Gravitational Lensing

Volker Perlick (perlick@zarm.uni-bremen.de)

Winter Term 2012/2013, U Bremen, ZARM, Room 1730

Lectures: Thu 12–14 and Fr 13–14, Tutorials: Fri 12–13

Video recordings of the lectures are available

English books:

P. Schneider, J. Ehlers, E. Falco: "Gravitational Lenses" Springer (1992)
careful exposition of all aspects of lensing

A. Petters, H. Levine, J. Wambsganss: "Singularity Theory and Gravitational Lensing" Birkhäuser (2001)
after a general introduction the book concentrates on the theory of caustics in the quasi-Newtonian approximation formalism

S. Mollerach, E. Roulet: "Gravitational Lensing and Microlensing" World Scientific (2002)
emphasis is on applications, not so much on the mathematical foundations

Russian books:

P. Blioch, A. Minakov: "Gravitational Lensing" Naukova Dumka, Kiev (1989)

A. Zakharov: "Gravitational Lenses and Microlenses" Janus, Moscow (1997)

Electronic Review Articles:

J. Wambsganss: "Gravitational Lensing in Astronomy" Living Rev. Relativity 1 (1998), 12, `http://www.livingreviews.org/lrr-1998-12`
restricted to the quasi-Newtonian approximation formalism, concentrates on applications

V. Perlick: "Gravitational Lensing from a Spacetime Perspective" Living Rev. Relativity 7 (2004), 9, `http://www.livingreviews.org/lrr-2004-9`
concentrates on the mathematical formalism of lensing in a spacetime setting, without quasi-Newtonian approximation

Contents:

1. Introduction

1.1 Historic Notes

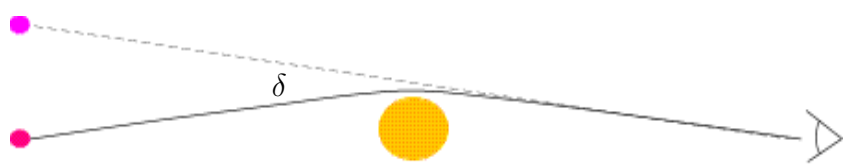
1704 I. Newton asks in Query 1 of his book *Opticks* if "bodies" have an influence on light particles.

1783 J. Michell speculates in a letter to H. Cavendish if there might exist "dark bodies" that are so dense that light cannot escape from their surface (see Worksheet 1).

1784 H. Cavendish calculates on a scrap of paper the deflection of light particles by the the Sun on the basis of Newtonian theory (see Worksheet 1). This calculation is found only after Cavendish's death.

1796 P. S. Laplace calculates, independently of J. Michell, under what condition the escape velocity from the surface of a body is bigger than the velocity of light.

1801 J. v. Soldner gives, independently of H. Cavendish, a detailed calculation of the deflection of a light particle by the Sun, on the basis of Newtonian theory. For a light ray grazing the surface of the Sun, he finds a deflection angle of $\delta \approx 0.87''$.



Soldner's long paper is published in *Bayrisches Jahrbuch der Astronomie* 1804, but it has apparently no impact on his contemporaries.

1911 A. Einstein calculates the deflection of light by the Sun, on the basis of the equivalence principle. He finds the same deflection angle as J. v. Soldner (of whom he knows nothing).

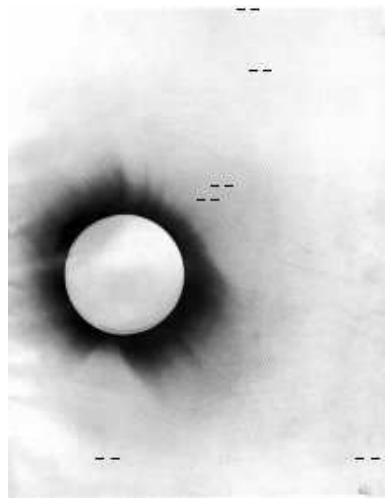
1912 In an entry in his notebook, A. Einstein estimates if light deflection by gravitational fields could lead to multiple images with a resolvable angular distance. He does not publish his results, probably because he thought that the effect would be unobservable. The notebook entry is found only in the 1990s.

1914 Triggered by A. Einstein, E. Freundlich leads an expedition to the Crimean peninsula. They want to measure the light deflection by the gravitational field of the Sun during a total Solar eclipse. The expedition team is arrested by Russian troops when World War I breaks out, so they cannot do any observations. Later they are exchanged against Russian prisoners of war and arrive safely back to Germany.

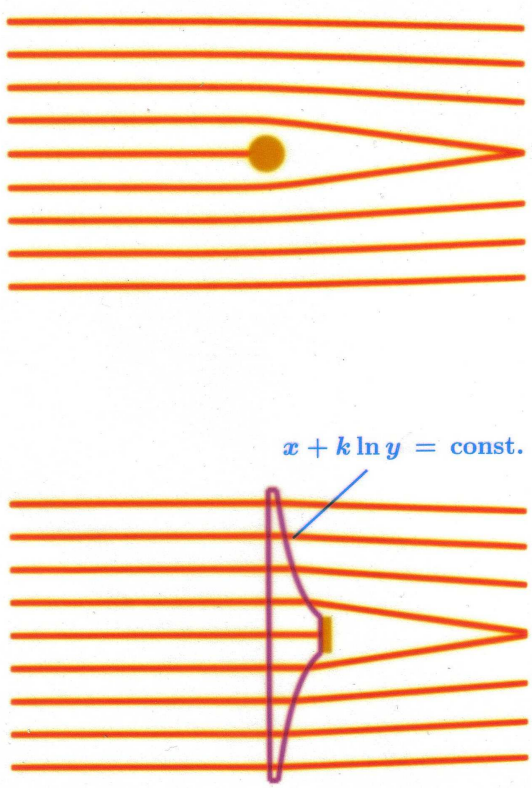
1915 A. Einstein calculates the light deflection by the Sun on the basis of his linearised field equation. He finds twice the Newtonian deflection angle, $\delta \approx 1.75''$. Shortly thereafter, K. Schwarzschild finds the exact spherically symmetric vacuum solution to Einstein's field equation. With the help of the Schwarzschild metric, the bending angle can be calculated precisely in terms of an elliptic integral.

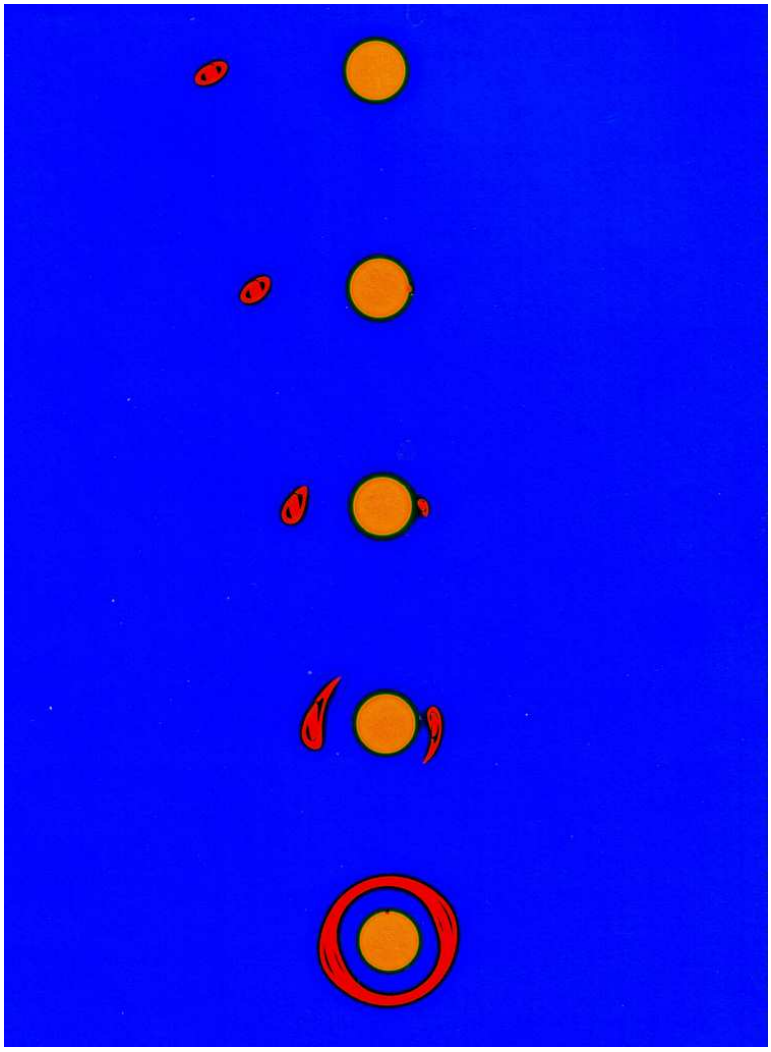
1919 A. Eddington verifies Einstein's prediction of light deflection by the Sun during a total Solar eclipse.

The picture shows a photographic plate taken by the Eddington expedition. Star positions are marked by horizontal lines. These positions were compared with the positions of the same stars on photographs taken half a year earlier, when the Sun was not in the viewing field.



1919 O. Lodge raises the question of whether a gravitational field influences light similarly to a lens. He comes to the conclusion that this is not the case, because the gravitational field does not focus light rays into a focal point, in contrast to a (convex) lens. However, one can mimick the effect of a spherically symmetric gravitational field onto light, according to the linearised Einstein theory, with the help of a logarithmically shaped concave lens. (The foot of a wine glass gives a reasonably good approximation.)





Deformation of Saturn according to the linearised Einstein theory. As the deflector we have taken a fictitious Sun whose radius is approximately 0.02 times the true radius of the Sun, while all other parameters are as in the real Solar system.

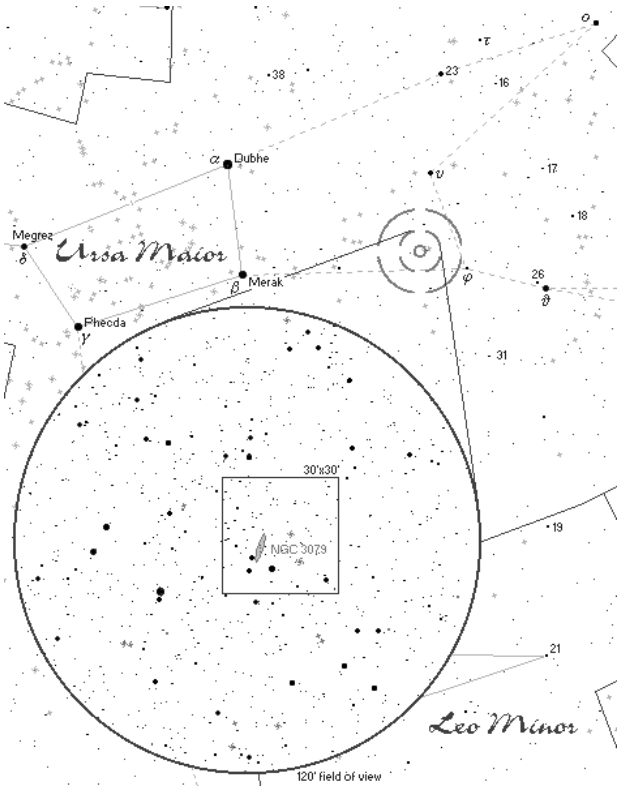
1924 O. Chwolson mentions in a short article the possibility that, in the case of axial symmetry, a gravitational field can produce ring-like images of a light source (nowadays called *Einstein rings*).

1936 Triggered by Bohemian engineer R. Mandl, A. Einstein publishes an article in Science in which he estimates the probability of observing star-star-lensing. He comes to a very pessimistic conclusion.

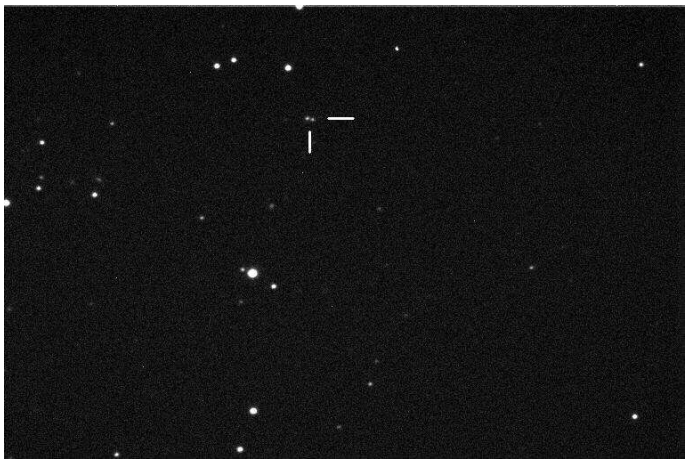
1937 Also triggered by R. Mandl, F. Zwicky estimates the probability of observing galaxy-galaxy-lensing. He comes to an encouraging result and starts searching in the sky for double galaxies as candidates for gravitational lensing, but without success.

1967 S. Refsdal establishes the quasi-Newtonian approximation formalism for lensing. It is based on the assumptions that gravitational fields are weak and that bending angles are small.

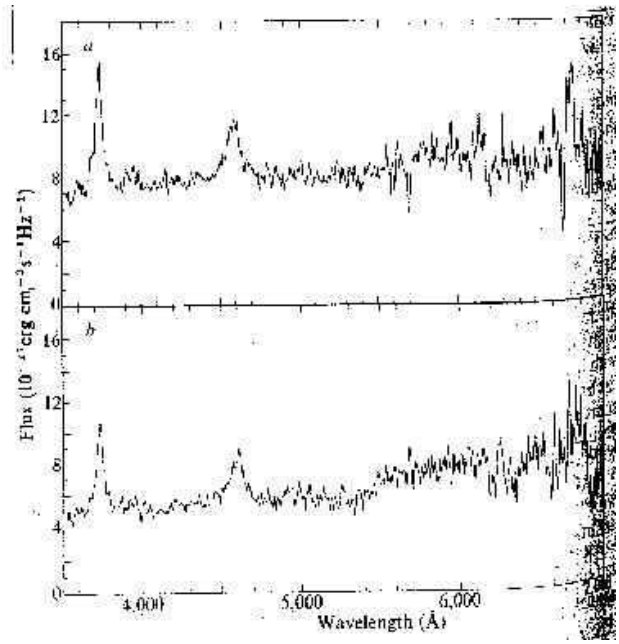
1979 D. Walsh, R.Carlswell and R. Weymann sugggest that the double quasar QSO 0957 +561 is actually only one quasar of whom we see two images produced by gravitational lensing.



Position of the double quasar QSO 0957 +561 in the constellation Ursa Maior.

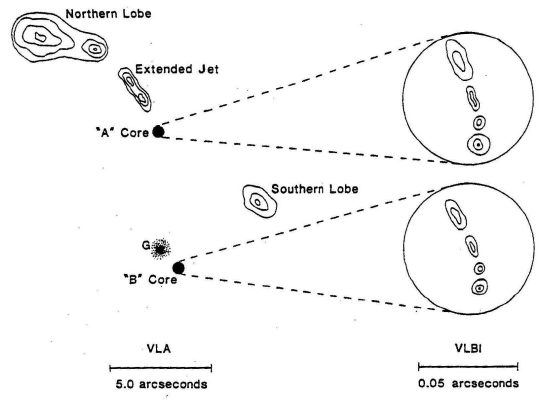


In an optical telescope, one sees two point images of 17th magnitude, the angular separation is 6″.

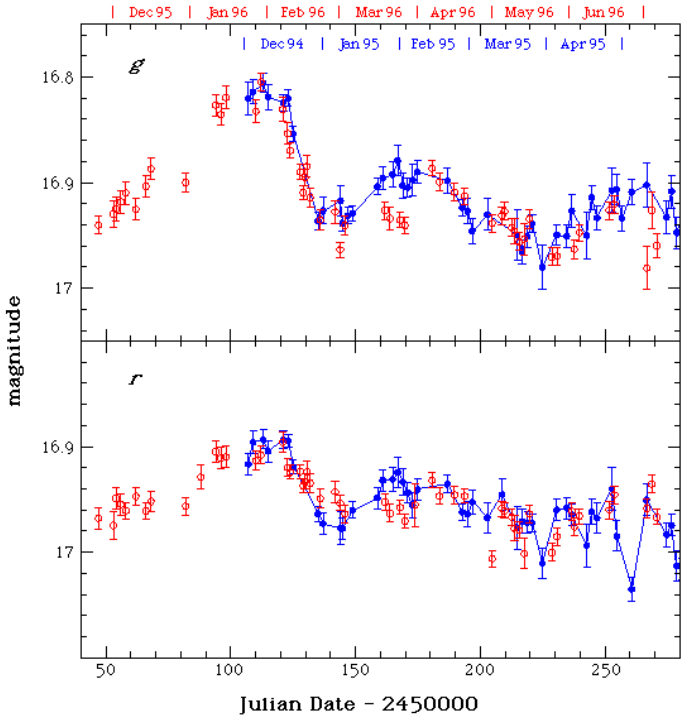


The spectra of the two quasar images are almost identical. This was what Walsh et al. led to the conjecture that they were two images of one and the same quasar. Small differences in the spectra are easily understood because the two images correspond to different light paths, passing through different interstellar gas clouds.

The deflecting mass is an intervening galaxy, seen in this picture as a fuzzy spot. The two quasar images have a redshift of $z_{\text{quasar}} = 1.4$, the galaxy has a redshift of $z_{\text{galaxy}} = 0.4$ and is thus closer to us.

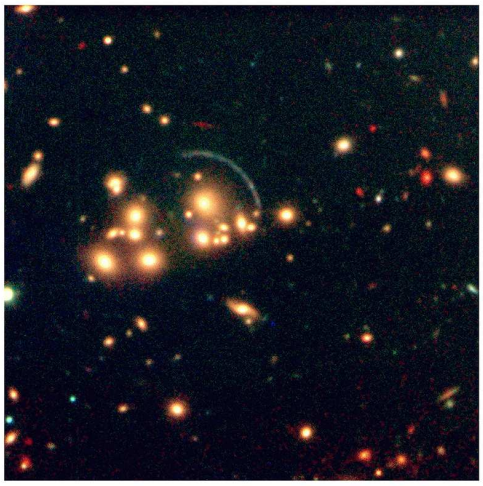


Radio images, produced with Very Long Baseline Interferometry, show substructures of the two quasar images. As they look very similar, this gives further support to the idea that one sees two images of one and the same object.



The strongest evidence for the lensing interpretation of QSO 0957 +561 comes from the light curves: Temporal changes of the intensity in one image (blue points) are also seen in the other image (red points), but with a time delay of a bit more than a year. The only reasonable explanation is that the two images correspond to light paths whose travel times differ by this amount.

1986 R. Lynds & V. Petrosian and G. Soucail et al. find the first giant luminous arcs.

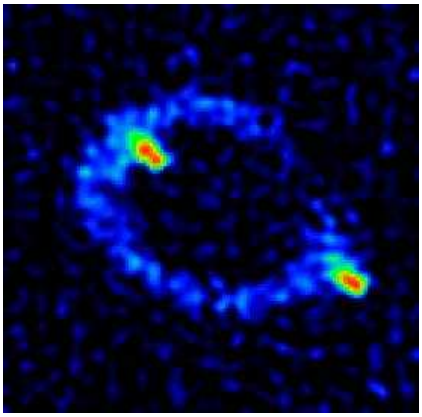


IR Colour Composite of Galaxy Cluster CL2244-02 with Gravitational Arcs
(VLT UT1 + ISAAC)

ESO PR Photo 46d/98 (26 November 1998) © European Southern Observatory

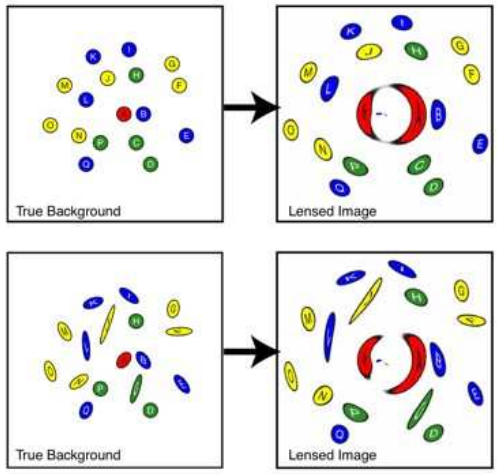
This giant luminous arc in the galaxy cluster Cl2244-02 was identified as a strongly deformed lensed image of a background galaxy by Lynds and Petrosian in 1986. Some of these arcs have an extension of more than 20 arcminutes.

1988 J. Hewitt et al. detect the first Einstein ring with the Very Large Array, a collection of radio telescopes in the US.



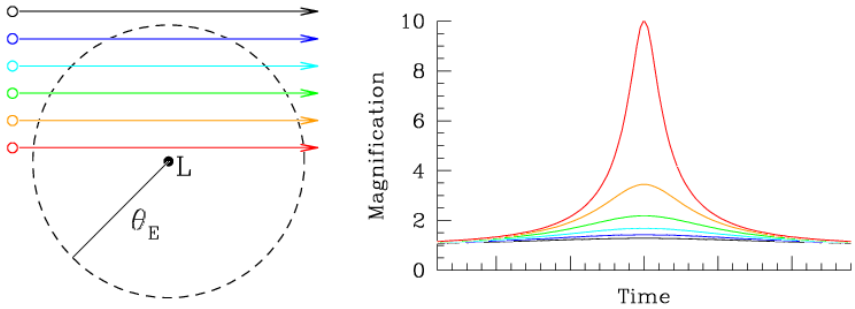
This is the radio source MG1131+0456 found by Jaqueline Hewitt et al., where a background (radio) galaxy is distorted into an almost closed ring. Later Einstein rings were found also in the infrared and even in the optical spectrum. Typically, the diameter of these rings is a few arcseconds or less.

1990 A. Tyson et al. establish the mathematical formalism of *weak lensing*. By evaluating the distribution of ellipticities of background galaxies, it allows to calculate the surface mass density (projected onto the surface perpendicular to the line of sight) in galaxy clusters.



This picture illustrates the basic idea of weak lensing. In the upper panel, it is shown how background galaxies would be distorted if they were perfectly spherical. From the observed distribution of elliptical images one could immediately determine the surface mass density. The situation is more complicated because galaxies have an intrinsic elliptic shape. By assuming that there is no prefered orientation of the semi-major axes of the background galaxies, one can determine the surface mass density with the help of statistical methods. This is illustrated in the lower panel.

1992 The first collaborations for observing microlensing events go into operation (MACHO, EROS, OGLE, ...) Microlensing was pioneered by Bohdan Paczyǹski.



If a star passes behind a dark compact object, the light curve of the star shows a maximum when it is closest to the line of sight. This is because the dark object deflects the light towards the observer. More than thousand microlensing events have been observed so far. In most cases, the lens is a dark object (e. g. a brown dwarf or an exoplanet) in our galaxy, in the Magellanic Clouds or in the Andromeda Galaxy. The typical time scale of a microlensing event is from a few days up to several months. Einstein was right when he concluded that star-star-lensing would not lead to resolvable multi-imaging. However, he did not think of the effect on the magnitude (brightness).

2000 The first indication for cosmic shear (image deformation on a cosmological scale) is found.

1.2 Status of observations

(a) Multiple quasars

Light source: quasar

Deflector: galaxy or several galaxies

First observation in 1979

Number of candidates: > 100
For a list of candidates, with detailed information, see the webpage of CASTLeS (CfA Arizona Space Telescope Lens Survey): `www.cfa.harvard.edu/glensdata`

Criteria for multiple imaging (in contrast to binary quasars):

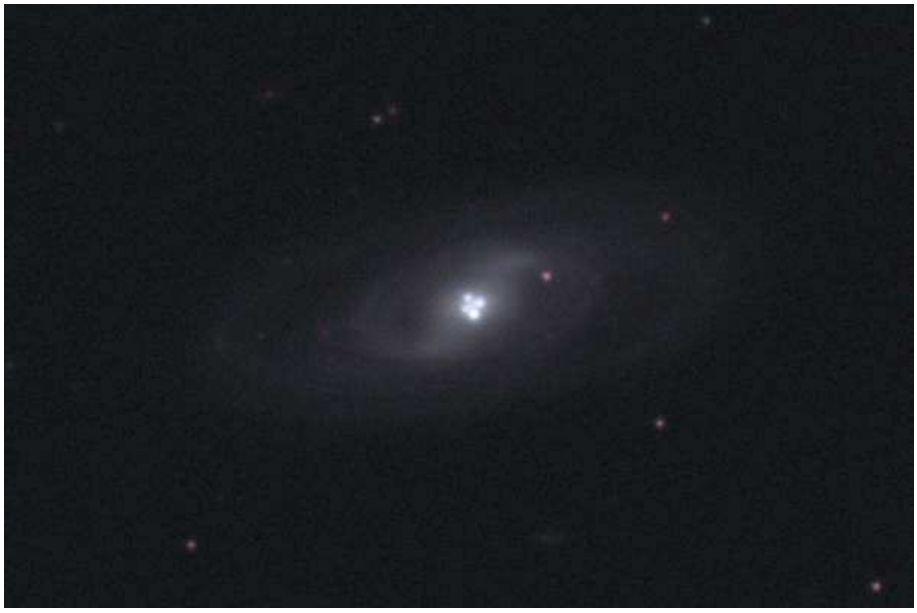
- candidate for deflector observable
- redshifts of quasar images (almost) equal, redshift of deflector smaller
- spectra of quasar images very similar
- light curves of quasar images identical up to time delay

Multiple quasars are observable in the optical and in the radio range.

Typical angular separation of quasar images: a few arcseconds

Typical magnitudes (brightness): 17^m or less

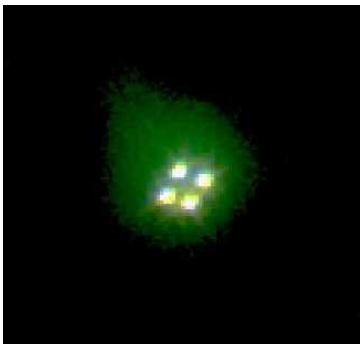
Typical time delay: several months

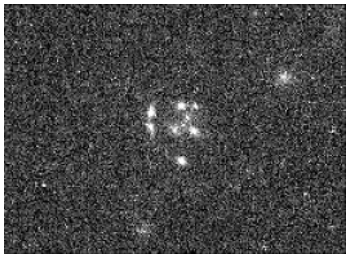


Composite picture of the *Einstein Cross* Q 2237+030, taken with the Wide Field and Planetary Camera (WFPC2) on the Hubble Space Telescope (HST).

.

HST photograph of the *Cloverleaf* H1413+1143, superimposed on picture of radio emission (greenish) from Hydrogen Cyanide gas in the deflector, as obvserved with the Very Large Array (VLA).





HST photograph of the six-fold imaged quasar B 1359+154, where the deflectors are three galaxies.

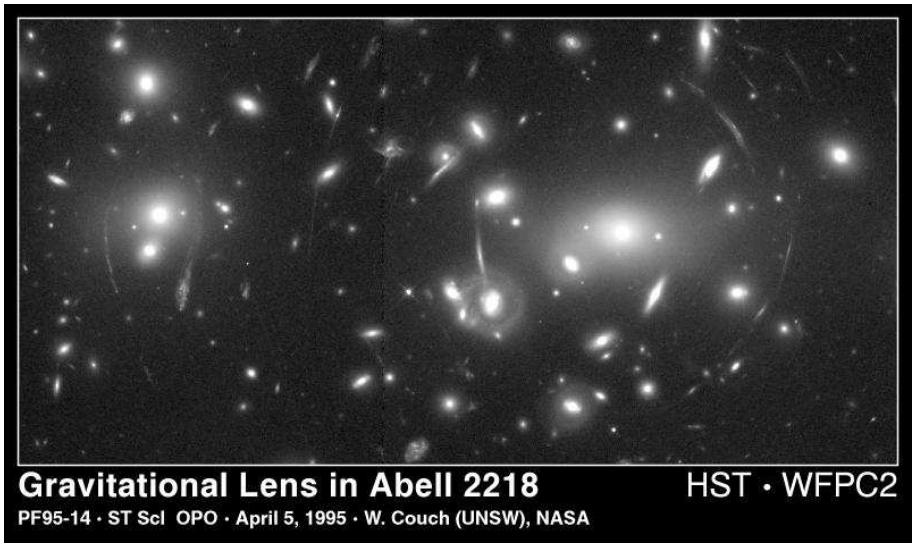
(b) Luminous arcs

Light source: galaxy
Deflector: cluster of galaxies

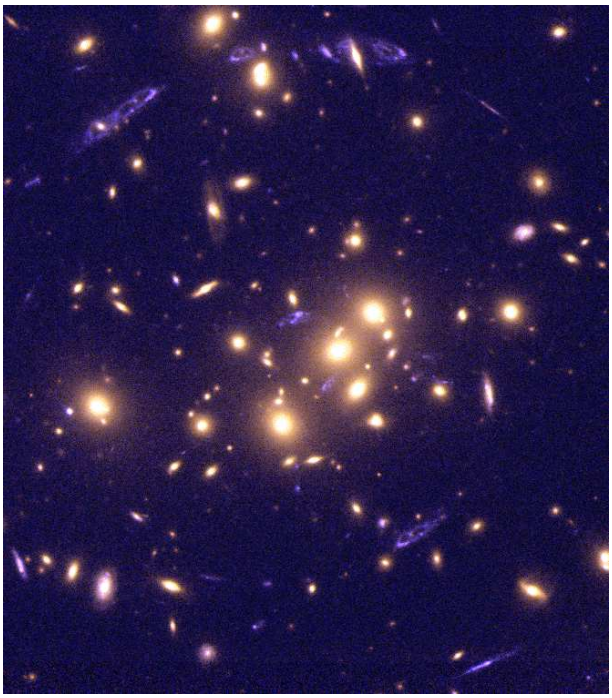
First observations in 1986

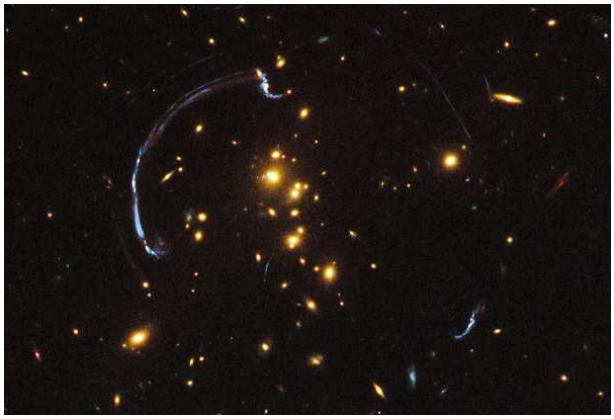
Number of galaxy clusters in which arcs are observed: several dozens

Luminous arcs are observed in the optical range.
Typical extension: up to more than 20 arcseconds.



HST photography of galaxy cluster 0024+1654 with several distorted images of blue background galaxies.





HST photography of galaxy cluster RCS2 032727-132623 with a blue background galaxy lensed into a giant luminous arc and a secondary image.

(c) Einstein rings

Light source: (radio) galaxy or lobe of a galaxy

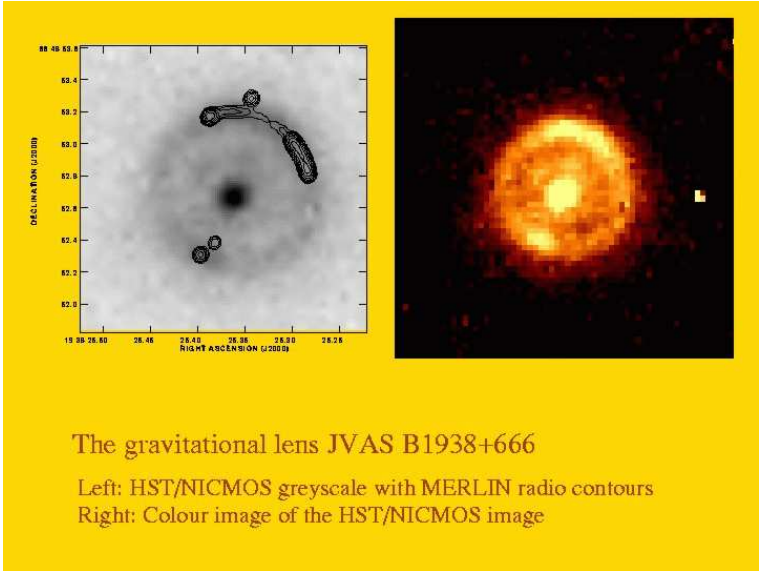
Deflector: galaxy

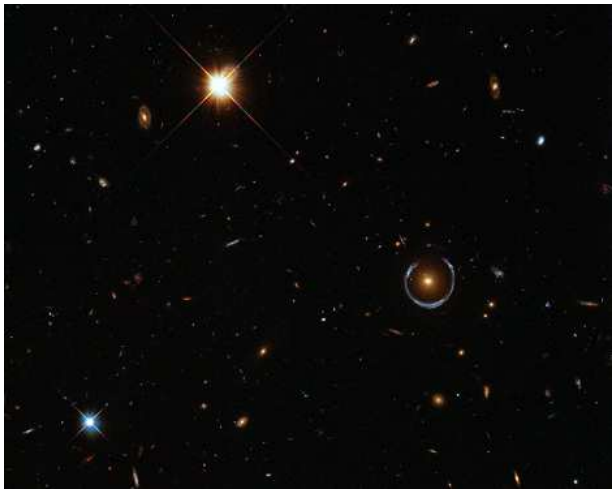
First observations in 1988

Number of observed rings: about a dozen

Einstein rings are observed primarily in the radio range, but some of them are also observable in the infrared or optical range.

