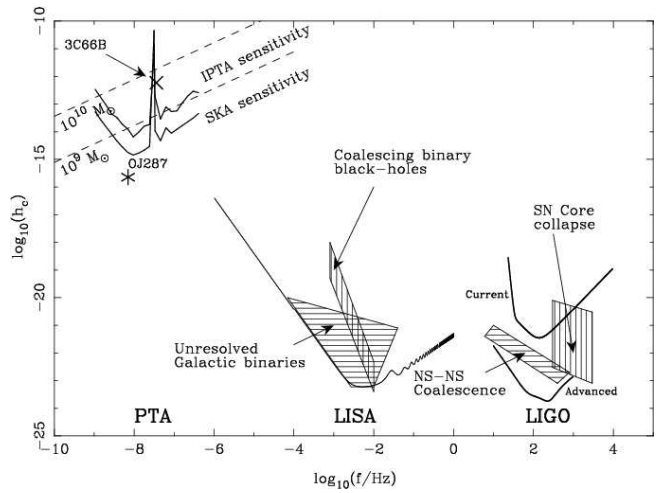


Fig. 6.26: from G. Hobbs et al., loc. cit.

Pulsar timing arrays are sensitive to gravitational waves at very low frequencies, between 10^{-6} and 10^{-9} Hz, with an amplitude of about 10^{-15}, see Fig. 6.26. Possible sources that could be detected with this method are merging supermassive black holes.

6.5 Influence of gravitational waves on electromagnetic waves

All the methods we have discussed so far were based on measuring the effect of a gravitational wave onto massive bodies, either onto vibrating masses or onto free test masses. Electromagnetic waves were used in some of these methods, but only as a tool for measuring the effect onto the massive bodies.

In this section we briefly discuss the possibility of detecting a gravitational wave by its effect onto an electromagnetic wave. One such method was suggested by M. Cruise ["An interaction between gravitational and electromagnetic waves" Mon. Not. Roy. Astron. Soc. 204, 485 (1983)]. It is based on the observation that a gravitational wave causes a rotation of the polarisation plane of an electromagnetic field.

In an arbitrary general-relativistic spacetime, the polarisation vector Π of a linearly polarised electromagnetic wave in vacuo is parallely transported along each ray. This can be deduced from Maxwell's equations in the high-frequency limit (i.e., in the geometric optics approximation). If we denote the tangent vector to the ray by K, the polarisation vector satisfies the equation

$$\nabla_K \Pi = 0$$

or, in coordinate notation,

$$K^\mu \partial_\mu \Pi^\rho + \Gamma^\rho{}_{\nu\tau} K^\nu \Pi^\tau = 0\,.$$

The vectors K and Π span the polarisation plane. This plane always contains the direction tangent to the ray, so the only thing it can do is to rotate about this direction. We see that, with respect to the coordinate system used, such a rotation is caused by the Christoffel symbols. For a gravitational wave in TT gauge, we already know that the Christoffel symbols read

$$\Gamma^\rho{}_{\nu\tau} \;=\; \frac{1}{2}\,\eta^{\rho\sigma}\,\mathrm{Re}\big\{\big(k_\nu A_{\sigma\tau} + k_\tau A_{\sigma\nu} - k_\sigma A_{\nu\tau}\big)\,i\,\mathsf{e}^{ik_\mu x^\mu}\big\}$$

where k_μ is the wave covector of the gravitational wave. According to this equation, a gravitational wave would cause a rotation of the polarisation plane of an electromagnetic wave.

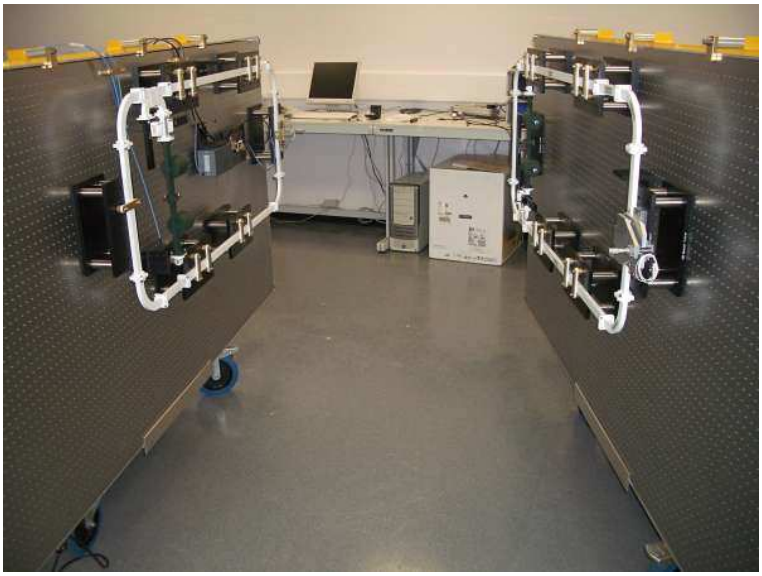


Fig. 6.27: from http://www.sr.bham.ac.uk/gravity

The expected rotation angle is tiny. Therefore, Cruise designed a gravitational wave detector that would enhance this rotation by making use of a resonance effect. The electromagnetic wave is a radio wave in a wave-guide that is bent into a loop. The resonance frequency of the system is 100 MHz. If a gravitational wave with the same frequency comes in, the polarisation plane is periodically kicked by a tiny rotation angle in such a way that these tiny rotations add up. Cruise built two such instruments in Birmingham so that he could look for coincidences, see Fig. 6.27. Note that these detectors would be sensitive at a frequency of about 100 MHz, i.e., at an extremely high frequency in comparison to all other gravitational wave detectors.

In addition to the possibility of constructing non-orthodox gravitational wave detectors, the effect of a gravitational wave onto electromagnetic waves is of crucial relevance in view of the cosmic background radiation. In recent years several experiments are analysing the polarisation of the cosmic background radiation. In analogy to decomposing a vector field into rotation-free and divergence-free parts, the Fourier components of the cosmic background radiation are decomposed into electric (E) and magnetic (B) modes. Primordial gravitational waves that have come into existence through quantum fluctuations in the very early universe would produce a specific signature of B modes. These B modes from primordial gravitational waves could have a measurable amplitude only if our universe underwent an inflationary period, i.e., a period in which the universe expanded exponentially.

In March 2014 it was announced that the BICEP2 experiment had found B modes from primordial gravitational waves in the cosmic background radiation. BICEP2 was a radio telescope near the South Pole that was operational from 2010 to 2012. If confirmed, the BICEP2 result would give strong support for the idea that quantum fluctuations in the early universe have produced gravitational waves and that there was an inflationary period. (The idea of primordial gravitational waves, resulting from quantum fluctuations, was developed already in the 1970s by L. Grishchuk and others. The idea of an inflationary universe, brought forward by A. Guth, F. Englert, A. Linde and others around 1980, allowed for an increase in the amplitudes of these primordial gravitational waves that could make them measurable.)

Fig. 6.28 summarises the different types of gravitational wave detectors, the frequency range in which they are sensitive and the types of sources they could detect.

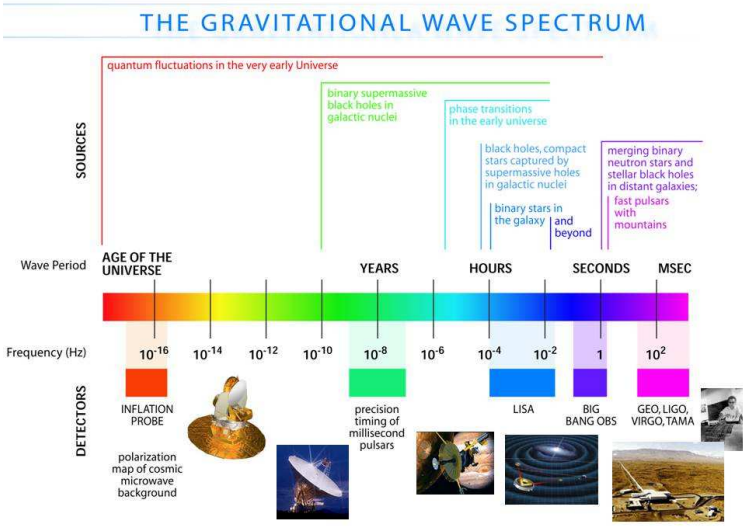


Fig. 6.28: from http://www.astro.gla.ac.uk

7. Gravitational waves in the linearised theory around curved spacetime

Up to now we have treated gravitational waves as small perturbations of Minkowski spacetime. We will now modify this approach by allowing for a curved background, still assuming that the gravitatonal wave is a small perturbation. Possible applications include the propagation of gravitational waves with a small amplitude near a black hole.

7.1 Linearised field equation around curved spacetime

We consider a metric of the form

$$g_{\mu\nu} = \overset{B}{g}_{\mu\nu} + h_{\mu\nu}$$

where $\overset{B}{g}_{\mu\nu}$ is an arbitray Lorentzian (background) metric and the perturbation is assumed to be so small that all terms of second or higher order with respect to $h_{\mu\nu}$ or its derivatives can be neglected. We want to work out the field equation (without a cosmological constant),

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \kappa T_{\mu\nu} \,,$$

in this linearised theory. As a first step, we have to calculate the Christoffel symbols.

For this chapter, we agree to raise and to lower indices with the background metric. Then, to within our linear approximation, the inverse metric is of the form

$$g^{\nu\rho} = \overset{B}{g}{}^{\nu\rho} - h^{\nu\rho} \,.$$

Proof: $g_{\mu\nu}\big(\overset{B}{g}{}^{\nu\rho} - h^{\nu\rho}\big) = \big(\overset{B}{g}_{\mu\nu} + h_{\mu\nu}\big)\big(\overset{B}{g}{}^{\nu\rho} - h^{\nu\rho}\big) = \delta^\rho_\mu + h_\mu{}^\rho - h_\mu{}^\rho = \delta^\rho_\mu$.

Let $\overset{B}{\Gamma}{}^\rho{}_{\mu\nu}$ denote the Christoffel symbols of the background metric and let $\overset{B}{\nabla}$ be the covariant derivative with respect to the background metric. Then the Christoffel symbols of the perturbed spacetime are

$$\begin{aligned}
\Gamma^\rho{}_{\mu\nu} \;&=\; \frac{1}{2} g^{\rho\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big) \\
&= \frac{1}{2}\big(\overset{B}{g}{}^{\rho\sigma} - h^{\rho\sigma}\big)\Big(\partial_\mu\big(\overset{B}{g}_{\sigma\nu} + h_{\sigma\nu}\big) + \partial_\nu\big(\overset{B}{g}_{\sigma\mu} + h_{\sigma\mu}\big) - \partial_\sigma\big(\overset{B}{g}_{\mu\nu} + h_{\mu\nu}\big)\Big) \\
= \frac{1}{2}\overset{B}{g}{}^{\rho\sigma}\big(\partial_\mu \overset{B}{g}_{\sigma\nu} + \partial_\nu \overset{B}{g}_{\sigma\mu} - \partial_\sigma \overset{B}{g}_{\mu\nu}\big) &+ \frac{1}{2}\overset{B}{g}{}^{\rho\sigma}\big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\big) - \frac{1}{2} h^{\rho\sigma}\big(\partial_\mu \overset{B}{g}_{\sigma\nu} + \partial_\nu \overset{B}{g}_{\sigma\mu} - \partial_\sigma \overset{B}{g}_{\mu\nu}\big) \\
&= \overset{B}{\Gamma}{}^\rho{}_{\mu\nu} + \frac{1}{2}\overset{B}{g}{}^{\rho\lambda}\big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\big) - \frac{1}{2}\overset{B}{g}{}^{\rho\lambda}\overset{B}{g}{}^{\sigma\tau} h_{\lambda\tau}\big(\partial_\mu \overset{B}{g}_{\sigma\nu} + \partial_\nu \overset{B}{g}_{\sigma\mu} - \partial_\sigma \overset{B}{g}_{\mu\nu}\big) \\
&= \overset{B}{\Gamma}{}^\rho{}_{\mu\nu} + \frac{1}{2}\overset{B}{g}{}^{\rho\lambda}\Big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} - 2\overset{B}{\Gamma}{}^\tau{}_{\mu\nu} h_{\lambda\tau}\Big) \\
&= \overset{B}{\Gamma}{}^\rho{}_{\mu\nu} + \frac{1}{2}\overset{B}{g}{}^{\rho\lambda}\Big(\overset{B}{\nabla}_\mu h_{\lambda\nu} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\mu\lambda} h_{\tau\nu}} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\mu\nu} h_{\lambda\tau}} + \overset{B}{\nabla}_\nu h_{\lambda\mu} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\nu\lambda} h_{\tau\mu}} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\nu\mu} h_{\lambda\tau}} \\
&\quad - \overset{B}{\nabla}_\lambda h_{\mu\nu} - \cancel{\overset{B}{\Gamma}{}^\tau{}_{\lambda\mu} h_{\tau\nu}} - \cancel{\overset{B}{\Gamma}{}^\tau{}_{\lambda\nu} h_{\mu\tau}} - \cancel{2\overset{B}{\Gamma}{}^\tau{}_{\mu\nu} h_{\lambda\tau}}\Big) \;=\; \overset{B}{\Gamma}{}^\rho{}_{\mu\nu} + \frac{1}{2}\Big(\overset{B}{\nabla}_\mu h^\rho{}_\nu + \overset{B}{\nabla}_\nu h^\rho{}_\mu - \overset{B}{\nabla}{}^\rho h_{\mu\nu}\Big)
\end{aligned}$$

We write this result as

$$\Gamma^{\rho}{}_{\mu\nu} \;=\; \overset{B}{\Gamma}{}^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}$$

where

$$\delta\Gamma^{\rho}{}_{\mu\nu} \;=\; \frac{1}{2}\Big(\overset{B}{\nabla}_{\mu} h^{\rho}{}_{\nu} + \overset{B}{\nabla}_{\nu} h^{\rho}{}_{\mu} - \overset{B}{\nabla}{}^{\rho} h_{\mu\nu}\Big)$$

is a tensor field. (Recall that the *difference* of the Christoffel symbols of two connections is a tensor field.)

Next we calculate the curvature tensor.

$$R^{\mu}{}_{\nu\rho\sigma} \;=\; \partial_{\nu}\Gamma^{\mu}{}_{\rho\sigma} \;-\; \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} \;+\; \Gamma^{\mu}{}_{\nu\kappa}\Gamma^{\kappa}{}_{\rho\sigma} \;-\; \Gamma^{\mu}{}_{\rho\kappa}\Gamma^{\kappa}{}_{\nu\sigma}$$

$$= \;\partial_{\nu}\Big(\overset{B}{\Gamma}{}^{\mu}{}_{\rho\sigma} + \delta\Gamma^{\mu}{}_{\rho\sigma}\Big) \;-\; \partial_{\rho}\Big(\overset{B}{\Gamma}{}^{\mu}{}_{\nu\sigma} + \delta\Gamma^{\mu}{}_{\nu\sigma}\Big)$$

$$+\Big(\overset{B}{\Gamma}{}^{\mu}{}_{\nu\kappa} + \delta\Gamma^{\mu}{}_{\nu\kappa}\Big)\Big(\overset{B}{\Gamma}{}^{\kappa}{}_{\rho\sigma} + \delta\Gamma^{\kappa}{}_{\rho\sigma}\Big) \;-\; \Big(\overset{B}{\Gamma}{}^{\mu}{}_{\rho\kappa} + \delta\Gamma^{\mu}{}_{\rho\kappa}\Big)\Big(\overset{B}{\Gamma}{}^{\kappa}{}_{\nu\sigma} + \delta\Gamma^{\kappa}{}_{\nu\sigma}\Big)$$

$$= \;\overset{B}{R}{}^{\mu}{}_{\nu\rho\sigma} \;+\; \partial_{\nu}\delta\Gamma^{\mu}{}_{\rho\sigma} \;-\; \partial_{\rho}\delta\Gamma^{\mu}{}_{\nu\sigma} \;+\; \overset{B}{\Gamma}{}^{\mu}{}_{\nu\kappa}\delta\Gamma^{\kappa}{}_{\rho\sigma} \;+\; \overset{B}{\Gamma}{}^{\kappa}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\nu\kappa} \;-\; \overset{B}{\Gamma}{}^{\mu}{}_{\rho\kappa}\delta\Gamma^{\kappa}{}_{\nu\sigma} \;-\; \overset{B}{\Gamma}{}^{\kappa}{}_{\nu\sigma}\delta\Gamma^{\mu}{}_{\rho\kappa} \;+\; \ldots$$

The term indicated by ellipses is of second order and will thus be neglected in the following. This gives us the curvature tensor in the form

$$R^{\mu}{}_{\nu\rho\sigma} \;=\; \overset{B}{R}{}^{\mu}{}_{\nu\rho\sigma} \;+\; \delta R^{\mu}{}_{\nu\rho\sigma}$$

where

$$\delta R^{\mu}{}_{\nu\rho\sigma} \;=\; \partial_{\nu}\delta\Gamma^{\mu}{}_{\rho\sigma} \;-\; \partial_{\rho}\delta\Gamma^{\mu}{}_{\nu\sigma} \;+\; \overset{B}{\Gamma}{}^{\mu}{}_{\nu\kappa}\delta\Gamma^{\kappa}{}_{\rho\sigma} \;+\; \overset{B}{\Gamma}{}^{\kappa}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\nu\kappa} \;-\; \overset{B}{\Gamma}{}^{\mu}{}_{\rho\kappa}\delta\Gamma^{\kappa}{}_{\nu\sigma} \;-\; \overset{B}{\Gamma}{}^{\kappa}{}_{\nu\sigma}\delta\Gamma^{\mu}{}_{\rho\kappa}$$

$$= \;\overset{B}{\nabla}_{\nu}\delta\Gamma^{\mu}{}_{\rho\sigma} \;-\; \overset{B}{\Gamma}{}^{\mu}{}_{\nu\kappa}\delta\Gamma^{\kappa}{}_{\rho\sigma} \;+\; \overset{B}{\Gamma}{}^{\kappa}{}_{\nu\rho}\delta\Gamma^{\mu}{}_{\kappa\sigma} \;+\; \overset{B}{\Gamma}{}^{\kappa}{}_{\nu\sigma}\delta\Gamma^{\mu}{}_{\kappa\rho}$$

$$-\overset{B}{\nabla}_{\rho}\delta\Gamma^{\mu}{}_{\nu\sigma} \;+\; \overset{B}{\Gamma}{}^{\mu}{}_{\rho\kappa}\delta\Gamma^{\kappa}{}_{\nu\sigma} \;-\; \overset{B}{\Gamma}{}^{\kappa}{}_{\rho\nu}\delta\Gamma^{\mu}{}_{\kappa\sigma} \;-\; \overset{B}{\Gamma}{}^{\kappa}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\kappa\nu}$$

$$+\overset{B}{\Gamma}{}^{\mu}{}_{\nu\kappa}\delta\Gamma^{\kappa}{}_{\rho\sigma} \;+\; \overset{B}{\Gamma}{}^{\kappa}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\nu\kappa} \;-\; \overset{B}{\Gamma}{}^{\mu}{}_{\rho\kappa}\delta\Gamma^{\kappa}{}_{\nu\sigma} \;-\; \overset{B}{\Gamma}{}^{\kappa}{}_{\nu\sigma}\delta\Gamma^{\mu}{}_{\rho\kappa}$$

$$= \;\overset{B}{\nabla}_{\nu}\delta\Gamma^{\mu}{}_{\rho\sigma} - \overset{B}{\nabla}_{\rho}\delta\Gamma^{\mu}{}_{\nu\sigma}\,.$$

With our earlier result for $\delta\Gamma^{\mu}{}_{\rho\sigma}$ this can be rewritten as

$$\delta R^{\mu}{}_{\nu\rho\sigma} \;=\; \frac{1}{2}\overset{B}{\nabla}_{\nu}\Big(\overset{B}{\nabla}_{\rho} h^{\mu}{}_{\sigma} + \overset{B}{\nabla}_{\sigma} h^{\mu}{}_{\rho} - \overset{B}{\nabla}{}^{\mu} h_{\rho\sigma}\Big) - \frac{1}{2}\overset{B}{\nabla}_{\rho}\Big(\overset{B}{\nabla}_{\nu} h^{\mu}{}_{\sigma} + \overset{B}{\nabla}_{\sigma} h^{\mu}{}_{\nu} - \overset{B}{\nabla}{}^{\mu} h_{\nu\sigma}\Big)$$

$$= \;\frac{1}{2}\Big(\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\rho} h^{\mu}{}_{\sigma} + \overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\sigma} h^{\mu}{}_{\rho} - \overset{B}{\nabla}_{\nu}\overset{B}{\nabla}{}^{\mu} h_{\rho\sigma} - \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\nu} h^{\mu}{}_{\sigma} - \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\sigma} h^{\mu}{}_{\nu} + \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}{}^{\mu} h_{\nu\sigma}\Big)\,.$$

Contraction gives the Ricci tensor

$$R_{\nu\sigma} \;=\; \overset{B}{R}_{\nu\sigma} \,+\, \delta R_{\nu\sigma}$$

where

$$\delta R_{\nu\sigma} \;=\; \delta R^{\mu}{}_{\nu\mu\sigma} \;=\; \frac{1}{2}\Big(\cancel{\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\mu} h^{\mu}{}_{\sigma}} + \overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\sigma} h - \cancel{\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}^{\mu} h_{\mu\sigma}} - \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu} h^{\mu}{}_{\sigma} - \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\sigma} h^{\mu}{}_{\nu} + \overset{B}{\Box} h_{\nu\sigma}\Big) .$$

Here we have introduced the trace of the perturbation,

$$h \;=\; h^{\mu}{}_{\mu} \;=\; \overset{B}{g}{}^{\mu\nu} h_{\mu\nu} \,,$$

and the wave operator of the background metric,

$$\overset{B}{\Box} \;=\; \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}{}^{\mu} \;=\; \overset{B}{g}{}^{\mu\nu}\overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu} \,.$$

If we contract another time we get the Ricci scalar

$$R \;=\; \overset{B}{R} \,+\, \delta R$$

with

$$\delta R \;=\; \delta R^{\nu}{}_{\nu} \;=\; \frac{1}{2}\Big(\overset{B}{\nabla}{}^{\nu}\overset{B}{\nabla}_{\nu} h - \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}{}^{\nu} h^{\mu}{}_{\nu} - \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu} h^{\mu\nu} + \overset{B}{\Box} h^{\nu}{}_{\nu}\Big) \;=\; \overset{B}{\Box} h \;-\; \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu} h^{\mu\nu} \,.$$

Finally, the Einstein tensor reads

$$G_{\mu\nu} \;=\; \overset{B}{G}_{\mu\nu} \,+\, \delta G_{\mu\nu}$$

where

$$\delta G_{\mu\nu} \;=\; \delta\Big(R_{\mu\nu} \,-\, \frac{1}{2}\, R\, g_{\mu\nu}\Big)$$

$$=\; \delta R_{\mu\nu} \,-\, \frac{1}{2}\, R\, \delta g_{\mu\nu} - \frac{1}{2}\, \delta R\, g_{\mu\nu} \;=\; \delta R_{\mu\nu} \,-\, \frac{1}{2}\, \overset{B}{R}\, h_{\mu\nu} - \frac{1}{2}\, \delta R\, \overset{B}{g}_{\mu\nu} \,+\, \ldots$$

$$=\; \frac{1}{2}\Big(\overset{B}{\Box} h_{\mu\nu} + \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu} h - \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\mu} h_{\nu}{}^{\rho} - \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\nu} h_{\mu}{}^{\rho} - \overset{B}{R}\, h_{\mu\nu} - \overset{B}{g}_{\mu\nu}\big(\overset{B}{\Box} h - \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\sigma} h^{\rho\sigma}\big)\Big) .$$

Now we use that the commutator of covariant derivatives can be expressed in terms of the curvature tensor,

$$\big(\overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\mu} - \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\rho}\big) h_{\nu}{}^{\rho} \;=\; \overset{B}{R}{}^{\tau}{}_{\rho\mu\nu} h_{\tau}{}^{\rho} - \overset{B}{R}{}^{\rho}{}_{\rho\mu\tau} h_{\nu}{}^{\tau} \,.$$

We find

$$\overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\mu} h_{\nu}{}^{\rho} + \overset{B}{\nabla}_{\rho}\overset{B}{\nabla}_{\nu} h_{\mu}{}^{\rho} \;=\; \overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\rho} h_{\nu}{}^{\rho} - \overset{B}{R}{}^{\tau}{}_{\mu\rho\nu} h_{\tau}{}^{\rho} + \overset{B}{R}{}^{\rho}{}_{\mu\rho\tau} h_{\nu}{}^{\tau} + \overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\rho} h_{\mu}{}^{\rho} - \overset{B}{R}{}^{\tau}{}_{\nu\rho\mu} h_{\tau}{}^{\rho} + \overset{B}{R}{}^{\rho}{}_{\nu\rho\tau} h_{\mu}{}^{\tau} \,.$$

This can be rewritten, with the curvature identities

$$\overset{B}{R}{}^{\tau}{}_{\mu\nu\tau} + \overset{B}{R}{}^{\tau}{}_{\nu\tau\mu} + \overset{B}{R}{}^{\tau}{}_{\tau\mu\nu} \;=\; 0 \,, \qquad \overset{B}{R}{}^{\tau}{}_{\mu\rho\nu} \;=\; -\, \overset{B}{R}{}^{\tau}{}_{\rho\mu\nu} \,,$$

as

$$\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\mu h_\nu{}^\rho+\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\nu h_\mu{}^\rho \;=\; \overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho h_\nu{}^\rho+\overset{B}{R}{}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho+\overset{B}{R}_{\mu\tau}h_\nu{}^\tau+\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho h_\mu{}^\rho+\big(\overset{B}{R}{}^\tau{}_{\rho\mu\nu}+\overset{B}{R}{}^\tau{}_{\mu\nu\rho}\big)h_\tau{}^\rho+\overset{B}{R}_{\nu\tau}h_\mu{}^\tau$$

$$=\; \overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho h_\nu{}^\rho+2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho+\overset{B}{R}_{\mu\tau}h_\nu{}^\tau+\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho h_\mu{}^\rho+\underbrace{\overset{B}{R}_{\tau\mu\nu\rho}h^{\tau\rho}}_{=0}+\overset{B}{R}_{\nu\tau}h_\mu{}^\tau$$

where the underbraced term vanishes because of the curvature identity $\overset{B}{R}_{\tau\mu\nu\rho}=-\overset{B}{R}_{\rho\mu\nu\tau}$. Inserting this expression into our result for $\delta G_{\mu\nu}$ gives

$$2\,\delta G_{\mu\nu} \;=\; \overset{B}{\Box}h_{\mu\nu}+\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\nu h-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho h_\nu{}^\rho-2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho-\overset{B}{R}_{\mu\tau}h_\nu{}^\tau$$

$$-\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho h_\mu{}^\rho-\overset{B}{R}_{\nu\tau}h_\mu{}^\tau-\overset{B}{R}\,h_{\mu\nu}-\overset{B}{g}_{\mu\nu}\Big(\overset{B}{\Box}h-\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\sigma h^{\rho\sigma}\Big)\,.$$

This is the general expression for $\delta G_{\mu\nu}$ on an arbitrary background spacetime. From now on we specify to the case that the background spacetime satisfies the vacuum field equation (without a cosmological constant),

$$\overset{B}{R}_{\mu\nu}=0\;.$$

Then

$$2\,\delta G_{\mu\nu} \;=\; \overset{B}{\Box}h_{\mu\nu}+\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\nu h-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho h_\nu{}^\rho-2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho-\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho h_\mu{}^\rho-\overset{B}{g}_{\mu\nu}\Big(\overset{B}{\Box}h-\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\sigma h^{\rho\sigma}\Big)\,.$$

This expression can be simplified if we use the gauge freedom. We follow our treatment in the case of a flat background (see p.8/9) as closely as possible. We introduce

$$\gamma_{\mu\nu} \;=\; h_{\mu\nu}\;-\;\frac{h}{2}\,\overset{B}{g}_{\mu\nu}$$

which implies

$$\gamma \;:=\; \overset{B}{g}{}^{\mu\nu}\gamma_{\mu\nu} \;=\; h\;-\frac{1}{2}\,4\,h \;=\; -\,h\;,$$

hence

$$h_{\mu\nu} \;=\; \gamma_{\mu\nu}\;-\;\frac{\gamma}{2}\,\overset{B}{g}_{\mu\nu}\;.$$

Then our expression for $\delta G_{\mu\nu}$ reads

$$2\,\delta G_{\mu\nu} \;=\; \overset{B}{\Box}\Big(\gamma_{\mu\nu}-\frac{\gamma}{2}\overset{B}{g}_{\mu\nu}\Big)-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\nu\gamma-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho\Big(\gamma_\nu{}^\rho-\frac{\gamma}{2}\delta^\rho_\nu\Big)$$

$$-2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}\Big(\gamma_\tau{}^\rho-\frac{\gamma}{2}\delta^\rho_\tau\Big)-\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho\Big(\gamma_\mu{}^\rho-\frac{\gamma}{2}\delta^\rho_\mu\Big)+\overset{B}{g}_{\mu\nu}\overset{B}{\Box}\gamma+\overset{B}{g}_{\mu\nu}\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\sigma\Big(\gamma^{\rho\sigma}-\frac{\gamma}{2}\overset{B}{g}{}^{\rho\sigma}\Big)$$

$$=\; \overset{B}{\Box}\gamma_{\mu\nu}-\cancel{\frac{1}{2}\overset{B}{g}_{\mu\nu}\overset{B}{\Box}\gamma}-\cancel{\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\nu\gamma}-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho\gamma_\nu{}^\rho+\cancel{\frac{1}{2}\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\nu\gamma}-2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}\gamma_\tau{}^\rho+\underbrace{\overset{B}{R}{}^\rho{}_{\rho\mu\nu}}_{=0}\gamma$$

$$-\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho\gamma_\mu{}^\rho+\frac{1}{2}\cancel{\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\mu\gamma}+\cancel{\overset{B}{g}_{\mu\nu}\overset{B}{\Box}\gamma}+\overset{B}{g}_{\mu\nu}\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\sigma\gamma^{\rho\sigma}-\frac{1}{2}\cancel{\overset{B}{g}_{\mu\nu}\overset{B}{\Box}\gamma}$$

$$=\overset{B}{\Box}\gamma_{\mu\nu}-\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho\gamma_\nu{}^\rho-2\overset{B}{R}{}^\tau{}_{\rho\mu\nu}\gamma_\tau{}^\rho-\overset{B}{\nabla}_\nu\overset{B}{\nabla}_\rho\gamma_\mu{}^\rho+\overset{B}{g}_{\mu\nu}\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\sigma\gamma^{\rho\sigma}$$

We can now make a coordinate transformation of the form

$$x^\mu\mapsto x^\mu+f^\mu(x)$$

where $f^\mu(x)$ is small of first order, i.e., so small that only terms linear in f^μ and its derivatives have to be kept. Then the metric transforms as

$$g_{\mu\nu}(x)dx^\mu dx^\nu\mapsto g_{\mu\nu}\big(x+f\big)\big(dx^\mu+\partial_\rho f^\mu(x)dx^\rho\big)\big(dx^\nu+\partial_\sigma f^\nu(x)dx^\sigma\big)$$

$$=g_{\mu\nu}(x)dx^\mu dx^\nu+\partial_\sigma g_{\mu\nu}(x)f^\sigma(x)dx^\mu dx^\nu+\partial_\rho f^\mu(x)g_{\mu\nu}(x)dx^\rho dx^\nu+\partial_\sigma f^\nu(x)g_{\mu\nu}(x)dx^\mu dx^\sigma+\ldots$$

$$=\Big(g_{\mu\nu}(x)+\partial_\sigma g_{\mu\nu}(x)f^\sigma(x)+\partial_\mu f^\rho(x)g_{\rho\nu}(x)+\partial_\nu f^\sigma(x)g_{\mu\sigma}(x)\Big)dx^\mu dx^\nu$$

i.e.

$$\overset{B}{g}_{\mu\nu}+h_{\mu\nu}\mapsto\overset{B}{g}_{\mu\nu}+h_{\mu\nu}+f^\sigma\partial_\sigma\overset{B}{g}_{\mu\nu}+\overset{B}{g}_{\rho\nu}\partial_\mu f^\rho+\overset{B}{g}_{\mu\sigma}\partial_\nu f^\sigma+\ldots$$

hence

$$h_{\mu\nu}\;\mapsto\;h_{\mu\nu}+f^\sigma\partial_\sigma\overset{B}{g}_{\mu\nu}+\overset{B}{g}_{\rho\nu}\partial_\mu f^\rho+\overset{B}{g}_{\mu\sigma}\partial_\nu f^\sigma+\ldots$$

$$=h_{\mu\nu}+f^\tau\partial_\tau\overset{B}{g}_{\mu\nu}+\overset{B}{g}_{\rho\nu}\big(\overset{B}{\nabla}_\mu f^\rho-\overset{B}{\Gamma}{}^\rho{}_{\mu\tau}f^\tau\big)+\overset{B}{g}_{\mu\sigma}\big(\overset{B}{\nabla}_\nu f^\sigma-\overset{B}{\Gamma}{}^\sigma{}_{\nu\tau}f^\tau\big)$$

$$=h_{\mu\nu}+\overset{B}{\nabla}_\mu f_\nu+\overset{B}{\nabla}_\nu f_\mu+f^\tau\Big(\partial_\tau\overset{B}{g}_{\mu\nu}-\frac{1}{2}\overset{B}{g}_{\rho\nu}\overset{B}{g}{}^{\rho\sigma}\big(\partial_\mu\overset{B}{g}_{\sigma\tau}+\partial_\tau\overset{B}{g}_{\sigma\mu}-\partial_\sigma\overset{B}{g}_{\mu\tau}\big)-\frac{1}{2}\overset{B}{g}_{\mu\sigma}\overset{B}{g}{}^{\sigma\rho}\big(\partial_\nu\overset{B}{g}_{\rho\tau}+\partial_\tau\overset{B}{g}_{\rho\nu}-\partial_\rho\overset{B}{g}_{\nu\tau}\big)\Big)$$

$$=h_{\mu\nu}+\overset{B}{\nabla}_\mu f_\nu+\overset{B}{\nabla}_\nu f_\mu+f^\tau\Big(\cancel{\partial_\tau\overset{B}{g}_{\mu\nu}}-\frac{1}{2}\big(\cancel{\partial_\mu\overset{B}{g}_{\nu\tau}}+\cancel{\partial_\tau\overset{B}{g}_{\nu\mu}}-\cancel{\partial_\nu\overset{B}{g}_{\mu\tau}}\big)-\frac{1}{2}\big(\cancel{\partial_\nu\overset{B}{g}_{\mu\tau}}+\cancel{\partial_\tau\overset{B}{g}_{\mu\nu}}-\cancel{\partial_\mu\overset{B}{g}_{\nu\tau}}\big)\Big)$$

$$=h_{\mu\nu}+\overset{B}{\nabla}_\mu f_\nu+\overset{B}{\nabla}_\nu f_\mu\,.$$

This implies

$$h\;\mapsto\;h+2\overset{B}{\nabla}_\sigma f^\sigma$$

or equivalently

$$\gamma\;\mapsto\;\gamma-2\overset{B}{\nabla}_\sigma f^\sigma\,,$$

and

$$\gamma_{\mu\nu}=h_{\mu\nu}-\frac{h}{2}\overset{B}{g}_{\mu\nu}\;\mapsto\;h_{\mu\nu}+\overset{B}{\nabla}_\mu f_\nu+\overset{B}{\nabla}_\nu f_\mu-\frac{1}{2}\big(h+2\overset{B}{\nabla}_\sigma f^\sigma\big)\overset{B}{g}_{\mu\nu}=\gamma_{\mu\nu}+\overset{B}{\nabla}_\mu f_\nu+\overset{B}{\nabla}_\nu f_\mu-\overset{B}{\nabla}_\sigma f^\sigma\overset{B}{g}_{\mu\nu}\,.$$

Hence, the covariant divergence of $\gamma_{\mu\nu}$ transforms according to

$$\overset{B}{\nabla}{}^{\mu}\gamma_{\mu\nu} \mapsto \overset{B}{\nabla}{}^{\mu}\gamma_{\mu\nu} + \overset{B}{\nabla}{}^{\mu}\overset{B}{\nabla}_{\mu}f_{\nu} + \overset{B}{\nabla}{}^{\mu}\overset{B}{\nabla}_{\nu}f_{\mu} - \overset{B}{\nabla}{}^{\mu}\overset{B}{\nabla}_{\sigma}f^{\sigma}\overset{B}{g}_{\mu\nu}$$

$$= \overset{B}{\nabla}{}^{\mu}\gamma_{\mu\nu} + \overset{B}{\Box}f_{\nu} + \overset{B}{\nabla}_{\sigma}\overset{B}{\nabla}_{\nu}f^{\sigma} - \overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\sigma}f^{\sigma} = \overset{B}{\nabla}{}^{\mu}\gamma_{\mu\nu} + \overset{B}{\Box}f_{\nu} + \underbrace{\overset{B}{R}{}^{\sigma}{}_{\sigma\nu\tau}}_{=0}f^{\tau}\,.$$

If f^{μ} is chosen such that

$$\overset{B}{\Box}f_{\nu} \;=\; -\overset{B}{\nabla}{}^{\mu}\gamma_{\mu\nu}\,, \qquad \text{(WE)}$$

the covariant divergence of $\gamma_{\mu\nu}$ is transformed to zero. (In general, a solution f^{μ} to the inhomogeneous wave equation (WE) does not exist globally on the spacetime manifold, unless the spacetime is what is called "globally hyperbolic". However, it does exist on a certain neighbourhood of a spacelike hypersurface, by giving initial data on this spacelike hypersurface. In the following we are satisfied with having coordinates with the desired property on such a neighbourhood.) Then

$$2\,\delta G_{\mu\nu} \;=\; \overset{B}{\Box}\gamma_{\mu\nu} - 2\overset{B}{R}{}^{\tau}{}_{\rho\mu\nu}\gamma_{\tau}{}^{\rho}\,.$$

We still have the freedom of making coordinate transformations $x^{\mu} \mapsto x^{\mu} + f^{\mu}(x)$ with $\overset{B}{\Box}f^{\mu} = 0$. Up to now we have assumed that the background metric satisfies the vacuum field equation. From now on we will assume, in addition, that the perturbed spacetime satisfies the vacuum field equation as well. Then we get

$$\overset{B}{\Box}\gamma_{\mu\nu} - 2\overset{B}{R}{}^{\tau}{}_{\rho\mu\nu}\gamma_{\tau}{}^{\rho} \;=\; 0\,. \qquad (*)$$

In this case we can use the remaining gauge freedom to make the trace $\gamma = -h$ to zero, so that $\gamma_{\mu\nu} = h_{\mu\nu}$.

> Proof: We can still make transformations $x^{\mu} \mapsto x^{\mu} + f^{\mu}(x)$ with $\overset{B}{\Box}f^{\mu} = 0$. This wave equation has a unique solution if initial data $f^{\mu}(x^0 = 0)$ and $\partial_0 f^{\mu}(x^0 = 0)$ are prescribed. Here we assume that $x^0 = 0$ is a spacelike hypersurface; then the solution is defined on a neighbourhood of this hypersurface. We have to show that we can choose the initial data such that the solution f^{μ} satisfies $2\overset{B}{\nabla}_{\mu}f^{\mu} - \gamma = 0$. We first observe that the expression $\psi = 2\overset{B}{\nabla}_{\mu}f^{\mu} - \gamma$ satisfies the wave equation:
>
> $$\overset{B}{\Box}\psi = 2\overset{B}{\nabla}{}^{\nu}\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\mu}f^{\mu} - \overset{B}{\Box}\gamma = 2\overset{B}{\nabla}{}^{\nu}\big(\overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu}f^{\mu} + \underbrace{\overset{B}{R}{}^{\mu}{}_{\nu\mu\sigma}}_{=0}f^{\sigma}\big) - \overset{B}{\Box}\gamma = 2\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}_{\mu}\overset{B}{\nabla}{}^{\nu}f^{\mu} - \overset{B}{\Box}\gamma$$
>
> $$= 2\big(\overset{B}{\nabla}_{\mu}\overset{B}{\nabla}_{\nu}\overset{B}{\nabla}{}^{\nu}f^{\mu} - \underbrace{R^{\nu}{}_{\nu\mu\sigma}}_{=0}\overset{B}{\nabla}{}^{\sigma}f^{\mu} - \underbrace{R^{\mu}{}_{\nu\mu\sigma}}_{=0}\overset{B}{\nabla}{}^{\nu}f^{\sigma}\big) - \overset{B}{\Box}\gamma = 2\overset{B}{\nabla}_{\mu}\underbrace{\overset{B}{\Box}f^{\mu}}_{=0} - \overset{B}{\Box}\gamma = 0$$
>
> where the last equality follows from taking the trace of $(*)$. This demonstrates that ψ is uniquely determined by its initial data $\psi(x^0 = 0)$ and $\partial_0\psi(x^0 = 0)$. If we can

choose the initial data for f^μ such that $\psi(x^0 = 0) = 0$ and $\partial_0\psi(x^0 = 0) = 0$, then ψ must vanish everywhere. As

$$\psi = 2\overset{B}{\nabla}_\mu f^\mu - \gamma = 2\overset{B}{\nabla}_0 f^0 + 2\overset{B}{\nabla}_i f^i - \gamma$$

does not involve x^0 derivatives of the f^i, and

$$\partial_0\psi = \overset{B}{\nabla}_0\psi = 2\overset{B}{g}_{00}\overset{B}{\nabla}^0\overset{B}{\nabla}_0 f^0 + 2\overset{B}{\nabla}_0\overset{B}{\nabla}_i f^i - \overset{B}{\nabla}_0\gamma = -2\overset{B}{g}_{00}\overset{B}{\nabla}^i\overset{B}{\nabla}_i f^0 + 2\overset{B}{\nabla}_0\overset{B}{\nabla}_i f^i - \overset{B}{\nabla}_0\gamma$$

does not involve an x^0 derivative of f^0, such a choice of initial data is indeed possible.

□

As the main result of this section, we have found that the linearised vacuum field equation reads

$$\overset{B}{\Box}h_{\mu\nu} - 2\overset{B}{R}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho \;=\; 0$$

in a gauge such that

$$\overset{B}{\nabla}^\mu h_{\mu\nu} \;=\; 0\,, \qquad h = 0\,.$$

We have shown that such a gauge is possible. Obviously, our gauge generalises the Hilbert gauge to the case of a curved background and it incorporates, in addition, the consition that the trace vanishes. Note, however, that it does not reproduce emphall properties of the TT gauge that we have discussed for plane-harmonic waves on flat background. In contrast to the TT gauge, is *not* in general possible to choose the coordinates on our curved background such that $h_{0\mu} = 0$.

We see that, even in the most convenient gauge, the linearised vacuum field equation on a curved background is not just the ordinary wave equation for $h_{\mu\nu}$ but it contains an additional curvature term. This term somewhat resembles the mass term in the Klein-Gordon equation, $\overset{B}{\Box}\psi - m^2\psi = 0$.

7.2 "Geometric optics" of gravitational waves on a curved background

We will now discuss how gravitational waves propagate, on a curved background, in the high-frequency limit. This is the gravitational-wave analogue of geometric optics. In the last section we have seen that, in a gauge with $\overset{B}{\nabla}^\mu h_{\mu\nu} = 0$ and $h = 0$, the linearised vacuum field equation reads

$$\overset{B}{\Box}h_{\mu\nu} - 2\overset{B}{R}^\tau{}_{\rho\mu\nu}h_\tau{}^\rho \;=\; 0\,.$$

On a flat background, we have found plane-wave solutions which are of the form

$$h_{\mu\nu}(x) = \mathrm{Re}\big\{A_{\mu\nu}\mathsf{e}^{k_\sigma x^\sigma}\big\}$$

with a constant complex amplitude $A_{\mu\nu}$ and a constant real wave covector k_μ. In the case of a curved background, exact solutions of this form do not exist. However, we can start out with the more general ansatz

$$h_{\mu\nu}(x) = \mathrm{Re}\big\{h^{(0)}_{\mu\nu}(x)\mathsf{e}^{i\phi(x)}\big\}$$

where $h^{(0)}_{\mu\nu}(x)$ is a complex-valued function of x and $\phi(x)$ is a real-valued function of x. With this ansatz we want to find *approximate* solutions that resemble plane-harmonic waves in sufficiently small spacetime regions. In the following we drop the "Re" sign to ease notation, but it is always understood that the real part is to be taken.

With our ansatz, we calculate

$$\overset{B}{\Box} h_{\mu\nu} = \overset{B}{\nabla}{}^{\sigma} \overset{B}{\nabla}_{\sigma} \big(h^{(0)}_{\mu\nu} \mathsf{e}^{i\phi} \big) = \overset{B}{\nabla}{}^{\sigma} \Big(\mathsf{e}^{i\phi} \big(\overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} + h^{(0)}_{\mu\nu} i \overset{B}{\nabla}_{\sigma} \phi \big) \Big)$$

$$= \mathsf{e}^{i\phi} \Big(\overset{B}{\nabla}{}^{\sigma} \overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} + \overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} i \overset{B}{\nabla}{}^{\sigma} \phi + \overset{B}{\nabla}{}^{\sigma} h^{(0)}_{\mu\nu} i \overset{B}{\nabla}_{\sigma} \phi + h^{(0)}_{\mu\nu} i \overset{B}{\nabla}{}^{\sigma} \overset{B}{\nabla}_{\sigma} \phi + h^{(0)}_{\mu\nu} i \overset{B}{\nabla}{}^{\sigma} \phi i \overset{B}{\nabla}_{\sigma} \phi \Big)$$

$$= \mathsf{e}^{i\phi} \Big(\overset{B}{\Box} h^{(0)}_{\mu\nu} - h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi \overset{B}{\nabla}_{\sigma} \phi + i \big(2 \overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi + h^{(0)}_{\mu\nu} \overset{B}{\Box} \phi \big) \Big) \,.$$

Feeding this expression into the linearised field equation yields

$$\cancel{\mathsf{e}^{i\phi}} \Big(\overset{B}{\Box} h^{(0)}_{\mu\nu} - h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi \overset{B}{\nabla}_{\sigma} \phi + i \big(2 \overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi + h^{(0)}_{\mu\nu} \overset{B}{\Box} \phi \big) \Big) - 2 \overset{B}{R}{}^{\tau}{}_{\rho\mu\nu} h^{(0)\rho}_{\tau} \cancel{\mathsf{e}^{i\phi}} \;=\; 0 \,,$$

$$h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi \overset{B}{\nabla}_{\sigma} \phi + i \big(2 \overset{B}{\nabla}_{\sigma} h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi + h^{(0)}_{\mu\nu} \overset{B}{\Box} \phi \big) + h^{(0)}_{\mu\nu} \overset{B}{\Box} \phi \big) + \overset{B}{\Box} h^{(0)}_{\mu\nu} - 2 \overset{B}{R}{}^{\tau}{}_{\rho\mu\nu} h^{(0)\rho}_{\tau} \;=\; 0 \,.$$

We now impose the approximations that allow to interpret our $h_{\mu\nu}(x)$ as an approximate plane-harmonic wave. To that end we assume that the phase varies more rapidly than the amplitude. We fix an event with coordinates x_0 and we consider a neighbourhood of x_0 that is so small that the amplitude can be treated as almost constant on this neighbourhood but ϕ varies such that it can be written, approximately, as

$$\phi(x) \approx \phi(x_0) + \overset{B}{\nabla}_{\mu} \phi(x_0) \big(x^{\mu} - x^{\mu}_0 \big) \,.$$

As the constant phase factor $\mathsf{e}^{i \left(\phi(x_0) - \overset{B}{\nabla}_{\mu} \phi(x_0) x^{\mu}_0 \right)}$ can be absorbed into the amplitude, this means that our $h_{\mu\nu}$ can be approximated as a plane-harmonic wave with wave covector

$$k_{\mu} = \overset{B}{\nabla}_{\mu} \phi(x_0) \,.$$

As this can be done around *any* point, the function $\overset{B}{\nabla}_{\mu} \phi$ can be interpreted as the covector *field* of our approximate plane-harmonic wave. If we split this into temporal and spatial parts, we may say that $\overset{B}{\nabla}_0 \phi$ gives the frequency (up to a factor of c) and that the $\overset{B}{\nabla}_i \phi$ give the components of the spatial wave covector.

If we adopt this approximation scheme, terms which are quadratic in the derivatives of ϕ are bigger than terms that are linear in the derivatives of ϕ, and the latter are bigger than terms which do not involve any derivatives of ϕ. This can be done, in a more formal way, by writing a book-keeping parameter α in front of the phase and then comparing equal orders of α. The highest-order term in the linearised field equation is a second-order term in α. Equating it to the zero on the right-hand side yields

$$h^{(0)}_{\mu\nu} \overset{B}{\nabla}{}^{\sigma} \phi \overset{B}{\nabla}_{\sigma} \phi = 0, \,. \qquad \text{(F0)}$$

Next we get two first-order terms which require

$$2\overset{B}{\nabla}_\sigma h^{(0)}_{\mu\nu}\overset{B}{\nabla}^\sigma\phi + h^{(0)}_{\mu\nu}\overset{B}{\Box}\phi = 0\,. \qquad \text{(F1)}$$

Finally, the zeroth-order terms give

$$\overset{B}{\Box}h^{(0)}_{\mu\nu} - 2\overset{B}{R}^\tau{}_{\rho\mu\nu}h^{(0)\rho}_\tau \;=\; 0\,. \qquad \text{(F2)}$$

As the amplitude has, of course, to be non-zero almost everywhere to give a non-trivial wave, (F0) says that the wave covector field $k_\mu = \overset{B}{\nabla}_\mu\phi$ is lightlike,

$$\overset{B}{g}{}^{\mu\nu}k_\mu k_\nu = 0\,.$$

This implies that the integral curves of the wave vector field $k^\nu = \overset{B}{g}{}^{\nu\mu}k_\mu = \overset{B}{g}{}^{\nu\mu}\overset{B}{\nabla}_\mu\phi$ are lightlike geodesics.