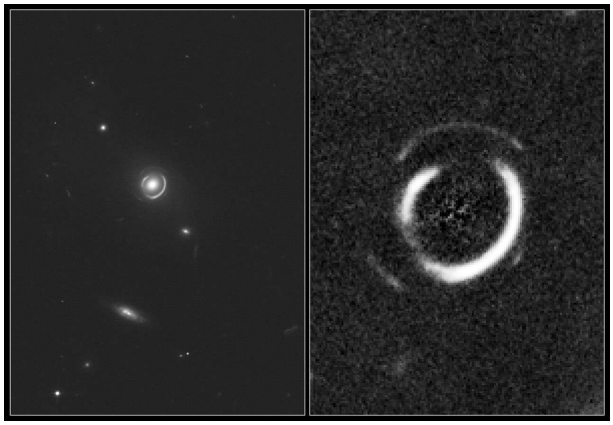
Typical diameter: about an arcsecond.





HST picture of the *Horseshoe* Einstein ring LRG 3-757.

HST picture of the double Einstein ring SDSSJ0946+1006, where two light sources are almost perfectly aligned with the deflector, one behind the other.



(d) Galactic microlensing

Light source: a star in our galaxy or in a neighbouring galaxy

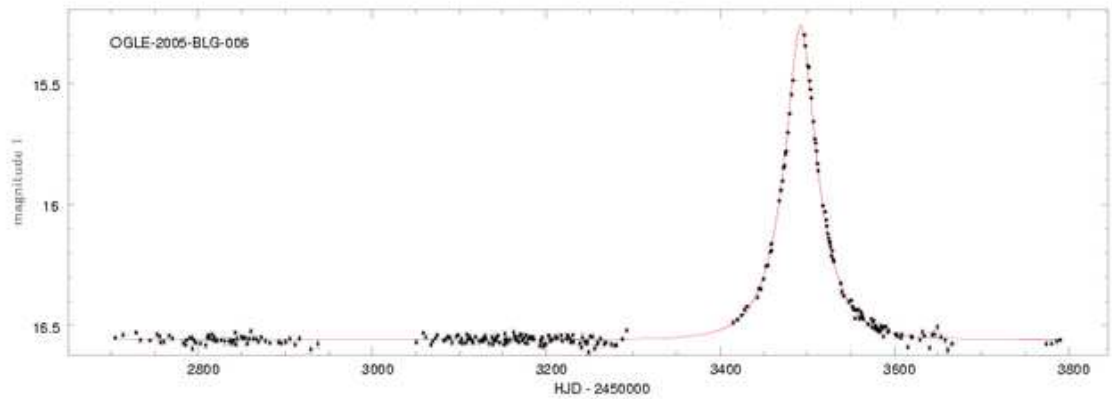
Deflector: a dark compact object (e.g. a brown dwarf or an exoplanet) in our galaxy or in a neighbouring galaxy

First observations in 1992 (collablorations MACHO, EROS, OGLE , . . .)

Number of observed events: > 1000

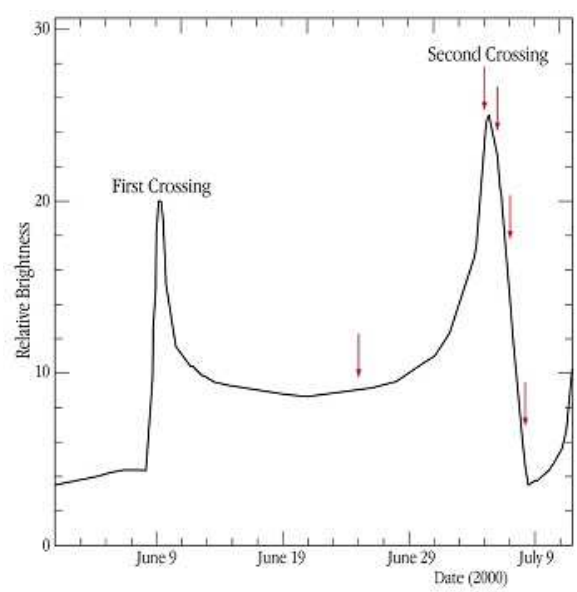
Criteria for distinguishing microlensing events from variable stars:

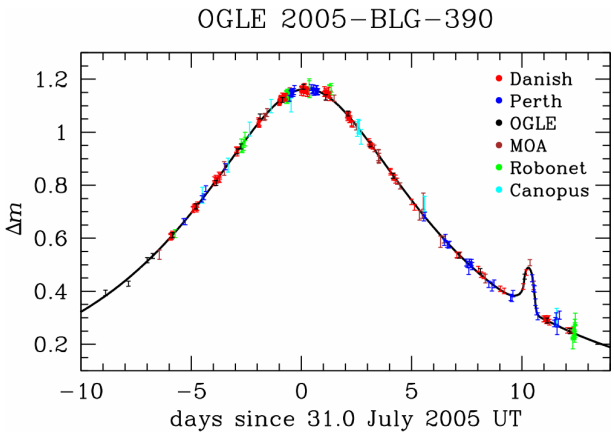
- achromatic light curve
- shape of the light curve



Light curve of a microlensing event where the deflector is a single compact object.

Light curve of the microlensing event EROS-BLG-2000-5 where the deflector is a binary.





Light curve of a microlensing event where the deflector is a star with a planet of approximately five Earth masses.

(e) Quasar microlensing

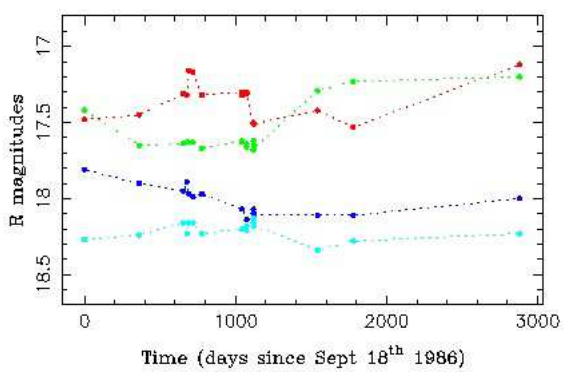
Light source: quasar

Deflectors: single stars in galaxies

First observations in 1989

Number of observed quasars that show microlensing variability: about a dozen

Variatiability, caused by microlensing, in the four quasar images of the Cloverleaf H 1413 + 117.



(f) Weak lensing

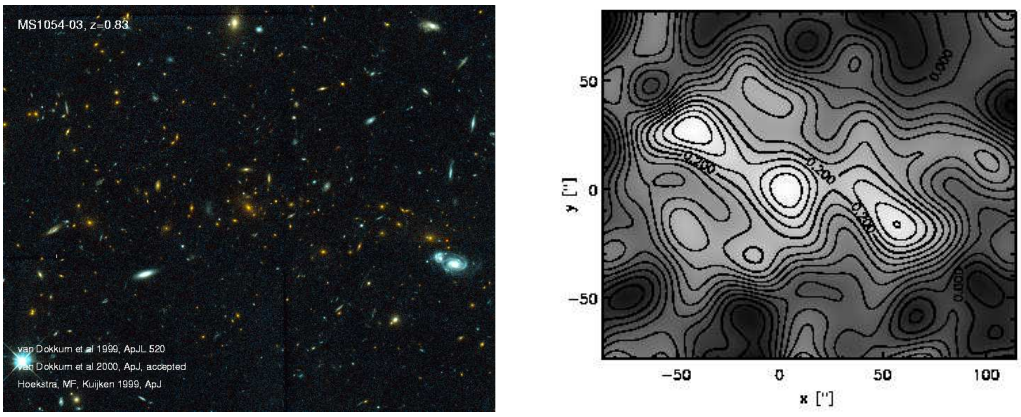
Light sources: elliptic galaxies

Deflector: dark matter

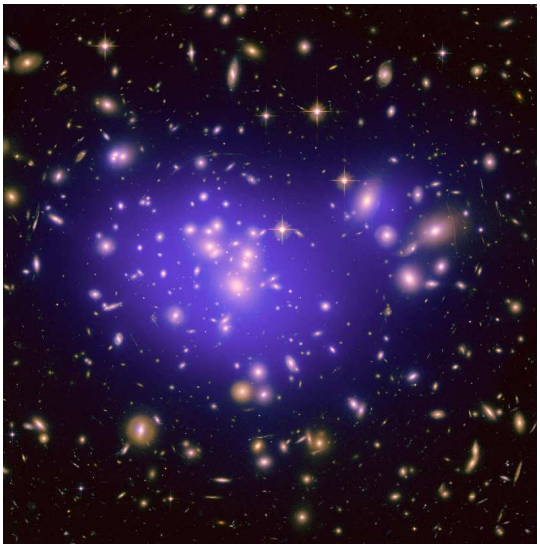
The distribution of the ellipticities of distant galaxies is statistically evaluated to get information on the surface density of intervening (dark) masses.

First observation of weak lensing by matter in galaxy clusters: 1989

First indication for weak lensing by large-scale structure ("cosmic shear"): 2000

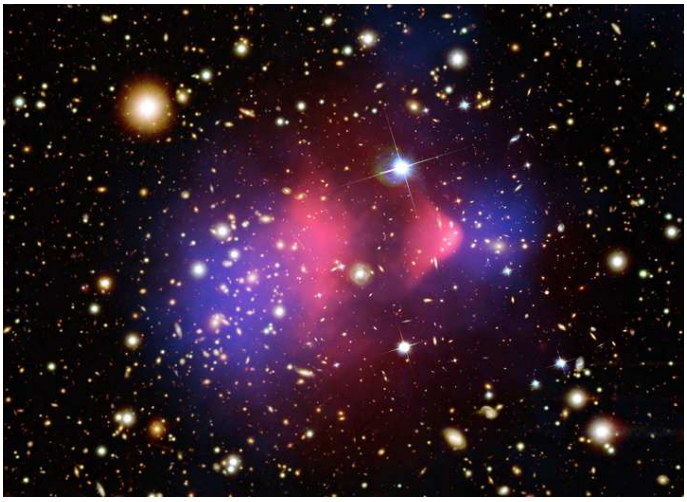


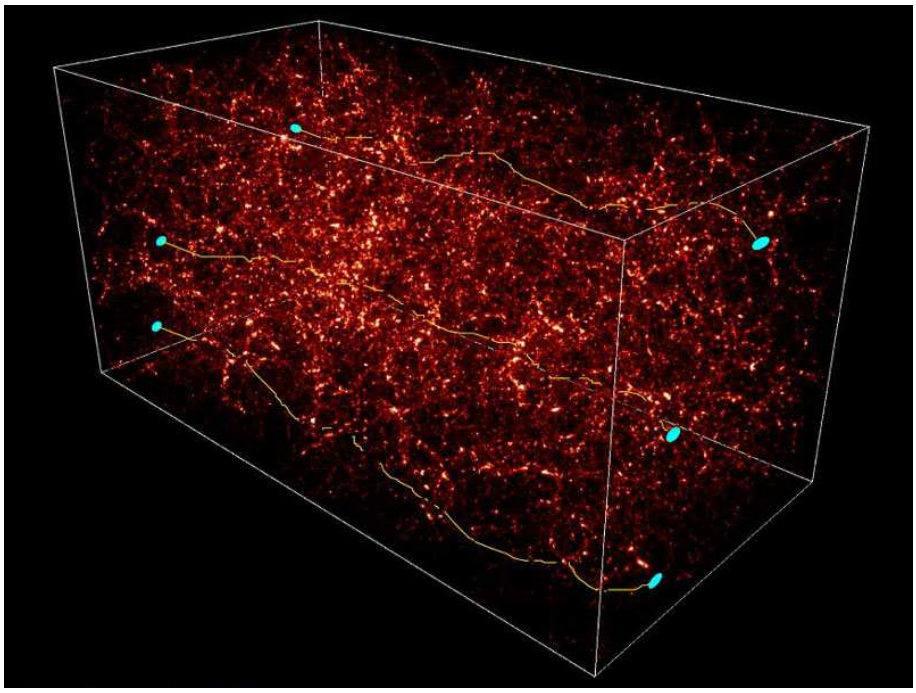
HST picture of the galaxy cluster MS 1054-03 (left) and contour lines of the surface mass density as calculated from weak lensing observations (right).



HST picture of the galaxy cluster Abell 1689, superimposed onto the distribution of matter (blue) as calculated from weak lensing observations.

HST picture of the *Bullet Cluster* 1E 0657-56, overlaid with the distribution of hot gases (red) as observed by the X-ray satellite Chandra and with the distribution of gravitating masses (blue) as calculated from weak lensing observations.





Result of a computer simulation by the CFHT team (Canada France Hawaii Telescope), based on weak lensing observations, showing the distribution of large-scale structure.

2. Mathematical Formalism

2.1 Brief review of general relativity

A general-relativistic spacetime is a pair (M, g) where:

M is a four-dimensional manifold; local coordinates will be denoted (x^0, x^1, x^2, x^3) and Einstein's summation convention will be used for greek indices $\mu, \nu, \sigma, \ldots = 0, 1, 2, 3$ and for latin indices $i, j, k, \ldots = 1, 2, 3$.

g is a Lorentzian metric on M, i.e. g is a covariant second-rank tensor field, $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, that is

(a) symmetric, $g_{\mu\nu} = g_{\nu\mu}$, and

(b) non-degenerate with Lorentzian signature, i.e., for any $p \in M$ there are coordinates defined near p such that $g|_p = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

We can, thus, introduce contravariant metric components by

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \ .$$

We use $g^{\mu\nu}$ and $g_{\sigma\tau}$ for raising and lowering indices, e.g.

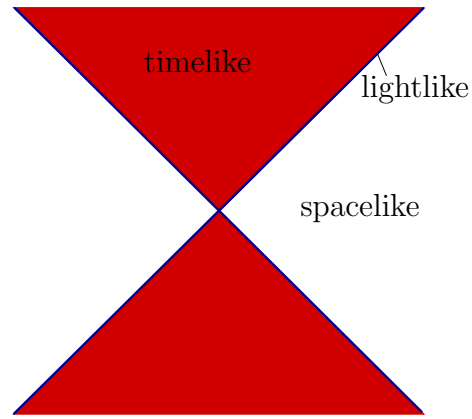
$$g_{\rho\tau}A^\tau = A_\rho\,, \qquad B_{\mu\nu}g^{\nu\tau} = B_\mu{}^\tau\ .$$

The metric contains all information about the spacetime geometry and thus about the gravitational field. In particular, the metric determines the following.

- The causal structure of spacetime:

  A curve $s \mapsto x(s) = \big(x^0(s), x^1(s), x^2(s), x^3(s)\big)$ is called

  $$\left.\begin{array}{r}\text{spacelike}\\ \text{lightlike}\\ \text{timelike}\end{array}\right\} \iff g_{\mu\nu}\big(x(s)\big)\dot{x}^\mu(s)\dot{x}^\nu(s) \left\{\begin{array}{l} > 0\\ = 0\\ < 0\end{array}\right.$$

  Timelike curves describe motion at subluminal speed and lightlike curves describe motion at the speed of light. Spacelike curves describe motion at superluminal speed which is forbidden for signals.

  The motion of a material continuum, e.g. of a fluid, can be described by a vector field $U = U^\mu\partial_\mu$ with $g_{\mu\nu}U^\mu U^\nu < 0$. The integral curves of U are to be interpereted as the worldlines of the fluid elements.

- The geodesics:

  By definition, the geodesics are the solutions to the Euler-Lagrange equations

  $$\frac{d}{ds}\frac{\partial\mathcal{L}(x,\dot{x})}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}(x,\dot{x})}{\partial x^\mu} = 0$$

  of the Lagrangian

  $$\mathcal{L}\big(x,\dot{x}\big) = \frac{1}{2}\,g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu\ .$$

  These Euler-Lagrange equations take the form

  $$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}(x)\dot{x}^\nu\dot{x}^\sigma = 0$$

  where

  $$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}\,g^{\mu\tau}\left(\partial_\nu g_{\tau\sigma} + \partial_\sigma g_{\tau\nu} - \partial_\tau g_{\nu\sigma}\right)$$

  are the socalled Christoffel symbols.

  The Lagrangian $\mathcal{L}(x,\dot{x})$ is constant along a geodesic (see Worksheet 2), so we can speak of timelike, lightlike and spacelike geodesics. Timelike geodesics ($\mathcal{L} < 0$) are to be interpreted as the worldlines of freely falling particles, and lightlike geodesics ($\mathcal{L} = 0$) are to be interpreted as light rays.

  The Christoffel symbols define a *covariant derivative* that makes tensor fields into tensor fields, e.g.

  $$\nabla_\nu U^\mu = \partial_\nu U^\mu + \Gamma^\mu{}_{\nu\tau}U^\tau\ ,$$
  $$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\rho{}_{\nu\mu}A_\rho\ .$$

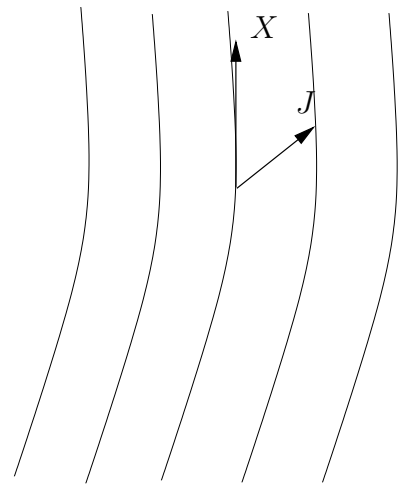
- The curvature:

  The Riemannian curvature tensor is defined, in coordinate notation, by

$$R^{\tau}{}_{\mu\nu\sigma} \;=\; \partial_\mu \Gamma^{\tau}{}_{\nu\sigma} \;-\; \partial_\nu \Gamma^{\tau}{}_{\mu\sigma} \;+\; \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\tau}{}_{\mu\rho} \;-\; \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\tau}{}_{\nu\rho}\,.$$

  The curvature tensor determines the relative motion of neighbouring geodesics: If $X = X^\mu \partial_\mu$ is a vector field whose integral curves are geodesics, and if $J = J^\nu \partial_\nu$ connects neighbouring integral curves of X (i.e., if the Lie bracket between X and J vanishes), then the *equation of geodesic deviation* or *Jacobi equation* holds:

$$\big(X^\mu \nabla_\mu\big)\big(X^\nu \nabla_\nu\big) J^\sigma \;=\; R^{\sigma}{}_{\mu\nu\rho} X^\mu J^\nu X^\rho\,.$$



  If the integral curves of X are timelike, they can be interpreted as worldlines of freely falling particles. In this case the curvature term in the Jacobi equation gives the *tidal force* produced by the gravitational field.

  If the integral curves of X are lightlike, they can be interpreted as light rays. In this case the curvature term in the Jacobi equation determines the influence of the gravitational field on the shapes of light bundles, i.e., image deformation and magnification. We will discuss this later in detail.

- Einstein's field equation:

  The fundamental equation that relates the spacetime metric (i.e., the gravitational field) to the distribution of energy is Einstein's field equation:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; \kappa\, T_{\mu\nu}$$

  where

  - $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$ is the *Ricci tensor*;
  - $R = R_{\mu\nu} g^{\mu\nu}$ is the *Ricci scalar*;
  - $T_{\mu\nu}$ is the *energy-momentum tensor* which gives the energy density $T_{\mu\nu} U^\mu U^\nu$ for any observer field with 4-velocity U^μ normalised to $g_{\mu\nu} U^\mu U^\nu = -c^2$;
  - Λ is the cosmological constant;
  - κ is Einstein's gravitational constant which is related to Newton's gravitational constant G through $\kappa = 8\pi G/c^4$.

  Einstein's field equation can be justified in the following way: One looks for an equation of the form $(\mathcal{D}g)_{\mu\nu} = T_{\mu\nu}$ where $\mathcal{D}$ is a differential operator acting on the metric. One wants to have $\mathcal{D}g$ satisfying the following two properties:

  (A) $\mathcal{D}g$ contains partial derivatives of the metric up to second order.

  (B) $\nabla^\mu (\mathcal{D}g)_{\mu\nu} = 0$.

Condition (A) is motivated by analogy to the Newtonian theory: The Poisson equation is a second-order differential equation for the Newtonian gravitational potential ϕ, and the metric is viewed as the general-relativistic analogue to ϕ . Condition (B) is motivated in the following way: For a closed system, in special relativity the energy-momentum satisfies the conservation law $\partial^\mu T_{\mu\nu} = 0$ in inertial coordinates. By the rule of minimal coupling, in general relativity the energy-momentum tensor of a closed system should satisfy $\nabla^\mu T_{\mu\nu} = 0$. For consistency, the same property has to hold for the left-hand side of the desired equation.

D. Lovelock has shown in 1972 that these two conditions (A) and (B) are satisfied if and only if $\mathcal{D}g$ is of the form

$$(\mathcal{D}g)_{\mu\nu} \;=\; \frac{1}{\kappa}\Big(R_\mu\nu - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu}\Big)$$

with some constants Λ and κ, i.e., if and only if the desired equation has indeed the form of Einstein's field equation.

For vacuum ($T_{\mu\nu} = 0$), Einstein's field equation reads

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\ .$$

By contraction with $g^{\mu\nu}$ this implies $R = 4\Lambda$, so the vacuum field equation reduces to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

For considerations which do not involve cosmological scales, Λ can be set equal to zero. Then the vacuum field equation takes the very compact form

$$R_{\mu\nu} = 0$$

which, however, is a complicated system of ten non-linear second-order partial differential equations for the ten independent components of the metric.

2.2 Example: Schwarzschild spacetime

The Schwarzschild metric is the unique spherically symmetric solution to Einstein's vacuum field equation without a cosmological constant, $R_{\mu\nu} = 0$. It describes the gravitational field around a spherically symmetric mass. It was found by K. Schwarzschild in 1916. It is usually written in the following form (which is due to D. Hilbert):

$$g \;=\; -\Big(1 - \frac{r_S}{r}\Big)\,c^2dt^2 \;+\; \frac{dr^2}{1 - \dfrac{r_S}{r}} \;+\; r^2\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\ .$$

Here ϑ and φ are standard coordinates on the two-dimensional sphere, r is a radius coordinate ranging from a constant r_* to ∞, and t is a time coordinate ranging over all of $\mathbb{R}$. r_S is an integration constant. By comparison with the Newtonian theory one finds that

$$r_S \;=\; \frac{2\,G\,M}{c^2}$$

where M is the mass of the central body. r_S is called the *Schwarzschild radius* of a spherically symmetric body with mass M.

- If $r_* > r_S$, the Schwarzschild metric describes the gravitational field around a star of radius r_*.
- If $r_* = 0$, the Schwarzschild metric describes a black hole, with a coordinate singularity at $r = r_S$ and a true singularity at $r = 0$. The coordinate singularity occurs at the horizon of the black hole; the metric becomes regular there if one changes, e.g., to Eddington-Finkelstein coordinates.

A star with $0 < r_* < r_S$ cannot be stable, i.e., r_* cannot be time-independent in this case. The star would collapse in a finite time into a (true) singularity at $r = 0$.

In the following we discuss in some detail the lensing features of the Schwarzschild metric. To that end we have to consider the lightlike geodesics. Because of spherical symmetry, it suffices to consider the equatorial plane $\vartheta = \pi/2$. The Lagrangian reads

$$\mathcal{L}(x,\dot{x}) \;=\; \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \;=\; \frac{1}{2}\Big\{ -\Big(1-\frac{r_S}{r}\Big)c^2\dot{t}^2 + \frac{\dot{r}^2}{1-\dfrac{r_S}{r}} + r^2\dot{\varphi}^2\Big\}\,.$$

Here the overdot means derivative with respect to the curve parameter s. The lightlike geodesics are determined by the following three equations:

- The t–component of the Euler-Lagrange equations:

$$0 \;=\; \frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{t}}\Big) - \frac{\partial L}{\partial t} \;=\; -\,c^2\,\frac{d}{ds}\Big(\big(1-\frac{r_S}{r}\big)\,\dot{t}\Big)\,,$$

$$\big(1-\frac{r_S}{r}\big)\,\dot{t} \;=\; E \;=\text{constant}\,. \qquad \text{(G1)}$$

- The φ–component of the Euler-Lagrange equations:

$$0 \;=\; \frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{\varphi}}\Big) - \frac{\partial L}{\partial \varphi} \;=\; \frac{d}{ds}\big(r^2\dot{\varphi}\big) \;=\; 0\,,$$

$$r^2\dot{\varphi} \;=\; L \;=\text{constant}\,. \qquad \text{(G2)}$$

- The condition of being lightlike:

$$\mathcal{L}(x,\dot{x}) \;=\; 0\,,$$

$$-\Big(1-\frac{r_S}{r}\Big)c^2\dot{t}^2 + \frac{\dot{r}^2}{1-\dfrac{r_S}{r}} + r^2\dot{\varphi}^2 \;=\; 0\,. \qquad \text{(G3)}$$

The r-component of the Euler-Lagrange equations gives no additional information. Dividing (G1) by (G2) results in

$$\frac{dt}{d\varphi} \;=\; \frac{\dot{t}}{\dot{\varphi}} \;=\; \frac{E\,r^2}{L\Big(1-\dfrac{r_S}{r}\Big)}\,. \qquad \text{(L1)}$$

Dividing (G3) by $\dot{\varphi}^2$ gives

$$-\Big(1-\frac{r_S}{r}\Big)\,\frac{c^2\,\dot{t}^2}{\dot{\varphi}^2} \;+\; \frac{1}{\Big(1-\dfrac{r_S}{r}\Big)}\,\frac{\dot{r}^2}{\dot{\varphi}^2} \;+\; r^2 \;=\; 0$$

and thus, with the help of (L1),

$$-\cancel{\left(1-\frac{r_S}{r}\right)}\frac{c^2E^2r^4}{L^2\left(1-\frac{r_S}{r}\right)^{\cancel{2}}}+\frac{1}{\left(1-\frac{r_S}{r}\right)}\left(\frac{dr}{d\varphi}\right)^2+r^2=0\,,$$

$$\left(\frac{dr}{d\varphi}\right)^2=\frac{c^2E^2r^4}{L^2}-r^2\left(1-\frac{r_S}{r}\right)\,. \qquad \text{(L2)}$$

The two equations (L1) and (L2) contain all the necessary information on the lightlike geodesics: (L2) gives the shape of the light orbit, (L1) gives the travel time. We will now evaluate these two equations.

- Circular lightlike geodesics:

  We consider the equation (L2)

  $$\left(\frac{dr}{d\varphi}\right)^2=\frac{c^2E^2r^4}{L^2}-r^2+r_S\,r$$

  and the φ-derivative of this equation,

  $$2\,\cancel{\frac{dr}{d\varphi}}\frac{d^2r}{d\varphi}=\left(\frac{4\,c^2E^2r^3}{L^2}-2\,r+r_S\right)\cancel{\frac{dr}{d\varphi}}\,.$$

  For a circular lightlike geodesic we must have $\frac{dr}{d\varphi}=0$ and $\frac{d^2r}{d\varphi^2}=0$, which gives us the following two equations:

  $$0=\frac{c^2E^2r^4}{L^2}-r^2+r_S\,r\,,$$
  $$0=\frac{4\,c^2E^2r^3}{L^2}-2\,r+r_S\,.$$

  To eliminate E^2/L^2, we multiply the first equation with $4/r$ and subtract the second equation. This results in

  $$0=-2\,r+3\,r_S \qquad\Longleftrightarrow\qquad r=\frac{3}{2}\,r_S=\frac{3GM}{c^2}\,.$$

  We have thus shown that there is a circular lightlike geodesic (or *photon circle*) at the radius value $3GM/c^2$. As we can choose *any* plane through the origin as our equatorial plane $\vartheta=\pi/2$, there is actually a *photon sphere* at this radius value in the sense that every great circle on this sphere is a lightlike geodesic.

  We will show in the 3rd worksheet that the photon circles at $r=3r_S/2$ are unstable in the following sense: A lightlike geodesic with an initial condition that deviates slightly from that of a photon circle at $r=3r_S/2$ will spiral away from $r=3r_S/2$ and either go to infinity or to the horizon.

  For later convenience, we also calculate the value of the constant of motion L^2/E^2 that corresponds to a photon circle: If we insert the value $r=r_S/2$ into the equation

  $$0=\frac{c^2E^2r^3}{L^2}-r+r_S$$

we find

$$\frac{c^2 E^2}{L^2} = \frac{\frac{3}{2}\, r_S - r_S}{\frac{27}{8}\, r_S^3} = \frac{4}{27\, r_S^2}\,,$$

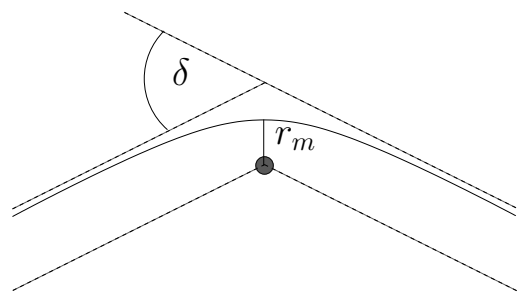
$$\frac{L^2}{E^2} = \frac{27\, c^2\, r_S^2}{4} = \frac{27\, c^2\, 4\, G^2\, M^2}{4\, c^4} = \frac{27\, G^2 M^2}{c^2}\,.$$

The photon sphere at $r = 3r_S/2$ does, of course, not exist for stars with $r_* > 3r_S/2$. It is relevant only for black holes and for (hypothetical) *ultracompact stars* where $r_S < r_* < 3r_S/2$.

- Formula for the deflection angle:

  We want to consider a light ray that comes in from infinity, goes through a minimum radius value at $r = r_m$ and then escapes back to infinity. We want to express the deflection angle δ in terms of r_m and the mass of the central body.

  We start out from (L2). E^2/L^2 is determined by the condition that



$$0 = \left(\frac{dr}{d\varphi}\right)^2\Big|_{r=r_m} = \frac{c^2 E^2}{L^2}\, r_m^4 - r_m^2 + r_S\, r_m$$

$$\Longrightarrow \qquad \frac{c^2 E^2}{L^2} = \frac{1}{r_m^2} - \frac{r_S}{r_m^3}\,.$$

  We can, thus, rewrite (L2) as

$$d\varphi = \frac{\pm dr}{\sqrt{\left(\frac{1}{r_m^2} - \frac{r_S}{r_m^3}\right) r^4 - r^2 + r_S\, r}}\,.$$

  Integration over the light ray results in

$$\int_{\varphi_0}^{\varphi_0+\pi+\delta} d\varphi = \left(- \int_{\infty}^{r_m} + \int_{r_m}^{\infty} \right) \frac{dr}{\sqrt{\left(\frac{1}{r_m^2} - \frac{r_S}{r_m^3}\right) r^4 - r^2 + r_S\, r}}$$

  where the signs of the two integrals on the right-hand side had to be chosen in angreement with the fact that φ is always increasing. We have thus found an exact formula,

$$\pi + \delta = 2 \int_{r_m}^{\infty} \frac{r_m\, dr}{\sqrt{\left(1 - \frac{r_S}{r_m}\right) r^4 - r_m^2\, r^2 + r_m^2\, r_S\, r}}\,,$$

  for the deflection angle δ in terms of an elliptic integral. From the derivation it is clear that the integrand has a singularity at the lower bound $r = r_m$, so the evaluation of the integral needs some care. A more detailed analysis shows that the integral is finite for all values of r_m that are bigger than $3r_S/2$. If we consider a sequence of light rays with r_m

approaching $3r_S/2$ from above, the deflection angle δ becomes bigger and bigger which means that the light rays make more and more turns around the centre. In the limit $r_m \to 3r_S/2$ the integral goes to infinity and the limiting light ray spirals asymptotically towards a circle at $r = 3r_S/2$. We will not prove these facts here, because we will later demonstrate that this is a general feature of lightlike geodesics in spherically symmetric and static spacetimes: If an unstable photon circle is approached, the deflection angle goes to infinity. We will also see that the singularity is always logarithmic.