Proof: Obviously,

$$\overset{B}{g}_{\mu\nu}k^\mu k^\nu = \overset{B}{g}{}^{\mu\nu}k_\mu k_\nu = 0\,,$$

so the integral curves of k^ν are lightlike. What we still have to show is that they are geodesics, i.e., that

$$k^\mu\overset{B}{\nabla}_\mu k^\rho = 0\,.$$

To that end, we apply the covariant derivative operator to the equation

$$0 = \overset{B}{g}{}^{\mu\nu}k_\mu k_\nu$$

which results in

$$0 = \overset{B}{\nabla}_\rho\big(\overset{B}{g}{}^{\mu\nu}k_\mu k_\nu\big) = 2\overset{B}{g}{}^{\mu\nu}\big(\overset{B}{\nabla}_\rho k_\mu\big)k_\nu = 2k^\mu\big(\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\mu\phi\big) = 2k^\mu\overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho\phi = 2k^\mu\overset{B}{\nabla}_\mu k_\rho$$

where we have used that, for a *scalar* field ϕ,

$$\overset{B}{\nabla}_\mu\phi = \partial_\mu\phi\,,$$

hence

$$\overset{B}{\nabla}_\rho\overset{B}{\nabla}_\mu\phi = \overset{B}{\nabla}_\rho\partial_\mu\phi = \partial_\rho\partial_\mu\phi - \Gamma^\sigma{}_{\rho\mu}\overset{B}{\nabla}_\sigma\phi = \partial_\mu\partial_\rho\phi - \Gamma^\sigma{}_{\mu\rho}\overset{B}{\nabla}_\sigma\phi = \overset{B}{\nabla}_\mu\partial_\rho\phi = \overset{B}{\nabla}_\mu\overset{B}{\nabla}_\rho\phi\,.$$

□

We have thus shown that, to within the linearised theory on a curved background, the rays of gravitational waves behave precisely as the rays of electromagnetic waves, i.e., that "gravitational waves propagate at the speed of light".

Now we evaluate the remaining equations (F1) and (F2). To that end we introduce the normalised amplitude

$$\hat{h}^{(0)}_{\mu\nu}(x) = \frac{h^{(0)}_{\mu\nu}(x)}{H(x)}\,, \qquad H(x) := \sqrt{h^{(0)\rho\tau}(x)h^{(0)}_{\rho\tau}(x)}\,.$$

From (F1), which can be rewritten as

$$2k^\sigma \overset{B}{\nabla}_\sigma h^{(0)}_{\mu\nu} = -h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda \overset{B}{\nabla}^\lambda \phi = -h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda \,,$$

we find

$$k^\sigma \overset{B}{\nabla}_\sigma H = k^\sigma \overset{B}{\nabla}_\sigma \sqrt{h^{(0)}_{\mu\nu} h^{(0)\mu\nu}} = \frac{2h^{(0)\mu\nu} k^\sigma \overset{B}{\nabla}_\sigma h_{(0)\mu\nu}}{2\sqrt{h^{(0)}_{\rho\tau} h^{(0)\rho\tau}}}$$

$$= \frac{-h^{(0)\mu\nu} h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda}{2H} = \frac{-H^2 \overset{B}{\nabla}_\lambda k^\lambda}{2H} = \frac{-H}{2} \overset{B}{\nabla}_\lambda k^\lambda \,.$$

With this information, we evaluate (F1) in the following way.

$$0 = 2k^\sigma \overset{B}{\nabla}_\sigma h^{(0)}_{\mu\nu} + h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda = 2k^\sigma \overset{B}{\nabla}_\sigma \big(H \hat{h}^{(0)}_{\mu\nu}\big) + h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda$$

$$= 2k^\sigma \big(H \overset{B}{\nabla}_\sigma \hat{h}^{(0)}_{\mu\nu} + \hat{h}^{(0)}_{\mu\nu} \overset{B}{\nabla}_\sigma H\big) + h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda = 2k^\sigma H \overset{B}{\nabla}_\sigma \hat{h}^{(0)}_{\mu\nu} + \cancel{2\hat{h}^{(0)}_{\mu\nu} \Big(\frac{-H}{2} \overset{B}{\nabla}_\lambda k^\lambda\Big)} + \cancel{h^{(0)}_{\mu\nu} \overset{B}{\nabla}_\lambda k^\lambda} \,.$$

As H is non-zero almost everywhere, the last equation implies

$$k^\sigma \overset{B}{\nabla}_\sigma \hat{h}^{(0)}_{\mu\nu} = 0 \,,$$

i.e., the normalised amplitude tensor is parallely transported along each ray. This result is again completely analogous to the electrodynamic case, where the polarisation vector is parallely transported along each ray.

Finally, we have to take (F2) into account. This equation implies that, on a non-flat background, the amplitude $h^{(0)}_{\mu\nu}$ is not in general constant.

We emphasise that our approximation scheme was based on the idea of comparing equal powers of derivatives of ϕ. By adopting this approximation scheme, we have not only assumed that the amplitude is varying slowly in comparison to the phase; we have also assumed that the curvature term in the linearised field equation is small in comparison to the square of the derivative of ϕ. By contrast, if the curvature term is of the same order as the square of the derivative of ϕ, it moves from (F2) to (F0), i.e., (F0) has to be replaced with

$$\overset{B}{g}^{\sigma\rho} \overset{B}{\nabla}^\sigma \phi \overset{B}{\nabla}_\sigma \phi - 2 \overset{B}{R}^\tau{}_{\rho\mu\nu} h^{(0)\rho}_\tau \;=\; 0 \,.$$

In this case, it is not true that that the rays of gravitational waves are lightlike geodesics; the background curvature produces a modification of the dispersion relation, i.e., of the relation between frequency and wavelength of a gravitational wave. The question of which approximation is justified depends on the physical situation: If the curvature of the spacetime is small in comparison to λ^{-2}, our original approximation is valid; if it is of the same order of magnitude, the latter approximation is valid. Here λ denotes the wavelength of the gravitational wave. (Note that the curvature tensor has dimension length^{-2}.)

7.3 Linearised field equation on Schwarzschild spacetime

In Section 7.1 we have derived the linearised vacuum field equation on a curved background in the form

$$\overset{B}{\Box} h_{\mu\nu} - 2\overset{B}{R}{}^{\tau}{}_{\rho\mu\nu} h_{\tau}{}^{\rho} \;=\; 0$$

which is valid only in a gauge such that $\overset{B}{\nabla}{}^{\mu} h_{\mu\nu} \;=\; 0$ and $h = 0$. As an alternative, we can write the linearised vacuum field equation in the form

$$0 \;=\; \delta R_{\nu\sigma} \;=\; \delta R^{\mu}{}_{\nu\mu\sigma} \;=\; \overset{B}{\nabla}_{\nu} \delta\Gamma^{\mu}{}_{\mu\sigma} - \overset{B}{\nabla}_{\mu} \delta\Gamma^{\mu}{}_{\nu\sigma}$$

which is true in *any* gauge, see p.65. In the following we will use the latter form because it leaves us the freedom of making arbitrary gauge transformations.

It is our goal to evaluate this equation for the case that the background metric is the Schwarzschild spacetime,

$$\overset{B}{g}_{\mu\nu} dx^{\mu} dx^{\nu} \;=\; -\Big(1 - \frac{r_S}{r}\Big) c^2 dt^2 \;+\; \frac{dr^2}{\Big(1 - \dfrac{r_S}{r}\Big)} \;+\; r^2 \big(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2\big) \,.$$

For later purpose, we list the non-vanishing Christoffel symbols of the Schwarzschild spacetime:

$$\overset{B}{\Gamma}{}^{t}_{rt} \;=\; \overset{B}{\Gamma}{}^{t}_{tr} \;=\; \frac{r_S}{2r^2\Big(1 - \dfrac{r_S}{r}\Big)} \,,$$

$$\overset{B}{\Gamma}{}^{r}_{rr} = \frac{-\, r_S}{2r^2\Big(1 - \dfrac{r_S}{r}\Big)} \,, \quad \overset{B}{\Gamma}{}^{r}_{tt} = \frac{c^2 r_S}{2r^2}\Big(1 - \frac{r_S}{r}\Big) \,, \quad \overset{B}{\Gamma}{}^{r}_{\varphi\varphi} = -r\Big(1 - \frac{r_S}{r}\Big)\sin^2\!\vartheta \,, \quad \overset{B}{\Gamma}{}^{r}_{\vartheta\vartheta} \;=\; -\, r\Big(1 - \frac{r_S}{r}\Big)$$

$$\overset{B}{\Gamma}{}^{\vartheta}_{r\vartheta} \;=\; \overset{B}{\Gamma}{}^{\vartheta}_{\vartheta r} \;=\; \frac{1}{r} \,, \quad \overset{B}{\Gamma}{}^{\vartheta}_{\varphi\varphi} \;=\; -\sin\vartheta\cos\vartheta$$

$$\overset{B}{\Gamma}{}^{\varphi}_{\vartheta\varphi} \;=\; \overset{B}{\Gamma}{}^{\varphi}_{\varphi\vartheta} \;=\; \cot\vartheta \,, \quad \overset{B}{\Gamma}{}^{\varphi}_{r\varphi} \;=\; \overset{B}{\Gamma}{}^{\varphi}_{\varphi r} \;=\; \frac{1}{r}$$

As the Schwarzschild spacetime is static and spherically symmetric, we can separate off the time part and the angle part so that, in the end, we are left with an ordinary differential equation for the radial part. The procedure is quite analogous to solving the Schrödinger equation with a time-independent spherically symmetric potential: One splits off the time part $\big(\sim \mathrm{e}^{i\omega t}\big)$ and the angle part $\big(\sim Y_{\ell m}(\vartheta,\varphi)\big)$ and is then left with an ordinary differential equation for the radial part; in the case of the Coulomb potential, this radial differential equation has the Laguerre polynomials as the solutions.

In the case at hand, the situation is considerably more complicated than in the case of the Schrödinger equation as our unknown function is not a scalar ψ but a tensor field $h_{\mu\nu}$. Therefore, we have to deal with vectorial and tensorial spherical harmonics in addition to the ordinary (scalar) spherical harmonics $Y_{\ell m}(\vartheta,\varphi)$. Also, the gauge freedom has to be taken into account. Instead of beginning with splitting off the time part, as one usual does with the Schrödinger equation, it is more convenient to do this at the very end. We proceed in the following five steps.

Step 1: Expand $h_{\mu\nu}(t, r, \vartheta, \varphi)$ in terms of spherical harmonics.

Step 2: Decompose $h_{\mu\nu}(t, r, \vartheta, \varphi)$ into parts that are even and odd, respectively, with respect to parity transformations.

Step 3: Restrict to the odd parts. Simplify $h_{\mu\nu}(t, r, \vartheta, \varphi)$ with the help of an appropriate gauge transformation.

Step 4: Insert this simplified expression into the linearised vacuum field equation. After an appropriate substitution this results in one equation for one scalar function $Q_{\ell m}(t, r)$, known as the time-dependent *Regge-Wheeler equation* [T. Regge, J. Wheeler: "Stability of a Schwarzschild singularity" Phys. Rev. 108, 1063 (1957)].

Step 5: Separate off the time part to get an ordinary differential equation for a funcion that depends on the radial variable only, known as the time-independent Regge-Wheeler equation.

The treatment of perturbations that are even with respect to parity transformations is considerably more difficult. We will not work this out here. In the end, also in this case one arrives at a Regge-Wheeler type equation for a radial function with an effective potential. It is called the *Zerilli equation* and was found only 13 years after the Regge-Wheeler paper [F. Zerilli: "Effective potential for even-parity Regge-Wheeler gravitational perturbation equations", Phys. Rev. Lett. 24, 737 (1970)].

Step 1: As a preparation for expanding the metric perturbation into spherical harmonics, we write it in the form

$$h_{\mu\nu}(t, r, \vartheta, \varphi)dx^\mu dx^\nu \;=\; h_{AB}(t, r, \vartheta, \varphi)dx^A dx^B$$
$$+\, 2\, h_{A\Sigma}(t, r, \vartheta, \varphi)dx^A dx^\Sigma \;+\; h_{\Sigma\Omega}(t, r, \vartheta, \varphi)dx^\Sigma dx^\Omega\,.$$

Here and in the following, indices $A, B, C, \ldots$ take values r and t while indices $\Sigma, \Omega, \Delta, \ldots$ take values ϑ and φ. Recall that two covector fields without a symbol between them means the symmetrised tensor product, $dx^A dx^\Sigma = \frac{1}{2}\big(dx^A \otimes dx^\Sigma + dx^\Sigma \otimes dx^A\big)$. We see that, with respect to the angular part, the perturbation splits into three scalar functions h_{AB}, two covector fields $h_{A\Sigma}dx^\Sigma$ and a symmetric second-rank tensor field $h_{\Sigma\Omega}dx^\Sigma dx^\Omega$. We do the expansion into spherical harmonics for these three cases separately.

Scalar part: For fixed (t, r), we have three scalar functions h_{tt}, $h_{rt} = h_{tr}$ and h_{rr} on the sphere. These can be expanded into the usual (scalar-valued) spherical harmonics. For non-negative m, they are defined as

$$Y_{\ell m}(\vartheta, \varphi) \;=\; C_{\ell m} P_{\ell m}\big(\cos\vartheta\big)\mathrm{e}^{im\varphi}$$

where the $P_{\ell m}$ are the associated Legendre polynomials,

$$P_{\ell m}(x) \;=\; (-1)^m (1 - x^2)^{m/2}\Big(\frac{d}{dx}\Big)^m P_\ell(x)\,,$$

the P_ℓ are the Legendre polynomials,

$$P_\ell(x) \;=\; \frac{1}{2^\ell \ell!}\Big(\frac{d}{dx}\Big)^\ell P_\ell\big(x^2 - 1\big)^\ell\,,$$

and the $C_{\ell m}$ are normalisation factors,

$$C_{\ell m} \;=\; \sqrt{\frac{(2\ell+1)(l-m)!}{4\pi(\ell+m)!}}\,.$$

The definition is extended to the case of negative m by requiring

$$Y_{\ell(-m)}(\vartheta,\varphi) \;=\; (-1)^m\overline{Y_{\ell m}(\vartheta,\varphi)}$$

where overlining means complex conjugation. ℓ runs over all integers from 0 to ∞ and m runs, for fixed ℓ, over all integers from $-\ell$ to ℓ.

Expanding with respect to spherical harmonics gives us the scalar parts of the metric perturbation as

$$h_{AB}(t,r,\vartheta,\varphi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_{AB\ell m}(t,r)Y_{\ell m}(\vartheta,\varphi)\,.$$

Covector part: For fixed (t,r), we have two covector fields $h_{t\Sigma}dx^\Sigma$ and $h_{r\Sigma}dx^\Sigma$ on the sphere. The coefficients $h_{A\Sigma}$ are scalar-valued functions on the sphere, so one could expand them in terms of the scalar-valued spherical harmonics $Y_{\ell m}$. However, this would not be meaningful because the $h_{A\Sigma}$ are not *invariant* scalar functions; they change if a new coordinate basis is chosen on the sphere. To get an expansion that respects the invariance properties of the mathematical objects, one needs (co)vector-valued spherical harmonics. As the sphere is two-dimensional, we need two such sets for a basis. There are different choices for such a basis. Here we choose the same basis as Regge and Wheeler: For the first set we choose the gradients of the $Y_{\ell m}$, which clearly have the correct transformation behaviour as covector fields,

$$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \overset{B}{\nabla}_{\!\Sigma}Y_{\ell m}(\vartheta,\varphi)dx^\Sigma\,.$$

The second set is constructed orthogonal to the first set,

$$\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \varepsilon_{\Sigma\Omega}\overset{B}{\nabla}{}^{\!\Omega}Y_{\ell m}(\vartheta,\varphi)dx^\Sigma\,,$$

where

$$\varepsilon_{\Sigma\Omega}dx^\Sigma\otimes dx^\Omega \;=\; r^2\sin\vartheta\big(d\vartheta\otimes d\varphi - d\varphi\otimes d\vartheta\big)$$

is the Levi-Civita tensor field (volume form) on the sphere. The latter is defined by the properties that it is anti-symmetric, $\varepsilon_{\Sigma\Omega} = -\varepsilon_{\Omega\Sigma}$, and that it evaluates to unity on an orthonormal basis,

$$\Big(\varepsilon_{\Sigma\Omega}dx^\Sigma\otimes dx^\Omega\Big)\Big(\frac{\partial_\vartheta}{\sqrt{\overset{B}{g}_{\vartheta\vartheta}}},\frac{\partial_\varphi}{\sqrt{\overset{B}{g}_{\varphi\varphi}}}\Big) \;=\; \frac{\varepsilon_{\vartheta\varphi}}{\sqrt{\overset{B}{g}_{\vartheta\vartheta}\,\overset{B}{g}_{\varphi\varphi}}} \;=\; \frac{r^2\sin\vartheta}{\sqrt{r^2r^2\sin^2\vartheta}} \;=\; 1\,.$$

As the covariant derivative of a scalar function is the same as the partial derivative, the (co)vector-valued spherical harmonics can be rewritten as

$$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;=\; \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\vartheta \;+\; \partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\varphi\,,$$

$$\begin{aligned}\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma \;&=\; \varepsilon_{\vartheta\varphi}\overset{B}{g}{}^{\varphi\varphi}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;+\; \varepsilon_{\varphi\vartheta}\overset{B}{g}{}^{\vartheta\vartheta}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi\\ &=\; \cancel{r^2}\cancel{\sin\vartheta}\,\frac{1}{\cancel{r^2}\cancel{\sin^2}\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;-\; \cancel{r^2}\sin\vartheta\,\frac{1}{\cancel{r^2}}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi\\ &=\; \frac{1}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta \;-\; \sin\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\varphi\,.\end{aligned}$$

We expand the covector parts of the metric perturbation with respect to this basis as

$$h_{A\Sigma}dx^\Sigma \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(\hat{v}_{A\ell m}(t,r)\Psi_{\ell m\Sigma} + v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}\Big)dx^\Sigma\,.$$

Tensor part: For fixed (t,r), we have a symmetric second-rank tensor field $h_{\Sigma\Omega}dx^\Sigma dx^\Omega$ on the sphere. On a two-dimensional space, a symmetric second-rank tensor has three independent components, so we need three sets of tensor-valued spherical harmonics for a basis. Again, we choose the basis in the same way as Regge and Wheeler. As the second covariant derivative of a scalar function gives a symmetric second-rank tensor field, we choose for the first set

$$\Psi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \Big(\overset{B}{\nabla}_\Sigma\overset{B}{\nabla}_\Omega Y_{\ell m}\Big)(\vartheta,\varphi)dx^\Sigma dx^\Omega\,.$$

As the angular part of the metric, $\overset{B}{g}_{\Sigma\Omega}$, is a symmetric second-rank tensor field proportional to r^2, we choose for the second set

$$\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \Big(Y_{\ell m}\,\frac{\overset{B}{g}_{\Sigma\Omega}}{r^2}\Big)(\vartheta,\varphi)dx^\Sigma dx^\Omega\,.$$

The third set we construct, in analogy to our procedure in the covector case, orthogonal to the $\Psi_{\ell m\Sigma\Omega}$,

$$\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \frac{1}{2}\Big(\varepsilon_{\Sigma\Delta}\overset{B}{\nabla}{}^\Delta\overset{B}{\nabla}_\Omega Y_{\ell m} + \varepsilon_{\Omega\Delta}\overset{B}{\nabla}{}^\Delta\overset{B}{\nabla}_\Sigma Y_{\ell m}\Big)(\vartheta,\varphi)\,dx^\Sigma\,dx^\Omega\,.$$

With the help of the Christoffel symbols listed at the beginning of this section, the tensor-valued spherical harmonics can be rewritten as

$$\begin{aligned}\Psi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;&=\; \overset{B}{\nabla}_\vartheta\overset{B}{\nabla}_\vartheta Y_{\ell m}(\vartheta,\varphi)d\vartheta^2 + 2\overset{B}{\nabla}_\vartheta\overset{B}{\nabla}_\varphi Y_{\ell m}(\vartheta,\varphi)d\vartheta d\varphi + \overset{B}{\nabla}_\varphi\overset{B}{\nabla}_\varphi Y_{\ell m}(\vartheta,\varphi)d\varphi^2\\ &=\; \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi)d\vartheta^2 + 2\Big(\partial_\vartheta\partial_\varphi Y_{\ell m} - \overset{B}{\Gamma}{}^\varphi{}_{\vartheta\varphi}\partial_\varphi Y_{\ell m}\Big)(\vartheta,\varphi)d\vartheta d\varphi + \Big(\partial_\varphi^2 Y_{\ell m} - \overset{B}{\Gamma}{}^\vartheta{}_{\varphi\varphi}\partial_\vartheta Y_{\ell m}\Big)(\vartheta,\varphi)d\varphi^2\\ &=\; \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi)d\vartheta^2 + 2\Big(\partial_\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta\,\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)d\vartheta d\varphi\\ &\quad + \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)d\varphi^2\,,\end{aligned}$$

$$\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; Y_{\ell m}(\vartheta,\varphi)\Big(\frac{\overset{B}{g}_{\vartheta\vartheta}}{r^2}d\vartheta^2 + \frac{\overset{B}{g}_{\varphi\varphi}}{r^2}d\varphi^2\Big) \;=\; Y_{\ell m}(\vartheta,\varphi)\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)\,,$$

$$\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \Big(\varepsilon_{\vartheta\varphi}\overset{B}{g}{}^{\varphi\varphi}\overset{B}{\nabla}_\varphi\overset{B}{\nabla}_\vartheta Y_{\ell m}\Big)(\vartheta,\varphi)d\vartheta^2 + \Big(\varepsilon_{\vartheta\varphi}\overset{B}{g}{}^{\varphi\varphi}\overset{B}{\nabla}_\varphi\overset{B}{\nabla}_\varphi Y_{\ell m}\Big)(\vartheta,\varphi)d\vartheta d\varphi$$

$$+\Big(\varepsilon_{\varphi\vartheta}\overset{B}{g}{}^{\vartheta\vartheta}\overset{B}{\nabla}_\vartheta\overset{B}{\nabla}_\vartheta Y_{\ell m}\Big)(\vartheta,\varphi)d\varphi d\vartheta + \Big(\varepsilon_{\varphi\vartheta}\overset{B}{g}{}^{\vartheta\vartheta}\overset{B}{\nabla}_\vartheta\overset{B}{\nabla}_\varphi Y_{\ell m}\Big)(\vartheta,\varphi)d\varphi^2$$

$$= \frac{\cancel{r^2}\sin\vartheta}{\cancel{r^2}\sin^{\not 2}\vartheta}\Big(\partial_\varphi\partial_\vartheta Y_{\ell m} - \overset{B}{\Gamma}{}^\varphi{}_{\varphi\vartheta}\partial_\varphi Y_{\ell m}\Big)(\vartheta,\varphi)d\vartheta^2 + \frac{\cancel{r^2}\sin\vartheta}{\cancel{r^2}\sin^{\not 2}\vartheta}\Big(\partial^2_\varphi Y_{\ell m} - \overset{B}{\Gamma}{}^\vartheta{}_{\varphi\varphi}\partial_\vartheta Y_{\ell m}\Big)(\vartheta,\varphi)d\vartheta d\varphi$$

$$-\frac{\cancel{r^2}\sin\vartheta}{\cancel{r^2}}\partial^2_\vartheta Y_{\ell m}(\vartheta,\varphi)d\vartheta d\varphi - \frac{\cancel{r^2}\sin\vartheta}{\cancel{r^2}}\Big(\partial_\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \overset{B}{\Gamma}{}^\varphi{}_{\vartheta\varphi}\partial_\varphi Y_{\ell m}\Big)d\varphi^2$$

$$= \frac{1}{\sin\vartheta}\Big(\partial_\varphi\partial_\vartheta Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta\,\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)\big(d\vartheta^2 - \sin^2\vartheta\,d\varphi^2\big)$$

$$+\frac{1}{\sin\vartheta}\Big(\partial^2_\varphi Y_{\ell m}(\vartheta,\varphi) - \sin^2\vartheta\,\partial^2_\vartheta Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)d\vartheta d\varphi\,.$$

We expand the tensorial part of the metric perturbation in the form

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi)dx^\Sigma dx^\Omega \;=$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(\hat w_{\ell m}(t,r)\Psi_{\ell m\Sigma\Omega}(\vartheta,\varphi) + \tilde w_{\ell m}(t,r)\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi) + w_{\ell m}(t,r)\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)\Big)dx^\Sigma dx^\Omega\,.$$

Step 2: We will now investigate the transformation behaviour of our various spherical harmonics with respect to parity transformations (i.e., reflections at the origin)

$$(\vartheta,\varphi) \;\mapsto\; (\pi-\vartheta,\varphi+\pi)\,.$$

Obviously, under such a transformation

$$\cos\vartheta \;\mapsto\; -\cos\vartheta\,,\quad \sin\vartheta \;\mapsto\; \sin\vartheta\,,\quad \mathsf{e}^{im\varphi} \;\mapsto\; \mathsf{e}^{im\varphi}\mathsf{e}^{im\pi} \;=\; \mathsf{e}^{im\varphi}\big(\mathsf{e}^{i\pi}\big)^m \;=\; \mathsf{e}^{im\varphi}(-1)^m\,,$$

$$d\vartheta \;\mapsto\; -\,d\vartheta\,,\quad d\varphi \;\mapsto\; d\varphi\,,\quad \partial_\vartheta \;\mapsto\; -\,\partial_\vartheta\,,\quad \partial_\varphi \;\mapsto\; \partial_\varphi\,,$$

We also need to know that

$$P_\ell(-x) \;=\; \frac{1}{2^\ell\ell!}\Big(-\frac{d}{dx}\Big)^\ell P_\ell\big(x^2-1\big)^\ell \;=\; (-1)^\ell P_\ell(x)\,,$$

$$P_{\ell m}(-x) \;=\; (-1)^m(1-x^2)^{m/2}\Big(-\frac{d}{dx}\Big)^m P_\ell(-x) \;=\; (-1)^m(-1)^\ell P_{\ell m}(x)\,.$$

We introduce the following terminology. A function $F_{\ell m}(\vartheta,\varphi)$ is said to be

- *even* under parity transformations if $F_{\ell m}(\pi-\vartheta,\varphi+\pi) = (-1)^\ell F_{\ell m}(\vartheta,\varphi)$,
- *odd* under parity transformations if $F_{\ell m}(\pi-\vartheta,\varphi+\pi) = (-1)^{\ell+1} F_{\ell m}(\vartheta,\varphi)$.

Instead of even/odd, some authors say *polar/axial*, *electric/magnetic* or *poloidal/toroidal*. With the help of the above transformation rules, we will now demonstrate that

$Y_{\ell m}(\vartheta,\varphi)$ is even,

$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma$ is even,

$\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma$ is odd,

$\Psi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega$ is even,

$\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega$ is even,

$\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega$ is odd.

Proof: Under a parity transformation,

$$Y_{\ell m}(\vartheta,\varphi) \;=\; C_{\ell m}P_{\ell m}\big(\cos\vartheta\big)\mathsf{e}^{im\varphi}$$
$$\mapsto\; C_{\ell m}P_{\ell m}\big(-\cos\vartheta\big)\mathsf{e}^{im\varphi}\mathsf{e}^{im\pi} \;=\; C_{\ell m}(-1)^m(-1)^\ell P_{\ell m}\big(\cos\vartheta\big)\mathsf{e}^{im\varphi}(-1)^m \;=\; (-1)^\ell Y_{\ell m}(\vartheta,\varphi)\,,$$

$$\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma = \partial_\vartheta Y_{\ell m}(\vartheta,\varphi)d\vartheta \,+\, \partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\varphi$$
$$\mapsto\; (-1)^\ell(-\partial_\vartheta)Y_{\ell m}(\vartheta,\varphi)(-d\vartheta) \,+\, (-1)^\ell\partial_\varphi Y_{\ell m}(\vartheta,\varphi)d\varphi \;=\; (-1)^\ell\Psi_{\ell m\Sigma}(\vartheta,\varphi)dx^\Sigma\,,$$

$$\Phi_{\ell m\Sigma}dx^\Sigma = \frac{1}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)dx^\vartheta \,-\, \sin\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)dx^\varphi$$
$$\mapsto\; \frac{(-1)^\ell}{\sin\vartheta}\partial_\varphi Y_{\ell m}(\vartheta,\varphi)(-dx^\vartheta) \,+\, \sin\vartheta\,(-\partial_\vartheta)Y_{\ell m}(\vartheta,\varphi)dx^\varphi \;=\; (-1)^{\ell+1}\Phi_{\ell m\Sigma}dx^\Sigma\,,$$

$$\Psi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi)d\vartheta^2$$
$$+2\Big(\partial_\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)d\vartheta d\varphi + \Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)d\varphi^2$$
$$\mapsto\; (-1)^\ell\partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi)d\vartheta^2 + 2(-1)^\ell\Big(-\partial_\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi) + \cot\vartheta\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)(-d\vartheta)d\varphi$$
$$+(-1)^\ell\Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)d\varphi^2 \;=\; (-1)^\ell\Psi_{\ell m\Sigma\Omega}dx^\Sigma dx^\Omega\,,$$

$$\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; Y_{\ell m}(\vartheta,\varphi)\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)$$
$$\mapsto\; (-1)^\ell Y_{\ell m}(\vartheta,\varphi)\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big) \;=\; (-1)^\ell\Phi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega\,,$$

$$\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \frac{1}{\sin\vartheta}\Big(\partial_\varphi\partial_\vartheta Y_{\ell m}(\vartheta,\varphi) - \cot\vartheta\,\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)\big(d\vartheta^2 - \sin^2\vartheta\,d\varphi^2\big)$$
$$+\frac{2}{\sin\vartheta}\Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) - \sin^2\vartheta\,\partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)d\vartheta d\varphi$$
$$\mapsto\; \frac{(-1)^\ell}{\sin\vartheta}\Big(-\partial_\varphi\partial_\vartheta Y_{\ell m}(\vartheta,\varphi) + \cot\vartheta\,\partial_\varphi Y_{\ell m}(\vartheta,\varphi)\Big)\big(d\vartheta^2 - \sin^2\vartheta\,d\varphi^2\big)$$
$$+\frac{2(-1)^\ell}{\sin\vartheta}\Big(\partial_\varphi^2 Y_{\ell m}(\vartheta,\varphi) - \sin^2\vartheta\,\partial_\vartheta^2 Y_{\ell m}(\vartheta,\varphi) + \sin\vartheta\cos\vartheta\,\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\Big)(-d\vartheta)d\varphi$$
$$=\; (-1)^{\ell+1}\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega$$

□

Step 3:

We restrict to odd metric perturbations,

$$h_{AB}(t,r,\vartheta,\varphi) \;=\; 0\,,$$

$$h_{A\Sigma}(t,r,\vartheta,\varphi)dx^\Sigma \;=\; \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}dx^\Sigma\,,$$

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi)dx^\Sigma dx^\Omega \;=\; \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell w_{\ell m}(t,r)\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega\,.$$

We fix ℓ and m, i.e., we consider one partial wave,

$$h_{\mu\nu}(t,r,\vartheta,\varphi)dx^\mu dx^\nu \;=\; 2\,v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^A dx^\Sigma + w_{\ell m}(t,r)\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)dx^\Sigma dx^\Omega\,.$$

This partial wave is determined by three scalar functions, $v_{t\ell m}(t,r)$, $v_{r\ell m}(t,r)$ and $w_{\ell m}(t,r)$. We will use the gauge freedom for transforming $w_{\ell m}(t,r)$ to zero. To that end, we recall that under a gauge transformation

$$x^\mu \mapsto x^\mu + f^\mu(x) = x^\mu + \overset{B}{g}{}^{\mu\nu}(x) f_\nu(x)$$

the metric perturbation changes according to

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \overset{B}{\nabla}_\mu f_\nu + \overset{B}{\nabla}_\nu f_\mu\,.$$

We choose a gauge function of the form

$$f_A(t,r,\vartheta,\varphi) = 0\,, \quad f_\Sigma(t,r,\vartheta,\varphi) = \Lambda_{\ell m}(t,r)\big(\varepsilon_\Sigma{}^\Omega \overset{B}{\nabla}_\Omega Y_{\ell m}\big)(\vartheta,\varphi)$$

with a function $\Lambda_{\ell m}(t,r)$ to be determined. Note that such a gauge transformation depends on ℓ and m, i.e., it is done for the chosen partial wave. Our gauge transformation preserves the equation $h_{AB}=0$, because with the Christoffel symbols listed at the beginning of this section we find

$$h_{AB} \mapsto h_{AB} + \overset{B}{\nabla}_A f_B + \overset{B}{\nabla}_B f_A = 0 + \partial_A f_B - \overset{B}{\Gamma}{}^\mu{}_{AB} f_\mu + \partial_B f_A - \overset{B}{\Gamma}{}^\mu{}_{BA} f_\mu \;=\; 0 - 2\underbrace{\overset{B}{\Gamma}{}^\Sigma{}_{AB}}_{=0} f_\Sigma \;=\; 0\,.$$

The tensorial part transforms as

$$h_{\Sigma\Omega}(t,r,\vartheta,\varphi) \;=\; w_{\ell m}(t,r)\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi) \;\mapsto\; \big(w_{\ell m}(t,r) + \Lambda_{\ell m}(t,r)\big)\chi_{\ell m\Sigma\Omega}(\vartheta,\varphi)\,.$$

Proof: We first observe that

$$\overset{B}{\nabla}_\Sigma\big(\Lambda_{\ell m}\varepsilon_\Omega{}^\Delta\big) \;=\; \varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Sigma \Lambda_{\ell m} + \Lambda_{\ell m}\overset{B}{\nabla}_\Sigma \varepsilon_\Omega{}^\Delta = \varepsilon_\Omega{}^\Delta \underbrace{\partial_\Sigma \Lambda_{\ell m}}_{=0} + \Lambda_{\ell m}\overset{B}{g}{}^{\Delta\Phi}\underbrace{\overset{B}{\nabla}_\Sigma \varepsilon_{\Omega\Phi}}_{=0} \;=\; 0\,.$$

Here we have used the fact that the scalar function $\Lambda_{\ell m}$ is independent of ϑ and φ and that the Levi-Civita tensor is covariantly constant,

$$\overset{B}{\nabla}_\Sigma \varepsilon_{\Omega\Delta} = 0\,.$$

The latter can be proven in the following way.

$$\overset{B}{\nabla}_\Sigma \varepsilon_{\Delta\Omega} = \partial_\Sigma \varepsilon_{\Delta\Omega} - \overset{B}{\Gamma}{}^{\Pi}{}_{\Sigma\Delta}\varepsilon_{\Pi\Omega} - \overset{B}{\Gamma}{}^{\Pi}{}_{\Sigma\Omega}\varepsilon_{\Delta\Pi}$$

demonstrates that the left-hand side is zero for $\Delta = \Omega$. This follows from the fact that then, because of the antisymmetry of $\varepsilon_{\Delta\Omega}$, the first term on the right-hand side vanishes and the other two compensate each other. Therefore, we only have to consider the case that $\Delta \neq \Omega$:

$$\begin{aligned} -\overset{B}{\nabla}_\vartheta \varepsilon_{\varphi\vartheta} = \overset{B}{\nabla}_\vartheta \varepsilon_{\vartheta\varphi} &= \partial_\vartheta \varepsilon_{\vartheta\varphi} - \overset{B}{\Gamma}{}^{\vartheta}{}_{\vartheta\vartheta}\varepsilon_{\vartheta\varphi} - \overset{B}{\Gamma}{}^{\varphi}{}_{\vartheta\varphi}\varepsilon_{\vartheta\varphi} \\ &= \partial_\vartheta\big(r^2 \sin\vartheta\big) - 0 - \cot\vartheta\, r^2 \sin\vartheta = r^2\cos\vartheta - r^2\cos\vartheta \;=\; 0\,, \end{aligned}$$

$$-\overset{B}{\nabla}_\varphi \varepsilon_{\varphi\vartheta} = \overset{B}{\nabla}_\varphi \varepsilon_{\vartheta\varphi} = \partial_\varphi \varepsilon_{\vartheta\varphi} - \overset{B}{\Gamma}{}^{\vartheta}{}_{\varphi\vartheta}\varepsilon_{\vartheta\varphi} - \overset{B}{\Gamma}{}^{\varphi}{}_{\varphi\varphi}\varepsilon_{\vartheta\varphi} = 0 - 0 - 0 = 0\,.$$

With this result at hand, it is now easy to complete the proof.

$$\begin{aligned} h_{\Sigma\Omega} \;&\mapsto\; h_{\Sigma\Omega} + \overset{B}{\nabla}_\Sigma f_\Omega + \overset{B}{\nabla}_\Omega f_\Sigma \\ &= w_{\ell m}\chi_{\ell m\Sigma\Omega} + \overset{B}{\nabla}_\Sigma\big(\Lambda_{\ell m}\varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Delta Y_{\ell m}\big) + \overset{B}{\nabla}_\Omega\big(\Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta \overset{B}{\nabla}_\Delta Y_{\ell m}\big) \\ &= w_{\ell m}\Big(\varepsilon_{\Sigma\Delta}\overset{B}{\nabla}{}^\Delta \overset{B}{\nabla}_\Omega Y_{\ell m} + \varepsilon_{\Omega\Delta}\overset{B}{\nabla}{}^\Delta \overset{B}{\nabla}_\Sigma Y_{\ell m}\Big) + \Lambda_{\ell m}\varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Sigma \overset{B}{\nabla}_\Delta Y_{\ell m} + \Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta \overset{B}{\nabla}_\Omega \overset{B}{\nabla}_\Delta Y_{\ell m} \\ &= w_{\ell m}\Big(\varepsilon_\Sigma{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Omega Y_{\ell m} + \varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Sigma Y_{\ell m}\Big) + \Lambda_{\ell m}\varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Sigma Y_{\ell m} + \Lambda_{\ell m}\varepsilon_\Sigma{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Omega Y_{\ell m} \\ &= \big(w_{\ell m} + \Lambda_{\ell m}\big)\Big(\varepsilon_\Sigma{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Omega Y_{\ell m} + \varepsilon_\Omega{}^\Delta \overset{B}{\nabla}_\Delta \overset{B}{\nabla}_\Sigma Y_{\ell m}\Big)\,. \end{aligned}$$

□

If we choose $\Lambda_{\ell m}(t,r) = -w_{\ell m}(t,r)$, the tensorial part is transformed to zero and in the new gauge the metric perturbation is determined by just two scalar functions $v_{t\ell m}(t,r)$ and $v_{r\ell m}(t,r)$,

$$h_{\mu\nu}(t,r,\vartheta,\varphi)dx^\mu dx^\nu \;=\; 2\,h_{A\Sigma}(t,r,\vartheta,\varphi)dx^A dx^\Sigma \;=\; 2\,v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}(\vartheta,\varphi)dx^A dx^\Sigma\,.$$

As the diagonal elements of $h_{\mu\nu}$ vanish, it is obvious that the condition of vanishing trace is satisfied

$$\overset{B}{g}{}^{\mu\nu}h_{\mu\nu} = 0\,.$$

The generalised Hilbert gauge condition is, however, *not* satisfied,

$$\overset{B}{\nabla}{}^\mu h_{\mu\nu} \neq 0$$

in general. Note that our choice of gauge, which is known as the *Regge-Wheeler gauge*, is done for a particular (ℓ, m).

Step 4:

Now comes the hard part of the construction. We plug our metric perturbation $h_{\mu\nu}$, whose only non-vanishing components are $h_{A\Sigma}(t,r,\vartheta,\varphi) = v_{A\ell m}(t,r)\Phi_{\ell m\Sigma}(\vartheta,\varphi)$, into the linearised field equation. We use the latter in the gauge-independent form

$$0 \;=\; \overset{B}{\nabla}_\nu \delta\Gamma^\mu{}_{\mu\sigma} - \overset{B}{\nabla}_\mu \delta\Gamma^\mu{}_{\nu\sigma}$$

where

$$2\delta\Gamma^\nu{}_{\rho\sigma} \;=\; \overset{B}{g}{}^{\nu\lambda}\Big(\overset{B}{\nabla}_\rho h_{\lambda\sigma} + \overset{B}{\nabla}_\sigma h_{\lambda\rho} - \overset{B}{\nabla}_\lambda h_{\rho\sigma}\Big)$$

$$= \overset{B}{g}{}^{\nu\lambda}\Big(\partial_\rho h_{\lambda\sigma} - \cancel{\overset{B}{\Gamma}{}^\tau{}_{\rho\lambda} h_{\tau\sigma}} - \overset{B}{\Gamma}{}^\tau{}_{\rho\sigma} h_{\lambda\tau} + \partial_\sigma h_{\lambda\rho} - \cancel{\overset{B}{\Gamma}{}^\tau{}_{\sigma\lambda} h_{\tau\rho}} - \overset{B}{\Gamma}{}^\tau{}_{\sigma\rho} h_{\lambda\tau} - \partial_\lambda h_{\rho\sigma} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\lambda\rho} h_{\tau\sigma}} + \cancel{\overset{B}{\Gamma}{}^\tau{}_{\lambda\sigma} h_{\rho\tau}}\Big)$$

$$= \overset{B}{g}{}^{\nu\lambda}\Big(\partial_\rho h_{\lambda\sigma} + \partial_\sigma h_{\lambda\rho} - \partial_\lambda h_{\rho\sigma} - 2\overset{B}{\Gamma}{}^\tau{}_{\rho\sigma} h_{\lambda\tau}\Big)\,.$$

We will show that, owing to our choice of gauge, $\delta\Gamma^\mu{}_{\mu\sigma} = 0$. We consider first the case $\sigma = A = t, r$, then the case $\sigma = \Sigma = \vartheta, \varphi$. With the help of the Christoffel symbols listed at the beginning of this section we find

$$\delta\Gamma^\mu{}_{\mu A} \;=\; \overset{B}{g}{}^{\mu\lambda}\Big(\cancel{\partial_\mu h_{\lambda A}} + \partial_A h_{\lambda\mu} - \cancel{\partial_\lambda h_{\mu A}} - 2\overset{B}{\Gamma}{}^\tau{}_{\mu A} h_{\lambda\tau}\Big)$$

$$= \overset{B}{g}{}^{CD}\underbrace{\partial_A h_{CD}}_{=0} + \overset{B}{g}{}^{\Sigma\Omega}\underbrace{\partial_A h_{\Sigma\Omega}}_{=0} - 2\overset{B}{g}{}^{CD}\underbrace{\overset{B}{\Gamma}{}^\Omega{}_{CA}}_{=0} h_{D\Omega} - 2\overset{B}{g}{}^{\Sigma\Omega}\underbrace{\overset{B}{\Gamma}{}^B{}_{\Sigma A}}_{=0} h_{\Omega B} \;=\; 0\,,$$

$$\delta\Gamma^\mu{}_{\mu\Sigma} \;=\; \overset{B}{g}{}^{\mu\lambda}\Big(\cancel{\partial_\mu h_{\lambda\Sigma}} + \partial_\Sigma h_{\lambda\mu} - \cancel{\partial_\lambda h_{\mu\Sigma}} - 2\overset{B}{\Gamma}{}^\tau{}_{\mu\Sigma} h_{\lambda\tau}\Big)$$

$$= \overset{B}{g}{}^{CD}\underbrace{\partial_\Sigma h_{CD}}_{=0} + \overset{B}{g}{}^{\Omega\Delta}\underbrace{\partial_\Sigma h_{\Omega\Delta}}_{=0} - 2\overset{B}{g}{}^{CD}\overset{B}{\Gamma}{}^\Delta{}_{C\Sigma} h_{D\Delta} - 2\overset{B}{g}{}^{\Omega\Delta}\overset{B}{\Gamma}{}^B{}_{\Omega\Sigma} h_{\Delta B}$$

$$= -2\overset{B}{g}{}^{CD}\frac{1}{r}\delta^r_C\delta^\Delta_\Sigma h_{D\Delta} + 2\overset{B}{g}{}^{\Omega\Delta} r\big(1 - \frac{r_S}{r}\big)\frac{\overset{B}{g}_{\Omega\Sigma}}{r^2}\delta^B_r h_{\Delta B}$$

$$= -2\overset{B}{g}{}^{rr}\frac{1}{r}h_{r\Sigma} + \frac{2}{r}\big(1 - \frac{r_S}{r}\big)h_{r\Sigma} \;=\; 0\,.$$

This means that the linearised field equation simplifies to

$$0 \;=\; \overset{B}{\nabla}_\mu \delta\Gamma^\mu{}_{\nu\sigma}\,.$$

As a preparation for working out the ten components of this tensor equation, we calculate the $\delta\Gamma^\nu{}_{\rho\sigma}$. In this calculation, we constantly use the Christoffel symbols of the Schwarzschild metric, and we will also need the eigenvalue equation of the angular momentum operator which reads

$$L^2 Y_{\ell m} = -\hbar^2(\sin\vartheta)^{-1}\Big(\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta Y_{\ell m}\big) + (\sin\vartheta)^{-1}\partial^2_\varphi Y_{\ell m}\Big) = \hbar^2\ell(\ell+1)Y_{\ell m}$$

in standard quantum mechanics notation. We find

$$2\delta\Gamma^{C}{}_{AB} \;=\; \overset{B}{g}{}^{CD}\Big(\underbrace{\partial_A h_{DB}}_{=0}+\underbrace{\partial_B h_{DA}}_{=0}-\underbrace{\partial_D h_{AB}}_{=0}-2\underbrace{\overset{B}{\Gamma}{}^{\Sigma}{}_{AB}}_{=0}h_{D\Sigma}\Big) \;=\; 0\,,$$

$$\begin{aligned}2\delta\Gamma^{\Delta}{}_{AB} \;&=\; \overset{B}{g}{}^{\Delta\Lambda}\Big(\partial_A h_{\Lambda B}+\partial_B h_{\Lambda A}-\underbrace{\partial_\Lambda h_{AB}}_{=0}-2\overset{B}{\Gamma}{}^{C}{}_{AB}h_{\Lambda C}\Big)\\ &=\; \big(\partial_A v_{B\ell m}+\partial_B v_{A\ell m}-2\overset{B}{\Gamma}{}^{C}{}_{AB}v_{C\ell m}\big)\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\\ &=\; \underbrace{\big(\partial_A v_{B\ell m}+\partial_B v_{A\ell m}-2\overset{B}{\Gamma}{}^{C}{}_{AB}v_{C\ell m}\big)}_{=:\,2\,q_{AB}}\overset{B}{g}{}^{\Delta\Lambda}\varepsilon_{\Lambda\Omega}\overset{B}{g}{}^{\Omega\Pi}\overset{B}{\nabla}_{\Pi}Y_{\ell m}\,,\end{aligned}$$

$$\begin{aligned}2\delta\Gamma^{C}{}_{A\Sigma} \;&=\; \overset{B}{g}{}^{CD}\Big(\partial_A h_{D\Sigma}+\underbrace{\partial_\Sigma h_{DA}}_{=0}-\partial_D h_{A\Sigma}-2\underbrace{\overset{B}{\Gamma}{}^{\Delta}{}_{A\Sigma}}_{\frac{1}{r}\delta^r_A\delta^\Delta_\Sigma}h_{D\Delta}\Big)\\ &=\; \underbrace{\overset{B}{g}{}^{CD}\big(\partial_A v_{D\ell m}-\partial_D v_{A\ell m}-\frac{2}{r}\delta^r_A v_{D\ell m}\big)}_{=:\,2\,p_{AD}}\Phi_{\ell m\Sigma}\,,\end{aligned}$$

$$\begin{aligned}2\delta\Gamma^{\Delta}{}_{A\Sigma} \;&=\; \overset{B}{g}{}^{\Delta\Lambda}\Big(\underbrace{\partial_A h_{\Lambda\Sigma}}_{=0}+\partial_\Sigma h_{\Lambda A}-\partial_\Lambda h_{A\Sigma}-2\underbrace{\overset{B}{\Gamma}{}^{C}{}_{A\Sigma}}_{=0}h_{\Lambda C}\Big) \;=\; v_{A\ell m}\overset{B}{g}{}^{\Delta\Lambda}\big(\partial_\Sigma\Phi_{\ell m\Lambda}-\partial_\Lambda\Phi_{\ell m\Sigma}\big)\\ &=\; v_{A\ell m}\overset{B}{g}{}^{\Delta\Lambda}\big(\partial_\vartheta\Phi_{\ell m\varphi}-\partial_\varphi\Phi_{\ell m\vartheta}\big)\big(\delta^\vartheta_\Sigma\delta^\varphi_\Lambda-\delta^\varphi_\Sigma\delta^\vartheta_\Lambda\big)\\ &=\; v_{A\ell m}\overset{B}{g}{}^{\Delta\Lambda}\Big(\partial_\vartheta\big(\varepsilon_{\varphi\vartheta}\overset{B}{g}{}^{\vartheta\vartheta}\partial_\vartheta Y_{\ell m}\big)-\partial_\varphi\big(\varepsilon_{\vartheta\varphi}\overset{B}{g}{}^{\varphi\varphi}\partial_\varphi Y_{\ell m}\big)\Big)\frac{\varepsilon_{\Sigma\Lambda}}{r^2\sin\vartheta}\\ &=\; v_{A\ell m}\overset{B}{g}{}^{\Delta\Lambda}\Big(-\partial_\vartheta\big(\sin\vartheta\partial_\vartheta Y_{\ell m}\big)-\partial_\varphi\big((\sin\vartheta)^{-1}\partial_\varphi Y_{\ell m}\big)\Big)\frac{\varepsilon_{\Sigma\Lambda}}{r^2\sin\vartheta}\\ &=\; v_{A\ell m}\overset{B}{g}{}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}\frac{1}{r^2}\ell(\ell+1)Y_{\ell m}\,,\end{aligned}$$

$$\begin{aligned}2\delta\Gamma^{C}{}_{\Sigma\Omega} \;&=\; \overset{B}{g}{}^{CD}\Big(\partial_\Sigma h_{D\Omega}+\partial_\Omega h_{D\Sigma}-\underbrace{\partial_D h_{\Sigma\Omega}}_{=0}-2\overset{B}{\Gamma}{}^{\Delta}{}_{\Sigma\Omega}h_{D\Delta}\Big)\\ =\; \overset{B}{g}{}^{CD}v_{D\ell m}\Big(\partial_\Sigma\Phi_{\ell m\Omega}+\partial_\Omega\Phi_{\ell m\Sigma}-2\overset{B}{\Gamma}{}^{\Delta}{}_{\Sigma\Omega}\Phi_{\ell m\Delta}\Big) \;&=\; \overset{B}{g}{}^{CD}v_{D\ell m}\Big(\overset{B}{\nabla}_\Sigma\Phi_{\ell m\Omega}+\overset{B}{\nabla}_\Omega\Phi_{\ell m\Sigma}\Big)\\ &=\; 2\overset{B}{g}{}^{CD}v_{D\ell m}\chi_{\ell m\Sigma\Omega}\,,\end{aligned}$$

$$\begin{aligned}2\delta\Gamma^{\Delta}{}_{\Sigma\Omega} \;&=\; \overset{B}{g}{}^{\Delta\Lambda}\Big(\underbrace{\partial_\Sigma h_{\Lambda\Omega}}_{=0}+\underbrace{\partial_\Omega h_{\Lambda\Sigma}}_{=0}-\underbrace{\partial_\Lambda h_{\Sigma\Omega}}_{=0}-2\overset{B}{\Gamma}{}^{C}{}_{\Sigma\Omega}h_{\Lambda C}\Big) \;=\; -2\overset{B}{g}{}^{\Delta\Lambda}\overset{B}{\Gamma}{}^{C}{}_{\Sigma\Omega}\Phi_{\ell m\Lambda}v_{C\ell m}\\ &=\; -2\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\Big(-v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\delta^\vartheta_\Sigma\delta^\vartheta_\Omega-v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\sin^2\vartheta\,\delta^\varphi_\Sigma\delta^\varphi_\Omega\Big)\\ &=\; 2\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}v_{r\ell m}r\big(1-\frac{r_S}{r}\big)\frac{\overset{B}{g}_{\Sigma\Omega}}{r^2} \;=\; 2\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}v_{r\ell m}\big(1-\frac{r_S}{r}\big)\frac{\overset{B}{g}_{\Sigma\Omega}}{r}\,.\end{aligned}$$