The asymptotic behaviour of light rays for r_m approaching $3r_S/2$ is relevant only for black holes and for (hypothetical) ultracompact stars. For an ordinary star, like our Sun, the possible values of r_m are much bigger than the Schwarzschild radius r_S. One can then restrict to a Taylor approximation with respect to r_S/r_m:

$$\delta + \pi = 2 \int_{r_m}^{\infty} \frac{r_m \, dr}{\sqrt{r^2 \left(r^2 - r_m^2\right) - \frac{r_S}{r_m} r \left(r^3 - r_m^3\right)}} =$$

$$= 2 \int_{r_m}^{\infty} \frac{r_m \, dr}{\sqrt{1 - \frac{r_S}{r_m} \frac{\left(r^3 - r_m^3\right)}{r \left(r^2 - r_m^2\right)}} \; r \sqrt{r^2 - r_m^2}} =$$

$$= 2 \int_{r_m}^{\infty} \left\{ 1 + \frac{1}{2} \frac{r_S}{r_m} \frac{\left(r^3 - r_m^3\right)}{r \left(r^2 - r_m^2\right)} + O\Big(\frac{r_S^2}{r_m^2}\Big) \right\} \frac{r_m \, dr}{r \sqrt{r^2 - r_m^2}} =$$

$$= 2 \underbrace{\int_{r_m}^{\infty} \frac{r_m \, dr}{r \sqrt{r^2 - r_m^2}}}_{= I_1} + \cancel{2} \frac{1}{\cancel{2}} \frac{r_S}{r_m} \underbrace{\int_{r_m}^{\infty} \frac{\left(r^3 - r_m^3\right) r_m \, dr}{r^2 \sqrt{r^2 - r_m^2}^{\,3}}}_{= I_2} + O\Big(\frac{r_S^2}{r_m^2}\Big) \, .$$

The integrals I_1 and I_2 are elementary and can be looked up in a table (or calculated with the help of a substitution), $I_1 = \pi/2$ and $I_2 = 2$; hence

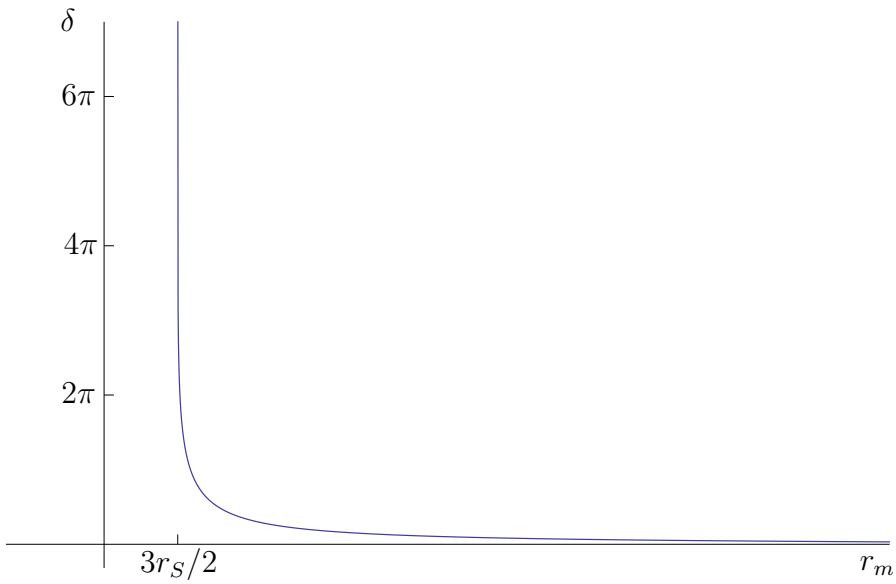
$$\pi + \delta = \pi + 2 \frac{r_S}{r_m} + O\Big(\frac{r_S^2}{r_m^2}\Big) \, .$$

If quadratic and higher-order terms are neglected we get *Einstein's deflection formula*

$$\delta = 2 \frac{r_S}{r_m} = \frac{4 \, G \, M}{c^2 \, r_m} \, .$$

To within the same approximation, the Newtonian deflection formula of Cavendish and Soldner gave half this value, recall Worksheet 1 .

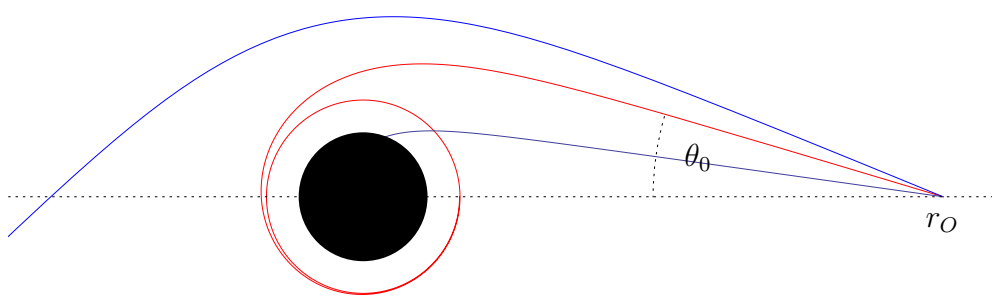
In the 3rd worksheet we will further discuss Einstein's deflection formula. We will see that bending according to this formula can be mimicked by an appropriatelty shaped plastic lens, and we will calculate the resulting *focal length* of a spherically symmetric deflector of a given mass and a given radius.



Plot of the exact bending angle δ against the minimum radius r_m.

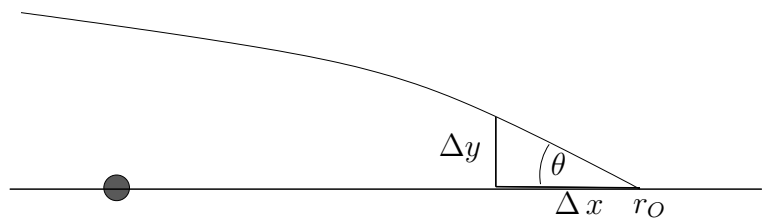
- Shadow of a Schwarzschild black hole:

  We fix an observer at radius r_O and consider all light rays that go from the position of this observer into the past. (To put this another way, we consider all light rays that *arrive* at the position of the observer.) They fall into two categories: Category I consists of light rays that go out to infinity, category II consists of light rays that go to the horizon at $r = r_S$. The borderline case that separates the two categories is given by light rays that asymptotically spiral towards the light sphere at $r = 3r_S/2$.



  Now assume that there are light sources distributed at large radii, but no other light sources. Then the initial directions of light rays of category I correspond to points at the observer's sky that are bright, and the initial directions of light rays of category II correspond to points at the observer's sky that are dark, known as the *shadow* of the black hole. The boundary of the shadow corresponds to light rays that spiral towards $r = 3r_S/2$. It is our goal to calculate the angular radius θ_0 of the shadow, in dependence of r_S and r_O.

For any light ray, the initial direction makes an angle θ with respect to the axis that is given, according to the picture, by



$$\tan\theta \;=\; \lim_{\Delta x\to 0}\frac{\Delta\, y}{\Delta\, x}\;.$$

From the Schwarzschild metric in the equatorial plane,

$$g \;=\; -\Big(1-\frac{r_S}{r}\Big)c^2dt^2 \;+\; \frac{dr^2}{1-\dfrac{r_S}{r}} \;+\; r^2d\varphi^2\,,$$

we can read the length Δx and Δy in the desired limit,

$$\tan\theta \;=\; \frac{r\,d\varphi}{\Big(1-\dfrac{r_S}{r}\Big)^{-1/2}dr}\Bigg|_{r=r_O}\;.$$

$dr/d\varphi$ can be expressed with the help of eq. (L2) from p.22, hence

$$\tan^2\theta \;=\; \frac{r_O^2\Big(1-\dfrac{r_S}{r_O}\Big)}{\dfrac{E^2r_O^4}{c^2L^2}-r_O^2+r_Sr_O} \;=\; \frac{r_O-r_S}{\dfrac{E^2r_O^3}{c^2L^2}-r_O+r_S}\;.$$

By elementary trigonometry,

$$\sin^2\theta \;=\; \frac{\sin^2\theta}{\sin^2\theta+\cos^2\theta} \;=\; \frac{1}{\cot^2\theta\;+\;1} \;=\; \frac{1}{(r_O-r_S)^{-1}\Big(\dfrac{E^2r_O^3}{c^2+L^2}-r_O+r_S\Big)\;+\;1} \;=$$

$$=\; \frac{r_O-r_S}{\dfrac{E^2r_O^3}{c^2L^2}\;-\cancel{r_O}+\cancel{r_S}+\cancel{r_O}-\cancel{r_S}} \;=\; \frac{c^2L^2\big(r_O-r_S\big)}{E^2r_O^3}\,.$$

The angular radius θ_0 of the shadow is given by the angle θ for a light ray that spirals towards $r=3r_S/2$. This light ray must have the same constants of motion E and L as a circular light ray at $r=3r_S/2$ (because the tangent vectors of these two light rays come arbitrarily close to each other),

$$\frac{c^2L^2}{E^2} \;=\; \frac{27}{4}r_S^2$$

as we have calculated on p.23.

This gives us θ_0 in dependence of $r_S = 2GM/c^2$ and r_O,

$$\sin^2\theta_0 \;=\; \frac{27\, r_S^2 (r_O - r_S)}{4\, r_O^3}\;.$$

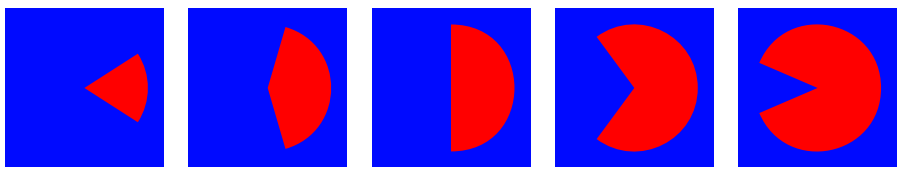
Note that

$r_O \to \infty$: $\theta_0 \to 0$ (i.e., the shadow vanishes).

$r_O = 3r_S/2$: $\theta_0 = \pi/2$ (i.e., the shadow covers half of the sky).

$r_O \to r_S$: $\theta_0 \to \pi$ (i.e., the shadow covers the whole sky).

The shadow is usually visualised in terms of the socalled *escape cones*. For each observer position, the red cone indicates the part of the sky that is bright:



$r_O = 1.05\, r_S$ $r_O = 1.3\, r_S$ $r_O = 3\, r_S/2$ $r_O = 2.5\, r_S$ $r_O = 6\, r_S$

For the black hole at the centre of our galaxy ($M \approx 4 \times 10^6\, M_\odot$, $r_O \approx 8\,\mathrm{kpc}$) the angular radius of the shadow is $\theta_0 \approx 15\,\mu$as. One expects that this shadow will be seen with radio telescopes in the near future, using *Very Long Baseline Interferometry* (VLBI). A dedicated project, called the *Event Horizon Telescope*, is now in the planning stage. The name is a bit misleading: First, it is not one telescope but rather a system of several existing and planned VLBI stations around the Earth. Second, it will not make the event horizon directly visible; what is meant is to reach an angular resolution comparable to the size of the event horizon.

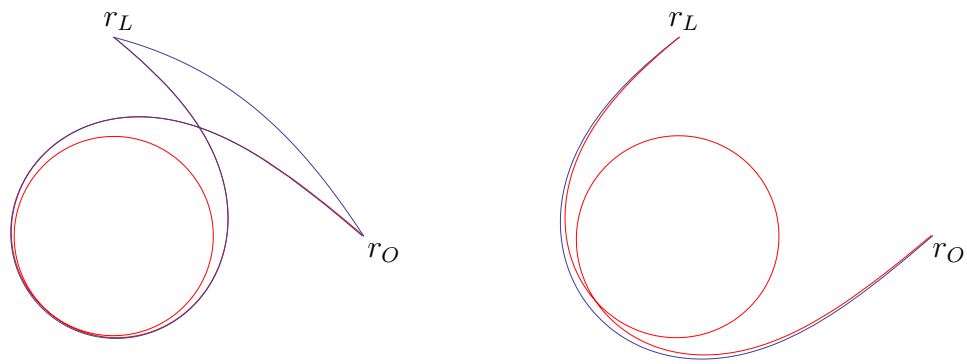
Note that the shadow would exist not only for a black hole, but in exactly the same way also for an ultracompact star ($r_S < r_* < 3r_S/2$), provided the star is dark.

Our calculation was based on the Schwarzschild metric, so it does not apply to a *rotating* black hole. The latter is to be described by the Kerr metric; then the shadow turns out to be non-circular. So the shape of the shadow tells directly if the black hole is rotating or not. There is some evidence that the black hole at the centre of our galaxy is rotating quite fast. However, our calculation with the Schwarzschild metric gives the correct order of magnitude for the size of the shadow.

- Multiple imaging of a Schwarzschild black hole:

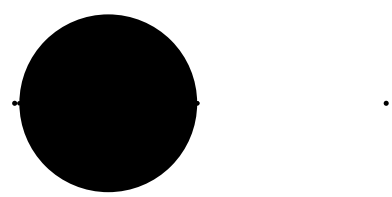
  In this section we will briefly discuss the qualitative properties of multiple imaging in the Schwarzschild spacetime. A more quantitative characterisation, including precise image positions and magnitudes, will be given later with the help of the exact lens map.

  We fix a static observer at radius r_O and a static light source at radius r_L. We exclude the case that observer and light source are exactly aligned (i.e., that they are on a straight line through the origin of the coordinate system). We want to determine how many images the observer sees of the light source. Clearly, every lightlike geodesic from the light source to the observer gives rise to an image.



The qualitative imaging features follow from the fact that the bending angle grows monotonically to infinity for light rays that approach the photon sphere at $r = 3r_S/2$. As a consequence, for any integer $n = 0, 1, 2, 3, \ldots$ there is a light ray from the light source to the observer that makes n full turns in the clockwise sense, and another light ray from the light source to the observer that makes n full turns in the counter-clockwise sense. Hence, there are two infinite sequences of light rays from the light source to the observer, one in the clockwise sense (left picture) and one in the counter-clockwise sense (right picture). Either sequence has as its limit curve a light ray that spirals asymptotically towards $r = 3r_S/2$. The pictures are not just qualitatively correct; they show numerically integrated lightlike geodesics in the Schwarzschild spacetime. One sees that for each sequence the light rays with $n = 1, 2, 3, \ldots$ lie practically on top of each other. Correspondingly, the observer sees infinitely many images on either side of the centre. Each sequence rapidly approaches the shadow.

In the picture, which is again the result of a calculation, the shadow is shown as a big black disk. On either side only the outermost image ($n = 0$) can be isolated, all the other ones clump together and they are very close to the boundary of the shadow.

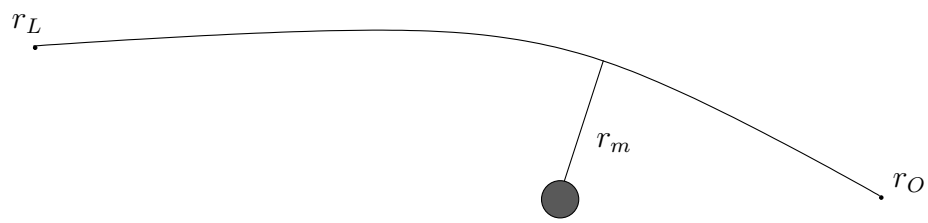


We will later calculate image positions and magnifications with the exact lens map. We will then see that the outermost images are brighter than all the other ones combined. Of the two outermost images, the brighter one is called the *primary image* and the other one is called the *secondary image*. All the other ones, which correspond to light rays that make at least one full turn, are known as *higher-order images* or *relativistic images*. The latter name is misleading as higher-order images also occur in the Newtonian theory of bending (in the sense of Cavendish and Soldner).

Higher-order images have not been observed so far. Just as with the observation of the shadow, there is some hope that they might be seen near the black hole at the centre of our galaxy in the near future. The *Very Large Telescope* in Chile will soon be equipped with a new instrument, called GRAVITY, for infrared interferometry. With this instrument it could be possible to observe higher-order images of stars orbiting the black hole.

- Travel time of light rays (Shapiro effect):

  In combination with eq.(L2), eq. (L1) allows to calculate the travel time of any light ray in the Schwarzschild spacetime. We will do this for a light ray that starts at a (big) radius r_L, passes through a minimum radius value r_m and terminates at a (big) radius r_O.



From equations (L1) and (L2) on p.21/22 we find

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\varphi}\right)^2 \left(\frac{d\varphi}{dt}\right)^2 = \left(\frac{c^2E^2r^4}{L^2} - r^2 + r_S r\right) \frac{L^2\left(1 - \frac{r_S}{r}\right)^2}{E^2r^4} =$$

$$= \left(\frac{c^2E^2r^3}{L^2} - r + r_S\right) \frac{L^2\left(r - r_S\right)^2}{E^2r^5} \,.$$

As before, we express c^2E^2/L^2 in terms of the minimum radius value r_m via

$$0 = \left(\frac{dr}{d\varphi}\right)^2\Big|_{r=r_m} = \frac{c^2E^2r_m^4}{L^2} - r_m^2 + r_S r_m \implies \frac{c^2E^2}{L^2} = \frac{r_m - r_S}{r_m^3} \,.$$

This results in

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{(r_m - r_S)\, r^3}{r_m^3} - r + r_S\right) \frac{c^2\left(r - r_S\right)^2 r_m^3}{(r_m - r_S) r^5} \,,$$

$$dt = \frac{\pm\sqrt{r_m - r_S}\, r^{5/2}\, dr}{c\,(r - r_S)\, r_m^{3/2} \sqrt{\frac{(r_m - r_S) r^3}{r_m^3} - r + r_S}} \,.$$

Integration over the light ray gives the travel time

$$\Delta t = \left(-\int_{r_L}^{r_m} + \int_{r_m}^{r_O}\right) \frac{\sqrt{r_m - r_S}\, r^{5/2}\, dr}{c\,(r - r_S)\, r_m^{3/2} \sqrt{\frac{(r_m - r_S) r^3}{r_m^3} - r + r_S}}$$

where the signs on the right-hand side had to be chosen in such a way that the time coordinate is always increasing along the light ray. This is an exact formula for the travel time in terms of an elliptic integral.

If $r_m \gg r_S$, we can make a Taylor approximation, in exactly the same way as we did it for the deflection formula:

$$\Delta t = \Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{\sqrt{r_m}\left(1-\dfrac{r_S}{r_m}\right)^{1/2} r^{5/2}\,dr}{c\,r\left(1-\dfrac{r_S}{r}\right) r_m^{3/2}\sqrt{\dfrac{r^3}{r_m^2}-r}\sqrt{1-\dfrac{r_S}{r_m}\dfrac{(r^3-r_m^3)}{r\,(r^2-r_m^2)}}} =$$

$$= \Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{r}{c\sqrt{r^2-r_m^2}} \left\{1 - \frac{r_S}{2r_m} + \frac{r_S}{r} + \frac{r_S}{2r_m}\frac{(r^3-r_m^3)}{r\,(r^2-r_m^2)} + \dots\right\} dr =$$

$$= \Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{r}{c\sqrt{r^2-r_m^2}} \left\{1 - \frac{r_S}{2r_m} + \frac{r_S}{r} + \frac{r_S\cancel{(r-r_m)}(r^2+r_m r+r_m^2)}{2r_m r\cancel{(r-r_m)}(r+r_m)} + \dots\right\} dr =$$

$$= \Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{r}{c\sqrt{r^2-r_m^2}} \left\{1 - \cancel{\frac{r_S}{2r_m}} + \frac{r_S}{r} + \cancel{\frac{r_S}{2r_m}} + \frac{r_S r_m}{2r(r+r_m)} + \dots\right\} dr =$$

$$= \frac{1}{c}\Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{r\,dr}{\sqrt{r^2-r_m^2}} + \frac{r_S}{c}\Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{dr}{\sqrt{r^2-r_m^2}} +$$

$$+ \frac{r_S r_m}{2c}\Big(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\Big) \frac{dr}{\sqrt{r-r_m}\,\sqrt{r+r_m}^{\,3}} + \dots$$

The integrals are elementary:

$$\Delta t = \frac{1}{c}\left(\sqrt{r_L^2-r_m^2} + \sqrt{r_O^2-r_m^2}\right) +$$

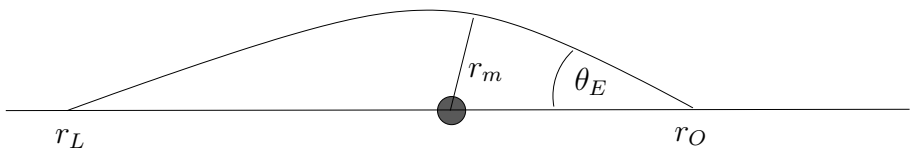
$$+ \frac{r_S}{2\,c}\left(2\ln\frac{r_L+\sqrt{r_L^2-r_m^2}}{r_m} + 2\ln\frac{r_O+\sqrt{r_O^2-r_m^2}}{r_m} + \sqrt{\frac{r_L-r_m}{r_L+r_m}} + \sqrt{\frac{r_O-r_m}{r_O+r_m}}\right) + \dots$$

The zeroth-order term is, of course, the Euclidean travel time for a light ray with speed c along a straight line. The deviation of the general-relativistic calculation from this zeroth-order term is known as the *Shapiro time delay*.

In 1964, I. Shapiro suggested to use this effect as a *fourth test of general relativity* (after perihelion precession, light deflection and gravitational redshift). In 1967 the first experiment was done: A strong radio signal was sent to Venus, when it was in opposition to the Earth, and the time was measured until the signal arrived back on the Earth, after being reflected at the Venus atmosphere. Later experiments were done with transponders on spacecraft, that sent the signal back with increased intensity. The best measurement to date was done with the Cassini spacecraft in 2002. The general-relativistic time-delay was verified to within an accuracy of 0.001 %.

- Angular radius of Einstein rings:

  When light source and observer are perfectly aligned (i.e., directly opposite to each other) in the Schwarzschild spacetime, an Einstein ring is seen.



  We want to determine the angular radius θ_E of the Einstein ring in dependence of the radius coordinate r_L of the light source, the radius coordinate r_O of the observer and, of course, the Schwarzschild radius r_S.

  We use the formula

$$d\varphi \;=\; \frac{\pm\, dr}{\sqrt{\dfrac{(r_m - r_S)r^4}{r_m^3} \;-\; r^2 \;+\; r_S r}}\,.$$

  which was derived on p.23 for a light ray in the equatorial plane of the Schwarzschild spacetime. We integrate over the light ray, but this time for the case that the observer is at finite radius r_O and the light source is at finite radius r_L. In the case of perfect alignment, integration over the light ray gives

$$\pi \;=\; \left(\int_{r_m}^{r_L} + \int_{r_m}^{r_O}\right) \frac{dr}{\sqrt{\dfrac{(r_m - r_S)r^4}{r_m^3} \;-\; r^2 \;+\; r_S r}}\,.$$

  This equation determines r_m as a function of r_L, r_O and r_S,

$$r_m \;=\; f(r_L, r_O, r_S)\,.$$

  With the help of the implicit function theorem, one can indeed, check that the above equation can be solved for r_m. However, one cannot write down the resulting function f explicitly as the unknown quantity r_m occurs both in the limits and in the integrand of an elliptic integral. It is, of course, possible, to determine r_m numerically for any given values of r_L, r_O and r_S with arbitrary accuracy.

  With r_m (implicitly) determined, the angular radius θ_E of the Einstein ring can now be calculated in a way that is just a repitition of what we have done for the calculation of the shadow:

$$\tan\theta_E \;=\; \left.\frac{r\, d\varphi}{\left(1 - \dfrac{r_S}{r}\right)^{-1/2} dr}\right|_{r=r_O} \;=\; \frac{r_O\left(1 - \dfrac{r_S}{r_O}\right)^{1/2}}{\sqrt{\dfrac{(r_m - r_S)r_O^4}{r_m^3} - r_O^2 + r_S r_O}}\,.$$

  On the right-hand side we have to substitute $r_m = f(r_L, r_O, r_S)$ to get θ_E as a function of r_L, r_O and r_S as desired.

  If all equations are linearised with respect to r_S/r_m, one gets a simple and quite explicit formula for θ_E, see Worksheet 4.

2.3 Fermat's principle

We discuss a version of Fermat's principle for light rays in general relativity that holds in an arbitrary spacetime (M, g). No symmetry assumption will be made, and the causal structure need not be restricted.

This version of Fermat's principle was first suggested, in a local version restricted to convex normal neighbourhoods of an event, by G. Temple, Proc. Roy. Soc. London A 168, 122 (1938).

It was independently rediscovered by I. Kovner, Astrophys. J. 351, 114 (1990). Kovner clearly indicated the relevance of this variational principle for gravitational lensing, but he did not give a complete proof.

For a complete proof see V. Perlick, Class. Quantum Grav. 3, 1319 (1990).

A nice discussion can be found in Schneider/Ehlers/Falco.

In an arbitrary general-relativistic spacetime (M, g), we fix an event $q = (q^0, q^1, q^2, q^3) \in M$ and a curve

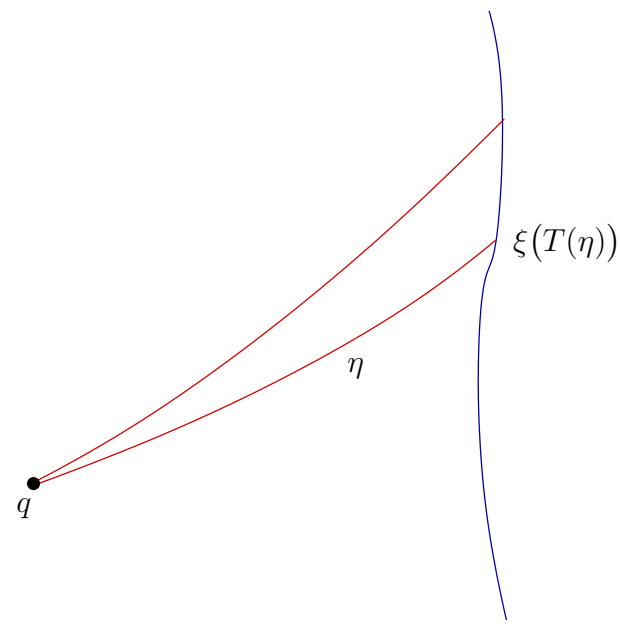
$$\xi : I \longrightarrow M$$

$$s \longmapsto \xi(\tau) = \big(\xi^0(\tau), \xi^1(\tau), \xi^2(\tau), \xi^3(\tau)\big)$$

that is timelike,

$$g_{\mu\nu} \frac{\xi^\mu(\tau)}{d\tau} \frac{\xi^\nu(\tau)}{d\tau} < 0 \,.$$

The parameter τ ranges over some open interval I. We may choose τ as the proper time, but any other smooth parametrisation would do as well.



We want to determine, by way of a variational principle, all light rays (i.e., all lightlike geodesics) that start at the event q and terminate on the worldline γ. To that end we need two things:

We have to choose the set of trial paths among which the solutions are to be sought, and we have to choose the functional that is to be extremised. Guided by Fermat's principle in ordinary optics, we choose as the space of trial paths the set of all worldlines that go from q to ξ at the speed of light. More precisely, we define the space of trial paths $\mathcal{L}_{q,\xi}$ as the set of all curves

$$\eta : [0, 1] \longrightarrow M \,, \qquad s \longmapsto \eta(s) = \big(\eta^0(s), \eta^1(s), \eta^2(s), \eta^3(s)\big)$$

such that:

(i) η starts at the event q, i.e. $\eta(0) = q$,

(ii) η terminates somewhere on the worldline ξ, i.e., there is a $T(\eta)$ such that $\eta(1) = \xi\big(T(\eta)\big)$,

(iii) η is the worldline of an object that moves at the speed of light, i.e., $g_{\mu\nu} \dfrac{d\eta^\mu(s)}{ds} \dfrac{d\eta^\nu(s)}{ds} = 0$ for all $s \in [0, 1]$.

(iv) η is future-oriented with respect to ξ, i.e., $g_{\mu\nu} \dfrac{d\eta^\mu(s)}{ds}\Big|_{s=1} \dfrac{d\xi^\nu(\tau)}{d\tau}\Big|_{\tau=T(\eta)} < 0$.

Condition (ii) defines the *arrival time functional*

$$T : \mathcal{L}_{q,\xi} \longrightarrow \mathbb{R}$$

$$\eta \longmapsto T(\eta)$$

This is the functional to be extremised. We can now state the following mathematical theorem which is to be interpreted as a general-relativistic version of Fermat's principle.

Theorem (Fermat's principle): For a curve $\eta \in \mathcal{L}_{q,\xi}$, the following equivalence holds:

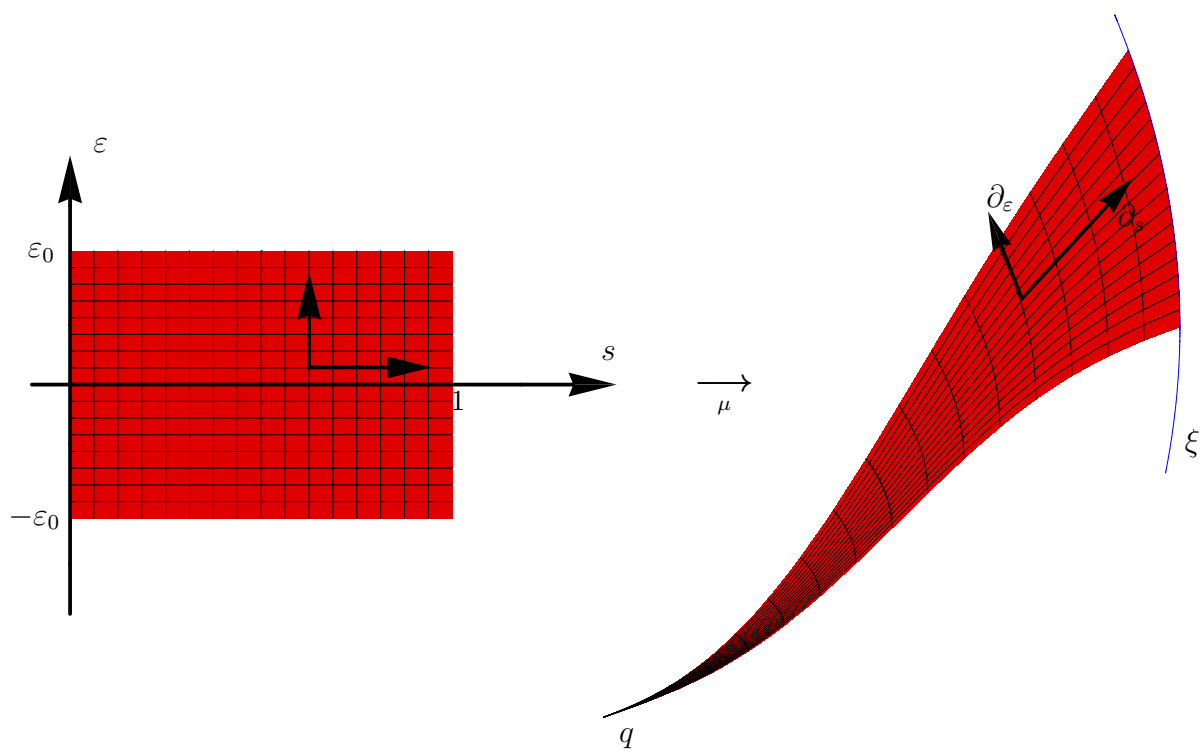
$$\left.\begin{array}{l}\eta \text{ is a stationary point (i.e., a min-}\\ \text{imum, a maximum or a saddle) of}\\ \text{the arrival time functional } T.\end{array}\right\} \quad \Longleftrightarrow \quad \eta \text{ is a geodesic.}$$

Proof of "$\Longleftarrow$": (A proof of "$\Longrightarrow$", under slightly more restrictive assumptions, can be found in Schneider/Ehlers/Falco.) Let $\eta \in \mathcal{L}_{q,\xi}$ be a geodesic. Consider a variation μ of η, i.e., a map

$$\mu : \ [0,1] \times \ ]-\varepsilon_0, \varepsilon_0[\ \longrightarrow \ M$$

$$(s, \varepsilon) \ \longmapsto \ \mu(s, \varepsilon)$$

such that $\mu(\,\cdot\,, 0) = \eta$ and $\mu(\,\cdot\,, \varepsilon) \in \mathcal{L}_{q,\xi}$ for all ε .



The image of μ is a 2-surface (possibly with self-intersections) in the spacetime manifold. This 2-surface is parametrised by s and ε. The partial derivatives with respect to these parameters define two vector fields (strictly speaking: "vector fields along the map μ") that are suggestively denoted ∂_s and ∂_ε.