We are now ready to calculate the covariant derivatives

$$\overset{B}{\nabla}_C \delta\Gamma^C{}_{\rho\sigma} = \partial_C \delta\Gamma^C{}_{\rho\sigma} + \overset{B}{\Gamma}{}^C{}_{C\mu}\delta\Gamma^\mu{}_{\rho\sigma} - \overset{B}{\Gamma}{}^\mu{}_{C\rho}\delta\Gamma^C{}_{\mu\sigma} - \overset{B}{\Gamma}{}^\mu{}_{C\sigma}\delta\Gamma^C{}_{\rho\mu}$$

and

$$\overset{B}{\nabla}_\Delta \delta\Gamma^\Delta{}_{\rho\sigma} = \partial_\Delta \delta\Gamma^\Delta{}_{\rho\sigma} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta\mu}\delta\Gamma^\mu{}_{\rho\sigma} - \overset{B}{\Gamma}{}^\mu{}_{\Delta\rho}\delta\Gamma^\Delta{}_{\mu\sigma} - \overset{B}{\Gamma}{}^\mu{}_{\Delta\sigma}\delta\Gamma^\Delta{}_{\rho\mu}$$

for all index combinations ρ, σ. On the right-hand sides we split the sum over $\mu = t, r, \vartheta, \varphi$ into two sums, over $A = t, r$ and $\Omega = \vartheta, \varphi$, and collect all non-zero terms. We find

$$\overset{B}{\nabla}_C \delta\Gamma^C{}_{AB} = 0\,,$$

$$\begin{aligned}
\overset{B}{\nabla}_\Delta \delta\Gamma^\Delta{}_{AB} &= \partial_\Delta \delta\Gamma^\Delta{}_{AB} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta\Omega}\delta\Gamma^\Omega{}_{AB} - \overset{B}{\Gamma}{}^\Omega{}_{\Delta A}\delta\Gamma^\Delta{}_{\Omega B} - \overset{B}{\Gamma}{}^\Omega{}_{\Delta B}\delta\Gamma^\Delta{}_{A\Omega} \\
&= \partial_\Delta\big(\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda} q_{AB}\big) + \overset{B}{\Gamma}{}^\Delta{}_{\Delta\Omega}\overset{B}{g}{}^{\Omega\Lambda}\Phi_{\ell m\Lambda} q_{AB} - \delta^r_A \frac{1}{r}\delta^\Omega_\Delta \delta\Gamma^\Delta{}_{\Omega B} - \delta^r_B \frac{1}{r}\delta^\Omega_\Delta \delta\Gamma^\Delta{}_{\Omega A} \\
&= q_{AB}\overset{B}{\nabla}_\Delta\big(\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\big) - \delta^r_A \frac{1}{r}\underbrace{\delta\Gamma^\Omega{}_{\Omega B}}_{=0} - \delta^r_B \tfrac{1}{r}\underbrace{\delta\Gamma^\Omega{}_{\Omega A}}_{=0} \\
&= q_{AB}\overset{B}{\nabla}_\Delta\big(\overset{B}{g}{}^{\Delta\Lambda}\varepsilon_{\Lambda\Sigma}\overset{B}{g}{}^{\Sigma\Pi}\overset{B}{\nabla}_\Pi Y_{\ell m}\big) = q_{AB}\underbrace{\varepsilon_{\Lambda\Sigma}}_{=-\varepsilon_{\Sigma\Lambda}}\overset{B}{g}{}^{\Delta\Lambda}\overset{B}{g}{}^{\Sigma\Pi}\underbrace{\overset{B}{\nabla}_\Delta\overset{B}{\nabla}_\Pi Y_{\ell m}}_{=\overset{B}{\nabla}_\Pi\overset{B}{\nabla}_\Delta Y_{\ell m}} = 0\,,
\end{aligned}$$

$$\begin{aligned}
&\overset{B}{\nabla}_C \delta\Gamma^C{}_{A\Sigma} = \partial_C \delta\Gamma^C{}_{A\Sigma} - \overset{B}{\Gamma}{}^B{}_{CA}\delta\Gamma^C{}_{B\Sigma} - \overset{B}{\Gamma}{}^\Omega{}_{C\Sigma}\delta\Gamma^C{}_{A\Omega} = \partial_C\big(\overset{B}{g}{}^{CD} p_{AD}\Phi_{\ell m\Sigma}\big) \\
&-\overset{B}{\Gamma}{}^B{}_{CA}\overset{B}{g}{}^{CD} p_{BD}\Phi_{\ell m\Sigma} - \delta^r_C \frac{1}{r}\delta^\Omega_\Sigma \overset{B}{g}{}^{CD} p_{AD}\Phi_{\ell m\Omega} = \Phi_{\ell m\Sigma}\Big(\partial_C\big(\overset{B}{g}{}^{CD} p_{AD}\big) - \overset{B}{\Gamma}{}^B{}_{CA}\overset{B}{g}{}^{CD} p_{BD} - \frac{1}{r}\overset{B}{g}{}^{rr} p_{Ar}\Big)
\end{aligned}$$

$$\begin{aligned}
\overset{B}{\nabla}_\Delta \delta\Gamma^\Delta{}_{A\Sigma} &= \underbrace{\partial_\Delta \delta\Gamma^\Delta{}_{A\Sigma} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta\Omega}\delta\Gamma^\Omega{}_{A\Sigma} - \overset{B}{\Gamma}{}^\Omega{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{A\Omega}}_{=\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\overset{B}{\nabla}_\Delta\left(\overset{B}{g}{}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}Y_{\ell m}\right)} - \overset{B}{\Gamma}{}^\Omega{}_{\Delta A}\delta\Gamma^\Delta{}_{\Omega\Sigma} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta C}\delta\Gamma^C{}_{A\Sigma} \\
-\overset{B}{\Gamma}{}^C{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{AC} &= \ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\overset{B}{g}{}^{\Delta\Lambda}\varepsilon_{\Sigma\Lambda}\overset{B}{\nabla}_\Delta Y_{\ell m} - \delta^r_A \frac{1}{r}\delta^\Omega_\Delta \overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\overset{B}{g}_{\Omega\Sigma}\frac{v_{r\ell m}}{r}\big(1 - \frac{r_S}{r}\big) \\
&\quad + \delta^r_C \frac{2}{r}\overset{B}{g}{}^{CD} p_{AD}\Phi_{\ell m\Sigma} + \delta^C_r \frac{1}{r}\big(1 - \frac{r_S}{r}\big)\overset{B}{g}_{\Delta\Sigma}\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda} q_{AC} \\
&= \ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2}\Phi_{\ell m\Sigma} - \delta^r_A \Phi_{\ell m\Sigma}\frac{v_{r\ell m}}{r^2}\big(1 - \frac{r_S}{r}\big) + \frac{2}{r}\overset{B}{g}{}^{rr} p_{Ar}\Phi_{\ell m\Sigma} + \frac{1}{r}\big(1 - \frac{r_S}{r}\big)\Phi_{\ell m\Sigma} q_{Ar} \\
&= \Phi_{\ell m\Sigma}\Big(\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2} - \delta^r_A \frac{v_{r\ell m}}{r^2}\big(1 - \frac{r_S}{r}\big) + \frac{2}{r}\overset{B}{g}{}^{rr} p_{Ar} + \frac{1}{r}\big(1 - \frac{r_S}{r}\big) q_{Ar}\Big)\,,
\end{aligned}$$

$$\begin{aligned}
\overset{B}{\nabla}_C \delta\Gamma^C{}_{\Sigma\Omega} &= \partial_C \delta\Gamma^C{}_{\Sigma\Omega} - \overset{B}{\Gamma}{}^\Delta{}_{C\Sigma}\delta\Gamma^C{}_{\Delta\Omega} - \overset{B}{\Gamma}{}^\Delta{}_{C\Omega}\delta\Gamma^C{}_{\Sigma\Delta} \\
&= \frac{1}{r}\delta^\Delta_\Omega \overset{B}{g}{}^{CD} v_{D\ell m}\chi_{\ell m\Sigma\Delta} = \chi_{\ell m\Sigma\Omega}\Big(\partial_C\big(\overset{B}{g}{}^{CD} v_{D\ell m}\big) - \frac{2}{r}g^{rr} v_{r\ell m}\Big)\,,
\end{aligned}$$

$$\overset{B}{\nabla}_\Delta \delta\Gamma^\Delta{}_{\Sigma\Omega} = \partial_\Delta \delta\Gamma^\Delta{}_{\Sigma\Omega} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta\Pi}\delta\Gamma^\Pi{}_{\Sigma\Omega} - \overset{B}{\Gamma}{}^\Pi{}_{\Delta\Sigma}\delta\Gamma^\Delta{}_{\Pi\Omega} - \overset{B}{\Gamma}{}^\Pi{}_{\Delta\Omega}\delta\Gamma^\Delta{}_{\Sigma\Pi} + \overset{B}{\Gamma}{}^\Delta{}_{\Delta C}\delta\Gamma^C{}_{\Sigma\Omega}$$

$$= \frac{v_{r\ell m}}{r}\big(1-\frac{r_S}{r}\big)\underbrace{\overset{B}{\nabla}_\Delta\big(\overset{B}{g}{}^{\Delta\Lambda}\Phi_{\ell m\Lambda}\overset{B}{g}_{\Sigma\Omega}\big)}_{=0} + \delta^r_C\frac{2}{r}\overset{B}{g}{}^{CD}v_{D\ell m}\chi_{\ell m\Sigma\Omega} = \frac{2}{r}\overset{B}{g}{}^{rr}v_{r\ell m}\chi_{\ell m\Sigma\Omega}\,.$$

Hence

$$\overset{B}{\nabla}_\mu \delta\overset{B}{\Gamma}{}^\mu{}_{AB} = 0\,,$$

$$\overset{B}{\nabla}_\mu \delta\overset{B}{\Gamma}{}^\mu{}_{A\Sigma} = \Phi_{\ell m\Sigma}\Big(\partial_C\big(\overset{B}{g}{}^{CD}p_{AD}\big) - \overset{B}{\Gamma}{}^B{}_{CA}\overset{B}{g}{}^{CD}p_{BD} - \cancel{\frac{1}{r}\overset{B}{g}{}^{rr}p_{Ar}}$$

$$+\ell(\ell+1)\frac{v_{A\ell m}}{2\,r^2} - \delta^r_A\frac{v_r}{r^2}\big(1-\frac{r_S}{r}\big) + \cancel{\frac{2}{r}}\overset{B}{g}{}^{rr}p_{Ar} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{Ar}\Big)\,,$$

$$\overset{B}{\nabla}_\mu \delta\overset{B}{\Gamma}{}^\mu{}_{\Sigma\Omega} = \chi_{\ell m\Sigma\Omega}\partial_C\big(\overset{B}{g}{}^{CD}v_{D\ell m}\big)\,.$$

We see that the (AB) components of the linearised field equation are identically satisfied. The $(t\Sigma)$ and $(r\Sigma)$ components read

$$0 = \partial_r\big(\overset{B}{g}{}^{rr}p_{tr}\big) - \overset{B}{\Gamma}{}^t{}_{rt}\overset{B}{g}{}^{rr}p_{tr} - \overset{B}{\Gamma}{}^r{}_{tt}\overset{B}{g}{}^{tt}p_{rt} + \ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} + \frac{1}{r}\overset{B}{g}{}^{rr}p_{tr} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{tr}$$

$$= \frac{1}{2}\partial_r\Big(\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big)\Big) - \frac{r_S}{4r^2}\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big) + \frac{r_S}{4r^2}\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)$$

$$+\ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} + \frac{1}{2r}\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} - \cancel{\partial_r v_{t\ell m}}\big) + \frac{1}{2r}\big(1-\frac{r_S}{r}\big)\big(\partial_t v_{r\ell m} + \cancel{\partial_r v_{t\ell m}} - 2\overset{B}{\Gamma}{}^t{}_{tr}v_{t\ell m}\big)$$

$$= \frac{1}{2}\big(1-\frac{r_S}{r}\big)\partial_r\big(\partial_t v_{r\ell m} - \partial_r v_{t\ell m}\big) + \big(1-\frac{r_S}{r}\big)\frac{\partial_t v_{r\ell m}}{r} + \ell(\ell+1)\frac{v_{t\ell m}}{2\,r^2} - \frac{r_S v_{t\ell m}}{r^3}\,, \qquad \text{(F1)}$$

$$0 = \partial_r\big(\overset{B}{g}{}^{rr}p_{rr}\big) + \partial_t\big(\overset{B}{g}{}^{tt}p_{rt}\big) - \overset{B}{\Gamma}{}^r{}_{rr}\overset{B}{g}{}^{rr}p_{rr} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2}\big(1-\frac{r_S}{r}\big) + \frac{1}{r}\overset{B}{g}{}^{rr}p_{rr} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)q_{rr}$$

$$= -\cancel{\partial_r\Big(\big(1-\frac{r_S}{r}\big)\frac{v_{r\ell m}}{r}\Big)} - \frac{\partial_t\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)}{2\,c^2\big(1-\frac{r}{r_S}\big)} - \frac{r_S v_{r\ell m}}{2\,r^3} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2}\big(1-\cancel{\frac{r_S}{r}}\big)$$

$$-\cancel{\big(1-\frac{r_S}{r}\big)\frac{v_{r\ell m}}{r^2}} + \frac{1}{r}\big(1-\frac{r_S}{r}\big)\big(\cancel{\partial_r v_{r\ell m}} - \overset{B}{\Gamma}{}^r{}_{rr}v_{r\ell m}\big)$$

$$= -\frac{\partial_t\big(\partial_r v_{t\ell m} - \partial_t v_{r\ell m} - \frac{2}{r}v_{t\ell m}\big)}{2\,c^2\big(1-\frac{r}{r_S}\big)} - \cancel{\frac{r_S v_{r\ell m}}{2\,r^3}} + \ell(\ell+1)\frac{v_{r\ell m}}{2\,r^2} - \frac{v_{r\ell m}}{r^2} + \cancel{\frac{r_S v_{r\ell m}}{2\,r^3}}$$

$$= -\frac{\partial_t\partial_r v_{t\ell m} - \partial_t^2 v_{r\ell m} - \frac{2}{r}\partial_t v_{t\ell m}}{2\,c^2\big(1-\frac{r}{r_S}\big)} + \frac{\big(\ell(\ell+1)-2\big)v_{r\ell m}}{2\,r^2}\,. \qquad \text{(F2)}$$

The $(\Sigma\Omega)$ component of the linearised field equation gives one equation,

$$0 = \partial_r\big(\overset{B}{g}{}^{rr} v_{r\ell m}\big) + \partial_t\big(\overset{B}{g}{}^{tt} v_{t\ell m}\big) = \partial_r\Big(\big(1-\frac{r_S}{r}\big)v_{r\ell m}\Big) - \frac{\partial_t v_{t\ell m}}{c^2\big(1-\frac{r_S}{r}\big)}\,. \qquad \text{(F3)}$$

The field equations (F1), (F2) and (F3) can be decoupled. To that end we replace $v_{r\ell m}$ with

$$Q_{\ell m} = \big(1-\frac{r}{r_S}\big)\frac{v_{r\ell m}}{r}$$

which allows to rewrite (F3) as

$$\frac{\partial_t v_{t\ell m}}{c^2} = \big(1-\frac{r_S}{r}\big)\partial_r\big(r\,Q_{\ell m}\big) = \big(1-\frac{r_S}{r}\big)Q_{\ell m} + \big(1-\frac{r_S}{r}\big)\,r\,\partial_r Q_{\ell m}\,. \qquad \text{(F3')}$$

Differentiation with respect to r yields

$$\frac{\partial_r\partial_t v_{t\ell m}}{c^2} = \frac{r_S}{r^2}Q_{\ell m} + \,2\,\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m} + \,r\,\partial_r\Big(\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m}\Big)\,. \qquad \text{(F3")}$$

With the help of (F3') and (F3"), (F2) can be rewritten as an equation for $Q_{\ell m}$ alone,

$$\begin{aligned}
0 &= \frac{1}{r}\big(1-\frac{r_S}{r}\big)\Big(\frac{\partial_t\partial_r v_{t\ell m}}{c^2} - \frac{\partial_t^2 v_{r\ell m}}{c^2} - \frac{2\partial_t v_{t\ell m}}{r\,c^2}\Big) - \big(1-\frac{r_S}{r}\big)^2\frac{\big(\ell(\ell+1)-2\big)v_{r\ell m}}{r^3} \\
&= \frac{1}{r}\big(1-\frac{r_S}{r}\big)\Big\{\frac{r_S}{r^2}Q_{\ell m} + \cancel{2\,\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m}} + \,r\,\partial_r\Big(\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m}\Big)\Big\} \\
&\quad -\frac{\partial_t^2 Q_{\ell m}}{c^2} - \frac{2}{r^2}\,\big(1-\frac{r_S}{r}\big)^2\Big(Q_{\ell m} + \cancel{r\,\partial_r Q_{\ell m}}\Big) - \big(1-\frac{r_S}{r}\big)\frac{\big(\ell(\ell+1)-2\big)Q_{\ell m}}{r^2} \\
&= \big(1-\frac{r_S}{r}\big)\partial_r\Big(\big(1-\frac{r_S}{r}\big)\partial_r Q_{\ell m}\Big) - \frac{\partial_t^2 Q_{\ell m}}{c^2} - \frac{1}{r^2}\,\big(1-\frac{r_S}{r}\big)\Big(\big(\ell(\ell+1) - \frac{3r_S}{r}\big)\Big)Q_{\ell m}\,.
\end{aligned}$$

If we introduce Wheelers *tortoise coordinate*

$$r_* = r + r_S\ln\Big(\frac{r}{r_S}-1\Big)\,, \qquad \partial_{r_*} = \Big(1-\frac{r_S}{r}\Big)\partial_r\partial_{r_*} = \Big(1-\frac{r_S}{r}\Big)\partial_r\,,$$

which shifts the horizon to $r_* = -\infty$, we have derived the standard form of the time-dependent *Regge-Wheeler equation*

$$\partial_{r_*}^2 Q_{\ell m} - \frac{1}{c^2}\partial_t^2 Q_{\ell m} - V_\ell(r_*)Q_{\ell m} = 0.$$

Here the *Regge-Wheeler potential* $V_\ell(r_*)$ is given implicitly by

$$V_\ell(r_*) = \frac{1}{r^2}\big(1-\frac{r_S}{r}\big)\Big(\ell(\ell+1) - \frac{3r_S}{r}\Big)\,.$$

Note that the potential depends on ℓ but not on m. This means that we could drop the index m on $Q_{\ell m}$, i.e., $Q_{\ell m'} = Q_{\ell m}$.

Step 5: Finally, we separate the time coordinate with the help of the ansatz

$$Q_{\ell m}(t, r_*) = \mathsf{e}^{-i\omega t} Z_{\ell m}(r_*)\,.$$

Inserting this expression into the time-dependent Regge-Wheeler equation yields

$$\frac{d^2 Z_{\ell m}}{dr_*^2}\mathsf{e}^{-i\omega t} + \frac{\omega^2}{c^2} Z_{\ell m}\mathsf{e}^{-i\omega t} - V_\ell(r_*) Z_{\ell m}\mathsf{e}^{-i\omega t} = 0, ,$$

$$-\frac{d^2 Z_{\ell m}}{dr_*^2} + V_\ell(r_*) Z_{\ell m} = \frac{\omega^2}{c^2} Z_{\ell m}\,.$$

This is the time-independent *Regge-Wheeler equation*. It is very similar to the radial part of the time-independent Schrödinger equation with a spherically symmetric potential. There are some differences, however. (i) The frequency occurs quadratic, rather than linear, because the Regge-Wheeler equation is of *second* order in time. (ii) The radius coordinate r_* ranges from $-\infty$ to ∞, rather than from 0 to ∞. (iii) We have to impose the condition on our complex function $Q_{\ell m}$ that the corresponding metric perturbations $h_{\mu\nu}$ are real. (The $v_{A\ell m}$ are complex because we expand the real $h_{\mu\nu}$ with respect to the complex spherical harmonics.) (iv) In contrast to the wave function in quantum mechanics, there is no physical reason why $Q_{\ell m}$ should have to satisfy a square-integrability condition; instead, one has to impose physically motivated boundary conditions.

From any solution $Q_{\ell m}(t, r_*) = Z_{\ell m}(r_*)\mathsf{e}^{-i\omega t}$ of the Regge-Wheeler equation we can construct the metric perturbations $v_{r\ell m}$ and $v_{t\ell m}$ in the following way. $v_{r\ell m}$ is given directly as

$$v_{r\ell m} = Q_{\ell m} r \big(1 - \frac{r_S}{r}\big)^{-1}$$

and $v_{t\ell m}$ follows if we plug the ansatz $v_{t\ell m}(t, r_*) = U_{\ell m}\mathsf{e}^{-i\omega t}$ into (F3'),

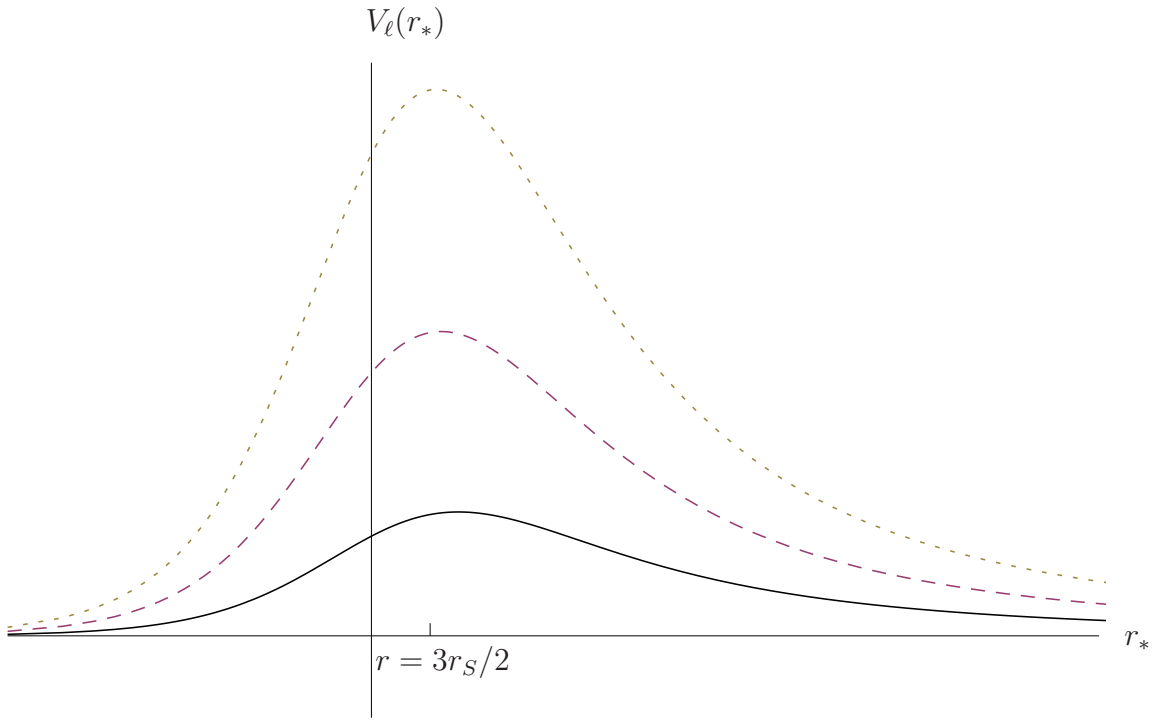
$$\frac{1}{c^2}\partial_t v_{t\ell m} = \big(1 - \frac{r_S}{r}\big) Q_{\ell m} + r\partial_{r_*} Q_{\ell m}\,,$$

$$-\frac{i\omega}{c^2} U_{\ell m} = \big(1 - \frac{r_S}{r}\big) Z_{\ell m} + r\frac{dZ_{\ell m}}{dr_*}\,.$$

It can be shown that then the field equation (F1), which has not been used so far, is automatically satisfied by $v_{r\ell m}$ and $v_{t\ell m}$.

From time-harmonic solutions to the Regge-Wheeler equation we can construct the *general* solution to odd linear perturbations of the Schwarzschild metric by forming superpositions of solutions with different ω and different (ℓ, m). Note that such a superposition is indeed possible, although we have chosen a gauge (i.e., a coordinate system) that depends on (ℓ, m); the reason is that in the representation $h_{\mu\nu}(x)dx^\mu dx^\nu$ of the metric perturbation the $x = (x^0, x^1, x^2, x^3)$ denote the *unperturbed* Schwarzschild coodinates. (As the $h_{\mu\nu}$ are small of first order, the difference between their values at the perturbed coordinates and at the unperturbed coordinates is of second order and hence to be neglected.) Therefore the $h_{\mu\nu}$ can be superimposed, even if we use different *perturbed* coordinates for different (ℓ, m).

The Regge-Wheeler potential describes a potential *barrier* rather than a potential *well*, see the picture below (and Worksheet 10). Correspondingly, there are no bound states. We know already that monopole perturbations ($\ell = 0$) and dipole perturbations ($\ell = 1$) cannot describe gravitational waves. Therefore, we omit these cases and plot the potential for $\ell = 2$ (solid), $\ell = 3$ (dashed) and $\ell = 4$ (dotted). The maximum of the potential is near the light sphere at $r = 3r_S/2$. In the limit $\ell \to \infty$ it approaches this value.



Two interesting types of problems are related with the Regge-Wheeler equation (and, analogously, with the Zerilli equation for even modes). Firstly, one can study the way in which an incoming wave is scattered by the black hole. To a large extent, mathematical techniques can be taken over from the quantum-mechanical scattering theory. We will not discuss this here. Secondly, one can study quasi-normal modes. The latter are defined as solutions of the time-independent Regge-Wheeler equation with a *complex* ω satisfying the boundary conditions that nothing is coming in from infinity or from the horizon. In physical terms, they describe what happens if a black hole is perturbed and then left alone. What one expects is that the perturbation dies down in the course of time. That is exactly what comes out. One speaks of quasi-normal modes, rather than of normal modes, because ω is non-real. Solutions with real ω cannot satisfy the prescribed boundary conditions. With our convention of writing the exponential factor as $\mathsf{e}^{-i\omega t}$, the imaginary part of ω describes exponential damping if it is negative and it describes exponential growth if it is positive. In other words, a positive imaginary part would indicate that a small perturbation of a Schwarzschild black hole becomes bigger and bigger in the course of time.

More precisely, quasi-normal modes are defined as solutions $Z_{\ell m}(r_*)$ to the time-independent Regge-Wheeler equation with a complex ω that satisfy the boundary conditions

$$\left|Z_{\ell m}(r_*) - B\,\mathsf{e}^{-i\omega r_*/c}\right| \to 0 \qquad \text{for} \quad r_* \to -\infty$$

$$\left|Z_{\ell m}(r_*) - C\,\mathsf{e}^{i\omega r_*/c}\right| \to 0 \qquad \text{for} \quad r_* \to \infty$$

where B and C are any constants. These conditions mean that the function $Q_{\ell m}(t, r_*) = Z_{\ell m}(r_*)\mathsf{e}^{-i\omega t}$ satisfies

$$\left|Q_{\ell m}(t, r_*) - B\,\mathsf{e}^{-i\omega(t+r_*/c))}\right| \to 0 \qquad \text{for} \quad r_* \to -\infty$$

$$\left|Q_{\ell m}(t, r_*) - C\,\mathsf{e}^{-i\omega(t-r_*/c))}\right| \to 0 \qquad \text{for} \quad r_* \to \infty\,,$$

i.e., that the solutions are purely ingoing (towards the horizon) for r near r_S and purely outgoing (towards infinity) for big r. (In Worksheet 10 we discuss a bit the behaviour of the general solution to the Regge-Wheeler equation near the horizon and near infinity.)

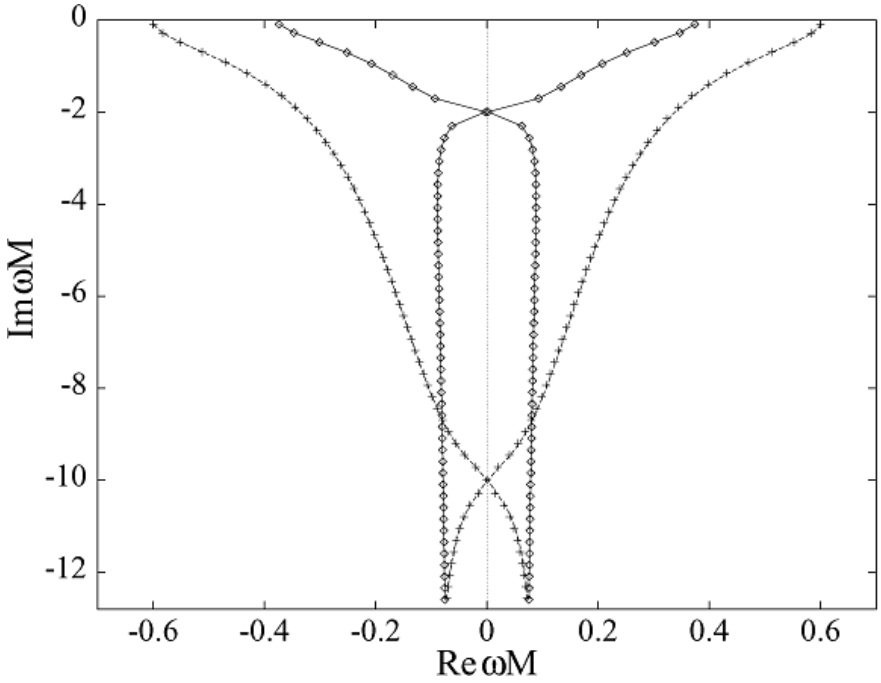
Quasi-normal modes cannot be determined analytically. However, they have been extensively studied numerically and with the help of analytical approximation methods. The following table shows the complex frequencies for the first four quasi-normal modes for $\ell = 2, 3, 4$, taken from a paper by E. Leaver ["An analytic representation for the quasi-normal modes of Kerr black holes", Proc. R. Soc. London, Ser. A, 402, 285298, (1985)].

n	$\ell = 2$	$\ell = 3$	$\ell = 4$
0	0.37367 -0.08896 i	0.59944 -0.09270 i	0.80918 -0.09416 i
1	0.34671 -0.27391 i	0.58264 -0.28130 i	0.79663 -0.28443 i
2	0.30105 -0.47828 i	0.55168 -0.47909 i	0.77271 -0.47991 i
3	0.25150 -0.70514 i	0.51196 -0.69034 i	0.73984 -0.68392 i

TABLE I: *The first four QNM frequencies (ωM) of the Schwarzschild black hole for $\ell = 2, 3,$ and 4[3].*

Natural units are chosen; for conversion into Hz one has to multiply with $2\pi \times 5142\,\mathrm{Hz} \times M_\odot/M$. We see that for fixed ℓ the real part of the frequency is *maximal* for the fundamental mode ($n = 0$). This is in contrast to normal modes where the frequency of the fundamental mode is minimal. For all quasi-normal modes, the imaginary part of the frequency is strictly negative. This demonstrates that every perturbation dies down in the cause of time, at least in the linear theory, i.e. that a Schwarzschild black hole is stable against linear perturbations. The damping time (i.e., the inverse of the imaginary part of ω) is surprisingly small: From the table we read that, for a black hole with a few solar masses, the frequency is in the order of Kilohertz and the damping time is in the order of Milliseconds!

The picture below shows the frequencies of the quasinormal modes of a Schwarzschild black hole in the complex plane.



This diagram, taken from N. Andersson and S. Linnaeus ["Quasinormal modes of a Schwarzschild black hole: Improved phase-integral treatment", Phys. Rev. D 46, 4179, (1992)], displays the values for the quasi-normal modes with $\ell = 2$ as diamonds and with $\ell = 3$ as crosses. A similar diagram can also be produced with the even (Zerilli) quasi-normal modes. One finds that they lie along the same curves but at different values.

Quasi-normal modes have also been calculated for charged black holes (i.e., for the Reissner-Nordstöm metric), and, with much greater difficulty, for rotating black holes (i.e., for the Kerr metric). The differences could be used, in principle, for distinguishing different types of black holes by the gravitational radiation they emit when they are perturbed.

8. Exact wave solutions of Einstein's field equation

Up to now we have treated gravitational waves as perturbations of a background spacetime that are so small that all equations can be linearised with respect to them. This is a viable theory for explaining any observations that are expected for the foreseeable future. Nonetheless it is helpful, and even necessary for a full understanding, to study gravitational waves at the level of the *full nonlinear Einstein equation*. It could well be that some of the observations made in the linear theory, e.g. about the polarisation states or about the multipole characters of gravitational waves, are just an artefact of the linearisation. In this chapter we are going to discuss three classes of *exact* wave solutions to Einstein's vacuum equations, known as Brinkmann solutions (or pp-waves), Einstein-Rosen solutions, and Robinson-Trautman solutions.

8.1 Brinkmann solutions (pp waves)

We begin with Minkowski spacetime in double null coordinates (x^1, x^2, u, v), where the u lines are the straight lightlike lines in negative x^3 direction and the v lines are the straight lightlike lines in positive x^3 direction. We then modify the spacetime in such a way that it is no longer flat but that the v lines remain lightlike, geodesic and orthogonal to planes. The idea is that the v lines can then be interpreted as the "rays" of a gravitational wave with planar wave surfaces, if the vacuum Einstein equation is satisfied.

From Minkowski spacetime in standard coordinates,

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,+\, (dx^3)^2 \,-\, (dx^0)^2\,,$$

we transform to "double-null coordinates" , $(x^1, x^2, x^3, x^0) \mapsto (x^1, x^2, u, v)$, defined by

$$x^0 \;=\; \frac{1}{\sqrt{2}}\,(v+u)\;, \qquad x^3 \;=\; \frac{1}{\sqrt{2}}\,(v-u)\;.$$

Then

$$(dx^0)^2 \,-\, (dx^3)^2 \;=\; \frac{1}{2}\,\big(dv+du\big)^2 \,-\, \frac{1}{2}\,\big(dv-du\big)^2 \;=$$

$$= \frac{1}{2}\big(\cancel{du^2} \,+\, 2\,dv\,du \,+\, \cancel{dv^2}\big) \,-\, \frac{1}{2}\big(\cancel{du^2} \,-\, 2\,dv\,du \,+\, \cancel{dv^2}\big) \;=\; 2\,dv\,du\;,$$

hence the Minkowski metric reads

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,-\, 2\,dv\,du\,.$$

We now add a term that makes the spacetime dynamic (time-dependent), but in such a way that ∂_v remains lightlike,

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,-\, 2\,dv\,du \,+\, H(x^1,x^2,u)\,du^2\,. \qquad \text{(B)}$$

with some function $H(x^1, x^2, u)$. The dependence of H on u (i.e., on $x^0 - x^3$) makes the metric time-dependent. The vector field ∂_u is no longer lightlike, but the vector field ∂_v still is. We will show in Worksheet 10 that, in addition, ∂_v is absolutely parallel, i.e., covariantly constant in *any* direction, hence in particular geodesic. This allows to interpret the v lines as the propagation direction of a wave that travels at the speed of light. Each x^1-x^2-surface (i.e., each surface $\{u = \text{constant}, v = \text{constant}\}$) is a Euclidean plane perpendicular to the propagation direction of the wave.

Below we will calculate the Christoffel symbols of the metric (B) from which one can easily determine the Ricci tensor. One finds that the vacuum Einstein equation $R_{\mu\nu} = 0$ holds if and only if H satisfies the Laplace equation with respect to the variables x^1 and x^2.

$$\delta^{AB}\partial_A\partial_B H \;=\; 0\,. \qquad \text{(T)}$$

If the condition (T) is satisfied, the metric (B) can be interpreted as a (pure) gravitational wave. For the case that (T) is not satisfied, one finds that the energy-momentum tensor has the form of that of an electromagnetic field; the metric can then be interpreted as a combination of a gravitational wave and an electromagnetic wave.

Metrics of the form (B) made their first appearence in a purely mathematical paper by H. Brinkmann ["Einstein spaces which are mapped conformally on each other" Math. Annalen 94, 119 (1925)]. The coordinates (x^1, x^2, u, v) are known as *Brinkmann coordinates*. A. Peres ["Some gravitational waves " Phys. Rev. Lett. 3, 571 (1959)] rediscovered these metrics and interpreted them as gravitational waves. They were studied in several papers by J. Ehlers and W. Kundt [see in particular J. Ehlers and W. Kundt: "Exact solutions of the gravitational field equations" in L. Witten (ed.) "Gravitation: an introduction to current research" Wiley, New York (1962) p.49] who called them *plane-fronted waves with parallel rays* or *pp-waves* for short. Obviously, "plane-fronted" refers to the (x^1, x^2)-surfaces and "parallel rays" refers to the v-lines.

We will now write down the geodesic equation for the metric (B) which will give us the Christoffel symbols. As usual, the most convenient way is to start from the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) \,=\, \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \,=\, \frac{1}{2}\,\Big((\dot{x}^1)^2+(\dot{x}^2)^2-2\,\dot{u}\,\dot{v}\,+\,H(x^1,x^2,u)\,\dot{u}^2\Big)$$

where an overdot means derivative with respect to an affine parameter s. From this we get the geodesics as the solutions to the Euler-Lagrange equation

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}\big(x,\dot{x}\big)}{\partial\dot{x}^\mu}\right) \,-\, \frac{\partial\mathcal{L}\big(x,\dot{x}\big)}{\partial x^\mu} \,=\, 0\,.$$

Doing this for $x^\mu = u$, v and x^A yields

$$\begin{aligned}
-\,\ddot{u} \,&=\, 0\,,\\
-\ddot{v}+\frac{d\big(H\,\dot{u}\big)}{ds} \,-\, \frac{1}{2}\,\partial_u H\,\dot{u}^2 \,&=\, 0\,,\\
\ddot{x}^A \,-\, \frac{1}{2}\,\delta^{AB}\partial_B H\,\dot{u}^2 \,&=\, 0\,,
\end{aligned}$$

hence

$$\begin{aligned}
\ddot{u} \,&=\, 0\,,\\
\ddot{v} \,-\, \frac{1}{2}\,\partial_u H\,\dot{u}^2 \,-\, \partial_A H\,\dot{u}\,\dot{x}^A \,&=\, 0\,,\\
\ddot{x}^A \,-\, \frac{1}{2}\,\delta^{AB}\partial_B H\,\dot{u}^2 \,&=\, 0\,.
\end{aligned}$$

From these equations we read that the only non-vanishing Christoffel symbols are

$$\Gamma^v{}_{uu} \,=\, -\,\frac{1}{2}\,\partial_u H \qquad \Gamma^v{}_{uA} \,=\, -\,\frac{1}{2}\,\partial_A H \qquad \Gamma^A{}_{uu} \,=\, -\,\delta^{AB}\,\partial_B H\,.$$

From the Christoffel symbols one can calculate the Ricci tensor

$$R_{\mu\sigma} = \partial_\mu\Gamma^\tau{}_{\tau\sigma} - \partial_\tau\Gamma^\tau{}_{\mu\sigma} + \Gamma^\rho{}_{\tau\sigma}\Gamma^\tau{}_{\mu\rho} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\tau{}_{\tau\rho}\,.$$

One finds that the only non-vanishing component of the Ricci tensor is

$$R_{uu} \,=\, \frac{1}{2}\,\delta^{AB}\partial_A\partial_B H$$

so that indeed the vacuum field equation $R_{\mu\nu} = 0$ is equivalent to the Laplace equation (T), as was already anticipated above.

In the following we will consider those pp-waves for which the function H is a quadratic form in the variables x^1 and x^2,

$$H(x^1, x^2, u) \;=\; h_{AB}(u) x^A x^B$$

with a symmetric (2×2) matrix $\big(h_{AB}(u)\big)$. If the vacuum field equation (T) is satisfied, i.e., if the matrix $\big(h_{AB}(u)\big)$ is trace-free,

$$\delta^{AB} h_{AB}(u) = 0\,,$$

these special pp-waves are called *plane gravitational waves*; otherwise they describe a coupled system of a plane gravitational wave and a plane electromagnetic wave. Both cases were first studied by O. Baldwin and G. Jeffery ["The relativity theory of plane waves", Proc. Roy. Soc. London A 111, 95 (1926)] who did not know about Brinkmann's earlier work on the larger class of what we now call pp-waves.

The condition of vanishing trace means that for a plane gravitational wave the matrix $h_{AB}(u)$ can be written as

$$\big(h_{AB}(u)\big) \;=\; \begin{pmatrix} f_+(u) & f_\times(u) \\ f_\times(u) & -\,f_+(u) \end{pmatrix}\,.$$

The *profile functions* $f_+(u)$ and $f_\times(u)$ determine the shape of the gravitational wave. The fact that (within the class of metrics considered) two scalar functions are necessary to determine the wave can be interpreted by saying that "a gravitational wave has two polarisation states". This is in perfect agreement with what we have found for plane harmonic waves in the linearised theory about Minkowski spacetime, where we also had two polarisation states, the *plus-mode* and the *cross-mode*).

For a plane gravitational wave the geodesic equation specifies to

$$\ddot{u} \;=\; 0\,,$$

$$\begin{aligned} \ddot{v} \;&=\; \frac{1}{2}\Big(f'_+(u)\big((x^1)^2 - (x^2)^2\big) + 2\, f'_\times(u) x^1 x^2 \Big)\, \dot{u}^2 \\ &+ \Big(f_+(u)\big(x^1\dot{x}^1 - x^2\dot{x}^2\big) + f_\times(u)\big(x^1\dot{x}^2 + x^2\dot{x}^1\big)\Big)\dot{u}\,, \end{aligned}$$

$$\begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} \;=\; \frac{1}{2} \begin{pmatrix} f_+(u) & f_\times(u) \\ f_\times(u) & -f_+(u) \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\,.$$

We see that there are geodesics that are completely contained in a lightlike hypersurface $u =$ constant. For them we have $u(s) = u_0$, $\dot{u}(s) = 0$ and $\ddot{u}(s) = 0$, so the u component of the geodesic equation is satisfied. The other components read

$$\ddot{v}(s) = 0\,,$$

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f_+(u_0) & f_\times(u_0) \\ f_\times(u_0) & -f_+(u_0) \end{pmatrix} \begin{pmatrix} x^1(s) \\ x^2(s) \end{pmatrix} ,$$

which can be integrated easily.

For all the other geodesics we have $\dot{u}(s) \neq 0$. Then the u component of the geodesic equation, $\ddot{u} = 0$, says that u can be used as the affine parameter. (Recall that the affine parametrisation along a geodesic is unique only up to a transformation of the form $s \mapsto as + b$ with a non-zero constant a.) With $u(s) = s$, the other components of the geodesic equation read

$$\ddot{v}(s) = \frac{1}{2} \Big(f'_+(s) \big((x^1(s))^2 - (x^2(s))^2 \big) + 2\, f'_\times(s) x^1(s) x^2(s) \Big)$$

$$+ \Big(f_+(s) \big(x^1(s) \dot{x}^1(s) - x^2(s) \dot{x}^2(s) \big) + f_\times(s) \big(x^1(s) \dot{x}^2(s) + x^2(s) \dot{x}^1(s) \big), ,$$

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f_+(s) & f_\times(s) \\ f_\times(s) & -f_+(s) \end{pmatrix} \begin{pmatrix} x^1(s) \\ x^2(s) \end{pmatrix} .$$

We see that the (x^1, x^2) equation decouples. After having solved this equation, $v(s)$ is determined by a straight-forward integration. Therefore we concentrate on the matrix differential equation for x^1 and x^2. This equation gives the motion of the geodesics in the (x^1, x^2) plane, i.e., in the plane orthogonal to the propagation direction of the wave. For the plus-mode, $f_\times = 0$, we have

$$\begin{pmatrix} \ddot{x}^1(s) \\ \ddot{x}^2(s) \end{pmatrix} = \frac{f_+(s)}{2} \begin{pmatrix} x^1(s) \\ -x^2(s) \end{pmatrix} .$$

At points where f_+ is positive, there is focussing in the x^1 direction and defocussing in the x^2 direction; at points where f_+ is negative, it is vice versa.

To discuss the cross mode, we may rotate the coordinates by 45^o,

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} , \qquad \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} .$$

Then

$$\begin{pmatrix} \ddot{y}^1 \\ \ddot{y}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{f_\times(s)}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$= \frac{f_\times(s)}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{f_\times(s)}{4} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{f_\times(s)}{2} \begin{pmatrix} y^1 \\ -y^2 \end{pmatrix},$$

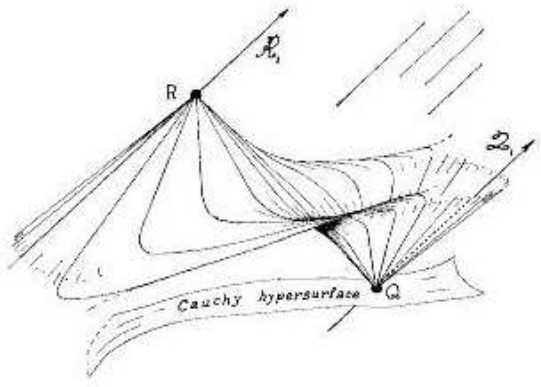
so we have the same focussing and defocussing properties as for the plus-mode, just rotated by 45^o.

This consideration holds for timelike, lightlike and spacelike geodesics. For timelike geodesics it gives the motion of freely falling test particles, in analogy to what we have discussed in the linearised theory. We see that the plus-mode and the cross-mode have the same physical interpretation for the exact plane gravitational waves, but now x^1 and x^2 may be arbitrarily large. To make the analogy with our treatment of the linearised theory perfect, we may Fourier-expand the matrix-valued function $h_{AB}(u)$ (i.e., the profile functions $f_+(u)$ and $f_\times(u)$). Then we get exactly the same expression for each Fourier mode

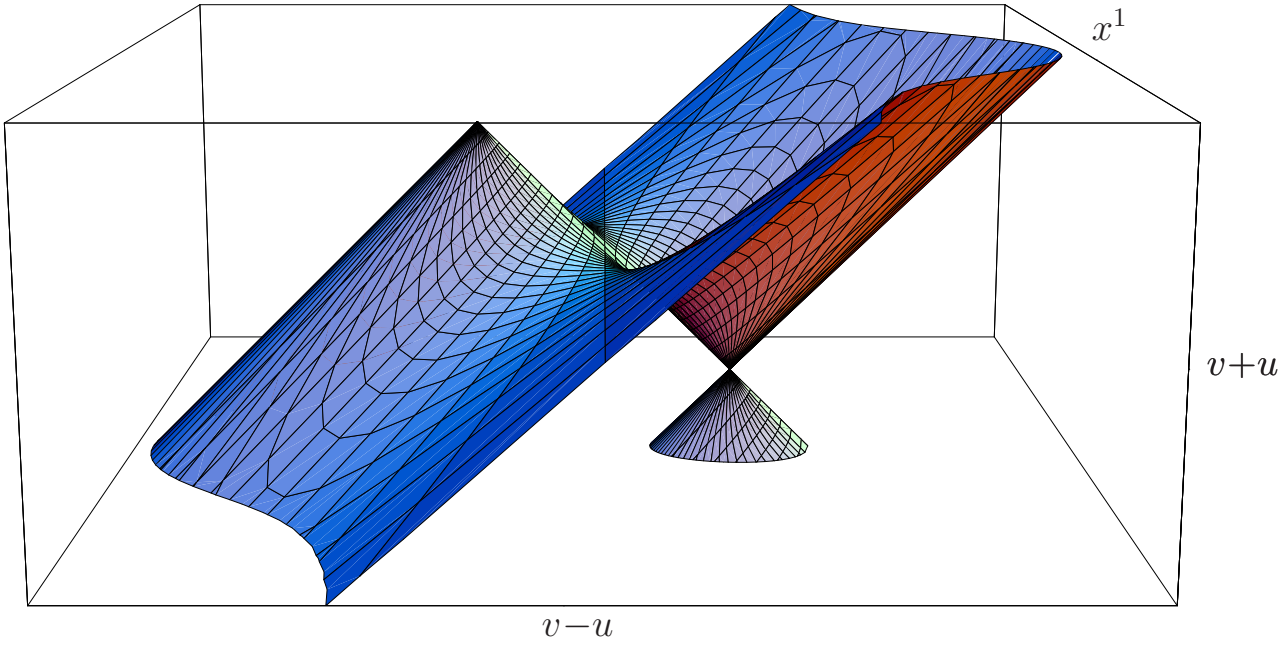
$$h_{AB}(u) \;=\; \mathrm{Re}\big\{h^0_{AB}\mathsf{e}^{-i\omega u/c}\big\}$$

as we had in the linearised theory.