As all trial paths start at q and terminate on ξ we have

$$\partial_\varepsilon\big|_{s=0} = 0 \qquad \text{and} \qquad \partial_\varepsilon\big|_{s=1} = \dot\xi\big(T(\mu(\cdot,\varepsilon))\big)\,\frac{d}{d\varepsilon}T\big(\mu(\cdot,\varepsilon)\big)\,.$$

It is our goal to prove that

$$\frac{d}{d\varepsilon}T\big(\mu(\,\cdot\,,\varepsilon)\big)\Big|_{\varepsilon=0} = 0\,.$$

As all trial paths are lightlike, we have $g(\partial_s,\partial_s)=0$ everywhere on the image of μ, i.e., for all values of s and ε. Differentiation with respect to ε and integration over a trial path yields, for any ε,

$$0 = \int_0^1 \partial_\varepsilon\Big(g(\partial_s,\partial_s)\Big)ds = \int_0^1 \nabla_{\partial_\varepsilon}\Big(g(\partial_s,\partial_s)\Big)ds =$$

$$= \int_0^1 \Big\{\Big(\underbrace{\nabla_{\partial_\varepsilon}g}_{=0}\Big)(\partial_s,\partial_s) + 2\,g\big(\nabla_{\partial_\varepsilon}\partial_s,\partial_s\big)\Big\}\,ds =$$

$$= 2\int_0^1 \Big\{\,g\big(\nabla_{\partial_s}\partial_\varepsilon + \underbrace{[\partial_\varepsilon,\partial_s]}_{=0},\partial_s\,\big)\Big\}\,ds =$$

$$= 2\int_0^1 \Big\{\,\partial_s\Big(g\big(\partial_\varepsilon,\partial_s\big)\Big) - g\big(\partial_\varepsilon,\nabla_{\partial_s}\partial_s\big)\,\Big\}\,ds =$$

$$= 2\,g\big(\partial_\varepsilon,\partial_s\big)\Big|_{s=1} - 2\,g\big(\partial_\varepsilon,\partial_s\big)\Big|_{s=0} - 2\int_0^1 g\big(\partial_\varepsilon,\nabla_{\partial_s}\partial_s\big)\,ds =$$

$$= 2\,g\Big(\dot\xi\big(T(\mu(\cdot,\varepsilon))\big),\partial_s\big|_{s=1}\Big)\,\frac{d}{d\varepsilon}T\big(\mu(\cdot,\varepsilon)\big) - 0 - 2\int_0^1 g\big(\partial_\varepsilon,\nabla_{\partial_s}\partial_s\big)\,ds\,. \qquad (*)$$

We evaluate this equation at $\varepsilon=0$. As η is a geodesic, we have $\nabla_{\partial_s}\partial_s=0$ for $\varepsilon=0$, so the integral in $(*)$ vanishes. As $\dot\xi$ is timelike and ∂_s is lightlike, the factor $g\Big(\dot\xi\big(T(\mu(\cdot,\varepsilon))\big),\partial_s\big|_{s=1}\Big)$ cannot be zero. (By condition (iv) of our definition of the space of trial paths, it is negative.) So $(*)$ implies that, indeed

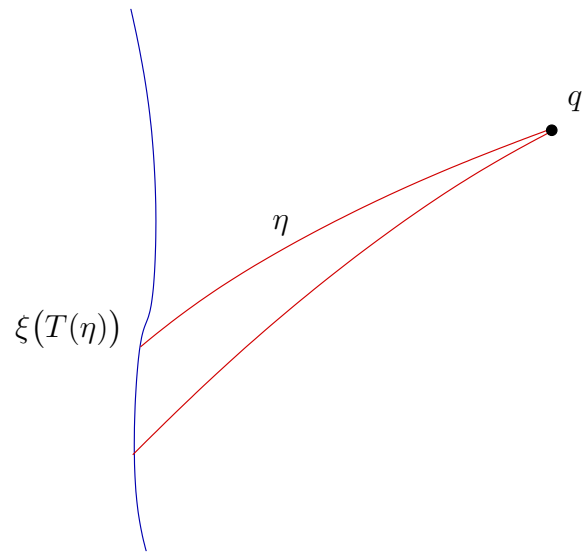
$$\frac{d}{d\varepsilon}T\big(\mu(\cdot,\varepsilon)\big)\Big|_{\varepsilon=0} = 0\,.$$

□

Remarks:

- We have proven that a geodesic must be a stationary point of the arrival time functional, i.e., a minimum, a maximum or a saddle. Actually, maxima do not occur. This is intuitively clear because one can always produce neighbouring trial paths with a longer travel time by putting wiggles into it. This is quite analogous to Fermat's principle in ordinary optics where also only minima and saddles occur, see e.g. M. Born and E. Wolf: *Principles of Optics*, 7th edition, Cambridge Univ. Press (1999), p.137.

- In applications to gravitational lensing, one chooses ξ with a *past*-oriented parametrisation. q is to be interpreted as an observation event and ξ as the worldline of a light source. The solution curves are all *past*-oriented light rays from q to ξ. They give all the images of the light source that are seen by the observer at the event q.



Fermat's principle can be reduced to a purely spatial variational principle in the case that the spacetime is conformally stationary. By definition, a spacetime is called *stationary* if it admits coordinates (x^0, x^1, x^2, x^3) such that $g_{00} < 0$ and $\partial_0 g_{\mu\nu} = 0$, i.e., such that the x^0-lines are timelike and the metric coefficients depend on (x^1, x^2, x^3) only. A spacetime is called *conformally stationary* if it can be made stationary by multiplying the metric with a scalar factor. (Recall that the multiplication of the metric with a scalar factor is called a "conformal transformation". The factor must be strictly positive to preserve the signature.) Hence, a conformally stationary metric is of the form

$$g \;=\; g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j \;=$$

$$= \,-\, g_{00}\Big\{ \,-\, (dx^0)^2 \,+\, 2\underbrace{\frac{g_{0i}}{-g_{00}}}_{=:\,\gamma_{0i}} dx^0dx^i \,+\, \underbrace{\frac{g_{ij}}{-g_{00}}}_{=:\,\gamma_{ij}} dx^idx^j \Big\}$$

with

$$g_{00} < 0\,, \quad \partial_0\gamma_{0i} = 0\,, \quad \partial_0\gamma_{ij} = 0\,.$$

In such a spacetime, we apply Fermat's principle to the case that ξ is an integral curve of ∂_0, parametrised with its proper time with respect to the conformally rescaled metric, i.e.

$$\big(\xi^0(\tau), \xi^1(\tau), \xi^2(\tau), \xi^3(\tau)\big) \;=\; \big(c\tau, x_0^1, x_0^2, x_0^3\big)$$

with constants x_0^1,x_0^2 and x_0^3. q may be any event in the spacetime. As each trial path $\eta \in \mathcal{L}_{q,\xi}$ is lightlike, we have $g_{\mu\nu}\dot\eta^\mu\dot\eta^\nu = 0$. After dividing by the (strictly positive) factor $-g_{00}$ this results in

$$0 \;=\; -\,(\dot\eta^0)^2 \,+\, 2\,\gamma_{0i}\dot\eta^0\dot\eta^i \,+\, \gamma_{ij}\dot\eta^i\dot\eta^j \qquad \Longrightarrow \qquad \dot\eta^0 \;=\; \gamma_{0i}\dot\eta^i \,\pm\, \sqrt{\big(\gamma_{0i}\gamma_{0j} + \gamma_{ij}\big)\dot\eta^i\dot\eta^j}\;.$$

By condition (iv) of the definition of the trial paths, the plus sign must be chosen. We integrate over the trial path from $s = 0$ to $s = 1$. As

$$\eta^0(s)\big|_{s=1} \;=\; c\,T(\eta) \qquad \text{and} \qquad \eta^0(s)\big|_{s=0} \;=\; \text{constant}\ ,$$

this gives us the arrival time functional as an integral,

$$T(\eta) \;=\; \frac{1}{c}\int_0^1 \Big\{ \gamma_{0i}\dot\eta^i \;+\; \sqrt{\big(\gamma_{0i}\gamma_{0j} + \gamma_{ij}\big)\dot\eta^i\dot\eta^j}\ \Big\}\, ds \;+\; \text{constant}\ .$$

(The constant is, of course, irrelevant for the variational problem and can be disregarded.) As only the spatial components (η^1, η^2, η^3) enter, and as γ_{0i} and γ_{ij} are independent of x^0, this reduces Fermat's principle to a variational problem for curves in 3-dimensional space.

An even simpler version of Fermat's principle arises if we further specify to the case that the γ_{0i} are zero, i.e., that the x^0-lines are orthogonal to the hypersurfaces $x^0 = \text{constant}$. In this case, the spacetime is called *conformally static*. The arrival time functional simplifies to

$$T(\eta) \;=\; \frac{1}{c}\int_0^1 \sqrt{\gamma_{ij}\,\dot\eta^i\dot\eta^j}\ ds\ \big(\ +\ \text{constant}\ \big)\ .$$

This is, up to the factor c, just the *length functional* of the positive definite Riemannian metric $\gamma_{ij}dx^i dx^j$ on 3-space. We call this metric the *Fermat metric* (or the *optical metric*) of our conformally static spacetime. The stationary points of the length functional are, of course, the geodesics. So in this case Fermat's principle reduces to the statement that the light rays are precisely the geodesics of the Fermat metric $\gamma_{ij}dx^i dx^j$ if projected to 3-space. This special version of Fermat's principle was found by H. Weyl already in 1917. (Weyl did not allow for a time-dependent conformal factor, i.e., he assumed $\partial_0 g_{00} = 0$, but this generalisation is fairly obvious.)

In the even more special case that the Fermat metric is conformal to the (flat) Euclidean metric,

$$\gamma_{ij}dx^i dx^j \;=\; n(x^1, x^2, x^3)^2\,\delta_{ij}dx^i dx^j\ ,$$

Fermat's principle takes precisely the same form as in ordinary optics for a medium with index of refraction $n(x^1, x^2, x^3)$,

$$T(\eta) \;=\; \frac{1}{c}\int_0^{\ell_0} n(x^1, x^2, x^3)d\ell\ \big(\ +\ \text{constant}\ \big)\ ,$$

where

$$d\ell = \sqrt{\delta_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\ ds$$

is the Euclidean length element. In the standard terminology of optics,

$$L(\eta) \;=\; \int_0^{\ell_0} n(x^1, x^2, x^3)d\ell$$

is called the *optical path length*. For conformally static spacetimes in which the Fermat metric is conformal to the flat metric, it is thus possible to mimic the light propagation by a medium with an appropriately chosen index of refraction. This goes under the name of "analogue models" for general-relativistic spacetimes. As an example, we will determine the index of refraction for the Schwarzschild metric in the 5th worksheet.

2.4 Example: Robertson-Walker spacetimes

By definition, a Robertson-Walker spacetime is a Lorentzian manifold with a metric of the form

$$g \;=\; -\,c^2 dt^2 \;+\; a(t)^2\,\frac{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)}{(1+kr^2)^2}\;.$$

Robertson-Walker spacetimes are the most basic cosmological spacetime models. They are homogeneous (no spatial points are distinguished from others) and isotropic (no spatial directions are distinguished from others).

The function $a(t)$ is known as the *scale factor*. The universe is expanding if $\dot{a}(t) > 0$ and contracting if $\dot{a}(t) < 0$. The constant k gives the spatial curvature of the spacetime. $k > 0$ means spatially positively curved, $k = 0$ means spatially flat, $k < 0$ means spatially negatively curved. By a (dimensional) rescaling of the coordinates one can make k to $+1$, 0 or -1.

Every Robertson-Walker metric satisfies Einstein's field equation

$$R_{\mu\nu} \;-\; \frac{R}{2}\,g_{\mu\nu} \;+\; \Lambda\, g_{\mu\nu} \;=\; \kappa\, T_{\mu\nu}$$

with the energy-momentum tensor of a perfect fluid,

$$T_{\mu\nu} \;=\; \big(\,\mu \;+\; \frac{p}{c^2}\,\big)U_\mu U_\nu \;+\; p\, g_{\mu\nu}\;, \qquad U^\rho \partial_\rho = \partial_t\;.$$

Only two of the ten components of Einstein's field equation give independent equations. They are known as the *Friedmann equations*, named after Russian mathematician Alexander Friedmann who found them in 1922:

$$\frac{\dot{a}(t)^2 \;+\; k\,c^2}{a(t)^2} \;=\; \frac{1}{3}\left(\kappa c^4 \mu(t) + \Lambda c^2\right)\;,$$

$$\frac{\ddot{a}(t)}{a(t)} \;=\; -\,\frac{\kappa c^4}{6}\left(\mu(t) \;+\; \frac{3\,p(t)}{c^2}\right) \;+\; \frac{\Lambda c^2}{3}\;.$$

Obviously, the second equation admits a time-independent solution, $a =$ constant, with $\mu > 0$ and $p \geq 0$ only if $\Lambda \neq 0$. In other words, Λ is necessary if one wants to construct time-independent homogeneous and isotropic cosmological models on the basis of general relativity. This is what Einstein, actually, motivated to introduce the cosmological constant in 1917. Nowadays we are no longer interested in time-independent cosmological models; nonetheless, the cosmological constant plays a crucial role in modern cosmology, in particular as a possible explanation for the *accelerated* expansion of the universe.

For each solution of the Friedmann equations, we can calculate the maximal time interval $]t_{\min}, t_{\max}[$ on which it is defined. We speak of a

- big-bang model if $t_{\min} \neq -\infty$,
- big-crunch model if $t_{\max} \neq \infty$.

To apply Fermat's principle to a Robertson-Walker universe, we rewrite the metric in the form

$$g \;=\; a(t)^2\left\{\,-\,\frac{c^2\,dt^2}{a(t)^2} \;+\; \frac{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)}{(1+kr^2)^2}\,\right\}.$$

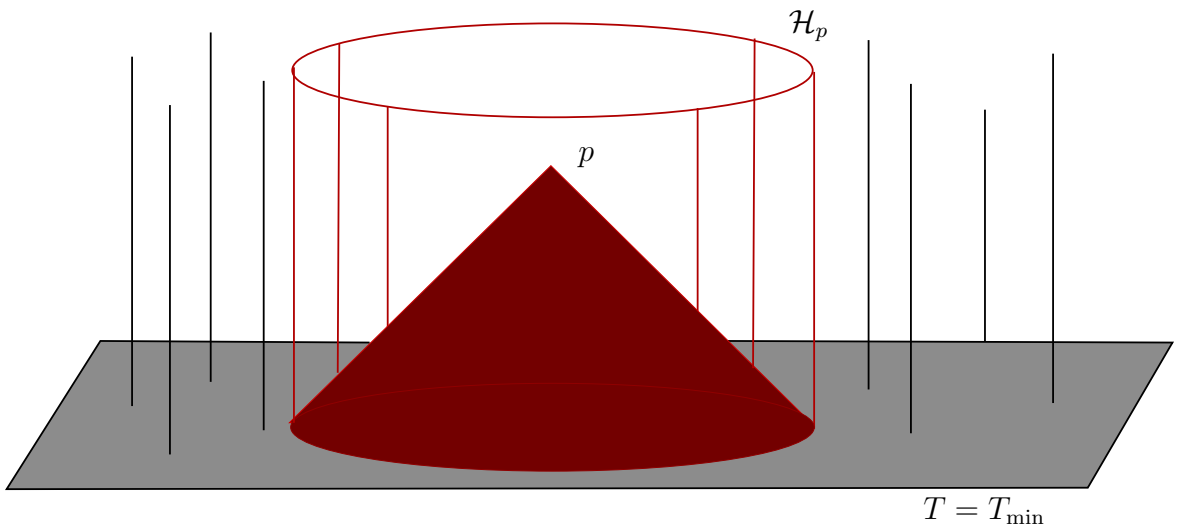
We introduce a new time coordinate

$$T = \int \frac{dt}{a(t)} ,$$

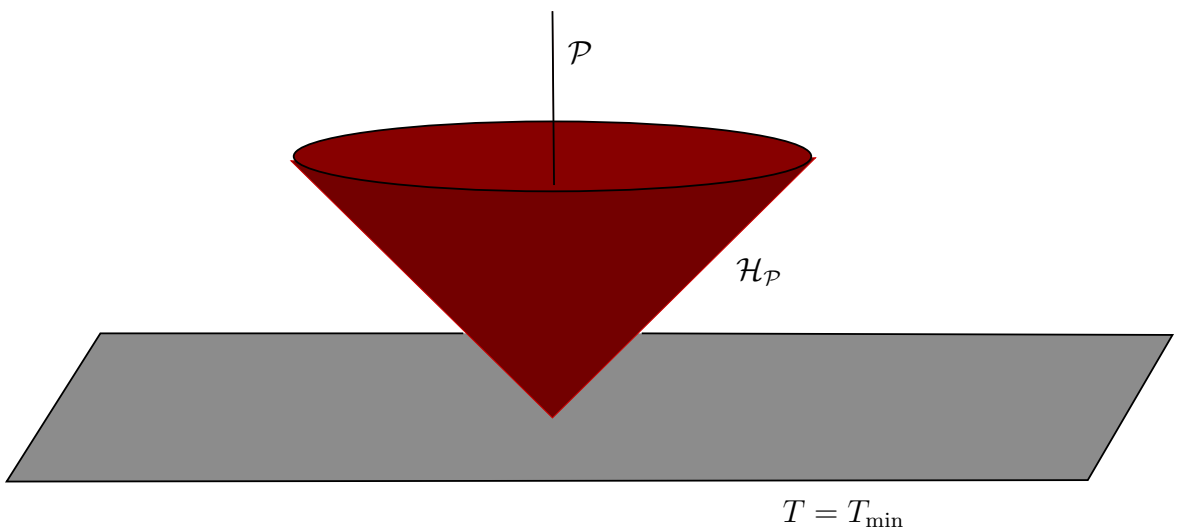
known as the "conformal time", such that

$$\frac{dt^2}{a(t)^2} = dT^2 .$$

The transformation $t \mapsto T$ maps the domain $]t_{\min}, t_{\max}[$ of the t-coordinate onto a maximal domain $]T_{\min}, T_{\max}[$ of the T-coordinate. If $T_{\min} \neq -\infty$, there are *particle horizons* and *event horizons*, even in the spatially flat case ($k = 0$) where the metric inside the curly bracket is just the Minkowski metric.



Integral curves of ∂_t ("particles") outside of $\mathcal{H}_p$ cannot causally influence the event p. Therefore, $\mathcal{H}_p$ is called the *particle horizon* of p.



Events outside of the light cone $\mathcal{H}_{\mathcal{P}}$ cannot be causally influenced by the particle $\mathcal{P}$. Therefore $\mathcal{H}_{\mathcal{P}}$ is called the *event horizon* of the particle $\mathcal{P}$.

To verify that our metric is, indeed, conformally static, we have to compare it with the form

$$g \;=\; -\,g_{00}\left\{\,-\;(dx^0)^2\;+\;\gamma_{ij}\,dx^i dx^j\;\right\}$$

and to verify that the γ_{ij} are indeed independent of x^0. Clearly, this is true with the identifications

$$g_{00} \;=\; -\,a(t)^2\;, \qquad x^0 \;=\; c\,T\;, \qquad \gamma_{ij}\,dx^i dx^j \;=\; \frac{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)}{(1+kr^2)^2}\;.$$

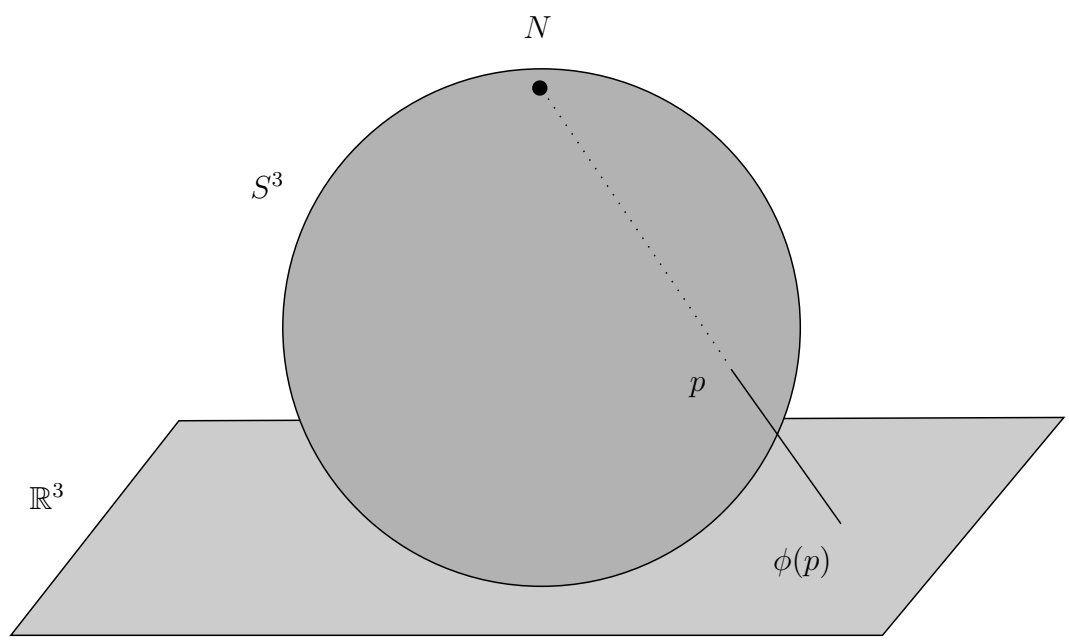
The Fermat metric is conformal to the flat metric,

$$\gamma_{ij}\,dx^i dx^j \;=\; n(r)^2\left(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\right)$$

with an index of refraction

$$n(r) \;=\; \frac{1}{1+kr^2}\;.$$

In a medium with such an r-dependent index of refraction the light rays move precisely in the same way as in a Robertson-Walker universe.



For $k = 1$, a medium with this index of refraction was considered already in the 19th century by James Maxwell. It is known as *Maxwell's fish-eye*. The Fermat metric (on $\mathbb{R}^3$) is related to the standard metric on the 3-sphere S^3 by stereographic projection $p \mapsto \phi(p)$. On S^3, the geodesics are great circles; hence, all the geodesics through a fixed point also pass through the antipodal point. It was this property that caught the interest of Maxwell: Maxwell's fish-eye is an optical system where the light rays issuing from a fixed point are perfectly refocused into another point. (In the $\mathbb{R}^3$-representation, this works for all points with the exception of the origin; the latter is refocused into the "point at infinity" that corresponds to the north pole N.)

2.5 Redshift

It is our goal to derive a redshift formula for observer and light source in arbitrary motion (with subluminal speed, of course) in an arbitrary spacetime.

> This formula was first given by W. Kermack, W. McCrea and E. Whittaker, Proc. R. Soc. Edinburgh 53, 31 (1932). The derivation we will give here follows D. Brill, in D. Farnsworth, J. Fink, J. Porter and A. Thompson (eds.), *Methods of local and global differential geometry in general relativity*, Springer (1972), p.45.

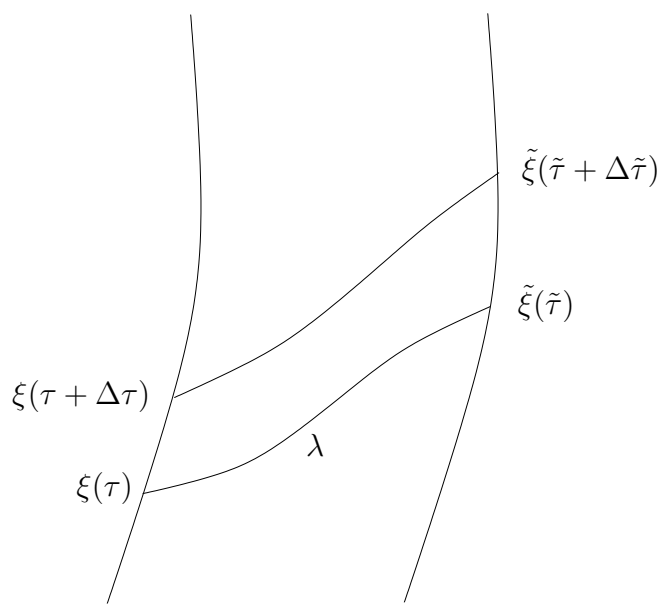
We consider, in an arbitrary general-relativistic spacetime, two timelike curves

$$\xi : I \longrightarrow M\,, \qquad \tau \longmapsto \xi\big(\tau\big) \qquad\qquad \text{and} \qquad\qquad \tilde{\xi} : I \longrightarrow M\,, \qquad \tilde{\tau} \longmapsto \tilde{\xi}\big(\tilde{\tau}\big)\,.$$

ξ is to be interpreted as the worldline of a light source (or emitter) and $\tilde{\xi}$ as the worldline of an observer (or receiver). The parameters τ and $\tilde{\tau}$ are arbitrary, for the time being.

Assume that the light source emits at time τ a light ray λ that arrives at time $\tilde{\tau}$ with the receiver, and a second light ray at time $\tau + \Delta\tau$ that arrives at time $\tilde{\tau} + \Delta\tilde{\tau}$. Then we define the frequency ratio

$$\frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau\to\infty} \frac{\Delta\tilde{\tau}}{\Delta\tau} =$$
$$= \frac{\omega_{\text{emitter}}}{\omega_{\text{receiver}}} = 1 + z\,.$$



As we have not required τ and $\tilde{\tau}$ to be proper time, here the word "frequency" does not necessarily refer to the physical frequency as it is measured with standard clocks. However, the mathematical construction works for any parametrisations. – The quantity

$$z = \frac{\omega_{\text{emitter}} - \omega_{\text{receiver}}}{\omega_{\text{receiver}}}$$

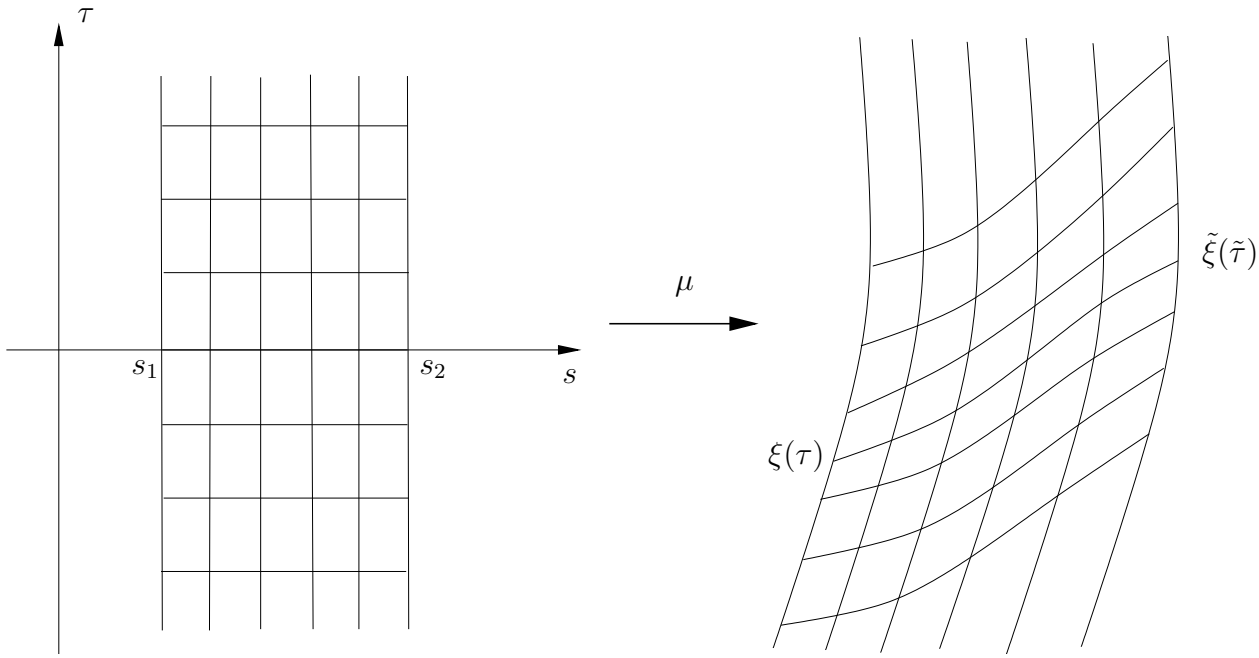
is called the *redshift*. Clearly, $z > 0$ means that the light signal arrives redshifted and $z < 0$ means that it arrives blueshifted.

It is now our goal to find a formula that allows to calculate z if ξ, $\tilde{\xi}$ and λ are known. To that end we consider a variation

$$\begin{aligned} \mu : [s_1, s_2] \times I &\longrightarrow M \\ (s, \varepsilon) &\longmapsto \mu(s, \varepsilon) \end{aligned}$$

such that

$$\mu(s_1, \tau) = \xi\big(\tau\big)\,, \qquad \mu(s_2, \tau) = \tilde{\xi}\big(\tilde{\tau}(\tau)\big)\,, \qquad \mu(\,\cdot\,, \tau) \text{ is a lightlike geodesic.}$$



We calculate

$$\partial_s\Big(g(\partial_s,\partial_\tau)\Big) \;=\; \nabla_{\partial_s}\Big(g\big(\partial_s,\partial_\tau\big)\Big) \;=$$

$$=\; \big(\underbrace{\nabla_{\partial_s}g}_{=0}\big)\big(\partial_s,\partial_\tau\big) \;+\; g\big(\underbrace{\nabla_{\partial_s}\partial_s}_{=0},\partial_\tau\big) \;+\; g\,\big(\partial_s\,,\underbrace{\nabla_{\partial_s}\partial_\tau}_{=\nabla_{\partial_\tau}\partial_s\,+\,[\partial_s,\partial_\tau]}\big) \;=$$

$$=\; g\big(\partial_s,\nabla_{\partial_\tau},\partial_s\big) \;+\; g\big(\partial_s,\underbrace{[\partial_s\,\partial_\tau]}_{=0}\big) \;=$$

$$=\; \frac{1}{2}\nabla_{\partial_\tau}\Big(\underbrace{g\big(\partial_s,\partial_s\big)}_{=0}\Big) \;-\; \frac{1}{2}\Big(\underbrace{\nabla_{\partial_\tau}g}_{=0}\Big)\big(\partial_s,\partial_s\big) \;=\; 0\;.$$

This demonstrates that the function $g\big(\partial_s,\partial_\tau\big)$ is constant along each of the light rays from ξ to $\tilde{\xi}$ that form our variation. In particular, this function takes the same value at the two end-points of the light ray,

$$g\big(\partial_s,\partial_\tau\big)\big|_{s=s_1,\tau} \;=\; g\big(\partial_s,\partial_\tau\big)\big|_{s=s_2,\tau}\,, \qquad (*)$$

$$g_{\mu\nu}\,\frac{d\lambda^\mu}{ds}\Big|_{s=s_1}\frac{d\xi^\nu}{d\tau} \;=\; g_{\mu\nu}\,\frac{d\lambda^\mu}{ds}\Big|_{s=s_2}\frac{d\tilde{\xi}^\nu}{d\tilde{\tau}}\,\frac{d\tilde{\tau}}{d\tau}\,.$$

This gives us the desired redshift formula

$$1\,+\,z\;=\;\frac{g_{\mu\nu}\,\dfrac{d\lambda^\mu}{ds}\Big|_{s=s_1}\dfrac{d\xi^\nu}{d\tau}}{g_{\rho\sigma}\,\dfrac{d\lambda^\rho}{ds}\Big|_{s=s_2}\dfrac{d\tilde{\xi}^\sigma}{d\tilde{\tau}}}\;.$$

We will now specialise this general redshift formula to the case that the spacetime is conformally stationary,

$$g \;=\; -\,g_{00}\Big\{\;-\;(dx^0)^2\;+\;2\underbrace{\frac{g_{0i}}{-g_{00}}}_{=:\,\gamma_{0i}}\,dx^0dx^i\;+\;\underbrace{\frac{g_{ij}}{-g_{00}}}_{=:\,\gamma_{ij}}\,dx^idx^j\;\Big\}$$

with

$$g_{00}<0\,,\quad \partial_0\gamma_{0i}=0\,,\quad \text{and}\quad \partial_0\gamma_{ij}=0\,,$$

recall p.35. The equation

$$e^{2f}\;=\;-\,g_{00}$$

defines a function f that, in general, depends on all four spacetime coordinates. The four-velocity of observers that move on x^0-lines is then given by

$$U\;=\;U^\mu\partial_\mu\;=\;c\,e^{-f}\partial_0\;,\qquad g_{\mu\nu}U^\mu U^\nu\;=\;-\,c^2\;.$$

We assume that emitter and observer move on x^0-lines. We can then apply our above result $(*)$ to the case that $\partial_\tau=\partial_0$,

$$g\Big(\partial_s\,,\frac{1}{c}\,e^f\,U\Big)\Big|_p\;=\;g\Big(\partial_s\,,\frac{1}{c}\,e^f\,U\Big)\Big|_{\tilde p}$$

where p is the event of emission and $\tilde p$ is the event of reception. We have, thus,

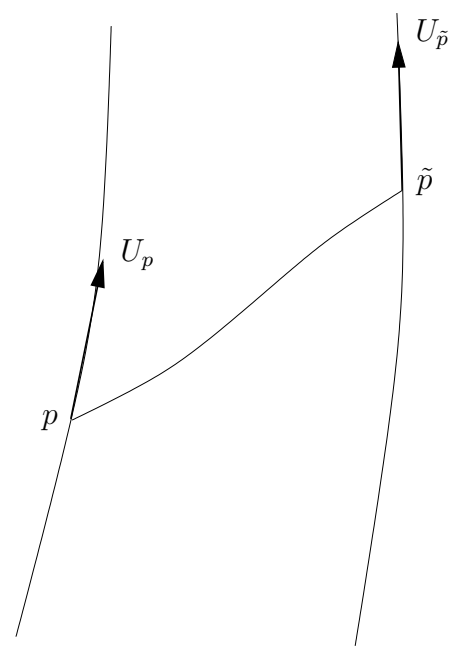
$$\frac{g\big(\partial_s,U\big)\big|_p}{g\big(\partial_s,U\big)\big|_{\tilde p}}\;=\;e^{f(\tilde p)-f(p)}\;.$$

According to the general redshift formula, the left-hand side is the frequency ratio $1+z$ for the case that emitter and receiver use standard clocks,

$$U_p\;=\;\frac{d\xi}{d\tau}\,,\qquad U_{\tilde p}\;=\;\frac{d\tilde\xi}{d\tilde\tau}\;.$$

For this case the redshift z is thus given by the simple formula

$$\ln(1+z)\;=\;f(\tilde p)-f(p)$$



which means that f is a *redshift potential*. We have thus shown that every conformally stationary spacetime admits a redshift potential. One can show that the converse is also true: A redshift potential exists *if and only if* the spacetime is conformally stationary.

In the stationary case, where f is independent of x^0, the above redshift formula gives us z directly in terms of the spatial coordinates (x^1,x^2,x^3) of the emitter and the spatial coordinates $(\tilde x^1,\tilde x^2,\tilde x^3)$ of the receiver. If f does depend on x^0, we have to find out which pairs of points p and $\tilde p$ on the worldlines of emitter and receiver can be connected by light rays. Only then can we calculate z with the above redshift formula, e.g., as a function of the receiver's proper time. A simple example of the latter case is given by the Robertson-Walker spacetimes, see Worksheet 5.

2.6 Example: Kottler spacetime

The Kottler spacetime is the generalisation of the Schwarzschild spacetime to the case that the cosmological constant is allowed to be non-zero. More precisely, the Kottler metric is the unique spherically symmetric solution of the vacuum Einstein equation with cosmological constant, $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$. It reads

$$g \;=\; -\big(1 \,-\, \frac{2GM}{c^2 r} \,-\, \frac{\Lambda}{3}\, r^2\big)\, c^2 dt^2 \,+\, \frac{dr^2}{1 \,-\, \dfrac{2GM}{c^2 r} \,-\, \dfrac{\Lambda}{3}\, r^2} \,+\, r^2\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\;.$$

It was found by F. Kottler in 1918. It is also known as the "Schwarzschild-deSitter metric" if $\Lambda > 0$ and as the "Schwarzschild-anti-deSitter metric" if $\Lambda < 0$. This terminology has its origin in the fact that the Kottler metric goes over into the deSitter metric if $M \to 0$ and $\Lambda > 0$, and into the anti-deSitter metric if $M \to 0$ and $\Lambda < 0$. For $\Lambda \to 0$, the Kottler metric is of course the Schwarzschild metric.

For all values of Λ, the Kottler metric has a true singularity at $r = 0$. Horizons occur at (real and positive) values of r where

$$1 \,-\, \frac{2GM}{c^2 r} \,-\, \frac{\Lambda}{3} r^2 \;=\; 0\;.$$

One finds

- one horizon if $\Lambda < 0$,
- two horizons if $0 < \Lambda < \Big(\dfrac{c^2}{3GM}\Big)^2$,
- no horizon if $\Big(\dfrac{c^2}{3GM}\Big)^2 < \Lambda$.

We rewrite the Kottler metric in the form

$$g \;=\; \big(1 \,-\, \frac{2GM}{c^2 r} \,-\, \frac{\Lambda}{3}\, r^2\big)\Big\{ -\, c^2 dt^2 \,+\, \frac{dr^2}{\Big(1 \,-\, \dfrac{2GM}{c^2 r} \,-\, \dfrac{\Lambda}{3}\, r^2\Big)^2} \,+\, \frac{r^2\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)}{1 \,-\, \dfrac{2GM}{c^2 r} \,-\, \dfrac{\Lambda}{3}\, r^2}\Big\}\;.$$

From this expression we read that the redshift potential f of the Kottler metric is given as

$$e^{2f} \;=\; 1 \,-\, \frac{2GM}{c^2 r} \,-\, \frac{\Lambda}{3}\, r^2\;.$$

This gives us immediately the redshift for the case that emitter and receiver move on x^0-lines and use standard clocks,

$$1 \,+\, z \;=\; \frac{\sqrt{1 \,-\, \dfrac{2GM}{c^2 \tilde{r}} \,-\, \dfrac{\Lambda}{3}\, \tilde{r}^2}}{\sqrt{1 \,-\, \dfrac{2GM}{c^2 r} \,-\, \dfrac{\Lambda}{3}\, r^2}}$$

where r and $\tilde{r}$ are the radius coordinates of emitter and receiver, respectively. Note that $z \to \infty$ if the emitter approaches the horizon.

2.7 Geometry of light bundles

In this chapter we discuss how the cross section of a light bundle changes in dependence of an affine parameter of a central light ray. This has relevance in view of

- the magnitude of images,
- the deformation of images.

A central role in this chapter will play the *Sachs equations* which are differential equations for the socalled *optical scalars*.

> The relevant equations for the cross sections of light bundles in general relativity were first published in German by J. Ehlers and R. Sachs, Abh. Mainzer Akad. Wissensch., Math.-Naturw. Klasse Nr. 1, 1 (1961). The Sachs equations were then published in English by R. Sachs, Proc. Roy. Soc. London A 264, 309 (1961), 270, 103 (1962).

Starting point is the geodesic deviation equation (also known as the Jacobi equation). We briefly recall how this equation is derived. Assume that X is geodesic, $\nabla_X X = 0$, and that $[X, Y] = 0$, i.e., that the integral curves of X and Y form a closed grid. Then one finds from the definition of the curvature tensor, $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, and from the fact that the torsion vanishes, $\nabla_Y Z - \nabla_Z Y + [Z, Y] = 0$, the geodesic deviation equation

$$\nabla_X \nabla_X Y \;=\; R(X, Y, X)\,.$$

This equation tells how, along any integral curve of the geodesic vector field X, the connecting vector Y to an infinitesimally close neighbouring integral curve of X changes.

We now specify to the case that the geodesic vector field X is lightlike, $g(X, X) = 0$. Then each integral curve of X is a light ray. We fix one of those integral curves of X and call it λ. A *bundle of light rays* (or, more precisely, an *infinitesimally thin bundle of light rays*) around λ is defined by any two vector fields Y_1 and Y_2 such that

(a) $\nabla_X \nabla_X Y_A \;=\; R(X, Y_A, X)$ for $A = 1, 2$;

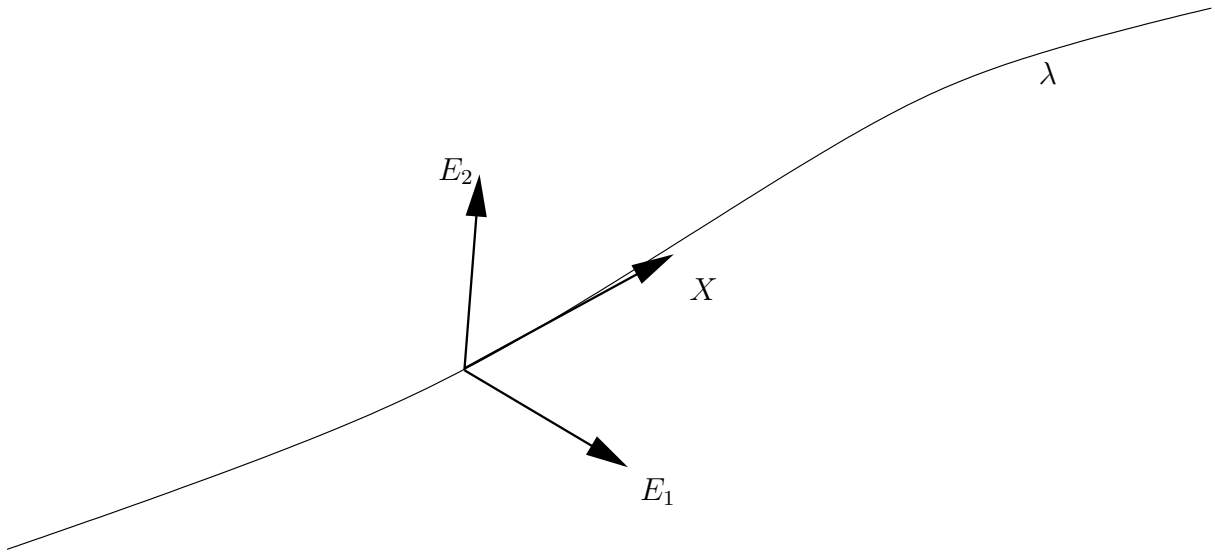
(b) $g(X, Y_A) \;=\; 0$ for $A = 1, 2$;

(c) Y_1, Y_2 and X are linearly independent almost everywhere.

The bundle is then the set

$$\mathcal{B} \;=\; \left\{\, c_1 Y_1 + c_2 Y_2 \,\middle|\, c_1^2 + c_2^2 = 1 \,\right\}\,.$$

Condition (a) means that "the tip of the arrow" Y_A describes an infinitesimally close neighbouring geodesic. Condition (b) makes sure that this neighbouring geodesic is again lightlike and spatially related to the central geodesic. Condition (c) requires that the bundle has a two-dimensional cross-section except, possibly, at some isolated ("focal") points were the bundle cross-section collapses into a line or a point.



We can rewrite the Jacobi equation conveniently as a matrix equation if we express Y_1 and Y_2 with respect to an appropriate basis. We say that a pair of vector fields (E_1, E_2) along λ is a *Sachs bein* if E_1 and E_2 are

- orthonormal, i.e. $g(E_A, E_B) = \delta_{AB}$;
- orthogonal to X, i.e., $\mathrm{g}(X, E_A) = 0$;
- parallel along λ, i.e., $\nabla_X E_A = 0$.

Note that, by the third condition, the Sachs bein is fixed along λ once it has been chosen at one point of λ. The first two conditions imply that, at each point of λ, the three vectors X, E_1 and E_2 span the orthocomplement of X. We can, thus, write Y_1 and Y_2 as a linear combination of them,

$$Y_A \,=\, D_A{}^B E_B \,+\, y_A X \;.$$

If we plug this into the Jacobi equation we get

$$\frac{d^2 D_A{}^B}{ds^2}\, E_B \,+\, \frac{d^2 y^A}{ds^2}\, X \,=\, g\big(X, D_A{}^B E_B, X\big) \;,$$

where we have used that $\nabla_X E_A = 0$ and $\nabla_X X = 0$. Applying the operator $g(E_C, \cdot\,)$ to this equation results in

$$\frac{d^2 D_A{}^B}{ds^2}\, \underbrace{g(E_C, E_B)}_{=\delta_{CB}} \,+\, \frac{d^2 y_A}{ds^2}\, \underbrace{g(E_C, X)}_{=0} \,=\, D_A{}^B\, g\big(E_C, R(X, E_B, X)\big) \;.$$

This gives us the *matrix Jacobi equation*

$$\frac{d^2}{ds^2}\boldsymbol{D} \,=\, \boldsymbol{D}\,\boldsymbol{Z}$$

where

$$\boldsymbol{D} \,=\, \big(D_A{}^B\big) \,, \qquad \boldsymbol{Z} \,=\, \big(Z_{CB}\big) \,, \qquad Z_{CB} \,=\, g\big(E_C, R(X, E_B, X)\big) \;.$$

Note that, owing to the symmetries of the Riemannian curvature tensor R, the *optical tidal matrix* $\boldsymbol{Z}$ is symmetric,

$$Z_{CB} \,=\, Z_{BC}\,.$$

We can, thus, decompose $\boldsymbol{Z}$ into a trace part and a trace-free part,

$$\boldsymbol{Z} \,=\, \begin{pmatrix} \Phi_{00} & 0 \\ 0 & \Phi_{00} \end{pmatrix} + \begin{pmatrix} \mathrm{Re}(\psi_0) & \mathrm{Im}(\psi_0) \\ \mathrm{Im}(\psi_0) & -\mathrm{Re}(\psi_0) \end{pmatrix}$$

where we have introduced the real quantity

$$\Phi_{00} \,=\, \frac{1}{2}\big(\,Z_{11} \,+\, Z_{22}\,\big) \,=\, \frac{1}{2}\,\mathrm{Ric}(X,X) \qquad\qquad (N1)$$

and the complex quantity

$$\psi_0 \,=\, \frac{1}{2}\big(\,Z_{11} \,-\, Z_{22}\,\big) \,+\, i\,Z_{12} \,=\, \frac{1}{2}g\big(E_1 + iE_2, C(X, E_1 + iE_2, X)\big)\,. \qquad\qquad (N2)$$

In $(N1)$, Ric denotes the Ricci tensor, $\mathrm{Ric}(X,X) = R_{\mu\nu}X^\mu X^\nu$, and in $(N2)$ C denotes the trace-free part of the curvature tensor which is known as the *Weyl tensor* or as the *conformal curvature tensor*. (The symbols Φ_{00} and ψ_0 have been chosen in agreement with the Newman-Penrose formalism, just for those readers who are familiar with it. For our purpose, however, this background is not relevant. Just view Φ_{00} and ψ_0 as being defined by the first equality sign in $N(1)$ and $(N2)$, respectively.)

To get a better understanding of the geometry behind the matrix Jacobi equation, we decompose the matrices $\boldsymbol{D}$ and $\boldsymbol{Z}$ in an appropriate way. Recall that *any* matrix can be written as a product of an orthogonal matrix and a symmetric matrix. (This result can be found in books on linear algebra under the name of "polar decomposition".) We can, thus, decompose our matrix $\boldsymbol{D}$ in the form

$$\boldsymbol{D} \,=\, \boldsymbol{R}\,\boldsymbol{M}$$

where $\boldsymbol{R}$ is orthogonal,

$$\boldsymbol{R}^T \,=\, \boldsymbol{R}^{-1}\,,$$

and $\boldsymbol{M}$ is symmetric,

$$\boldsymbol{M}^T \,=\, \boldsymbol{M}\,.$$

Now we use the well-known fact that any symmetric matrix M can be diagonalised. We can, thus, write the matrix $\boldsymbol{M}$ in the form

$$\boldsymbol{M} \,=\, \boldsymbol{T}^{-1}\,\boldsymbol{\Delta}\,\boldsymbol{T}$$

where $\boldsymbol{T}$ is orthogonal and $\boldsymbol{\Delta}$ is diagonal. This puts $\boldsymbol{D}$ into the form

$$\boldsymbol{D} \,=\, \underbrace{\boldsymbol{R}\boldsymbol{T}^{-1}}_{\text{orthogonal}}\,\boldsymbol{\Delta}\,\underbrace{\boldsymbol{T}}_{\text{orthogonal}}\,.$$

As an orthogonal (2×2)-matrix is nothing but a rotation matrix, this gives us the following parametrisation for the matrix $\boldsymbol{D}$:

$$\boldsymbol{D} \;=\; \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} .$$

Multiplication with the inverse of the first rotation matrix results in

$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \boldsymbol{D} \;=\; \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} .$$

This matrix equation can be applied to the "two-column vector" whose components are the vector fields E_1 and E_2,

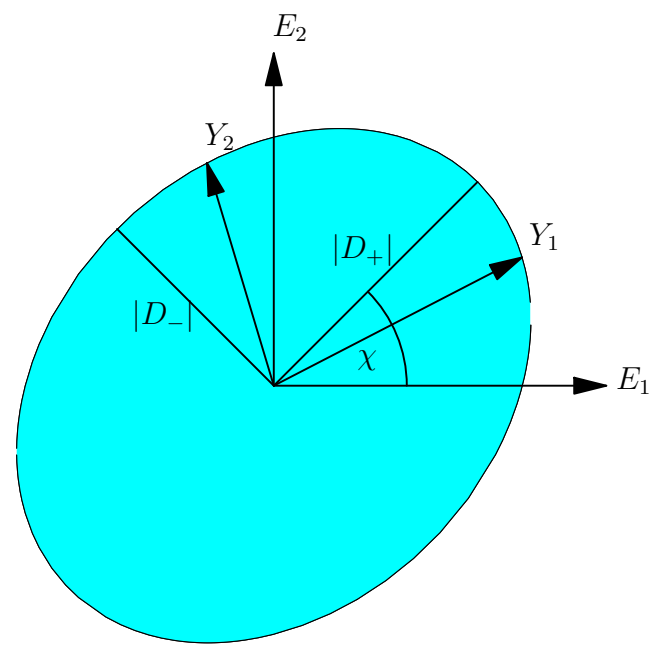
$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \simeq \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

where $\simeq$ means equality up to multiples of X. This gives us two vector equations,

$$Y_+ \;:=\; \cos\psi\, Y_1 \;+\; \sin\psi\, Y_2 \;\simeq\; D_+ \cos\chi\, E_1 \;+\; D_+ \sin\chi\, E_2 \;,$$

$$Y_- \;:=\; -\sin\psi\, Y_1 \;+\; \cos\psi\, Y_2 \;\simeq\; -\, D_- \sin\chi\, E_1 \;+\; D_- \cos\chi\, E_2 \;.$$

From these equations we read that the eigenvalues D_+ and D_- determine the semi-major and the semi-minor axis of the bundle and the angle χ gives the rotation of these axes with respect to the Sachs bein. In this way D_+, D_- and χ determine the shape of the bundle. The angle ψ is less relevant; it just tells how Y_1 and Y_2 are rotated with respect to the principal axes.



With $\boldsymbol{D}$ parametrised by D_+, D_-, χ and ψ, the matrix Jacobi equation has now become a coupled system of second order differential equations for these four scalar quantities. It is convenient to reduce this to first order form. To that end we introduce the socalled *deformation matrix*

$$\boldsymbol{S} \;=\; \boldsymbol{D}^{-1} \frac{d}{ds} \boldsymbol{D} \;.$$

With

$$\frac{d}{ds}\boldsymbol{D} \;=\; \boldsymbol{D}\boldsymbol{S} \;, \qquad (D)$$

the matrix Jacobi equation reads

$$\boldsymbol{D}\,\boldsymbol{Z} \;=\; \frac{d}{ds}(\boldsymbol{D}\boldsymbol{S}) \;=\; \boldsymbol{D}\,(\frac{d}{ds}\boldsymbol{S}) \;+\; (\frac{d}{ds}\boldsymbol{D})\,\boldsymbol{S} \;=\; \boldsymbol{D}\,(\frac{d}{ds}\boldsymbol{S}) \;+\; \boldsymbol{D}\,\boldsymbol{S}\,\boldsymbol{S} \;.$$

After multiplication with $\boldsymbol{D}^{-1}$ from the left, we find

$$\frac{d}{ds}\boldsymbol{S} + \boldsymbol{S}\,\boldsymbol{S} = \boldsymbol{Z}\,. \qquad (S)$$

This is the *Sachs equation* in matrix form. The two first-order matrix differential equations (D) and (S) are equivalent to the second-order matrix Jacobi equation.

We decompose $\boldsymbol{S}$ into antisymmetric part, trace part and symmetric trace-free part,

$$\boldsymbol{S} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{pmatrix}\,.$$

ω gives the rotation of the bundle, θ gives the expansion and (σ_1, σ_2) gives the shear. With this notation, (D) reads

$$\frac{d}{ds}\begin{pmatrix} \theta+\sigma_1 & -\omega+\sigma_2 \\ \omega+\sigma_2 & \theta-\sigma_1 \end{pmatrix} + \begin{pmatrix} (\theta+\sigma_1)^2+\sigma_2^2-\omega^2 & -2\theta\omega+2\theta\sigma_2 \\ 2\theta\omega+2\theta\sigma_2 & (\theta-\sigma_1)^2+\sigma_2^2-\omega^2 \end{pmatrix} =$$
$$= \begin{pmatrix} \Phi_{00}+\mathrm{Re}(\psi_0) & \mathrm{Im}(\psi_0) \\ \mathrm{Im}(\psi_0) & \Phi_{00}-\mathrm{Re}(\psi_0) \end{pmatrix}\,.$$

This matrix equation gives us four scalar equations

$$\frac{d}{ds}\theta \pm \frac{d}{ds}\sigma_1 + \theta^2 \pm 2\,\theta\,\sigma_1 + \sigma_1^2 + \sigma_2^2 - \omega^2 = \Phi_{00} \pm \mathrm{Re}(\psi_0)\,, \quad (A_\pm)$$

$$\pm\frac{d}{ds}\omega + \frac{d}{ds}\sigma_2 \pm 2\,\theta\,\omega + 2\,\theta\,\sigma_2 = \mathrm{Im}(\psi_0)\,, \quad (B_\pm)$$

We write the complex combination $(A_\pm) \pm i(B_\pm)$ of these equations.

$$\frac{d}{ds}(\theta+i\omega) \pm \frac{d}{ds}(\sigma_1+i\sigma_2) + \theta^2 + 2\,i\,\theta\,\omega \pm 2\,\theta\,(\sigma_1+i\sigma_2) + \sigma_1^2 + \sigma_2^2 - \omega^2 = \Phi_{00} \pm \psi_0\,. \quad (S_\pm)$$

After introducing the complex *optical scalars*

$$\varrho = \theta + i\,\omega\,, \qquad \sigma = \sigma_1 + i\,\sigma_2\,,$$

the equations $(S_+)+(S_-)$ and $(S_+)-(S_-)$ read

$$\frac{d}{ds}\varrho = -\,\varrho^2 - |\sigma|^2 + \Phi_{00}\,, \qquad (S1)$$

$$\frac{d}{ds}\sigma = -\left(\varrho+\overline{\varrho}\right)\sigma + \psi_0\,. \qquad (S2)$$

This system of first-order differential equations is known as the *Sachs equations*. It is equivalent to the matrix equation (S). By the first Sachs equation, the Ricci-tensor term Φ_{00} influences the real part of ϱ, i.e., the expansion. This effect is known as *Ricci focussing*. Note, however, that the conformal curvature term ψ_0 produces shear, by the second Sachs equation, and that the shear also influences the expansion, by the first Sachs equation. In this indirect way, ψ_0 also has a focussing effect.

Similarly to the matrix equation (S), we can also decompose the matrix equation (D) into two complex equations. We use the parametrisation of $\boldsymbol{D}$ in terms of D_+, D_-, χ and ψ, and we use the parametrisation of $\boldsymbol{S}$ in terms of θ, σ_1, σ_2 and ω. Then the equation (D) reads

$$\frac{d}{ds}\left\{\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix}\right\} =$$

$$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix}\begin{pmatrix}\theta+\sigma_1 & -\omega+\sigma_2\\ \omega+\sigma_2 & \theta-\sigma_1\end{pmatrix}.$$

On the left-hand side, we use the product rule; on the right-hand side we multiply out the the last three matrices.

$$\frac{d\psi}{ds}\begin{pmatrix}-\sin\psi & -\cos\psi\\ \cos\psi & -\sin\psi\end{pmatrix}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix} +$$

$$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\left\{\frac{d}{ds}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\right\}\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix} +$$

$$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\begin{pmatrix}-\sin\chi & \cos\chi\\ -\cos\chi & -\sin\chi\end{pmatrix}\frac{d\chi}{ds} =$$

$$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}D_+\Big(\cos\chi(\theta+\sigma_1)+\sin\chi(\omega+\sigma_2)\Big) & D_+\Big(\cos\chi(-\omega+\sigma_2)+\sin\chi(\theta-\sigma_1)\Big)\\ D_-\Big(-\sin\chi(\theta+\sigma_1)+\cos\chi(\omega+\sigma_2)\Big) & D_-\Big(\sin\chi(\omega-\sigma_2)+\cos\chi(\theta-\sigma_1)\Big)\end{pmatrix}.$$

We multiply from the left by $\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}$ and from the right by $\begin{pmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{pmatrix}$.

Then we get

$$\frac{d\psi}{ds}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix} + \frac{d}{ds}\begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix} + \begin{pmatrix}D_+ & 0\\ 0 & D_-\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{d\chi}{ds} =$$

$$\begin{pmatrix}D_+\Big(\theta+\sigma_1\cos(2\chi)+\sigma_2\sin(2\chi)\Big) & D_+\Big(-\omega-\sigma_1\sin(2\chi)+\sigma_2\cos(2\chi)\Big)\\ D_-\Big(\omega-\sigma_1\sin(2\chi)+\sigma_2\cos(2\chi)\Big) & D_-\Big(\theta-\sigma_1\cos(2\chi)-\sigma_2\sin(2\chi)\Big)\end{pmatrix}$$

where we have used the familiar trigonometric identities

$$\cos^2\chi-\sin^2\chi=\cos(2\chi) \qquad \text{and} \qquad 2\sin\chi\cos\chi=\sin(2\chi)\,.$$

The last matrix equation gives 4 real scalar equations.

$$\frac{dD_+}{ds} \;=\; D_+\,\big(\theta + \sigma_1\cos(2\chi) + \sigma_2\sin(2\chi)\big)\;,$$

$$D_+\,\frac{d\chi}{ds} \;-\; D_-\,\frac{d\psi}{ds} \;=\; D_+\,\big(-\,\omega - \sigma_1\sin(2\chi) + \sigma_2\cos(2\chi)\big)\;,$$

$$\frac{dD_-}{ds} \;=\; D_-\,\big(\theta - \sigma_1\cos(2\chi) - \sigma_2\sin(2\chi)\big)\;,$$

$$D_-\,\frac{d\chi}{ds} \;-\; D_+\,\frac{d\psi}{ds} \;=\; D_-\,\big(-\,\omega + \sigma_1\sin(2\chi) - \sigma_2\cos(2\chi)\big)\;.$$

Using the optical scalars $\varrho = \theta + i\omega$ and $\sigma = \sigma_1 + i\sigma_2$ these equations can be conveniently comprised into the following complex form.

$$\frac{dD_\pm}{ds} \;+\; i\,D_\pm\,\frac{d\chi}{ds} \;-\; D_\mp\,\frac{d\psi}{ds} \;=\; D_\pm\big(\overline{\varrho} \;\pm\; \sigma\,e^{-2i\chi}\big)\;. \qquad\qquad (D_\pm)$$

This result clearly demonstrates the advantage of the $D_\pm$ notation which might have appeared somewhat unmotivated until now. (This notation goes back to R. Kantowski.)

The two complex first-order equations (D_+) and (D_-) and the two complex first-order equation $(S1)$ and $(S2)$ are equivalent to the second-order matrix Jacobi equation

$$\frac{d}{ds^2}\boldsymbol{D} \;=\; \boldsymbol{D}\,\boldsymbol{Z}\;.$$

Before working out some examples and demonstrating the relevance of this formalism to image magnification and deformation we will derive a conservation law from the matrix Jacobi equation.

Claim: Any two bundles $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ along the same lightlike geodesic satisfy

$$\Big(\frac{d}{ds}\boldsymbol{D}_1\Big)\,\boldsymbol{D}_2^T \;-\; \boldsymbol{D}_1\,\Big(\frac{d}{ds}\boldsymbol{D}_2^T\Big) \;=\; \textbf{constant}$$

where $(\,\cdot\,)^T$ denotes the transpose of a matrix.

Proof:

$$\frac{d}{ds}\Big\{\Big(\frac{d}{ds}\boldsymbol{D}_1\Big)\,\boldsymbol{D}_2^T \;-\; \boldsymbol{D}_1\,\Big(\frac{d}{ds}\boldsymbol{D}_2^T\Big)\Big\} \;=$$

$$\Big(\frac{d^2}{ds^2}\boldsymbol{D}_1\Big)\,\boldsymbol{D}_2^T \;+\; \cancel{\Big(\frac{d}{ds}\boldsymbol{D}_1\Big)\Big(\frac{d}{ds}\boldsymbol{D}_2^T\Big)} \;-\; \cancel{\Big(\frac{d}{ds}\boldsymbol{D}_1\Big)\Big(\frac{d}{ds}\boldsymbol{D}_2^T\Big)} \;-\; \boldsymbol{D}_1\,\Big(\frac{d^2}{ds^2}\boldsymbol{D}_2^T\Big) \;=$$

$$\boldsymbol{D}_1\,\boldsymbol{Z}\,\boldsymbol{D}_2^T \;-\; \boldsymbol{D}_1\,\big(\boldsymbol{D}_2\,Z\big)^T \;=\; \boldsymbol{D}_1\big(\boldsymbol{Z} - \boldsymbol{Z}^T\big)\boldsymbol{D}_2^T\;.$$

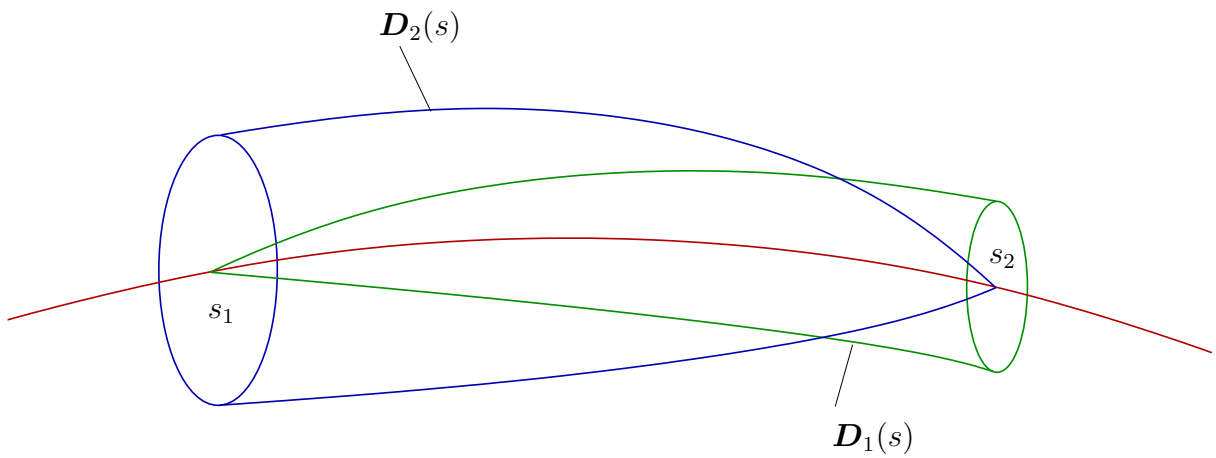
By symmetries of the curvature tensor, the optical tidal matrix is symmetric, $\boldsymbol{Z} = \boldsymbol{Z}^T$, so the last expression is indeed zero. $\square$

This result has two important consequences

- Consider two bundles $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ along the same lightlike geodesic with the properties that $\boldsymbol{D}_1$ has a vertex at the affine parameter value s_1 and $\boldsymbol{D}_2$ has a vertex at the affine parameter value s_2, i.e.

$$\boldsymbol{D}_1(s_1) = \mathbf{0}\,, \qquad \frac{d}{ds}\boldsymbol{D}_1(s_1) = \mathbf{1}\,,$$

$$\boldsymbol{D}_2(s_2) = \mathbf{0}\,, \qquad \frac{d}{ds}\boldsymbol{D}_2(s_2) = \mathbf{1}\,.$$



Then the conservation law

$$\left\{\left(\frac{d}{ds}\boldsymbol{D}_1\right)\boldsymbol{D}_2^T \,-\, \boldsymbol{D}_1\left(\frac{d}{ds}\boldsymbol{D}_2^T\right)\right\}\Big|_{s_1} \,=\, \left\{\left(\frac{d}{ds}\boldsymbol{D}_1\right)\boldsymbol{D}_2^T \,-\, \boldsymbol{D}_1\left(\frac{d}{ds}\boldsymbol{D}_2^T\right)\right\}\Big|_{s_2}$$

implies that

$$\mathbf{1}\,\boldsymbol{D}_2^T(s_1) \,-\, \mathbf{0}\,\frac{d}{ds}\boldsymbol{D}_2^T(s_1) \,=\, \frac{d}{ds}\boldsymbol{D}_1(s_2)\,\mathbf{0} \,-\, \boldsymbol{D}_1(s_2)\,\mathbf{1}\,,$$

hence

$$\boldsymbol{D}_2^T(s_1) \,=\, -\,\boldsymbol{D}_1(s_2)\,.$$

This remarkable relation is known as the "reciprocity law". It was proven by I. Etherington, Philos. Mag. and J. of Science 15, 761 (1933). We will later discuss important consequences of the reciprocity law for distance measures that are used, in particular, in cosmology.