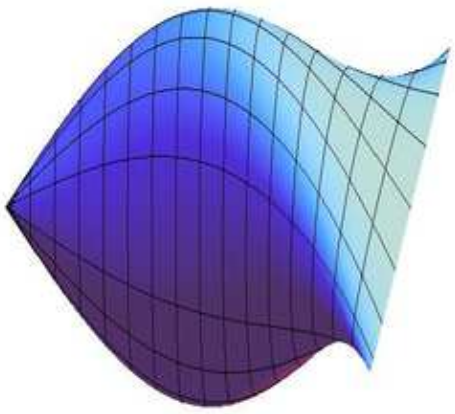
We now turn to the lightlike geodesics. The picture on the right shows the past light-cone of an event R, in a famous hand-drawing by Roger Penrose [" A remarkable property of plane waves in general relativity" Rev. Modern Phys. 37, 215 (1965)]. One sees that, with the exception of the v-line through R (which is a straight line), all light rays that are issuing from the event R into the past are refocussed into another event Q. Actually, taking the fourth dimension into account which is missing in the picture, a pure gravitational wave refocusses light rays into a line ("astigmatic focussing"). A combined gravitational and electromagnetic wave can refocus light rays into a point ("anastigmatic focussing"). The picture also indicates that a plane-wave spacetime cannot admit a Cauchy hypersurfaces, i.e., a hypersurface that intersects any causal curve exactly once: Such a hypersurface would have to intersect the v-line through R. But then some of the other past-oriented lightlike geodesics from R to Q have to be intersected twice.



The following picture of the light cone was produced with Mathematica. The profile functions were chosen as $f_\times(u) = 0$ and $f_+(u) = k^2\chi(u)$, where k is a non-zero constant and $\chi(u)$ is the characteristic function of a finite interval (i.e., the gravitational wave is "sandwiched" between two flat spacetime regions, bounded by hypersurfaces u = constant). The x^2 dimension is omitted. The similarity with the Penrose drawing is striking.



The picture on the right gives a purely spatial view of the light-cone above. Now both spatial dimensions x^1 and x^2 are shown and the temporal dimension, $u+v$, is omitted. One clearly sees the astigmatic focussing: There is focussing in one spatial dimension and defocussing in the other spatial dimension, so that the light-like geodesics are refocussed in a line.



8.1 (Beck-)Einstein-Rosen solutions

In this section we want to discuss a class of excact wave-like solutions to Einstein's vacuum equation with cylindrical symmetry. These solutions are usually called *Einstein-Rosen waves* although they were found by Austrian physicist Guido Beck already 12 years before Einstein and Rosen [G. Beck: "Zur Theorie binärer Gravitationsfelder" Zeitschr. f. Physik 33, 713 (1925)].

Beck started out from known results on axisymmetric and static metrics which had been found by H. Weyl in 1917. A metric is axisymmetric and static if it can be written in cylindrical polar coordinates (t, ρ, φ, z) such that the $g_{\mu\nu}$ are independent of t, independent of φ, and invariant under a transformation $\varphi \mapsto -\varphi$. Such metrics describe the gravitational fields of time-independent non-rotating bodies with axial symmetry. (If the invariance under the transformation $\varphi \mapsto -\varphi$ is dropped one speaks of axisymmetric stationary metrics; then rotating bodies are included.) Beck took the known axisymmetric and static metrics and performed the formal substitution $t \mapsto iz$, $z \mapsto it$. Then the metric is still axisymmetric but, instead of being time-independent, it is now invariant under translations in z-direction. In this way one gets time-dependent metrics (waves) with cylindrical symmetry.

The ansatz for the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = \mathsf{e}^{2\gamma-2\psi}\big(d\rho^2 - c^2dt^2\big) + \mathsf{e}^{-2\psi}W^2d\varphi^2 + \mathsf{e}^{2\psi}dz^2$$

where γ, ψ and W are functions of t and ρ. This is precisely the same ansatz, with the above-mentioned substitution, as it is used for the axisymmetric and static metrics; in the latter context, one speaks of *Weyl canonical coordinates*. This is the most general form of a cylindrically symmetric metric apart from the fact that we have assumed invariance under $\varphi \mapsto -\varphi$ (in analogy to the axisymmetric static case). Note that the ansatz of the metric in the (t, ρ) plane being proportional to $\big(d\rho^2 - c^2dt^2\big)$ is no restriction as every two-dimensional metric is conformal to the flat metric.

To find vacuum solutions with the prescribed symmetry we have to calculate the Ricci tensor. As usual, the easiest way to find the Christoffel symbols is by starting from the Lagrangian for the geodesics,

$$\mathcal{L}(x\dot{x}) \;=\; \frac{1}{2}\Big(\mathsf{e}^{2\gamma-2\psi}\big(\dot\rho^2 - c^2\dot t^2\big) + \mathsf{e}^{-2\psi}W^2\dot\varphi^2 + \mathsf{e}^{2\psi}\dot z^2\Big)$$

where the overdot means differentiation with respect to an affine parameter s. The Euler-Lagrange equations give the four components of the geodesic equation.

After some elementary algebra they take the following form.

$$\ddot{z} + 2\,\partial_\rho\psi\,\dot{\rho}\,\dot{z} + 2\,\partial_t\psi\,\dot{t}\,\dot{z} = 0\,,$$

$$\ddot{\varphi} + 2\Big(\frac{\partial_\rho W}{W} - \partial_\rho\psi\Big)\dot{\rho}\,\dot{\varphi} + 2\Big(\frac{\partial_t W}{W} - \partial_t\psi\Big)\dot{t}\,\dot{\varphi} = 0\,,$$

$$\begin{aligned}\ddot{\rho} + \big(\partial_\rho\gamma - \partial_\rho\psi\big)\dot{\rho}^2 - 2\big(\partial_t\gamma - \partial_t\psi\big)\dot{\rho}\,\dot{t} + \big(\partial_\rho\gamma - \partial_\rho\psi\big)c^2\dot{t}^2\\ -\mathsf{e}^{-2\gamma}W^2\Big(\frac{\partial_\rho W}{W} - \partial_\rho\psi\Big)\dot{\varphi}^2 - \mathsf{e}^{-2\gamma+4\psi}\partial_\rho\psi\,\dot{z}^2 = 0\,,\end{aligned}$$

$$\begin{aligned}\ddot{t} + 2\big(\partial_\rho\gamma - \partial_\rho\psi\big)\dot{\rho}\,\dot{t} + \big(\partial_t\gamma - \partial_t\psi\big)\dot{t}^2 + \big(\partial_t\gamma - \partial_t\psi\big)\frac{1}{c^2}\dot{\rho}^2\\ +\mathsf{e}^{-2\gamma}\frac{W^2}{c^2}\Big(\frac{\partial_t W}{W} - \partial_t\psi\Big)\dot{\varphi}^2 + \mathsf{e}^{-2\gamma+4\psi}\partial_t\psi\,\frac{1}{c^2}\,\dot{z}^2 = 0\,.\end{aligned}$$

From these equations we can read the Christoffel symbols and, thereupon, calculate the Ricci tensor. We find

$$R_{zz} = \mathsf{e}^{-2\gamma+4\psi}\Big(\partial_\rho^2\psi - \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho W}{W}\partial_\rho\psi - \frac{\partial_t W}{c^2 W}\partial_t\psi\Big)\,,$$

$$R_{\varphi\varphi} = W^2\mathsf{e}^{-2\gamma}\Big(-\partial_\rho^2\psi + \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho^2 W}{W} - \frac{\partial_t^2 W}{c^2 W} - \frac{\partial_\rho W}{W}\partial_\rho\psi + \frac{\partial_t W}{c^2 W}\partial_t\psi\Big)\,,$$

$$\begin{aligned}R_{tt} = \partial_t^2\gamma - c^2\partial_\rho^2\gamma - \partial_t^2\psi + c^2\partial_\rho^2\psi + \frac{\partial_t^2 W}{W} - c^2\frac{\partial_\rho W}{W}\big(\partial_\rho\gamma - \partial_\rho\psi\big)\\ -\frac{\partial_t W}{W}\big(\partial_t\gamma + \partial_t\psi\big) + 2\big(\partial_t\psi\big)^2\,,\end{aligned}$$

$$\begin{aligned}R_{\rho\rho} = \partial_\rho^2\gamma - \frac{1}{c^2}\partial_t^2\gamma - \partial_\rho^2\psi + \frac{1}{c^2}\partial_t^2\psi + \frac{\partial_\rho^2 W}{W} - \frac{\partial_\rho W}{W}\big(\partial_\rho\gamma + \partial_\rho\psi\big)\\ -\frac{\partial_t W}{c^2 W}\big(\partial_t\gamma - \partial_t\psi\big) + 2\big(\partial_\rho\psi\big)^2\,,\end{aligned}$$

$$R_{\rho t} = R_{t\rho} = \frac{\partial_\rho\partial_t W}{W} - \frac{\partial_\rho W}{W}\partial_t\gamma - \frac{\partial_t W}{W}\partial_\rho\gamma + 2\partial_\rho\psi\,\partial_t\psi\,,$$

The other components of the Ricci tensor are zero. This reduces the vacuum field equation to five scalar equations. The first two equations, $R_{zz} = 0$ and $R_{\varphi\varphi} = 0$, are equivalent to the two equations

$$\partial_\rho^2 W - \frac{1}{c^2}\partial_t^2 W = 0\,, \qquad \text{(B1)}$$

$$\partial_\rho^2 \psi - \frac{1}{c^2}\partial_t^2 \psi + \frac{\partial_\rho W}{W}\partial_\rho \psi - \frac{\partial_t W}{c^2 W}\partial_t \psi = 0\,. \qquad \text{(B2)}$$

Similarly, the equations $R_{\rho\rho} = 0$ and $R_{tt} = 0$ are equivalent to the two equations

$$\frac{\partial_t^2 W}{2c^2 W} + \frac{\partial_\rho^2 W}{2W} - \frac{\partial_\rho W}{W}\partial_\rho \gamma - \frac{\partial_t W}{c^2 W}\partial_t \gamma + \frac{1}{c^2}\big(\partial_t \psi\big)^2 + \big(\partial_\rho \psi\big)^2 = 0\,, \qquad \text{(B3)}$$

$$\partial_\rho^2 \gamma - \frac{1}{c^2}\partial_t^2 \gamma - \frac{1}{c^2}\big(\partial_t \psi\big)^2 + \big(\partial_\rho \psi\big)^2 = 0\,. \qquad \text{(B4)}$$

The last component requires

$$\frac{\partial_\rho \partial_t W}{W} - \frac{\partial_\rho W}{W}\partial_t \gamma - \frac{\partial_t W}{W}\partial_\rho \gamma + 2\partial_\rho \psi\, \partial_t \psi = 0\,. \qquad \text{(B5)}$$

We will solve these equations for two cases.

Case A : $\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2 > 0$

This condition, which says that the gradient of the function W is spacelike, guarantees, in particular, that this gradient has no zeros. We can, therefore, use

$$\tilde{\rho} = W(t, \rho)$$

as a new coordinate. We use this freedom for performing a coordinate transformation $(t, \rho) \mapsto \big(\tilde{t}, \tilde{\rho}\big)$ such that

$$d\tilde{t} = \partial_\rho W\, dt + \frac{\partial_t W}{c^2} d\rho\,, \qquad d\tilde{\rho} = \partial_\rho W\, d\rho + \partial_t W\, dt\,.$$

The second equation is just the differential version of the definition of $\tilde{\rho}$. We have to check if the first equation defines, indeed, a function $\tilde{t}$. The equations

$$\partial_t \tilde{t} = \partial_\rho W\,, \qquad \partial_\rho \tilde{t} = \frac{\partial_t W}{c^2}$$

can be satisfied only if the integrability condition

$$\partial_\rho^2 W = \frac{\partial_t^2 W}{c^2}$$

is satisfied. This, however, is guaranteed by the field equation, see (B1). Our new coordinates satisfy

$$d\tilde{\rho}^2 - c^2 d\tilde{t}^2 = \Big(\partial_\rho W\, d\rho + \partial_t W\, dt\Big)^2 - c^2\Big(\partial_\rho W\, dt + \frac{\partial_t W}{c^2} d\rho\Big)^2$$

$$= \Big(\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2\Big)\big(d\rho^2 - c^2 dt^2\big)\,.$$

Note that the factor on the right-hand side is positive by assumption. Therefore, we can replace the function γ by a new function $\tilde{\gamma}$, defined by

$$\mathsf{e}^{2\tilde{\gamma}} \;=\; \frac{\mathsf{e}^{2\gamma}}{\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2}\,.$$

Then the metric reads

$$g_{\mu\nu} dx^\mu dx^\nu = \mathsf{e}^{2\tilde{\gamma}-2\psi}\big(d\tilde{\rho}^2 - c^2 d\tilde{t}^2\big) + \mathsf{e}^{-2\psi}\tilde{\rho}^2 d\varphi^2 + \mathsf{e}^{2\psi} dz^2\,.$$

In the following we drop the tildas. Now we have to evaluate our field equations (B1) to (B5) with $W(t,\rho) = \rho$. (B1) is automatically satisfied. (B2) becomes

$$\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi - \frac{1}{c^2}\partial_t^2\psi = 0\,. \qquad \text{(B2')}$$

(B3) and (B5) can be solved for the partial derivatives of γ,

$$\partial_\rho\gamma = \rho\Big(\big(\partial_\rho\psi\big)^2 + \frac{1}{c^2}\big(\partial_t\psi\big)^2\Big)\,, \qquad \text{(B3')}$$

$$\partial_t\gamma \;=\; 2\,\rho\,\partial_\rho\psi\,\partial_t\psi\,. \qquad \text{(B5')}$$

(B4) is then automatically satisfied. Note that (B2') is a differential equation for ψ alone. We can solve this equation with a standard separation ansatz. After splitting off the time part we are left with the radial part of the Laplace equation in cylindrical coordinates which is the well-known Bessel equation. Therefore the general solution to (B2') is

$$\psi(t,\rho) \;=\; A\,J_0(\omega\rho)\cos(\omega t) + B\,Y_0(\omega\rho)\sin(\omega t)$$

where J_0 and Y_0 are the Bessel functions of first and second kind, respectively. While J_0 is regular everywhere, Y_0 goes to $-\infty$ for $\rho \to 0$. If we want to have a solution that is regular on the axis we have to choose $B = 0$. Having solved (B2'), we can determine γ from (B3') and (B5'). It is obvious

that the solution γ is unique up to an additive constant. Existence of the solution is less trivial. We have to check if the integrability condition is satisfied:

$$\partial_t\Big\{\rho\Big(\big(\partial_\rho\psi\big)^2+\frac{1}{c^2}\big(\partial_t\psi\big)^2\Big)\Big\} \stackrel{?}{=} \partial_\rho\big\{2\,\rho\,\partial_\rho\psi\,\partial_t\psi\big\}\,,$$

$$\rho\Big(\cancel{2\partial_\rho\psi\,\partial_t\partial_\rho\psi}+\frac{2}{c^2}\partial_t\psi\partial_t^2\psi\Big) \stackrel{?}{=} 2\,\partial_\rho\psi\,\partial_t\psi+2\,\rho\,\partial_\rho^2\psi\,\partial_t\psi+\cancel{2\,\rho\,\partial_\rho\psi\,\partial_\rho\partial_t\psi}\,,$$

$$0 \stackrel{!}{=} 2\,\rho\,\partial_t\psi\Big(\frac{1}{\rho}\partial_\rho\psi+\partial_\rho^2\psi-\frac{1}{c^2}\partial_t^2\psi\Big)\,.$$

We see that the integrability condition of (B3') and (B5') is just the equation (B2'). This guarantees that to every solution of (B2') we find a corresponding γ such that all components of the vacuum field equation are satisfied.

This class of solutions describes gravitational waves with cylindrical symmetry. For $B = 0$ they are well-defined, as source-free vacuum solutions, on all of $\mathbb{R}^4$. There is a coordinate singularity on the axis, as always when using cylindrical polar coordinates, but no curvature singularity. This class of vacuum solutions was (re-)discovered by A. Einstein and N. Rosen ["On gravitational waves" J. Franklin Inst. 223, 43 (1937)]. In an earlier version of this paper, Einstein and Rosen had interpreted the coordinate we called φ as a non-periodic, Cartesian-like coordinate and, correspondingly, the waves as planar rather than as cylindrical. The (coordinate) singularity at $\rho = 0$ gave them the impression that this solution is unphysical and they even concluded from this observation that gravitational waves do not exist in the full non-linear theory. Einstein and Rosen submitted their paper with this (completely false) conclusion to Physical Review. The Editor sent the article to a referee (which had never been happened to Einstein before) who pointed out that the conclusion was erroneous and that, actually, the solutions are cylindrical. Einstein was so angry about the fact that his article had been sent for refereeing that he withdraw the paper and decided never again to publish in Physical Review. After H. P. Robertson (who, as we know now, was the referee) explained to him his error, Einstein wrote a completely new version of the article (N. Rosen had left for Russia by that time) which was then published in the Journal of the Franklin Institute. The cylindrical solutions presented in this paper are now known as Einstein-Rosen waves although they had already been found by Beck 12 years earlier.

Case B : $\big(\partial_\rho W\big)^2 - \frac{1}{c^2}\big(\partial_t W\big)^2 = 0\,, \quad \partial_\rho W \neq 0$

This condition says that the gradient of the function W is lightlike and non-zero. Then we have

$$\partial_\rho W = \pm\frac{1}{c}\partial_t W\,.$$

Here and in the following, either the upper sign or the lower sign holds. The components (B3) and (B5) of the field equations read

$$\frac{\partial_\rho^2 W}{W} - \frac{\partial_\rho W}{W}\partial_\rho\gamma \mp \frac{\partial_\rho W}{cW}\partial_t\gamma + \frac{1}{c^2}\big(\partial_t\psi\big)^2 + \big(\partial_\rho\psi\big)^2 = 0\,,$$

$$\frac{\partial_\rho^2 W}{W} \mp \frac{\partial_\rho W}{cW}\partial_t\gamma - \frac{\partial_\rho W}{W}\partial_\rho\gamma \pm \frac{2}{c}\partial_\rho\psi\partial_t\psi = 0\,.$$

Subtracting the second equation from the first yields

$$\partial_\rho\psi = \pm\frac{1}{c}\partial_t\psi\,,$$

i.e.,

$$\psi(t,\rho) = f(ct \pm \rho)\,.$$

Upon inserting this result into (B4), and using that $\partial_\rho W$ has no zeros, we find

$$\partial_\rho^2\gamma - \frac{1}{c^2}\partial_t^2\gamma = 0\,,$$

i.e.,

$$\gamma(t,\rho) = p(ct \pm \rho) + q(ct \mp \rho)\,.$$

With these results our metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = -\Big(e^{2p-2f}(cdt \pm d\rho)\Big)\Big(e^{2q}(cdt \mp d\rho)\Big) + e^{-2f}W^2 d\varphi^2 + e^{2f}dz^2\,.$$

We replace t and ρ by new coordinates $\big(\tilde{u},\tilde{v}\big)$ such that

$$d\tilde{u} = \frac{1}{\sqrt{2}}\,e^{2p-2f}(cdt \pm d\rho)\,, \qquad d\tilde{v} = \frac{1}{\sqrt{2}}\,e^{2q}(cdt \mp d\rho)\,.$$

This is possible as the integrability conditions are obviously satisfied. Then the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = 2d\tilde{u}d\tilde{v} + C_{11}(\tilde{u})d\varphi^2 + C_{22}(\tilde{u})dz^2$$

where we have used that f and W depend on $t \pm \rho$ only which, in turn, can be expressed in terms of $\tilde{u}$ alone. Metrics of the form

$$g_{\mu\nu}dx^\mu dx^\nu = -2d\tilde{u}d\tilde{v} + C_{AB}(\tilde{u})d\tilde{x}^A d\tilde{X}^B$$

are known as *Rosen waves*. They were discussed in a paper by N. Rosen which he wrote after he had left Princeton for the Soviet Union [N. Rosen: "Plane polarized waves in the general theory of relativity". Phys. Z. Soviet Union 12, 366 (1937)]. With our metric ansatz we have found only those Rosen waves for which the matrix $C_{AB}(\tilde{u})$ is diagonal; one gets the general class if one drops the assumption of the φ lines being orthogonal to the z lines.

The Rosen waves are actually locally isometric to the plane waves we have studied in the preceding section in the Brinkmann coordinates. We demonstrate this for the case that the matrix C_{AB} is diagonal.

We start out from the metric in Rosen coordinates with

$$\big(C_{AB}\big) = \begin{pmatrix} e_1(\tilde{u}^2)^2 & 0 \\ 0 & e_2(\tilde{u}^2)^2 \end{pmatrix}.$$

We express the Rosen coordinates $\big(\tilde{u}, \tilde{v}, \tilde{x}^1, \tilde{x}^2\big)$ in terms of new coordinates (which will turn out to be the Brinkmann coordinates) (u, v, x^1, x^2) by

$$\tilde{u} = u\,, \qquad \tilde{v} = v - \frac{1}{2}\Big(\frac{\dot{e}_1(u)}{e_1(u)}(x^1)^2 + \frac{\dot{e}_2(u)}{e_2(u)}(x^2)^2\Big)\,,$$

$$\tilde{x}^1 = \frac{x^1}{e_1(u)}\,, \qquad \tilde{x}^2 = \frac{x^2}{e_2(u)}\,.$$

Then

$$g_{\mu\nu}dx^\mu dx^\nu = -2d\tilde{u}d\tilde{v} + e_1^2\big(d\tilde{x}^1\big)^2 + e_2^2\big(d\tilde{x}^2\big)^2$$

$$= -2du\Big\{dv - \frac{1}{2}\frac{\dot{e}_1}{e_1}2x^1dx^1 - \frac{1}{2}\frac{\dot{e}_2}{e_2}2x^2dx^2 - \frac{1}{2}\Big(\frac{\dot{e}_1}{e_1}\Big)^{\cdot}(x^1)^2du + \frac{1}{2}\Big(\frac{\dot{e}_2}{e_2}\Big)^{\cdot}(x^2)^2du\Big\}$$

$$+e_1^2\Big(\frac{dx^1}{e_1} - \frac{\dot{e}_1}{e_1^2}x^1du\Big)^2 + e_2^2\Big(\frac{dx^2}{e_2} - \frac{\dot{e}_2}{e_2^2}x^2du\Big)^2$$

$$= -2dudv + dudx^1\Big\{\cancel{2\frac{\dot{e}_1}{e_1}x^1} - \cancel{2\frac{\dot{e}_1}{e_1}x^1}\Big) + dudx^2\Big(\cancel{2\frac{\dot{e}_2}{e_2}x^2} - \cancel{2\frac{\dot{e}_2}{e_2}x^1}\Big\}$$

$$+du^2\Big\{\Big(\frac{\ddot{e}_1}{e_1} - \cancel{\frac{\dot{e}_1^2}{e_1^2}}\Big)(x_1)^2 + \Big(\frac{\ddot{e}_1}{e_2} - \cancel{\frac{\dot{e}_2^2}{e_2^2}}\Big)(x_2)^2 + \cancel{\frac{\dot{e}_1^2}{e_1^2}(x^1)^2} + \cancel{\frac{\dot{e}_2^2}{e_2^2}(x^2)^2}\Big) + (dx^1)^2 + (dx^2)^2\,.$$

This is precisely the form of a plane wave in Brinkmann coordinates,

$$g_{\mu\nu}dx^\mu dx^\nu = -2dudv + h_{AB}(u)x^A x^B du^2 + (dx^1)^2 + (dx^2)^2\,,$$

with

$$\big(h_{AB}(u)\big) = \begin{pmatrix} \dfrac{\ddot{e}_1(u)}{e_1} & 0 \\ 0 & \dfrac{\ddot{e}_2(u)}{e_2} \end{pmatrix}\,.$$

The vacuum field equation requires

$$\frac{\ddot{e}_1(u)}{e_1} + \frac{\ddot{e}_2(u)}{e_2} = 0\,.$$

The other cases, where the gradient of W is timelike, or where it changes its causal character from point to point, will not be treated here. The latter case is of relevance for *colliding* waves.

8.3 Robinson-Trautman solutions

While plane waves are associated with bounded sources only approximately, at a large distance from the sources, and cylindrical waves are not associated with bounded sources at all, we will finally study a class of solutions that do give a valid description of gravitational radiation from bounded sources. It was constructed by I. Robinson and A. Trautman ["Some spherical gravitational waves in general relativity" Proc. Roy. Soc. London A 265, 463 (1962)] in analogy to the Liénard-Wiechert field from electrodynamics. The latter is the electromagnetic field of an accelerated point charge in Minkowski spacetime. The radiation field propagates along the lightlike geodesics (i.e., lightlike straight lines) that issue from the worldline of the point charge into the future. These lightlike geodesics, which generate the future light-cones from the events of the worldline of the charge, are hypersurface-orthogonal, shear-free and expanding. The basic idea of Robinson and Trautman was to construct vacuum solutions to Einstein's field equation which admit a family of lightlike geodesics with the same properties. One could then interpret these lightlike geodesics as the rays of gravitational radiation.

We begin by writing down the general form of a spacetime that is foliated into lightlike hypersurfaces. These hypersurfaces, which generalise the light-cones in

Minkowski spacetime, can be written as hypersurfaces $\sigma =$ constant where σ is a scalar function with a lightlike gradient,

$$g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 0\,.$$

We define a vector field $K^\mu\partial_\mu$ by

$$K^\mu = g^{\mu\nu}\partial_\nu\sigma\,.$$

Clearly, this vector field is lightlike,

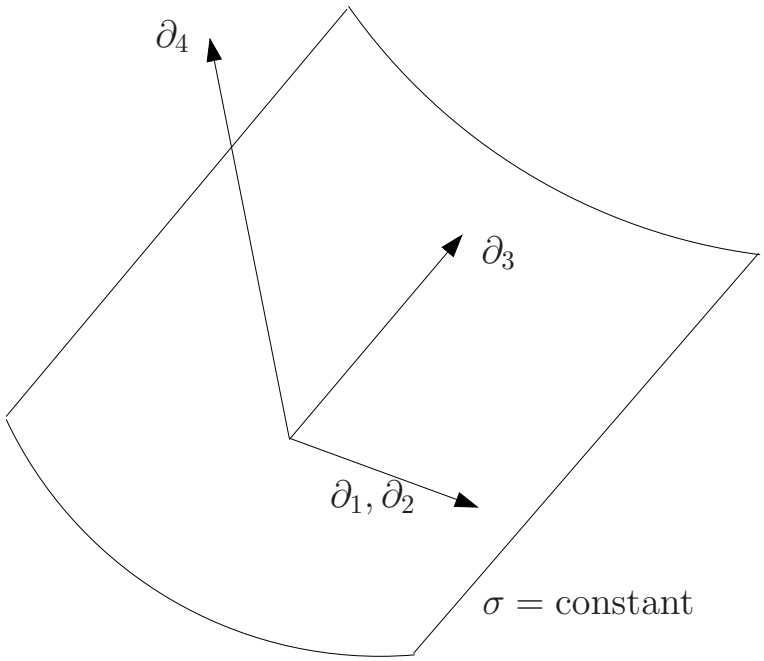
$$g_{\mu\nu}K^\mu K^\nu = g_{\mu\nu}g^{\mu\rho}\partial_\rho\sigma g^{\nu\lambda}\partial_\lambda\sigma = g^{\rho\lambda}\partial_\rho\sigma\partial_\lambda\sigma = 0\,,$$

and geodesic,

$$0 = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \quad \Longrightarrow \quad 0 = \nabla_\lambda\big(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\big) = 2g^{\mu\nu}\partial_\mu\nabla_\lambda\partial_\nu\sigma$$

$$= 2K^\nu\big(\partial_\lambda\partial_\nu\sigma - \Gamma^\tau{}_{\lambda\nu}\sigma\big) = 2K^\nu\big(\partial_\nu\partial_\lambda\sigma - \Gamma^\tau{}_{\nu\lambda}\sigma\big) = 2K^\nu\nabla_\nu\partial_\lambda\sigma = 2K^\nu\nabla_\nu K_\lambda\sigma\,.$$

Note that the vector field $K^\mu\partial_\mu$ is tangent to the hypersurfaces $\sigma =$ constant and at the same time orthogonal to them.



We can choose coordinates $x^1 = \xi$, $x^2 = \eta$, $x^3 = \rho$ and $x^4 = \sigma$ in such a way that

$$\frac{\partial}{\partial\rho} = \partial_3 = K^\mu\partial_\mu\,.$$

This can be achieved by assigning the value $\rho = \rho_0$ to a hypersurface that is transverse to the hypersurfaces $\sigma =$ constant and dragging it along with the

flow of $K^\mu\partial_\mu$ to get the other hypersurfaces $\rho =$ constant; the coordinates ξ and η have to be chosen transverse to σ, but arbitrarily otherwise, on the initial hypersurface $\rho = \rho_0$ and are then again fixed by dragging them along with the flow of $K^\mu\partial_\mu$. By construction,

$$g^{14} = g^{41} = g^{\mu\nu}\delta^4_\mu\delta^1_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\xi = K^\nu\partial_\nu\xi = \frac{\partial\xi}{\partial\rho} = 0\,,$$

$$g^{24} = g^{42} = g^{\mu\nu}\delta^4_\mu\delta^2_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\eta = K^\nu\partial_\nu\eta = \frac{\partial\eta}{\partial\rho} = 0\,,$$

$$g^{34} = g^{43} = g^{\mu\nu}\delta^4_\mu\delta^3_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\rho = K^\nu\partial_\nu\rho = \frac{\partial\rho}{\partial\rho} = 1\,,$$

$$g^{44} = g^{\mu\nu}\delta^4_\mu\delta^4_\nu = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 0\,.$$

This demonstrates that the contravariant components of the metric can be written as

$$\big(g^{\mu\nu}\big) = \begin{pmatrix} P^2\gamma^{11} & P^2\gamma^{12} & a & 0 \\ P^2\gamma^{12} & P^2\gamma^{22} & b & 0 \\ a & b & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{with} \qquad \det\begin{pmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{12} & \gamma^{22} \end{pmatrix} = 1\,.$$

Here we have used that the two-surfaces parametrised by ξ and η are spacelike, so the determinant of $\big(g^{AB}\big)$ must be positive. (As before, capital indices $A, B, \ldots$ take values 1 and 2.) From the minors of the matrix $(g^{\mu\nu})$ we read that $g_{31} = g_{32} = g_{33} = 0$, hence

$$\delta^B_A = g_{A\mu}g^{\mu B} = g_{AC}g^{CB} = g_{AC}P^2\gamma^{CB} \quad \Longrightarrow \quad (\gamma^{-1})_{AC} = P^2 g_{AC}\,.$$

We will now add the condition that $K^\mu\partial_\mu$ should be shear-free and expanding. Twist, expansion and shear of the lightlike vector field $K^\mu\partial_\mu$ are defined as

$$\text{twist}: \quad \Omega_{AB} = \frac{1}{2}\big(\nabla_A K_B - \nabla_B K_A\big)\,,$$

$$\text{expansion}: \quad \Theta = \nabla_A K^A\,,$$

$$\text{shear}: \quad \Sigma_{AB} = \frac{1}{2}\big(\nabla_A K_B + \nabla_B K_A\big) - \frac{\Theta}{2}g_{AB}\,.$$

In the case at hand,

$$\nabla_\mu K_\nu = \nabla_\mu\partial_\nu\sigma = \partial_\mu\partial_\nu\sigma - \Gamma^\lambda{}_{\mu\nu}\partial_\lambda\sigma = 0 - \frac{1}{2}g^{\lambda\tau}\big(\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}\big)\delta^4_\lambda$$

$$= -\frac{1}{2}\underbrace{g^{43}}_{=1}\big(\partial_\mu \underbrace{g_{3\nu}}_{=0} + \partial_\nu \underbrace{g_{3\mu}}_{=0} - \partial_3 g_{\mu\nu}\big) = \frac{1}{2}\partial_3 g_{\mu\nu}\,.$$

Hence, the twist vanishes, $\Omega_{AB} = 0$. (Quite generally, the property of being hypersurface-orthogonal is equivalent to being twist-free.) To calculate the expansion, we observe that the Jacobi formula

$$\partial_3\big(\det(\boldsymbol{\gamma})\big) = \mathrm{trace}\big(\boldsymbol{\gamma}^{-1}\partial_3\boldsymbol{\gamma}\big)$$

applied to the matrix $\boldsymbol{\gamma} = (\gamma^{AB})$ results in

$$0 \;=\; \big(\gamma^{-1}\big)_{AB}\partial_3\gamma^{AB}\,,$$

hence

$$\Theta = g^{AB}\nabla_A K_B = \frac{1}{2}g^{AB}\partial_3 g_{AB} = -\frac{1}{2}g_{AB}\partial_3 g^{AB} = -\frac{1}{2}P^{-2}(\gamma^{-1})_{AB}\partial_3(P^2\gamma^{AB})$$

$$= -\frac{1}{2}P^{-2}(\gamma^{-1})_{AB}\gamma^{AB}2P\partial_3 P = -2P^{-1}\partial_3 P\,.$$

Finally, we find the shear as

$$\Sigma_{AB} = \frac{1}{2}\partial_3 g_{AB} + P^{-1}\partial_3 P g_{AB} = \frac{1}{2}\partial_3\big(P^{-2}(\gamma^{-1})_{AB}\big) + P^{-3}(\gamma^{-1})_{AB}\partial_3 P$$

$$= \frac{1}{2}P^{-2}\partial_3\big((\gamma^{-1})_{AB}\big)\,.$$

We assume that the shear vanishes, i.e., that $\partial_3\big((\gamma^{-1})_{AB}\big) = 0$. This condition is equivalent to $\partial_3\gamma^{AB} = 0$. If we choose the coordinates ξ and η such that $\gamma^{AB} = \delta^{AB}$ on the initial hypersurface $\rho = \rho_0$, this condition will hold everywhere, so the contravariant components of the metric simplify to

$$\big(g^{\mu\nu}\big) = \begin{pmatrix} P^2 & 0 & a & 0 \\ 0 & P^2 & b & 0 \\ a & b & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\,.$$

This matrix can be easily inverted,

$$\big(g_{\mu\nu}\big) = \begin{pmatrix} P^{-2} & 0 & 0 & -P^{-2}a \\ 0 & P^{-2} & 0 & -P^{-2}b \\ 0 & 0 & 0 & 1 \\ -P^{-2}a & -P^{-2}b & 1 & -c + P^{-2}a^2 + P^{-2}b^2 \end{pmatrix}\,,$$

so the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = P^{-2}\Big(\big(d\xi - a\,d\sigma\big)^2 + \big(d\eta - b\,d\sigma\big)^2\Big) + 2\,d\rho\,d\sigma - c\,d\sigma^2\,.$$

This is the general form of a metric that admits a hypersurface-orthogonal, shear-free geodesic lightlike vector field.

Finally, we add the conditions that the expansion is non-zero and that the vacuum field equation $R_{\mu\nu} = 0$ holds. We begin with the 33-component of the field equation.

$$\begin{aligned}
0 = R_{33} = R^\tau{}_{\mu\tau\nu}K^\mu K^\nu &= K^\mu\big(\nabla_\mu\nabla_\tau K^\tau - \nabla_\tau\nabla_\mu K^\tau\big)\\
&= K^\mu\nabla_\mu\nabla_\tau K^\tau - \nabla_\tau\big(\underbrace{K^\mu\nabla_\mu K^\tau}_{=0}\big) + \nabla_\mu K^\mu\nabla_\tau K^\tau\\
= \partial_3\big(\nabla_A K^A + 0\big) + \nabla_A K^B\nabla_B K^A + 0 &= \partial_3\Theta + \frac{1}{4}g^{BC}g^{AD}\partial_3 g_{AC}\partial_3 g_{BD}\\
&= \partial_3\Theta + \frac{P^4}{4}\delta^{BC}\delta^{AD}\partial_3\big(P^{-2}\delta_{AC}\big)\partial_3\big(P^{-2}\delta_{BD}\big)\\
&= \partial_3\Theta + \frac{P^4}{4}\big(\underbrace{-2P^{-3}\partial_3 P}_{=P^{-2}\Theta}\big)^2\underbrace{\delta^B_A\delta^A_B}_{=2} = \frac{\partial\Theta}{\partial\rho} + \frac{\Theta^2}{2}\,.
\end{aligned}$$

Quite generally, evaluating the expression $R_{\mu\nu}K^\mu K^\nu$ results in a differential equation for the expansion Θ along the integral curves of $K^\mu\partial_\mu$ which is known as the *Raychudhuri equation*. In the case at hand, assuming $R_{33} = 0$, it simply reads

$$\frac{\partial\Theta}{\partial\rho} = \frac{\Theta^2}{2}\,.$$

Now we use our assumption that $\Theta \neq 0$. Then we can integrate the Raychudhuri equation,

$$-\frac{2}{\Theta^2}\frac{\partial\Theta}{\partial\rho} = 1 \qquad\Longrightarrow\qquad 2\frac{\partial\Theta^{-1}}{\partial\rho} = 1 \qquad\Longrightarrow\qquad \frac{2}{\Theta} = \rho + f(\xi,\eta,\sigma)\,.$$

As ρ was introduced by assigning a value ρ_0 to an arbitrary hypersurface transverse to the lightlike hypersurfaces $\sigma =$ constant, we are free to make a coordinate transformation $\rho \mapsto \rho - f(\xi,\eta,\sigma)$. Then we have

$$\rho = \frac{2}{\Theta} = -P\left(\frac{\partial P}{\partial\rho}\right)^{-1} \qquad\Longrightarrow\qquad \frac{\partial P}{\partial\rho} = -\frac{P}{\rho}$$

$$\Longrightarrow \qquad \frac{\partial\big(\rho\,P\big)}{\partial\rho} = P + \rho\,\frac{\partial P}{\partial\rho} = P - \rho\,\frac{P}{\rho} = 0\,.$$

So our assumption that (at least the 33-component of) the vacuum field equation holds and that $\Theta \neq 0$ has led to the conclusion that

$$p := \rho\,P \quad \text{satisfies} \quad \frac{\partial p}{\partial\rho} = 0\,.$$

So we can write the metric as

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho^2}{p^2}\Big(\big(d\xi - a\,d\sigma\big)^2 + \big(d\eta - b\,d\sigma\big)^2\Big) + 2\,d\rho\,d\sigma - c\,d\sigma^2 \qquad \text{with} \quad \partial_3 p = 0\,.$$

For evaluating the vacuum field equation, we now have to calculate the other components of the Ricci tensor for this metric. This is straight-forward but rather tedious. Mathematica gives the following results.

$$R_{13} = \frac{\partial_3(\rho^4\partial_3 a)}{2p^2\rho^2}\,,$$

$$R_{23} = \frac{\partial_3(\rho^4\partial_3 b)}{2p^2\rho^2}\,,$$

$$R_{11} - R_{22} = \frac{\rho}{2p^4}\Big(\rho^2(\partial_3 a)^2 - \rho^2(\partial_3 b)^2 - 2p^2\big(2\partial_2 b + \rho\partial_2\partial_3 b - 2\partial_1 a - \rho\partial_1\partial_3 a\big)\Big)\,,$$

$$R_2 = \frac{\rho}{2p^4}\Big(\rho^2\partial_3 a\,\partial_3 b + p^2\big(2\partial_2 a + \rho\partial_2\partial_3 a + 2\partial_1 b + \rho\partial_1\partial_3 b\big)\Big)\,.$$

The first two and the last two equations can be combined in complex form, respectively, if we introduce the complex function $z := a + ib$,

$$R_{13} + iR_{23} = \frac{\partial_3(\rho^4\partial_3 z)}{2p^2\rho^2}\,,$$

$$R_{11} - R_{22} + 2iR_{12} = \frac{\rho}{2p^4}\Big(\rho^2(\partial_3 z)^2 + 2p^2\big(\partial_1 + i\partial_2\big)\big(2z + \rho\partial_3 z\big)\Big)\,.$$

The vacuum field equation requires $R_{13} + iR_{23} = 0$, hence

$$\rho^4\partial_3 z = v \qquad \Longrightarrow \qquad z = u - \frac{v}{3\rho^3} \qquad \text{with} \quad \partial_3 u = \partial_3 v = 0\,.$$

Inserting this result into the equation $R_{11} - R_{22} + 2iR_{12} = 0$ yields

$$0 = v^2 + 2p^2\big(\partial_1 + i\partial_2\big)\big(2u\rho^6 + \frac{v}{3}\rho^3\big)\,.$$

By comparing equal powers of ρ we find

$$v = 0 \quad \text{and} \quad (\partial_1 + i\partial_2)u = 0\,,$$

i.e., the function $z = a + ib = u$ is independent of $\rho = x^3$ and analytic in the complex variable $\xi + i\eta = x^1 + ix^2$,

$$\partial_3 z = 0\,, \quad (\partial_1 + i\partial_2)z = 0\,.$$

The second condition means that, if real and imaginary parts are written separately, the Cauchy-Riemann equations

$$\frac{\partial a}{\partial \xi} = \frac{\partial b}{\partial \eta}\,, \qquad \frac{\partial a}{\partial \eta} = -\frac{\partial b}{\partial \xi}$$

hold.

On the basis of these observations we will now show that a and b can be transformed to zero. To that end we perform a coordinate transformation of the form

$$\xi = \alpha(\tilde{\xi}, \tilde{\eta}, \tilde{\sigma})\,, \quad \eta = \beta(\tilde{\xi}, \tilde{\eta}, \tilde{\sigma})\,, \quad \sigma = \gamma(\tilde{\sigma})\,, \quad \rho = \frac{\tilde{\rho}}{\gamma'(\tilde{\sigma})}$$

where $\alpha + i\beta$ is an analytic function of $\tilde{\xi} + i\tilde{\eta}$, i.e.,

$$\frac{\partial \alpha}{\partial \tilde{\xi}} = \frac{\partial \beta}{\partial \tilde{\eta}}\,, \qquad \frac{\partial \alpha}{\partial \tilde{\eta}} = -\frac{\partial \beta}{\partial \tilde{\xi}}\,.$$

We choose α and β such that

$$a\gamma'(\tilde{\sigma}) = \frac{\partial \alpha}{\partial \tilde{\sigma}}\,, \qquad b\gamma'(\tilde{\sigma}) = \frac{\partial \beta}{\partial \tilde{\sigma}}\,.$$

Such a choice is possible because a and b are independent of ρ and satisfy the Cauchy-Riemann equations, which guarantees that the necessary integrability conditions are satisfied,

$$\frac{\partial}{\partial \tilde{\xi}}\left(a\,\gamma'(\tilde{\sigma})\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial a}{\partial \xi}\frac{\partial \alpha}{\partial \tilde{\xi}} + \frac{\partial a}{\partial \eta}\frac{\partial \beta}{\partial \tilde{\xi}}\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial b}{\partial \eta}\frac{\partial \beta}{\partial \tilde{\eta}} + \frac{\partial b}{\partial \xi}\frac{\partial \alpha}{\partial \tilde{\eta}}\right) = \frac{\partial}{\partial \tilde{\eta}}\left(b\,\gamma'(\tilde{\sigma})\right),$$

$$\frac{\partial}{\partial \tilde{\eta}}\left(a\,\gamma'(\tilde{\sigma})\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial a}{\partial \xi}\frac{\partial \alpha}{\partial \tilde{\eta}} + \frac{\partial a}{\partial \eta}\frac{\partial \beta}{\partial \tilde{\eta}}\right) = \gamma'(\tilde{\sigma})\left(\frac{\partial b}{\partial \eta}\frac{\partial \beta}{\partial \tilde{\xi}} + \frac{\partial b}{\partial \xi}\frac{\partial \alpha}{\partial \tilde{\xi}}\right) = \frac{\partial}{\partial \tilde{\xi}}\left(b\,\gamma'(\tilde{\sigma})\right).$$

Under such a transformation the form of the metric is preserved, with

$$\frac{1}{\tilde{p}^2} = \frac{1}{p^2}\left(\left(\frac{\partial \alpha}{\partial \xi}\right)^2 + \left(\frac{\partial \beta}{\partial \xi}\right)^2\right) = \frac{1}{p^{-2}}\left(\left(\frac{\partial \alpha}{\partial \eta}\right)^2 + \left(\frac{\partial \beta}{\partial \eta}\right)^2\right)$$

$$\tilde{a} = 0\,, \qquad \tilde{b} = 0\,, \qquad \tilde{c} = c\gamma'(\tilde{\sigma})^2\,.$$

If we perform such a coordinate transformation, and then drop the tildas, the metric takes the simple form

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho^2}{p^2}\big(d\xi^2 + d\eta^2\big) + 2\,d\rho\,d\sigma - c\,d\sigma^2 \qquad \text{(RT)}$$

with functions $p(\xi, \eta, \sigma)$ and $c(\xi, \eta, \rho, \sigma)$.
For this metric we now calculate the remaining components of the Ricci tensor. Again with Mathematica, we find

$$R_{11} + R_{22} = \frac{2}{p}\Big(\partial_3(\rho c) - 4\rho\,\partial_4 \ln p - \Delta \ln p\Big)$$

where we introduced the modified Laplace operator

$$\Delta = p^2\big(\partial_1^2 + \partial_2^2\big)\,.$$

Integration of the equation $R_{11} + R_{22} = 0$ yields

$$c = 2\rho\partial_4 \ln p + \Delta \ln p - \frac{2m}{\rho} \quad \text{with} \quad \partial_3 m = 0\,. \qquad (*)$$

With this input we find that $R_{34} = 0$ is satisfied while

$$R_{14} = \frac{\partial_1 m}{\rho^2}\,, \qquad R_{24} = \frac{\partial_2 m}{\rho^2}\,.$$

Hence, the equations $R_{14} = 0$ and $R_{24} = 0$ require m to be a function of σ only. Finally, the remaining component of the Ricci tensor is

$$R_{44} = \frac{1}{2\rho^2}\Big(\Delta^2 \ln p + 12m\partial_4 \ln p - 4\frac{dm}{d\sigma}\Big)\,.$$

The condition $R_{44} = 0$ gives a fourth-order differential equation for p which is known as the *Robinson-Trautman equation*,

$$\Delta^2 \ln p + 12m\partial_4 \ln p - 4\frac{dm}{d\sigma} = 0\,.$$

We can now summarise the procedure of how to construct a Robinson-Trautman vacuum solution. We choose a function $m(\sigma)$. With this function, we have to find a solution p to the Robinson-Trautman equation. With this p and the chosen m, we define a function c via $(*)$. Then the metric (RT) is a solution to

Einstein's vacuum equation with the integral curves of $\partial/\partial\rho$ being a twist-free, shear-free, geodesic lightlike congruence with non-zero expansion.

Clearly, the Schwarzschild solution must be included. To verify this, choose for m a positive constant. Then the Robinson-Trautman equation is solved by

$$p = 1 + \frac{1}{4}\big(\xi^2 + \eta^2\big)\,,$$

because

$$\partial_4 \ln p = 0\,, \qquad \Delta \ln p = 1\,.$$

In this case the function c reads

$$c = \Delta \ln p - \frac{2m}{\rho} = 1 - \frac{2m}{\rho}\,.$$

We express the coordinates ξ and η in terms of new coordinates ϑ and φ via

$$\xi + i\eta = \, 2\tan\tfrac{\vartheta}{2}\, e^{i\varphi}$$

which is the stereographic projection mapping from a sphere to a plane. Then

$$d\xi^2 + d\eta^2 = \frac{4\sin^2\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}\,d\varphi^2 + \, d\vartheta^2}{\cos^4\frac{\vartheta}{2}} \; = \; \frac{\sin^2\vartheta\,d\varphi^2 + d\vartheta^2}{\cos^4\frac{\vartheta}{2}}\,,$$

$$p = 1 + \frac{1}{4}\big(\xi^2 + \eta^2\big) = 1 + \tan^2\tfrac{\vartheta}{2} = \frac{1}{\cos^2\frac{\vartheta}{2}}\,,$$

so the metric reads

$$g_{\mu\nu}dx^\mu dx^\nu = \rho^2\big(\sin^2\vartheta\,d\varphi^2 + d\vartheta^2\big) + 2d\rho d\sigma - \Big(1 - \frac{2m}{\rho}\Big)d\sigma^2\,.$$

If we rename $(\rho, \sigma) \mapsto (r, \pm c\tilde{t})$ we recognise the Schwarzschild metric in ingoing and outgoing Eddington-Finkelstein coordinates, respectively.

The Robinson-Trautman class of solutions also contains the socalled C-metric which describes a *uniformly accelerated* black hole. It can be viewed as the gravitational analogue of the *Born-Schott electromagnetic field* produced by a uniformly accelerated charge. Just as an accelerated charge produces *stationary* electromagnetic radiation, the C-metric describes *stationary* gravitational radiation.

Other Robinson-Trautman solutions describe *non-stationary* gravitational radiation produced by bounded sources. At the level of exact solutions to Einstein's field equation, the Robinson-Trautman metrics are the most realistic

models of gravitational radiation we have. As they do not include any (over-idealised) symmetry assumptions, their variety is much richer than that of the Brinkmann or (Beck-)Einstein-Rosen solutions. For a detailed discussion of Robinson-Trautman metrics, including the C-metric, see J. Griffiths and J. Podolský: "Exact Space-Times in General Relativity" Cambridge University Press, 2009.

There are several generalisations of the Robinson-Trautman solutions. In particular, the condition of the rays being hypersurface-orthogonal (twist-free) has been dropped. This is important to include *rotating* sources. A twisting null congruence can be rather complicated. In Roger Penrose's *twistor* formalism any twistor is associated with a certain twisting, shear-free, geodesic null congruence, called a "Robinson congruence", on (complexified, compactified) Minkowski spacetime. The picture below is a hand-drawing by Roger Penrose. It shows a time-slice of a Robinson congruence.