- The conservation law is, of course, also true in the case that $\boldsymbol{D}_1 = \boldsymbol{D}_2 = \boldsymbol{D}$. In this special case it reads

$$\left(\frac{d}{ds}\boldsymbol{D}\right)\boldsymbol{D}^T \;-\; \boldsymbol{D}\left(\frac{d}{ds}\boldsymbol{D}^T\right) \;=\; \textbf{constant}\;,$$

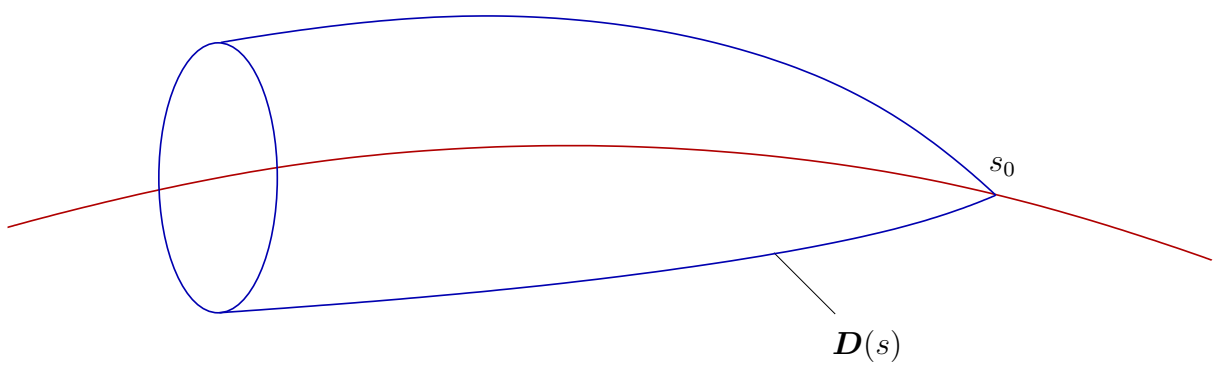
$$\boldsymbol{D}\,\boldsymbol{S}\,\boldsymbol{D}^T \;-\; \boldsymbol{D}\left(\boldsymbol{D}\,\boldsymbol{S}\right)^T \;=\; \textbf{constant}\;,$$

$$\boldsymbol{D}\,\boldsymbol{S}\,\boldsymbol{D}^T \;-\; \boldsymbol{D}\,\boldsymbol{S}^T\,\boldsymbol{D}^T \;=\; \textbf{constant}\;,$$

$$\boldsymbol{D}\big(\boldsymbol{S} - \boldsymbol{S}^T\big)\,\boldsymbol{D}^T \;=\; \textbf{constant}\;,$$

$$\boldsymbol{D}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\boldsymbol{D}^T \;=\; \textbf{constant}\;.$$

Now assume that the bundle has a vertex, i.e., that at one value of the affine parameter $s = s_0$ the matrix $\boldsymbol{D}$ is the zero matrix. (Such a bundle is called a "homocentric bundle" or a "pencil".)



Then the left-hand side of the above equation is the zero matrix at $s = s_0$; as it is constant, it must then be the zero matrix for all values of s. However, the matrix $\boldsymbol{D}$ was assumed to have a non-zero determinant almost everywhere (recall that we assumed that the bundle has a two-dimensional, non-degenerate cross-section for almost all values of s). So the left-hand side of the above equation can be identically zero only if ω is identically zero. We have thus proven the following result: *A homocentric bundle is necessarily twist-free.* For applications to lensing, in most cases one considers bundles with a vertex at a (pointlike) observer or at a (pointlike) light source. This is the reason why the twist of light bundles is rather irrelevant in view of applications to lensing.

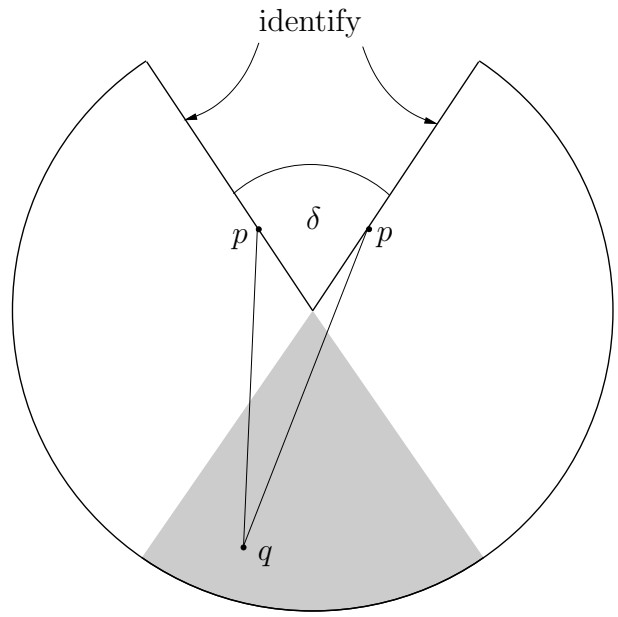
2.8 Example: Spacetime of a straight string

We modify the Minkowski metric, written in cylindrical polar cooordinates, by a factor k^2 in front of the angular part:

$$g \;=\; -\,c^2\,dt^2 \;+\; dz^2 \;+\; d\rho^2 \;+\; k^2\,\rho^2\,d\varphi^2\;.$$

The coordinates have their usual range, $t \in \mathbb{R}$, $z \in \mathbb{R}$, $\rho \in \mathbb{R}_+$ and $\varphi \in \mathbb{R} \bmod 2\pi$.

For $0 < k < 1$, the geometry in the plane t = constant, z = constant can be visualised in the following way. While the radial length measure is the same as in flat space, the circumference of a circle at radius ρ is no longer equal to $2\pi\rho$, as in flat space, but rather equal to $2\pi k\rho$. There is, thus, a deficit angle $\delta = 2\pi(1-k)$. (For $k > 1$ we get a negative δ, i.e., a surplus angle.) We get a valid representation of this geometry by taking a (flat) sheet of paper, cutting a piece of opening angle δ away and gluing the two boundaries together. What we get is a cone, i.e. a geometry that is flat everywhere but has a ("conic") singularity at $\rho = 0$.



As the (t, z)-plane is obviously flat, this observation implies that the entire metric is flat except at $\rho = 0$ where it is singular. Here "flat" is a local concept, meaning that the curvature tensor vanishes. The global structure of the metric differs from Minkowski spacetime in two respects. First, there is the conic singularity at $\rho = 0$ which cannot be removed by a coordinate transformation. Second, the circumference of a circle at radius ρ is not equal to $2\pi\rho$ but rather to $2\pi k\rho$.

As the curvature tensor vanishes, the metric satisfies the vacuum field equation without a cosmological constant, $R_{\mu\nu} = 0$, everywhere away from the axis $\rho = 0$ where the conic singularity sits. So it is to be interpreted as the gravitational field of a souce that is concentrated on a line and surrounded by vacuum. Such a source is called a (cosmic) *string*. The existence of cosmic strings was predicted first by T. Kibble: J. Phys. A 9, 1387 (1976) as the result of phase transitions in the early universe. They have not been detected until now. The infinitely long, straight and static string considered here is, of course, a highly idealised model of a realistic cosmic string. This particular "flat spacetime with a gravitational field" was first considered by L. Marder: Proc. Roy. Soc. London A 252, 45 (1959).

With the cone cut open and flattened, we can immediately deduce the multiple-imaging properties in this string spacetime. Light rays (geodesics of the Fermat metric) are then represented by straight lines. For the case $0.5 < k < 1$ we read from the picture that the point p can be connected by two light rays with any point in the shaded region, and by just one light ray with any point in the unshaded region. So the string produces a double image in the case that observer and light source are sufficiently well aligned.

The double galaxy CSL-1 was considered as a candidate for lensing by a cosmic string for a while, but then it was found that it actually is a physical pair of galaxies, not a double-image of one galaxy, see E. Agol, C. Hogan, R. Plotkin: Phys.Rev. D73, 087302 (2006).

As the string spacetime has vanishing curvature tensor, the optical tidal matrix $\boldsymbol{Z}$ is zero, along any light ray. Hence, the matrix Jacobi equation simply reads

$$\frac{d^2}{ds^2}\boldsymbol{D} \,=\, \mathbf{0}$$

which obviously has the solution

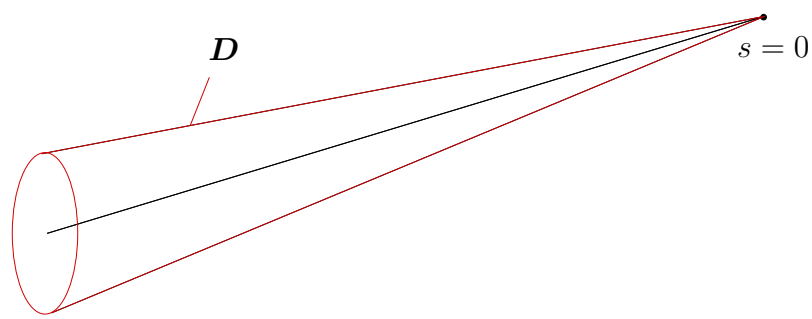
$$\boldsymbol{D}(s) \,=\, \boldsymbol{D}(0) \,+\, s\Big(\frac{d}{ds}\boldsymbol{D}\Big)(0)\,.$$

For a bundle with a vertex at $s = 0$,

$$\boldsymbol{D}(0) \,=\, \mathbf{0} \qquad \text{and} \qquad \Big(\frac{d}{ds}\boldsymbol{D}\Big)(0) \,=\, \mathbf{1}\,,$$

the solution further simplifies to

$$\boldsymbol{D}(s) \,=\, s\,\mathbf{1}\,.$$



If we parametrise $\boldsymbol{D}$ by the shape parameters D_+, D_-, χ and ψ, the last equation requires

$$D_+(s) \,=\, s\,, \qquad D_-(s) \,=\, s\,, \qquad \chi(s) \,-\, \psi(s) \,=\, 0\,.$$

The deformation matrix $\boldsymbol{S}$ takes the form

$$\boldsymbol{S}(s) \,=\, \boldsymbol{D}(s)^{-1}\,\frac{d}{ds}\boldsymbol{D}(s) \,=\, \frac{1}{s}\,\mathbf{1}\,\frac{d}{ds}\Big(s\,\mathbf{1}\Big) \,=\, \frac{1}{s}\,\mathbf{1}\,.$$

Decomposition into expansion, shear and twist,

$$\boldsymbol{S}(s) \,=\, \begin{pmatrix} \theta(s) + \sigma_1(s) & -\omega(s) + \sigma_2(s) \\ \omega(s) + \sigma_2(s) & \theta(s) - \sigma_1(s) \end{pmatrix}\,,$$

results in

$$\theta(s) \,=\, \frac{1}{s}\,, \qquad \omega(s) \,=\, \sigma_1(s) \,=\, \sigma_2(s) \,=\, 0\,.$$

These equations hold in any spacetime with vanishing curvature tensor, whatever the global structure of the spacetime may be.

2.9 Distance measures and brightness of images

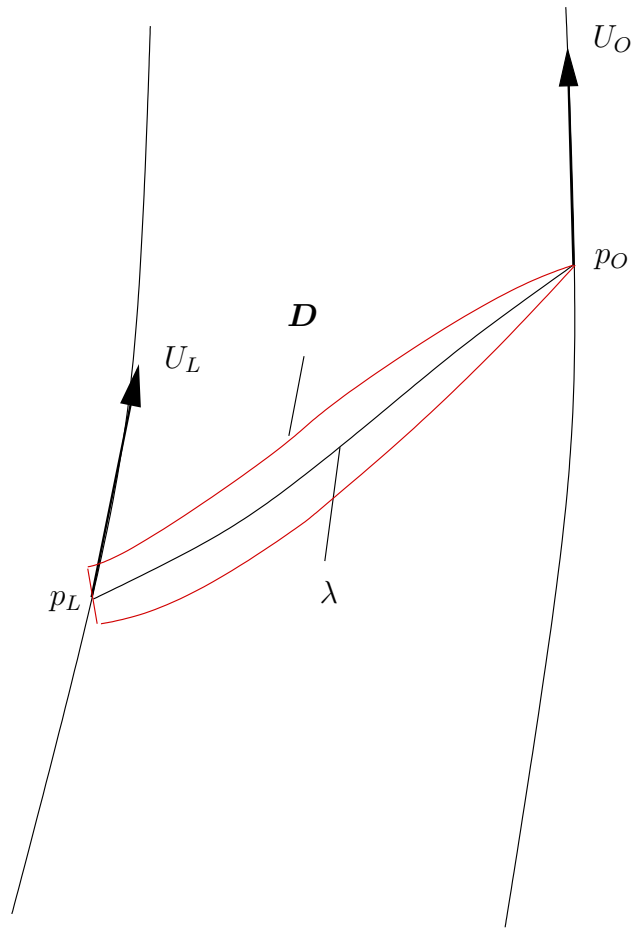
The first notion we want to discuss is the socalled *area distance* (or *angular diameter distance*). It is based on the intuitive idea that

$$\text{area at light source} \;=\; (\text{solid angle at observer}) \times \text{distance}^2 \;.$$

To translate this idea into a mathematical formula, we consider an observer at p_O with 4-velocity U_O and a light ray λ parametrised in a past-oriented sense such that $\lambda(0) = p_O$ and

$$g\big(U_O, X_O\big) \;=\; c$$

where $X_O = \dot\lambda(0)$ is the tangent vector to the light ray at the observer. The last condition fixes the ambiguity in the choice of the affine parametrisation. Assume that there is a light source at $p_L = \lambda(s_L)$ with 4-velocity U_L.



Let $\boldsymbol{D}$ be the solution to the matrix Jacobi equation with initial condition

$$\boldsymbol{D}(0) = \mathbf{0}\,, \qquad \Big(\frac{d}{ds}\boldsymbol{D}\Big)(0) \;=\; \mathbf{1}\,.$$

This gives us a bundle with vertex at the observer. With the help of the shape parameters D_+ and D_- of this bundle we define, according to the above idea, the *area distance* as

$$D_{\text{area}}(s_L) \;=\; \sqrt{\big|D_+(s_L)D_-(s_L)\big|}\;.$$

In flat spacetimes we have (recall Sect.2.8)

$$D_+(s) \;=\; s \qquad \text{and} \qquad D_-(s) \;=\; s\;,$$

hence

$$D_{\text{area}}(s) \;=\; s\;.$$

In curved spacetimes, however, D_{area} need not be monotonic. It may even take the value $D_{\text{area}} = 0$ for $s \neq 0$, if the homocentric bundle from p_O is refocused into p_L. This happens, e.g., in a spatially closed Robertson-Walker spacetime if observer and light source are antipodal.

With the help of the area distance, we will now derive the socalled *focussing theorem*. As a preparation, we prove the following result.

Claim : The area distance satisfies the differential equation

$$\frac{d}{ds}D_{\text{area}} \;=\; \theta\, D_{\text{area}}\;.$$

Proof : As a homocentric bundle is twist-free, $\omega = 0$, the differential equations $(D_\pm)$ from p.50 read

$$\frac{dD_\pm}{ds} + i\,D_\pm \frac{d\chi}{ds} - D_\mp \frac{d\psi}{ds} = D_\pm\big(\theta \pm \sigma\, e^{-2i\chi}\big)\,.$$

Taking the real part gives us the following two equations.

$$\frac{dD_+}{ds} = D_+\Big(\theta + \mathrm{Re}\big(\sigma\, e^{-2i\chi}\big)\Big)\,,$$

$$\frac{dD_-}{ds} = D_-\Big(\theta - \mathrm{Re}\big(\sigma\, e^{-2i\chi}\big)\Big)\,.$$

Now multiply the first equation with D_- and the second one with D_+. The sum of the resulting two equations gives

$$\frac{d}{ds}\big(D_+D_-\big) = 2\,D_+D_-\theta\,,$$

hence

$$\frac{d}{ds}D_{\mathrm{area}}^2 = 2\,\theta\,D_{\mathrm{area}}^2\,,$$

$$\cancel{2}\,\cancel{D_{\mathrm{area}}}\,\frac{d}{ds}D_{\mathrm{area}} = \cancel{2}\,\theta\,D_{\mathrm{area}}^{\cancel{2}}\,,$$

□

By differentiating another time with respect to s, we find

$$\frac{d^2}{ds^2}D_{\mathrm{area}} = \frac{d}{ds}\Big(\theta\,D_{\mathrm{area}}\Big) = \frac{d\theta}{ds}\,D_{\mathrm{area}} + \theta\,\frac{dD_{\mathrm{area}}}{ds}\,.$$

For the first term we use the Sachs equation (S1).

$$\frac{d^2}{ds^2}D_{\mathrm{area}} = \Big(-\,\theta^2 - |\sigma|^2 - \frac{1}{2}\,R_{\mu\nu}\dot\lambda^\mu\dot\lambda^\nu\Big)\,D_{\mathrm{area}} + \theta^2\,D_{\mathrm{area}}\,.$$

With Einstein's field equation

$$R_{\mu\nu} - \frac{R}{2}\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = \kappa\,T_{\mu\nu}$$

the Ricci tensor can be expressed with the energy-momentum tensor, using the fact that $\dot\lambda$ is lightlike,

$$\frac{d^2}{ds^2}D_{\mathrm{area}} = \Big(-\cancel{\theta^2} - |\sigma|^2 - \frac{\kappa}{2}\,T_{\mu\nu}\dot\lambda^\mu\dot\lambda^\nu + \cancel{\theta^2}\Big)\,D_{\mathrm{area}}\,.$$

Assume that the *weak energy condition* is satisfied, i.e., that the energy density is positive for all observers,

$$T_{\mu\nu}V^\mu V^\nu \geq 0 \qquad \text{if} \quad g_{\mu\nu}V^\mu V^\nu < 0\,.$$

Then we have by continuity

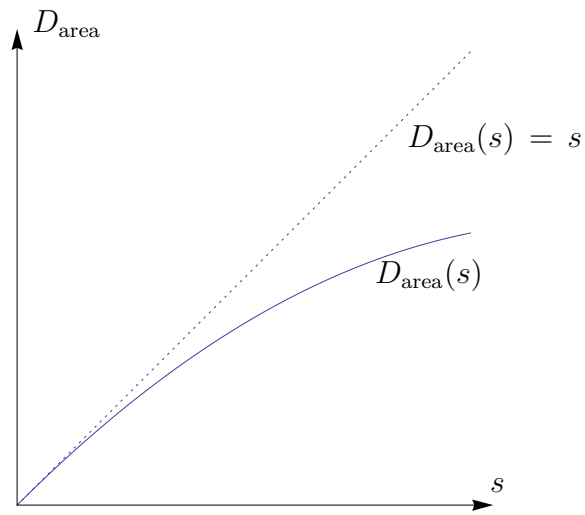
$$T_{\mu\nu}\dot{\lambda}^\mu\dot{\lambda}^\nu \;\geq\; 0$$

for the lightlike vector $\dot{\lambda}$, hence

$$\frac{d^2}{ds^2}D_{\text{area}} \;\leq\; 0\;.$$

This may be properly called the *focussing inequality*. Near $s = 0$, our initial conditions make sure that the graph of the function $D_{\text{area}}(s)$ is tangential to the flat-space solution $D_{\text{area}}(s) = s$. Now the focussing inequality says that the graph of this function can curve away from the flat-space solution only in the downward direction, i.e., that it stays *below* (or on) the flat-space solution.

We have thus proven the *focussing theorem*: If Einstein's field equation holds with an energy-momentum tensor that satisfies the weak energy condition, then gravity has a focussing effect (as opposed to a defocussing one).



Strictly speaking we have proven that, under the stated energy condition, the focussing inequality holds up to the first conjugate point, i.e., up to the first point $s > 0$ where $D_{\text{area}} = 0$ (if any). It does not necessarily hold beyond the first conjugate point because the definition of D_{area} involves a square root which is not differentiable at zero.

The definition of the area distance was based on the idea that the size of an object decreases with the square of its distance. Similarly, we could use the idea that the intensity of a light source decreases with the square of its distance, i.e.

$$\text{intensity} \;\sim\; (\text{solid angle at light source}) \times \text{distance}^2\;.$$

This leads to an analogous construction as we used for the area distance, but this time with a bundle that has a vertex at the light source. We call this bundle $\tilde{\boldsymbol{D}}$. By the reciprocity theorem, the cross-sectional area of the bundle $\tilde{\boldsymbol{D}}$ at the observer would be equal to the cross-sectional area of the bundle $\boldsymbol{D}$ at the light source if we would use the same parametrisation of the light ray for both constructions. However, as the solid angle at the light source is to be measured in the rest system of the light source, the condition $g(U_O, X_O) = c$ has now to be replaced by the condition $g(U_L, X_L) = c$ where X_L is the tangent vector to the light ray at the light source. Therefore, we define the *corrected luminosity distance* as

$$D'_{\text{lum}} \;=\; \frac{g(U_O, X_O)}{g(U_L, X_L)}\, D_{\text{area}}\;.$$

With the help of the general redshift formula (recall Section 2.5), this can be rewritten as

$$D'_{\text{lum}} \;=\; \big(1 + z\big)\, D_{\text{area}}\;.$$

If we think of the light source as emitting photons isotropically in the whole solid angle of 4π, then the inverse square of the corrected luminosity distance is proportional to the *number flux* (photons per area per time) at the observer. If one wants to determine the *energy flux*, one has to take into account that each photon undergoes a redshift. We have thus to replace the corrected luminosity distance $D'_{\rm lum}$ by the ("uncorrected") *luminosity distance*

$$D_{\rm lum} \;=\; (1+z)\, D'_{\rm lum} \;=\; (1+z)^2\, D_{\rm area}\;.$$

This quantity is related to the energy flux F at the observer by

$$F \;=\; \frac{L}{4\pi D_{\rm lum}^2}$$

where L is the (bolometric) luminosity of the light source.

Astronomers describe the brightness of images in terms of *magnitudes*. As human senses respond logarithmically to a physical stimulus ("Weber-Fechner law"), magnitudes are proportional to the log of the energy flux. The precise definition of magnitudes goes back to british astronomer N. Pogson (1829 – 1891) and reads, in our notation,

$$m \;=\; -\,2.5\log_{10}(L) \;+\; 2.5\log_{10}\!\big(D_{\rm lum}^2\big) \;+\; m_0$$

where m_0 is a universal constant. The factor $2.5 \approx \sqrt[5]{100}$ was chosen such that the scale approximately coincides with the classification of stars ("first magnitude" to "sixth magnitude") that was used in the star catalogue of Hipparchos (129 BC).

The above consideration gives the brightness of images for point sources. For extended sources it is appropriate to introduce the *surface brightness*

$$B \;=\; \frac{L}{\text{area at light source}}$$

and the *intensity* (or *irradiance*)

$$I \;=\; \frac{F}{\text{solid angle at observer}}\;.$$

Then

$$I \;=\; \frac{F}{\text{solid angle at observer}} \;=\; \frac{B}{4\pi D_{\rm lum}^2}\,\frac{(\text{area at light source})}{(\text{solid angle at observer})} \;=\; \frac{B\, D_{\rm area}^2}{4\pi D_{\rm lum}^2} \;=\; \frac{B}{4\pi(1+z)^4}\;.$$

Surprisingly, all distance measures have dropped out. For a source of known surface brightness, the intensity is completely determined by the redshift. Also note that the angle between the viewline and the surface of the light source does not enter. (This corresponds to *Lambert's law* in ordinary optics.) The formula was derived under the assumption that all the light comes from the surface of the light source and that each surface element radiates isotropically. This is not true, e.g., for our Sun where the radiation is produced in a layer of finite thickness (the socalled *photosphere*). For this reason, in the case of the Sun the relation between intensity and surface brightness does depend on the viewing angle. This leads to the phenomenon of "Randverdunklung", i.e., the Sun appears considerably darker near the rim than near the centre.

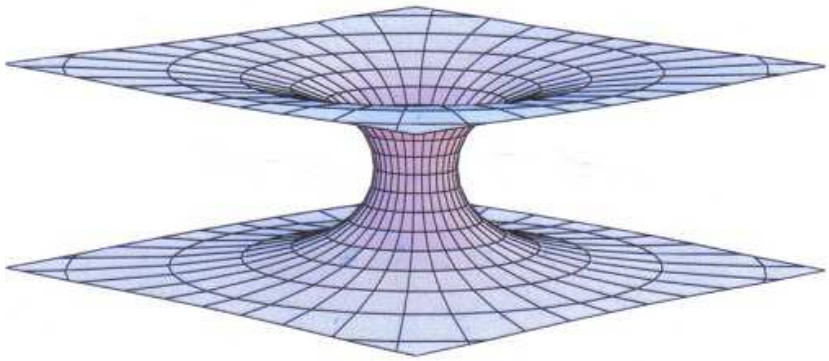
2.10 Example: Ellis wormhole

The best known example of a wormhole is the Ellis wormhole. Its metric reads

$$g \;=\; -\,c^2\,dt^2 \;+\; dr^2 \;+\; (r^2+a^2)\,\big(d\vartheta^2 + \sin^2\!\vartheta\,d\varphi^2\big)$$

with a constant $a > 0$. The coordinate ranges are $t \in \mathbb{R}$, $r \in \mathbb{R}$, $(\vartheta, \varphi) \in S^2$. This metric was first considered by H. Ellis [J. Math. Phys. 14, 104 (1973)] who called it a "drainhole". More precisely, the given metric is a special case of a class of wormhole metrics discussed in the Ellis paper.

Note that $r = 0$ is not a point but a regular sphere with area $4\pi a^2$. The 2-surface $t =$ constant, $\vartheta = \pi/2$ can be visualised as a tube that opens out into asymptotically flat ends for $r \to \infty$ and for $r \to -\infty$. The minimal diameter of this tube is at $r = 0$ which is known as the "neck" or the "throat" of the wormhole.



The name "wormhole", which was coined by John Wheeler, refers to the topology that results by gluing the two asymptotic ends together. However, we do *not* make this identification in the following.

We will demonstrate below that the Ellis wormhole is *traversible*, i.e., an observer can move through the throat, from one asymptotic end to the other, with subluminal velocity.

By symmetry, $r = 0$ is a *light sphere* (or *photon sphere*), i.e., a light ray sent tangentially to this sphere stays on the sphere. This light sphere is unstable, just as the light sphere at $r = 3r_S/2$ in the Schwarzschild spacetime, so light rays can asymptotically spiral towards it. Hence, the qualitative lensing features of an Ellis wormhole are very similar to the qualitative lensing features of a Schwarzschild black hole. The radii $r = \infty$, $r = 0$ and $r = -\infty$ in the Ellis spacetime correspond to the radii $r = \infty$, $r = 3r_S/2$, $r = r_S$ in the Schwarzschild spacetime. In particular, an Ellis wormhole has a shadow, just as the Schwarzschild black hole (or an ultracompact Schwarzschild star), whose boundary corresponds to light rays that asymptotically spiral towards the light sphere at the neck of the wormhole, see Worksheet 8. Also, an Ellis wormhole produces two families of infinitely many images of every light source, just as shown in the picture on p.28 for the Schwarzschild case.

The Ricci tensor of the Ellis spacetime has only one non-vanishing component,

$$R_{rr} = \frac{-2a^2}{(r^2+a^2)^2}\,.$$

We will now show that, by Einstein's field equation, this gives a negative energy density for some observers. Assume that the observer moves radially, i.e., that his four-velocity is of the form

$$V \;=\; \alpha\partial_t \;+\; \beta\partial_r\;.$$

The normalisation condition on the four-velocity requires

$$g_{\mu\nu}V^\mu V^\nu = \alpha^2 g_{tt} + \beta^2 g_{rr} = -\alpha^2 c^2 + \beta^2 = -c^2$$

which is equivalent to

$$\beta^2 \;=\; (\alpha^2 - 1)\, c^2 \,.$$

α can be chosen arbitrarily. We assume that Einstein's field equation holds, allowing for a non-zero cosmological constant. Then the energy density of our observer is

$$T_{\mu\nu}V^\mu V^\nu \;=\; \frac{1}{\kappa}\Big(R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu}\Big)V^\mu V^\nu \;=$$

$$= \frac{1}{\kappa}\Big(R_{rr}V^r V^r - \frac{R_{\sigma\tau}g^{\sigma\tau}}{2}g_{\mu\nu}V^\mu V^\nu + \Lambda g_{\mu\nu}V^\mu V^\nu\Big) \;=$$

$$= \frac{1}{\kappa}\Big(R_{rr}\beta^2 - \frac{R_{rr}g^{rr}}{2}(-c^2) + \Lambda(-c^2)\Big) \;=$$

$$= \frac{1}{\kappa}\Big(\, R_{rr}\,\big(\beta^2 + \frac{c^2}{2}\big) \;-\; \Lambda\, c^2\Big) \;=$$

$$= \frac{1}{\kappa}\Big(\frac{-2a^2}{(r^2+a^2)^2}\,\big(\alpha^2 - \frac{1}{2}\big)c^2 \;-\; \Lambda\, c^2\Big)\,.$$

This expression is negative for α sufficiently large, whatever Λ may be. One says that the *weak energy condition* holds in a spacetime if the energy density is non-negative for all observers. We have thus shown that the Ellis wormhole violates the weak energy condition. It is a matter of debate if negative energy densities exist. In any case, all known kind of matter has a non-negative energy density. So we see that some sort of "exotic matter" is needed to produce an Ellis wormhole. – It can be shown that for a rather large class of traversible wormholes, known as the Morris-Thorne class, the energy density must be negative for some observers. Here "traversible" means that an observer can move through the throat from one asymptotic end to the other at subluminal speed. It is easy to see that the Ellis wormhole is, indeed, traversible: For a radial light ray we have

$$0 \;=\; -\,c^2\,dt^2 \;+\; dr^2 \qquad \Longleftrightarrow \qquad \frac{dr}{dt} \;=\; \pm\, c\,.$$

Hence, the travel time T for such a light ray from $r = r_O > 0$ to $r = r_L < 0$ is given by

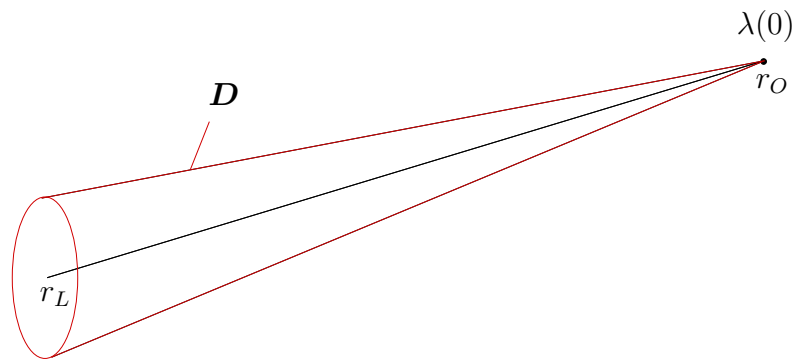
$$c\int_0^T dt \;=\; \int_{r_L}^{r_O} dr \qquad \Longleftrightarrow \qquad c\,T = r_O - r_L = |r_O| + |r_L|\,.$$

As a massive body can follow a light ray arbitrarily closely, it can move from $r = r_O > 0$ to $r = r_L < 0$ in a finite travel time

$$T = \frac{1}{c}\Big(|r_O| + |r_L| + \varepsilon\Big)$$

with arbitrarily small $\varepsilon > 0$.

Let us now discuss the focussing theorem in the Ellis spacetime for the case that observer and light source are static and on the same radial line, $\vartheta_O = \vartheta_L$ and $\varphi_O = \varphi_L$.



$\boldsymbol{D}$ is the solution to the matrix differential equation

$$\frac{d^2}{ds^2}\boldsymbol{D} \,=\, \boldsymbol{D}\,\boldsymbol{Z}$$

with initial condition

$$\boldsymbol{D}(0) \,=\, \mathbf{0}\,, \qquad \Big(\frac{d}{ds}\boldsymbol{D}\Big)(0) \,=\, \mathbf{1}\,.$$

The affine parameter s is fixed by the condition

$$g\big(U_O, \dot\lambda(0)\big) \,=\, c\,.$$

We need to determine the relation between s and r along the past-oriented radial light ray λ connecting the observer and the light source. We assume $r_O > r_L$. Then the tangent vector $\dot\lambda$ must be of the form

$$\dot\lambda(s) \,=\, u(s)\,\partial_r \,+\, v(s)\,\partial_t$$

with $u(s) < 0$ and $v(s) < 0$, hence

$$0 \,=\, g\big(\dot\lambda(s), \dot\lambda(s)\big) \,=\, u(s)^2\, g_{rr} \,+\, v(s)^2\, g_{tt} \,=\, u(s)^2 \,-\, c^2\, v(s)^2\,,$$

$$c \,=\, g\big(U_O, \dot\lambda(0)\big) \,=\, g\big(\partial_t, u(0)\partial_r + v(0)\partial_t\big) \,=\, u(0)\, g_{tr} \,+ v(0)\, g_{tt} \,=\, -v(0)\, c^2\,.$$

As

$$E \,=\, -\frac{\partial\mathcal{L}}{\partial\dot t} \,=\, c^2\,\dot t \,=\, c^2 v$$

is a constant of motion,

$$v(s) \,=\, v(0) \,=\, -\frac{1}{c}\,,$$

$$u(s) \,=\, c\,v(s) \,=\, -1\,.$$

This results in

$$\dot\lambda(s) \,=\, -\,\partial_r \,-\, \frac{1}{c}\partial_t\,.$$

In particular, s and r are related along the light ray by

$$\frac{dr}{ds} \;=\; -1\;.$$

With the initial condition $r(0) = r_O$ integration yields

$$r(s) \;=\; r_O \;-\; s\;.$$

After these preparations, we now turn to the focussing equation,

$$\frac{d^2}{ds^2} D_{\text{area}} \;=\; \Big(\;-\; |\sigma|^2 \;-\; \frac{1}{2}\, R_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu \Big)\, D_{\text{area}}\;.$$

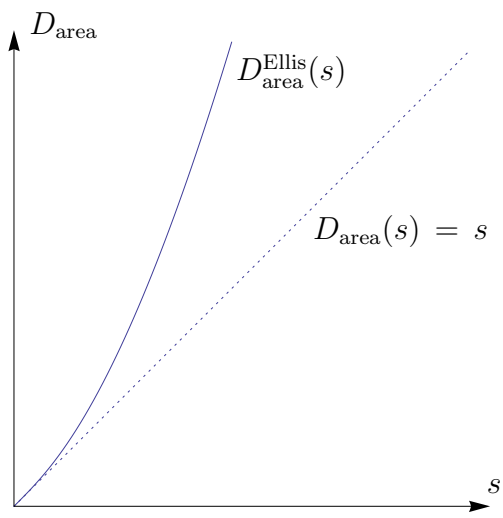
As the spacetime is rotationally symmetric about the radial light ray λ, the bundle $\boldsymbol{D}$ has a circular cross-section, hence $\sigma = 0$. The Ricci term is

$$R_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu \;=\; R_{rr} \dot{\lambda}^r \dot{\lambda}^r \;=\; \frac{-2a^2(-1)^2}{(r^2+a^2)^2} \;=\; \frac{-2a^2}{\big((r_O - s)^2 + a^2\big)^2}\;,$$

hence the focussing equation reduces to

$$\frac{d^2}{ds^2} D_{\text{area}} \;=\; \frac{a^2\, D_{\text{area}}}{\big((r_O - s)^2 + a^2\big)^2} \;>\; 0\;.$$

In contrast to spacetimes that satisfy the weak-energy condition, the Ellis geometry has a *defocussing* effect on the light bundle.



2.11 Image deformation

In the following we use the parametrisation of bundles $\boldsymbol{D}$ in terms of D_+, D_-, χ and ψ, see p.47. We define the *ellipticity* of the bundle as

$$\epsilon(s) \;=\; \Big(\frac{D_+(s)}{D_-(s)} \;-\; \frac{D_-(s)}{D_+(s)} \Big)\, e^{2i\chi(s)}\;.$$

The complex quantity ϵ gives the shape and the orientation of the elliptic bundle cross-section, as can be read from the picture on p.47. In particular, $\epsilon = 0$ holds if and only if $D_+^2 = D_-^2$, i.e., if and only if the bundle has a circular cross-section. $|\epsilon| \to \infty$ means that either $|D_+/D_-|$ or $|D_-/D_+|$ goes to infinity, i.e., that the ellipse degenerates into a line.

We will now derive a differential equation for ϵ. As we are mainly interested in homocentric bundles, which are necessarily twist-free, we restrict to the case that $\omega = 0$.

Claim: For a twist-free bundle ($\omega = 0$), the ellipticity ϵ satisfies the differential equation

$$\frac{d\epsilon}{ds} = \pm 2\,\sigma\,\sqrt{|\epsilon|^2 + 4}\,,$$

where the sign must be chosen in accordance with the sign of D_+D_-. (D_+D_- changes sign at every conjugate point of multiplicity one, recall Worksheet 6.)

Proof: We start out from the differential equations for D_+, D_-, χ and ψ which where combined into the complex equations ($D_\pm$) on p.50,

$$\frac{dD_\pm}{ds} + i\,D_\pm\,\frac{d\chi}{ds} - i\,D_\mp\,\frac{d\psi}{ds} = D_\pm\big(\overline{\varrho} \pm \sigma\,e^{-2i\chi}\big)\,.$$

For a twist-free bundle, the optical scalar $\varrho = \theta + i\omega$ reduces to the real quantity θ. We write the real parts of our differential equations,

$$\frac{dD_+}{ds} = D_+\big(\theta + \mathrm{Re}(\sigma\,e^{-2i\chi})\big)\,, \qquad (1)$$

$$\frac{dD_-}{ds} = D_-\big(\theta - \mathrm{Re}(\sigma\,e^{-2i\chi})\big)\,, \qquad (2)$$

and the imaginary parts,

$$D_+\,\frac{d\chi}{ds} - D_-\,\frac{d\psi}{ds} = D_+\mathrm{Im}\big(\sigma\,e^{-2i\chi}\big)\,,$$

$$D_-\,\frac{d\chi}{ds} - D_+\,\frac{d\psi}{ds} = -\,D_-\mathrm{Im}\big(\sigma\,e^{-2i\chi}\big)\,.$$

We divide the last two equations by D_- and D_+, respectively, and take the difference. This results in

$$\Big(\frac{D_+}{D_-} - \frac{D_-}{D_+}\Big)\frac{d\chi}{ds} = \Big(\frac{D_+}{D_-} + \frac{D_-}{D_+}\Big)\mathrm{Im}\big(\sigma\,e^{-2i\chi}\big)\,. \qquad (3)$$

Now we calculate

$$\frac{d\epsilon}{ds} = \frac{d}{ds}\Big\{\Big(\frac{D_+}{D_-} - \frac{D_-}{D_+}\Big)e^{2i\chi}\Big\} =$$

$$= e^{2i\chi}\Big\{\frac{1}{D_-}\frac{dD_+}{ds} - \frac{D_+}{D_-^2}\frac{dD_-}{ds} - \frac{1}{D_+}\frac{dD_-}{ds} + \frac{D_-}{D_+^2}\frac{dD_+}{ds} + 2i\frac{d\chi}{ds}\Big(\frac{D_+}{D_-} - \frac{D_-}{D_+}\Big)\Big\} =$$

$$= e^{2i\chi}\Big\{\frac{dD_+}{ds}\Big(\frac{1}{D_-} + \frac{D_-}{D_+^2}\Big) - \frac{dD_-}{ds}\Big(\frac{1}{D_+} + \frac{D_+}{D_-^2}\Big) + 2i\frac{d\chi}{ds}\Big(\frac{D_+}{D_-} - \frac{D_-}{D_+}\Big)\Big\}\,.$$

With equations (1), (2) and (3), this yields

$$\frac{d\epsilon}{ds} = e^{2i\chi}\Big\{\Big(\theta+\mathrm{Re}\big(e^{-2i\chi}\sigma\big)\Big)\Big(\frac{D_+}{D_-}+\frac{D_-}{D_+}\Big)-\Big(\theta-\mathrm{Re}\big(e^{-2i\chi}\sigma\big)\Big)\Big(\frac{D_-}{D_+}+\frac{D_+}{D_-}\Big)+2\,i\,\mathrm{Im}\big(e^{-2i\chi}\sigma\big)\Big(\frac{D_+}{D_-}+\frac{D_-}{D_+}\Big)\Big\} =$$

$$= e^{2i\chi}\Big\{2\,\mathrm{Re}\big(e^{-2i\chi}\sigma\big) + 2\,i\,\mathrm{Im}\big(e^{-2i\chi}\sigma\big)\Big\}\Big(\frac{D_+}{D_-}+\frac{D_-}{D_+}\Big) =$$

$$= \cancel{e^{2i\chi}}\,2\,\cancel{e^{-2i\chi}}\,\sigma\Big(\frac{D_+}{D_-}+\frac{D_-}{D_+}\Big)\,.$$

This completes the proof, since

$$\sqrt{|\epsilon|^2+4} = \sqrt{\Big(\frac{D_+}{D_-}-\frac{D_-}{D_+}\Big)^2+4} =$$

$$= \sqrt{\frac{D_+^2}{D_-^2}-2+\frac{D_-^2}{D_+^2}+4} == \sqrt{\frac{D_+^2}{D_-^2}+2+\frac{D_-^2}{D_+^2}} == \pm\Big(\frac{D_+}{D_-}+\frac{D_-}{D_+}\Big)\,.$$

□

Note that this differential equation admits the solution $\epsilon = 0$ if and only if the shear vanishes along the whole ray. Clearly, a bundle whose cross-section is everywhere circular must be shear-free.

We have now all the relevant equations at our disposal for determining the apparent shape of spherical objects: We have to solve the coupled system of differential equations

$$\frac{d\theta}{ds} = -\,\theta^2 - |\sigma|^2 - \Phi_{00}\,,$$

$$\frac{d\sigma}{ds} = 2\,\theta\,\sigma + \psi_0\,,$$

$$\frac{d\epsilon}{ds} = 2\,\sigma\,\sqrt{|\epsilon|^2+4}\,,$$

with initial conditions

$$\frac{1}{\theta}(0) = 0\,, \qquad \sigma(0) = 0\,, \qquad \epsilon(s_L) = 0\,.$$

The first two conditions have to hold for a bundle with a vertex at the observer position $s = 0$, recall Problem 2 of Worksheet 6, the third condition requires that the bundle has a circular cross-section at the position of the light source, $s = s_L$. After the solution $\big(\theta(s), \sigma(s), \epsilon(s)\big)$ to this initial value problem has been found, we can calculate the quantity $d\epsilon/ds(0)$ which gives us the apparent shape of the object.

Note that image deformation is produced by the conformal curvature term ψ_0. More precisely, ψ_0 produces shear, and the shear produces a change in the ellipticity.

2.12 Example: Plane gravitational wave

As an example for image deformation produced by a gravitational field, we consider an exact plane gravitational wave solution of Einstein's field equation. To construct the relevant metric, we begin with Minkowski spacetime,

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,+\, (dx^3)^2 \,-\, (dx^0)^2 \,.$$

We perform a coordinate transformation, $(x^1, x^2, x^3, x^0) \mapsto (x^1, x^2, u, v)$, to "double-null coordinates" via

$$x^0 \;=\; \frac{1}{\sqrt{2}}\,(v+u)\;, \qquad x^3 \;=\; \frac{1}{\sqrt{2}}\,(v-u)\;.$$

Then

$$(dx^0)^2 \,-\, (dx^3)^2 \;=\; \frac{1}{2}\big(dv+du\big)^2 \,-\, \frac{1}{2}\big(dv-du\big)^2 \;=$$

$$=\; \frac{1}{2}\big(\cancel{du^2} \,+\, 2\,dv\,du \,+\, \cancel{dv^2}\big) \,-\, \frac{1}{2}\big(\cancel{du^2} \,-\, 2\,dv\,du \,+\, \cancel{dv^2}\big) \;=\; 2\,dv\,du\;,$$

hence the Minkowski metric reads

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,-\, 2\,dv\,du\,.$$

To describe a wave propagating at the speed of light, we add a term that leaves ∂_v lightlike,

$$g \;=\; (dx^1)^2 \,+\, (dx^2)^2 \,-\, 2\,dv\,du \,+\, h_{AB}(u)x^A x^B du^2\,. \qquad (B)$$

Here and in the following, we use again the summation convention for capital indices $A, B, \cdots = 1, 2$, and $\big(h_{AB}(u)\big)$ may be any symmetric 2×2 matrix, depending on u. Each x^1-x^2-surface (i.e., each surface $\{u = \text{constant}, v = \text{constant}\}$) is a Euclidean plane perpendicular to the propagation direction of the wave.

Calculating the Ricci tensor shows that the vacuum Einstein equation $R_{\mu\nu} = 0$ holds if and only if $h_{AB}(u)$ is trace-free,

$$h_{AB}(u)\delta^{AB} \;=\; 0\,. \qquad (T)$$

Metrics of the form (B) made their first appearence in a purely mathematical paper by H. Brinkmann ["Einstein spaces which are mapped conformally on each other" Math. Annalen 94, 119 (1925)]. The coordinates (x^1, x^2, u, v) are known as *Brinkmann coordinates*. If the condition (T) is satisfied, the metric (B) can be interpreted as a (pure) gravitational wave. For the case that (T) is not satisfied, it can be shown that the energy-momentum has the form of that of an electromagnetic field; the metric can then be interpreted as a combination of a gravitational wave and an electromagnetic wave.

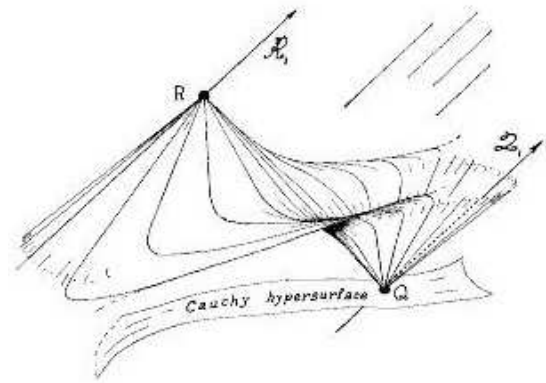
In the following we assume that (T) holds, i.e., that we have a (pure) gravitational wave. $h_{AB}(u)$ can then be written in the form

$$\big(h_{AB}(u)\big) \;=\; \begin{pmatrix} f(u) & g(u) \\ g(u) & -\,f(u) \end{pmatrix}\;.$$

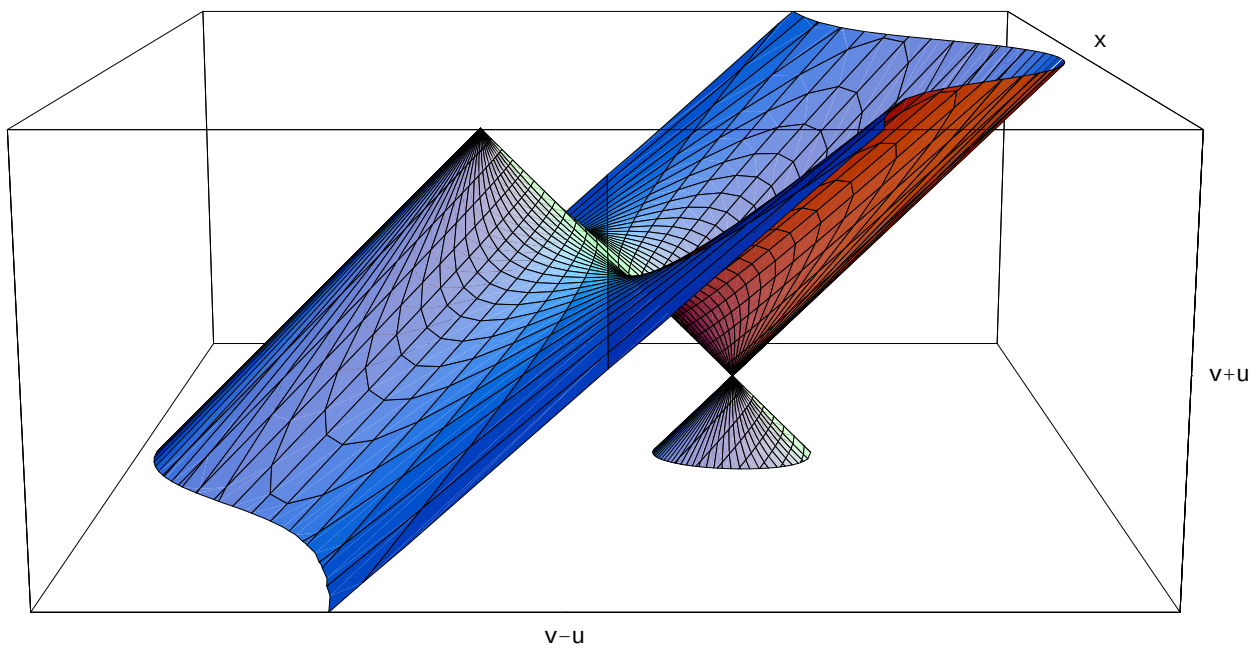
The *profile functions* $f(u)$ and $g(u)$ determine the shape of the gravitational wave. The fact that (within the class of metrics considered) two scalar functions are necessary to determine the wave can be interpreted by saying that "a gravitational wave has two polarisation states". This is in agreement with the well-known result that in the linearised Einstein theory there are two polarisation states (the *plus-mode* and the *cross-mode*) of plane harmonic waves.

Before we discuss image deformation we look at a light cone (all lightlike geodesics issuing from a chosen event) in our exact gravitational wave spacetime.

The picture on the right shows a famous hand-drawing by Roger Penrose [“ A remarkable property of plane waves in general relativity” Rev. Modern Phys. 37, 215 (1965)]. One sees that, with the exception of a single light ray that is a straight line, all light rays that are issuing from the event R into the past are refocussed into another event Q. Actually, taking the fourth dimension into account which is missing in the picture, a pure gravitational wave refocusses light rays into a line (“astigmatic focussing”). A combined gravitational and electromagnetic wave can refocus light rays into a point (“anastigmatic focussing”).



The following picture of the light cone was produced with Mathematica. The profile functions were chosen as $g(u) = 0$ and $f(u) = k^2$ with a non-zero constant k. The x^2 dimension is omitted. The similarity with the Penrose drawing is striking.



We now turn to the calculation of image deformation. We choose, as our central light ray, the u-line at $(x^1 = 0, x^2 = 0, v = v_0)$ with some constant v_0. It is easy to check that the u component of the geodesic equation yields

$$\frac{du}{ds} = \text{constant}\,.$$

We may thus choose the affine parameter such that $u = -s$ along the ray. The minus sign makes sure that the light ray is parametrised in the past-oriented sense, which is our usual convention. The tangent vector to the light ray is then $K = -\partial_u$.

To calculate image deformation, we consider a bundle determined by the matrix Jacobi equation

$$\frac{d^2\boldsymbol{D}}{ds^2} \;=\; \boldsymbol{D}\,\boldsymbol{Z}$$

with initial conditions

$$\boldsymbol{D}(0) \;=\; \boldsymbol{0}\;, \qquad \frac{d\boldsymbol{D}}{ds}(0) \;=\; \boldsymbol{1}\;.$$

Recall that the optical tidal matrix $\boldsymbol{Z} = (Z_{AB})$ is given by

$$Z_{AB} \;=\; g\big(E_A, R(K, E_B, K)\big)$$

where K is the tangent vector to the ray (i.e., $K = \partial_u$ in the case at hand), and (E_1, E_2) is a Sachs basis. Owing to the symmetry of the problem it is obvious that we can choose $E_1 = \partial_1$ and $E_2 = \partial_2$. Calculation of the curvature tensor yields

$$Z_{AB}(s) \;=\; h_{AB}(-s)$$

where we have used that $u = -s$. We will explicitly determine $\boldsymbol{D}(s)$ for the case that

$$g(u) \;=\; 0\;, \qquad f(u) \;=\; k^2 \;=\; \text{constant}\,.$$

In this case, the matrix Jacobi equation reads

$$\frac{d^2\boldsymbol{D}}{ds} \;=\; \boldsymbol{D}\begin{pmatrix} k^2 & 0 \\ 0 & -\,k^2 \end{pmatrix}\;.$$

Then the solution matching our initial conditions is obviously diagonal,

$$\boldsymbol{D}(s) \;=\; \begin{pmatrix} D_+(s) & 0 \\ 0 & D_-(s) \end{pmatrix}\;.$$

The resulting differential equations

$$\frac{d^2 D_+(s)}{ds^2} \;=\; k^2\, D_+(s)\;,$$

$$\frac{d^2 D_-(s)}{ds^2} \;=\; -\,k^2\, D_-(s)\;,$$

can be easily solved:

$$D_+(s) \;=\; A\cosh(ks) \;+\; B\sinh(ks)\;,$$

$$D_-(s) \;=\; E\cos(ks) \;+\; F\sin(ks)\;.$$

The initial conditions require

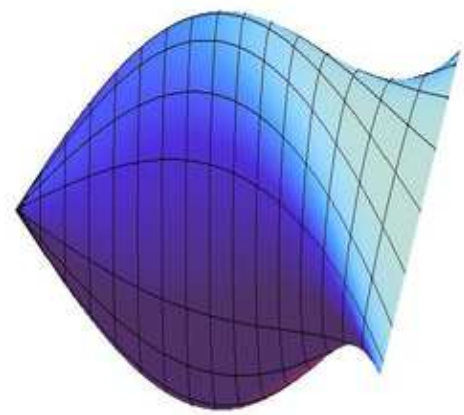
$$A \;=\; E \;=\; 0\;, \qquad E\,k \;=\; F\,k \;=\; 1\;,$$

hence

$$D_+(s) \;=\; \frac{1}{k}\sinh(ks)\;,$$

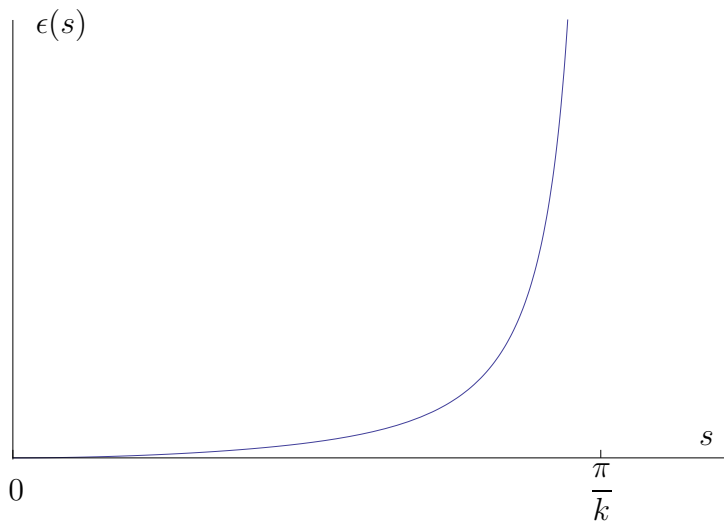
$$D_-(s) \;=\; \frac{1}{k}\sin(ks)\;.$$

The picture shows the bundle from the vertex at $s = 0$ until the first conjugate point is reached at $s = \pi/k$. The gravitational wave has a defocussing effect in the x^1 direction and a focussing effect in the x^2 direction. At $s = \pi/k$ the bundle collapses into a line, demonstrating that there is a conjugate point of multiplicity one.



The ellipticity is, quite generally,

$$\epsilon(s) \;=\; \Big(\frac{D_+(s)}{D_-(s)} - \frac{D_-(s)}{D_+(s)}\Big)\, e^{2i\chi(s)} \;.$$



In our case, $\boldsymbol{D}(s)$ is diagonal, hence $\chi(s) = 0$. With our results for $D_+(s)$ and $D_-(s)$ inserted, we find

$$\epsilon(s) \;=\; \frac{\sinh(ks)}{\sin(ks)} - \frac{\sin(ks)}{\sinh(ks)} \;.$$

With the Bernoulli-l'Hôpital rule one finds that $\epsilon(s) \to 0$ for $s \to 0$, as it must be at a vertex, while $\epsilon(s) \to \infty$ for $s \to \pi/k$, as it must be for a conjugate point of multiplicity one.

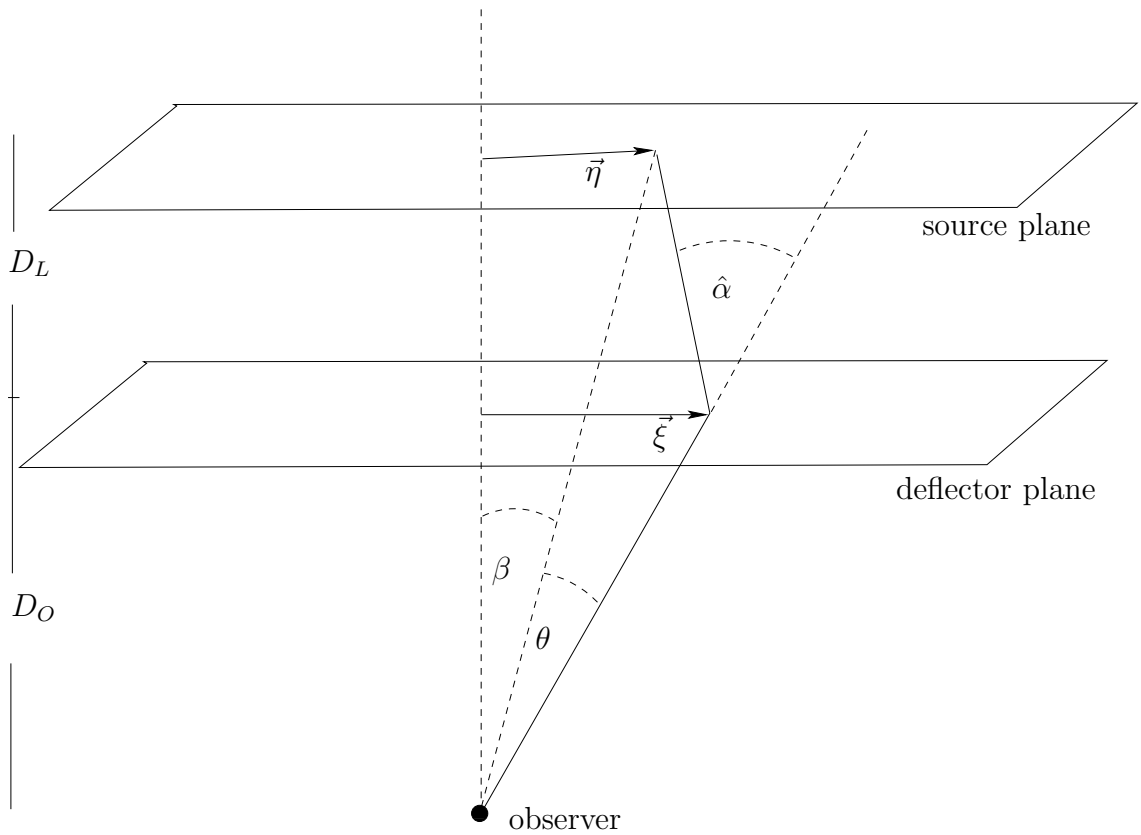
2.13 Lens equations

In the first part of this section we review the lens equation of the quasi-Newtonian approximation formalism which was introduced by S. Refsdal ["The gravitational lens effect", Mon. Not. R. Astron. Soc. 128, 295 (1964)].

The only input from general relativity is Einstein's approximation formula for the bending angle

$$\hat{\alpha} \;=\; \frac{2r_S}{r_m} \;=\; \frac{4GM}{c^2 r_m} \;.$$

Assume a time-independent situation, with a deflecting mass concentrated in a plane, and the laws of Euclidean geometry (flat space) valid outside of this plane:



Further assume that the angles β, θ and $\hat{\alpha}$ are so small that they can be identified with their sine or with their tangent. Then we read from the picture that

$$\vec{\eta} = \frac{D_L + D_O}{D_O}\,\vec{\xi} - D_L\,\vec{\hat{\alpha}}\,,$$

This is called the *lens equation* of the quasi-Newtonian approximation formalism. For given distances, D_L and D_O, the lens equation gives the *lens map* $\vec{\xi} \mapsto \vec{\eta}$ if the bending angle $\vec{\hat{\alpha}}$ is known.

It is often convenient to rewrite the lens equation in dimensionless form. To that end, we divide by $D_L + D_O$.

$$\underbrace{\frac{\vec{\eta}}{D_L + D_O}}_{=:\vec{\beta}} = \underbrace{\frac{\vec{\xi}}{D_O}}_{=:\vec{\theta}} - \underbrace{\frac{D_L}{D_L + D_O}\,\vec{\hat{\alpha}}}_{=:\vec{\alpha}}\,.$$

Then the lens equation takes the simple form

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}\,.$$

Clearly, in either form the lens equation is good for nothing as long as we don't know what the bending angle $\vec{\hat{\alpha}}$ (or the reduced bending angle $\vec{\alpha}$) is. That's where Einstein's formula comes in. For a point mass M at $\vec{\xi} = \vec{\xi}'$ Einstein's formula gives

$$\vec{\hat{\alpha}} = \frac{4GM}{c^2}\,\frac{\left(\vec{\xi} - \vec{\xi}'\right)}{\left|\vec{\xi} - \vec{\xi}'\right|^2}\,.$$

For a surface mass density $\Sigma(\vec{\xi'})$, we assume that the superposition principle holds,

$$\vec{\hat{\alpha}} \;=\; \frac{4G}{c^2}\int_{\mathbb{R}^2}\frac{(\vec{\xi}\,-\,\vec{\xi'})}{|\vec{\xi}\,-\,\vec{\xi'}|^2}\Sigma(\vec{\xi'})\,d^2\vec{\xi'}\;.$$

Here $d^2\vec{\xi'} \;=\; d\xi_1' d\xi_2'$ is the (Euclidean) surface element in the deflector plane. The superposition principle holds in Newtonian theory, i.e., the Newtonian potential of a sum of mass distributions is the sum of the individual potentials; according to Einstein's theory, which is non-linear, the superposition principle can be used only as an approximation for weak fields. In combination with the lens equation, the (approximate) formula for $\vec{\hat{\alpha}}$ as a function of $\vec{\xi}$ allows to derive the light deflection properties for any given surface mass density $\Sigma(\vec{\xi'})$ in the deflector plane.

The following properties hold.

- If the mass distribution is rotationally symmetric, $\Sigma(\vec{\xi'}) \;=\; f(|\vec{\xi'}|)$, then $\vec{\hat{\alpha}}$ and hence $\vec{\eta}$ are parallel to $\vec{\xi}$, so the lens equation becomes a scalar equation

$$\eta \;=\; \frac{D_L + D_O}{D_O}\,\xi \;-\; D_L\,\hat{\alpha}\;.$$

- The two-dimensional vector field $\vec{\xi} \mapsto \vec{\hat{\alpha}}(\xi)$ is a gradient and hence curl-free,

$$\delta_{AB}\hat{\alpha}^B \;=\; \frac{\partial}{\partial\xi^A}V(\vec{\xi})$$

with the *lensing potential*

$$V(\vec{\xi}) \;=\; \frac{4G}{c^2}\int_{\mathbb{R}^2}\Sigma(\vec{\xi'})\ln|\vec{\xi}-\vec{\xi'}|\,d^2\vec{\xi'}\;.$$

Proof:

$$\frac{\partial}{\partial\xi^A}\frac{4G}{c^2}\int_{\mathbb{R}^2}\Sigma(\vec{\xi'})\ln|\vec{\xi}-\vec{\xi'}|\,d^2\vec{\xi'} \;=$$

$$=\;\frac{4G}{c^2}\int_{\mathbb{R}^2}\Sigma(\vec{\xi'})\,\frac{1}{|\vec{\xi}-\vec{\xi'}|}\,\frac{\partial}{\partial\xi^A}\sqrt{(\xi^B-\xi'^B)(\xi^C-\xi'^C)\delta_{BC}}\,d^2\vec{\xi'} \;=$$

$$\frac{4G}{c^2}\int_{\mathbb{R}^2}\Sigma(\vec{\xi'})\,\frac{\cancel{2}(\xi^B-\xi'^B)\delta_{BA}}{\cancel{2}\,|\vec{\xi}-\vec{\xi'}|^2}\,d^2\vec{\xi'} \;=\; \delta_{AB}\hat{\alpha}^B(\xi)\;.$$

□

- The matrix $\left(\delta_{AB}\dfrac{\partial\beta^B}{\partial\xi^C}\right)$ is symmetric, hence it has two real eigenvalues.

Proof:

$$\delta_{AB}\hat{\alpha}^B \;=\; \frac{\partial}{\partial\xi^A}V \qquad\Longrightarrow\qquad \delta_{AB}\eta^B \;=\; \frac{D_L+D_O}{D_O}\delta_{AB}\xi^B \;-\; D_L\,\frac{\partial}{\partial\xi^A}V(\vec{\xi})$$

$$\Longrightarrow\qquad \delta_{AB}\frac{\partial\eta^B}{\partial\xi^C} \;=\; \frac{D_L+D_O}{D_O}\delta_{AC} \;-\; D_L\,\frac{\partial}{\partial\xi^C}\frac{\partial}{\partial\xi^A}V(\vec{\xi}) \;=\; \delta_{CB}\frac{\partial\eta^B}{\partial\xi^A}\;.$$

□

- If we use the dimensionless form of the lens equation, $\vec{\beta} = \vec{\theta} - \vec{\alpha}$, the matrix $\left(\delta_{BA}\frac{\partial\beta^A}{\partial\theta^C}\right)$ is symmetric, hence it has two real eigenvalues.

  Proof:

$$\beta^A = \theta^A - \frac{D_L}{(D_L + D_O)}\hat{\alpha}^A \qquad \Longrightarrow$$

$$\frac{\partial\beta^A}{\partial\theta^C} = \delta^A_C - \frac{D_L}{(D_L + D_O)}\frac{\partial}{\partial\theta^C}\frac{\partial}{\partial\xi^D}V\left(\vec{\xi}\right)\delta^{AD} =$$

$$= \delta^A_C - \frac{D_L D_O}{(D_L + D_O)}\frac{\partial^2}{\partial\theta^C\partial\theta^D}V\left(D_O\vec{\theta}\right)\delta^{AD} \qquad \Longrightarrow$$

$$\delta_{BA}\frac{\partial\beta^A}{\partial\theta^C} = \delta_{BC} - \frac{D_L D_O}{(D_L + D_O)}\frac{\partial^2}{\partial\theta^C\partial\theta^B}V\left(D_O\vec{\theta}\right) .$$

  The last expression is obviously symmetric with respect to the indices B and C. □

All lensing features are coded in the lens equation $\vec{\beta} = \vec{\theta} - \vec{\alpha}$:

- Multiple imaging is determined by how many $\theta_1, \ldots, \theta_n$ are mapped by the lens equation onto the same β.

- Brightness of images is given by the magnification μ which is defined as

$$\mu^{-1} = \det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right)$$

  where

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \left(\delta_{AB}\frac{\partial\beta^B}{\partial\theta^C}\right) .$$

  In other words, μ^{-1} is the determinant of the Jacobi matrix of the lens map. It relates the area of a domain in the deflector plane to the area of its image under the lens map in the source plane. The bigger μ, the brighter the image. Note that $\mu = 1$ if $\hat{\alpha} = 0$. In this sense, μ compares the brightness with the brightness of an "unlensed image".

  As the two eigenvalues of the symmetric matrix $\partial\vec{\beta}/\partial\vec{\theta}$ can have different signs, μ can be negative. An image with negative μ is side-inverted in comparison to an imgage with positive μ. This is observable for an extended non-symmetric source, e.g. for a galaxy with jets.

- Caustic points are characterised by $\mu = \infty$. This happens, for instance, in the case of an Einstein ring. Of course, the image isn't really infinitely bright. This infinity in the mathematical formalism just indicates that the ray optical treatment breaks down. A wave optical treatment would give a finite brightness.

The quasi-Newtonian lens map we have discussed so far relies on a number of approximations. In particular, it assumes that the gravitational field is weak and that bending angles are small. We will now investigate if a lens map can be formulated without such approximations, in the full formalism of general relativity.

The quasi-Newtonian lens map is a map from a deflector plane to a source plane. In an arbitrary general-relativistic spacetime, an exact lens map can be set up by replacing

deflector plane $\longmapsto$ celestial sphere S_{p_O} of observer,

source plane $\longmapsto$ three-dimensional submanifold $\mathcal{T}$ of spacetime, ruled by timelike curves.

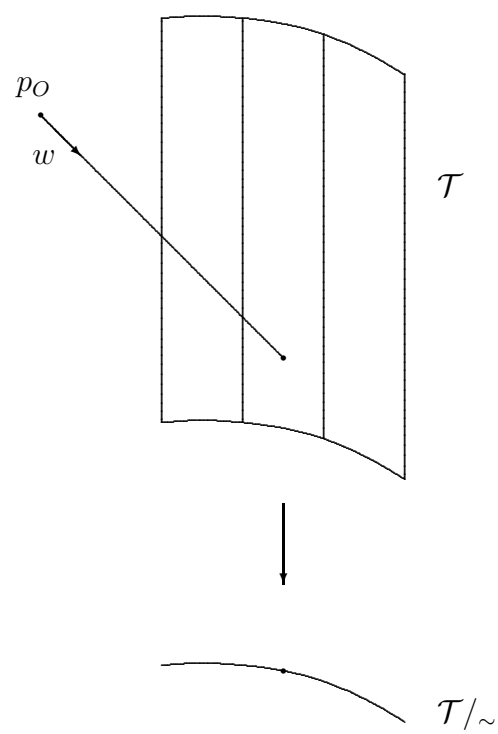
The timelike curves that rule $\mathcal{T}$ are to be interpreted as the worldlines of light sources. Of course, in an unspecified general-relativistic spacetime, there is no natural choice for $\mathcal{T}$. *Any* two-parameter family of light sources may be chosen.

We consider two events on $\mathcal{T}$ as equivalent if they are situated on the same worldline, and we denote the two-dimensional quotient manifold by $\mathcal{T}/_\sim$. In other words, the points in $\mathcal{T}/_\sim$ are in a one-to-one relation with the chosen light sources.

The lens map

$$\sigma : S_{p_O} \longrightarrow \mathcal{T}/_\sim$$

is defined in the following way: Given an element in S_{p_O}, i.e. a point on the celestial sphere of the observer, we consider the past-oriented lightlike geodesic with this initial direction. We follow this lightlike geodesic until it hits $\mathcal{T}$, then we project to $\mathcal{T}/_\sim$.



Clearly, for any chosen point in S_{p_O}, neither existence nor uniqueness of the image point in $\mathcal{T}/_\sim$ is guaranteed: Existence may fail because it might happen that the corresponding lightlike geodesic never meets $\mathcal{T}$. Uniqueness may fail because it might happen that it intersects $\mathcal{T}$ several times. For this reason, one has to investigate for each case individually what is the maximal domain on which σ is defined and whether or not it is single-valued.

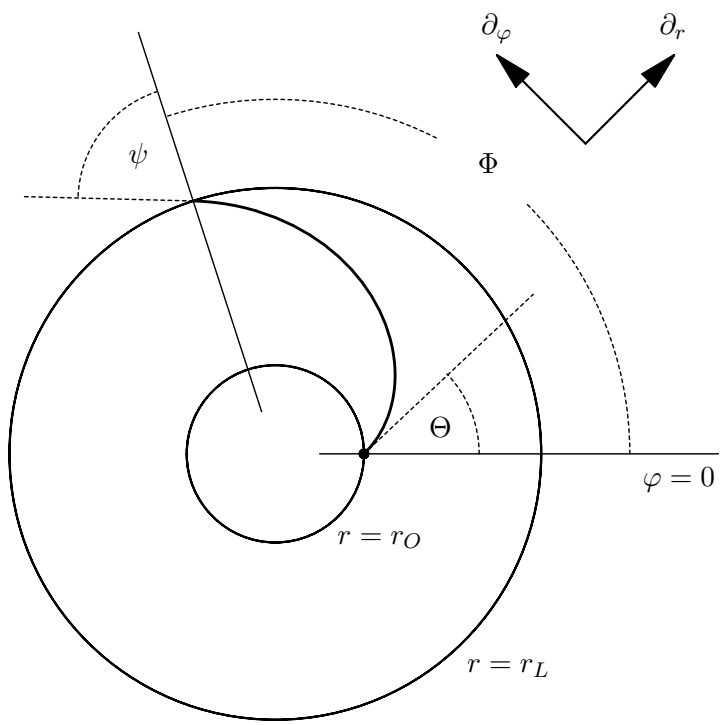
This exact gravitational lens map was introduced in S. Frittelli and E. T. Newman, "Exact gravitational lens equation", Phys. Rev. D 59, 124001 (1999). It has been used for investigating some general features of gravitational lensing, but only a few examples have been worked out. All of them concern spacetimes with symmetries, where a natural choice for the source surfaces $\mathcal{T}$ can be made. In the following we consider spherically symmetric and static spacetimes where, because of the symmetry, the exact lens map simplifies considerably.

In the case of a spherically symmetric and static spacetime, the metric reads

$$g \,=\, e^{2f(r)}\Big(\,-\,c^2dt^2\,+\,S(r)^2dr^2\,+\,R(r)^2\big(\,d\vartheta^2\,+\,\sin^2\vartheta\,d\varphi^2\,\big)\Big)\,.$$

We know already a couple of examples of spherically symmetric and static spacetimes: Schwarzschild spacetime, Kottler spacetime, and the Ellis wormhole.

In a spherically symmetric and static spacetime, it is very natural to choose a hypersurface $r = r_L$ for the source surface $\mathcal{T}$, which is ruled by the t-lines. The two-dimensional manifold $\mathcal{T}/_\sim$ is then just a sphere, parametrised by ϑ and φ. In other words, the lens map becomes a map from a sphere to a sphere. Because of the rotational symmetry about the radial line, it is completely determined by a map from an angle Θ to an angle Φ, as illustrated in the diagram.



The picture is purely spatial. The angle Θ determines, on the celestial sphere of the observer at radius r_O, the initial direction of a light ray with respect to the radial direction. The angle Φ is the angle swept out by the φ coordinate on the path of the light ray from the observer until it meets the sphere $r = r_L$. Note that Φ may take any value in $\mathbb{R}$, because a light ray can make arbitrarily many turns before arriving at $r = r_L$. This construction gives us, for every choice of r_O and r_L in a spherically symmetric and static spacetime, a lens map

$$\begin{aligned}]-\pi,\pi\,[\;&\longrightarrow\; \mathbb{R} \\ \Theta \;&\longmapsto\; \Phi \end{aligned}$$

However, it is not guaranteed that the lens map is defined on the whole interval $]-\pi,\pi\,[$ because it might happen that a light ray never arrives at the sphere $r = r_L$. Also, the lens map may be multi-valued because it might happen that a light ray intersects this sphere more than once.

We will now write down a "lens equation" that determines the lens map $\Theta \longmapsto \Phi$. We will see that it is given by an integral that involves the metric coefficients $S(r)$ and $R(r)$.

We start out from the Lagrangian for geodesics in the equatorial plane. As we are only interested in the paths of lightlike geodesics, we can ignore the factor $e^{2f(r)}$. (Recall that such a conformal factor changes only the parametrisations, but not the paths, of lightlike geodesics.) From this Lagrangian,

$$\mathcal{L} \,=\, \frac{1}{2}\Big(-\, c^2\dot{t}^2 \,+\, S(r)^2\dot{r}^2 \,+\, R(r)^2\dot{\varphi}^2 \Big)\,,$$

we get the constants of motion

$$E \,=\, -\,\frac{\partial\mathcal{L}}{\partial\dot{t}} \,=\, c^2\,\dot{t}\,, \qquad L \,=\, \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \,=\, R(r)^2\dot{\varphi}\,.$$

For lightlike geodesics we must have

$$0 \,=\, -\,c^2\dot{t}^2 \,+\, S(r)^2\dot{r}^2 \,+\, R(r)^2\dot{\varphi}^2\,,$$

$$0 \,=\, -\,c^2\frac{\dot{t}^2}{\dot{\varphi}^2} \,+\, S(r)^2\frac{\dot{r}^2}{\dot{\varphi}^2} \,+\, R(r)^2,$$

$$0 \,=\, -\,c^2\frac{E^2R(r)^4}{c^4\,L^2} \,+\, S(r)^2\Big(\frac{dr}{d\varphi}\Big)^2 \,+\, R(r)^2,$$

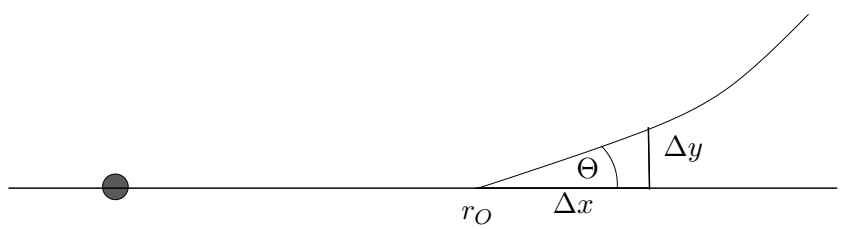
$$\Big(\frac{dr}{d\varphi}\Big)^2 \,=\, \frac{R(r)^2}{S(r)^2}\Big(\frac{E^2R(r)^2}{c^2\,L^2} \,-\, 1\Big)\,. \qquad (G)$$

L/E can be expressed in terms of the angle Θ in the following way:

The picture shows that

$$\tan\Theta \,=\, \lim_{\Delta x\to 0}\frac{\Delta\,y}{\Delta\,x} \,=$$

$$=\, \frac{R(r)d\varphi}{S(r)dr}\Big|_{r=r_O}$$



where the second equality was read from the metric. With (G) inserted, this results in

$$\frac{1}{\tan^2\Theta} \,=\, \frac{\cancel{S(r_O)^2}}{\cancel{R(r_O)^2}}\,\frac{\cancel{R(r_O)^2}}{\cancel{S(r_O)^2}}\Big(\frac{E^2R(r_O)^2}{c^2\,L^2} \,-\, 1\Big)\,,$$

$$\frac{E^2R(r_O)^2}{c^2\,L^2} \,=\, \frac{1}{\tan^2\Theta} \,+\, 1 \,=\, \frac{\cos^2\Theta + \sin^2\Theta}{\sin^2\Theta} \,=\, \frac{1}{\sin^2\Theta}\,.$$

This allows us to replace L/E in (G),

$$\Big(\frac{dr}{d\varphi}\Big)^2 \;=\; \frac{R(r)^2}{S(r)^2}\Big(\frac{R(r)^2}{R(r_O)^2\sin^2\Theta} \;-\; 1\Big)\,,$$

$$d\varphi \;=\; \frac{\pm\, R(r_O)\sin\Theta\, S(r)\, dr}{R(r)\,\sqrt{R(r)^2 \;-\; R(r_O)^2\sin^2\Theta}}\,.$$

Integration of this equation over the light ray gives Φ as a function of Θ. In general, the integration must be done piecewise. The plus or minus sign has to be chosen in such a way that, for $0 \le \Theta \le \pi$, the angle φ is always increasing. We consider two examples: If r is increasing monotonically from r_O to r_L, the integration yields

$$\Phi \;=\; \int_0^{\Phi} d\varphi \;=\; R(r_O)\sin\Theta \int_{r_O}^{r_L} \frac{S(r)\, dr}{R(r)\,\sqrt{R(r)^2 \;-\; R(r_O)^2\sin^2\Theta}}\,.$$

If, on the other hand, r decreases from r_O to a minimum value $r_m(\Theta)$ and then increases again to r_L, the integration yields

$$\Phi \;=\; \int_0^{\Phi} d\varphi \;=\; R(r_O)\sin\Theta\,\Bigg(\int_{r_m(\Theta)}^{r_O} + \int_{r_m(\Theta)}^{r_L}\Bigg)\, \frac{S(r)\, dr}{R(r)\,\sqrt{R(r)^2 \;-\; R(r_O)^2\sin^2\Theta}}\,,$$

where $r_m(\Theta)$ is determined by the equation

$$R\big(r_m(\Theta)\big)^2 \;=\; R(r_O)^2\sin^2\Theta\,.$$

The exact lens map for spherically symmetric and static spacetimes was introduced in V. Perlick, "On the exact gravitational lens equation in spherically symmetric and static spacetimes", Phys. Rev. D 69, 064917 (2004).

2.14 Example: Barriola-Vilenkin monopole

Monopoles have not been detected sofar. There is some speculation that they might have come into existence during phase transitions in the early universe. The mathematically simplest example of a monopole was introduced in M. Barriola and A. Vilenkin, "Gravitational field of a global monopole", Phys. Rev. Lett. 63, 341 (1989). The metric reads

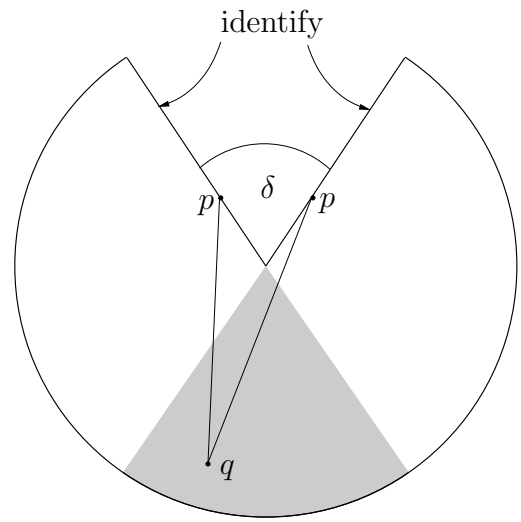
$$g \;=\; -\,c^2 dt^2 \;+\; dr^2 \;+\; k^2\, r^2\,\big(\, d\vartheta^2 \;+\; \sin^2\vartheta\, d\varphi^2\,\big)$$

with a constant $k < 1$. In the equatorial plane $\vartheta = \pi/2$, the metric

$$g \;=\; -\,c^2\, dt^2 \;+\; dr^2 \;+\; k^2\, r^2\, d\varphi^2$$

coincides precisely with the metric of a straight string in a plane perpendicular to the string axis, recall Section 2.8.

So we can use, for the sake of illustration, the same picture with a deficit angle δ that we have used for the string spacetime. However, now this picture is to be thought of as rotationally symmetric with respect to a vertical axis. With all three spatial dimensions taken into account, δ indicates a deficit *solid* angle which implies that the Barriola-Vilenkin metric is non-flat. (Only the restriction to the equatorial plane is flat.) Calculation of the Ricci tensor yields



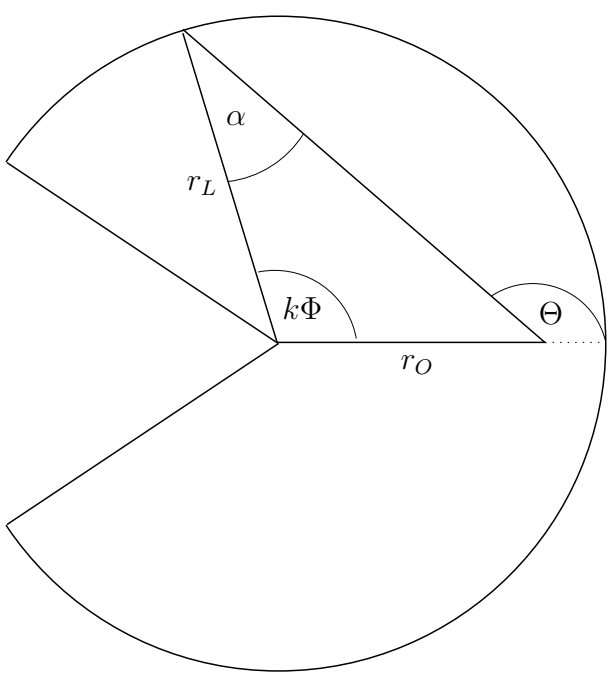
$$R_{\vartheta\vartheta} = 1 - k^2 , \qquad R_{\varphi\varphi} = \left(1 - k^2\right) \sin^2\vartheta ,$$

and zero for all other components $R_{\mu\nu}$. From this we can easily calculate the Ricci scalar,

$$R = R_{\mu\nu} g^{\mu\nu} = R_{\vartheta\vartheta} g^{\vartheta\vartheta} + R_{\varphi\varphi} g^{\varphi\varphi} = \frac{1-k^2}{k^2\, r^2} + \frac{(1-k^2)\,\cancel{\sin^2\vartheta}}{k^2\, r^2\, \cancel{\sin^2\vartheta}} = \frac{2(1-k^2)}{k^2\, r^2} .$$

Clearly, $R \to \infty$ for $r \to 0$ which means that there is a curvature singularity at $r = 0$.

The exact lens map $\Theta \mapsto \Phi$ in the Barriola-Vilenkin spacetime can be derived with the help of elementary geometry. We use the fact that in the equatorial plane the metric is the flat metric of a cone.



As the angular sum in a triangle equals π, we read from the diagram that

$$\alpha + k\Phi + \pi - \Theta = \pi ,$$

hence

$$\alpha = \Theta - k\Phi .$$

On the other hand, the sine theorem yields

$$\frac{\sin\alpha}{r_O} = \frac{\sin(\pi - \Theta)}{r_L} .$$

Inserting the first equation into the second gives the lens equation

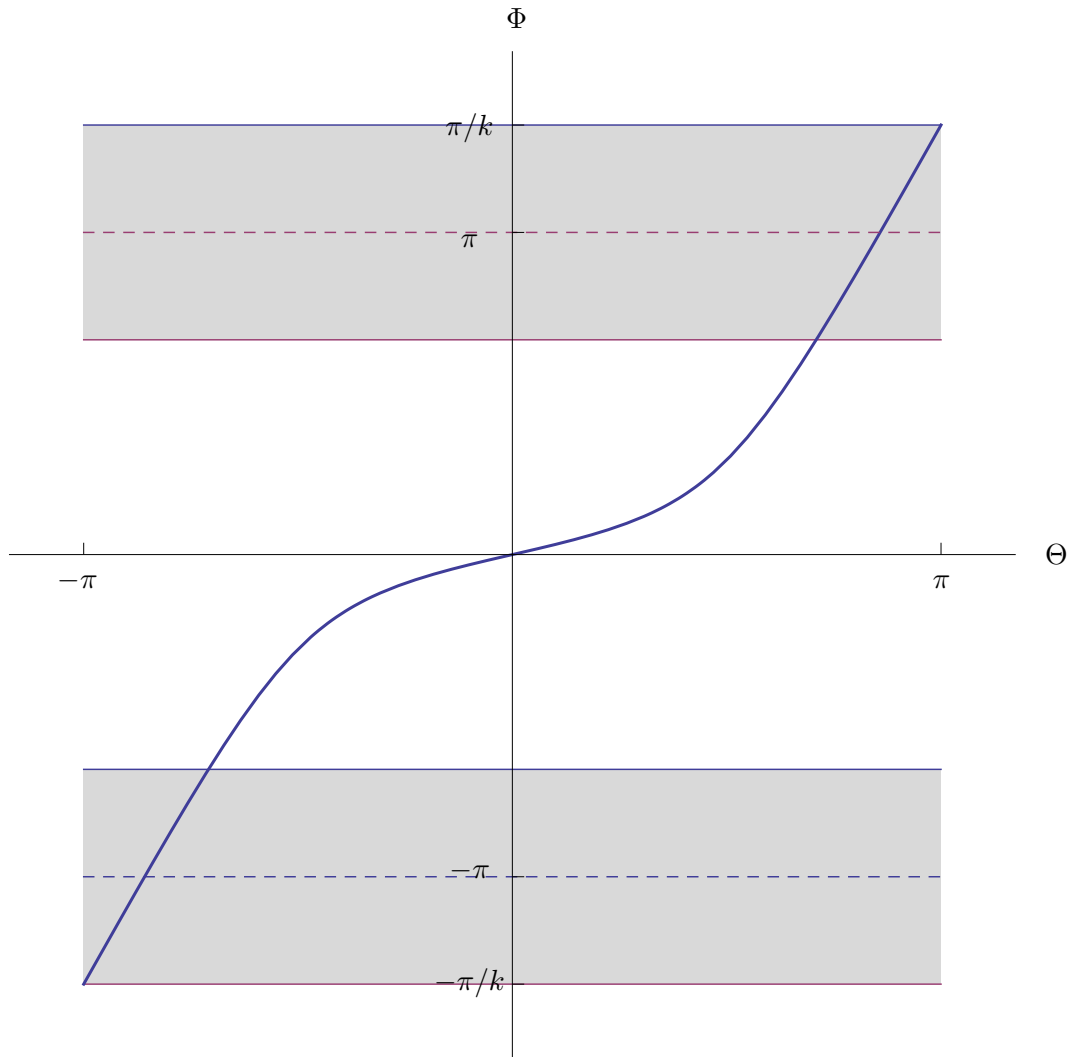
$$r_L \sin(\Theta - k\Phi) = r_O \sin\Theta$$

by which the lens map $\Theta \mapsto \Phi$ is (implicitly) determined.

Integrating the equation

$$d\varphi \;=\; \frac{\pm\, R(r_O)\sin\Theta\, dr}{R(r)\,\sqrt{R(r)^2 \;-\; R(r_O)^2\sin^2\Theta}}$$

over the light ray, taking into account that for the Barriola-Vilenkin spacetime the metric coefficients are $S(r) = 1$ and $R(r) = kr$, gives the same lens equation $r_L \sin(\Theta - k\Phi) \;=\; r_O \sin\Theta$.

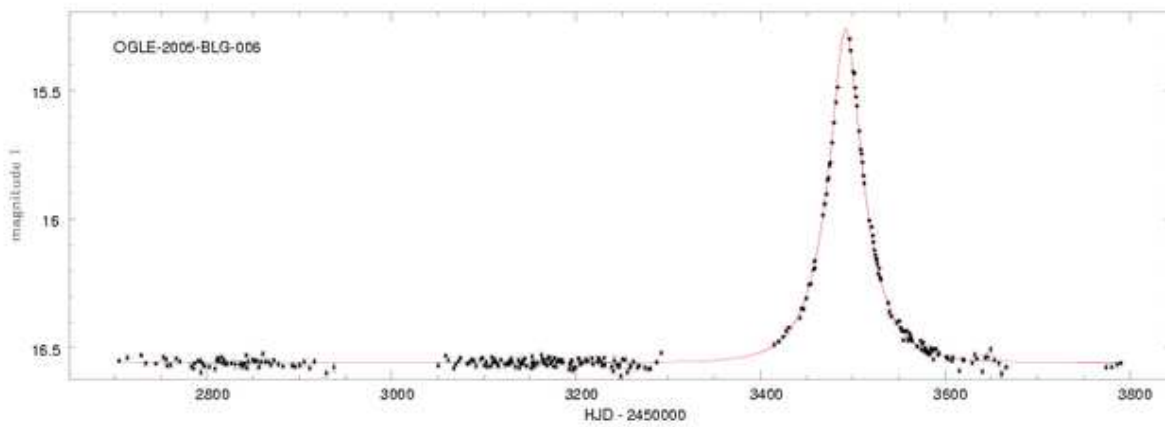


If $r_O \leq r_L$, the angle Θ ranges over the maximal domain from $-\pi$ to π. On this domain, Φ is monotonically increasing from $-\pi/k$ to π/k. The plot shows the lens map for the case $k = 0.75$ and $r_O = 0.75 r_L$. From the diagram we can read the number of images and the occurrence of Einstein rings. Recall that Θ_1 and Θ_2 are image positions of the same light source if and only if $\Phi(\Theta_2) - \Phi(\Theta_1)$ is an integer multiple of 2π. We see that there is a double imaging region (shaded in the diagram) and a single-imaging region. Einstein rings occur with an angular radius Θ_E if and only if $\Phi(\pm\Theta_E)$ is an integer multiple of π. We see that in the case at hand the values $\Phi = \pm\pi$ occur, indicated by dashed lines, but no other multiples of π. So there is one Einstein ring whose angular radius Θ_E is found by projecting onto the Θ axis from the intersection of the graph of the lens map with the dashed line at $\Phi = \pi$.

3. Applications to astrophysics

3.1 Microlensing

In this section we want to apply the mathematical formalism we have developed to the observed light curves of microlensing events. Recall from Chapter 1 that different sorts of such light curves are observed. Here we restrict to the simplest case of a light curve which has only one maximum and is symmetric with respect to this maximum. A typical example is the following.



We first try the simplest mathematical model we can think of: We use the quasi-Newtonian lens equation for a point mass as the deflector. The lens equation reads

$$\vec{\eta} = \frac{\left(D_L + D_O\right)}{D_O}\,\vec{\xi} - D_L\,\vec{\hat{\alpha}}$$

and for a point mass at $\xi = \vec{0}$ the bending angle is given by

$$\vec{\hat{\alpha}} = \frac{4GM}{c^2}\,\frac{\vec{\xi}}{\left|\vec{\xi}\right|^2}\,,$$

recall Section 2.13. Dividing the lens equation by $D_O + D_L$ gives

$$\frac{\vec{\eta}}{D_L + D_O} = \frac{\vec{\xi}}{D_O} - \frac{D_L}{\left(D_L + D_O\right)}\,\frac{4GM}{c^2}\,\frac{\vec{\xi}}{\left|\xi\right|^2}\,.$$

In terms of the dimensionless (angular) variables

$$\vec{\beta} = \frac{\vec{\eta}}{D_O + D_L}\,, \qquad \vec{\theta} = \frac{\vec{\xi}}{D_O}\,,$$

this can be rewritten as

$$\vec{\beta} = \vec{\theta} - \frac{D_L}{\left(D_L + D_O\right)}\,\frac{4GM}{c^2}\,\frac{\cancel{D_O}\,\vec{\theta}}{D_O^{\cancel{2}}\left|\vec{\theta}\right|^2}\,.$$

If we write $\vec{\beta} = \beta\,\vec{e}$ with a unit vector $\vec{e}$ and $\beta \geq 0$, the lens equation becomes a scalar equation,

$$\beta \;=\; \theta - \frac{4GMD_L}{c^2\big(D_L + D_O\big)\,D_O}\,\frac{1}{\theta}$$

where

$$\vec{\theta} \;=\; \theta\,\vec{e}\,.$$

Note that θ can be positive or negative.

An Einstein ring occurs if the light source is on the axis, $\beta = 0$. This happens if $\theta^2 = \theta_E^2$, where

$$\theta_E \;:=\; \sqrt{\frac{4GMD_L}{c^2\big(D_O + D_L\big)\,D_O}}\,.$$

With this abbreviation, the scalar lens equation reads

$$\beta \;=\; \theta - \frac{\theta_E^2}{\theta}\,.$$

For given β, this is a quadratic equation for θ,

$$\theta^2 \;-\; \beta\,\theta \;-\; \theta_E^2 \;=\; 0\,,$$

which has two solutions

$$\theta_\pm \;=\; \frac{\beta}{2} \;\pm\; \sqrt{\frac{\beta^2}{4} \;+\; \theta_E^2}\,. \qquad\qquad (I)$$

This demonstrates that for each light source that is not on the axis there are precisely two images.

The magnification μ of each image is given as the inverse of the determinant of the Jacobi matrix of the lens map. From

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \;=\; \Big(1 \;-\; \frac{\theta_E^2}{\theta_1^2 + \theta_2^2}\Big)\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

we get the Jacobi matrix,

$$\Big(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\Big) \;=\; \begin{pmatrix} 1 - \dfrac{\theta_E^2}{\theta^2} + \dfrac{2\theta_E^2\theta_1^2}{\theta^4} & \dfrac{2\theta_E^2\theta_1\theta 2}{\theta^4} \\[2ex] \dfrac{2\theta_E^2\theta_1\theta 2}{\theta^4} & 1 - \dfrac{\theta_E^2}{\theta^2} + \dfrac{2\theta_E^2\theta_2^2}{\theta^4} \end{pmatrix}\,.$$

Calculation of the determinant yields

$$\mu^{-1} \;=\; \Big(1 - \frac{\theta_E^2}{\theta^2}\Big)^2 + \Big(1 - \frac{\theta_E^2}{\theta^2}\Big)\frac{2\theta_E^2\big(\theta_1^2 + \theta_2^2\big)}{\theta^4} + \cancel{\frac{4\theta_E^4\theta_1^2\theta_e^2}{\theta^8}} - \cancel{\frac{4\theta_E^2\theta_1^2\theta_2^2}{\theta^8}} \;=$$

$$= \Big(1 - \frac{\theta_E^2}{\theta^2}\Big)\Big(1 - \cancel{\frac{\theta_E^2}{\theta^2}} + 2\frac{\theta_E^2}{\theta^2}\Big) \;=\; 1 - \frac{\theta_E^4}{\theta^4}\,.$$

So the magnification of the image at $\theta_\pm$ is

$$\mu_\pm \;=\; \frac{1}{1-\dfrac{\theta_E^4}{\theta_\pm^4}} \;=\; \frac{\theta_\pm^4}{\left(\theta_\pm^2-\theta_E^2\right)\left(\theta_\pm^2+\theta_E^2\right)}\,.$$

As (I) implies $\theta_+\theta_- \;=\; -\theta_E^2$, this can be rewritten as

$$\mu_\pm \;=\; \frac{\theta_\pm^4}{\left(\theta_\pm^2+\theta_+\theta_-\right)\left(\theta_\pm^2-\theta_+\theta_-\right)} \;=\; \frac{\pm\,\theta_\pm^{\cancel{4}2}}{\cancel{\theta_\pm}\left(\theta_+ + \theta_-\right)\cancel{\theta_\pm}\left(\theta_+ - \theta_-\right)} \;=\; \frac{\pm\left(\dfrac{\beta}{2}\pm\sqrt{\dfrac{\beta^2}{4}+\theta_E^2}\right)^2}{\beta\,2\sqrt{\dfrac{\beta^2}{4}+\theta_E^2}}\,.$$

With the abbreviation

$$u \;=\; \frac{\beta}{\theta_E}$$

this yields

$$\mu_\pm \;=\; \frac{\pm\left(\dfrac{u^2}{2}\pm u\sqrt{\dfrac{u^2}{4}+1}\;+\;1\right)}{2\,u\sqrt{\dfrac{u^2}{4}\;+\;1}} \;=\; \frac{1}{2}\;\pm\;\frac{u^2+2}{2u\sqrt{u^2+4}}\,.$$

Note that $\mu_+ > 0$ while $\mu_- < 0$. This indicates that the second image is side-inverted with respect to the first image. For an extended light source with a non-symmetric shape this is observable.

In microlensing situations, the two images cannot be resolved. What can be observed is the total magnification

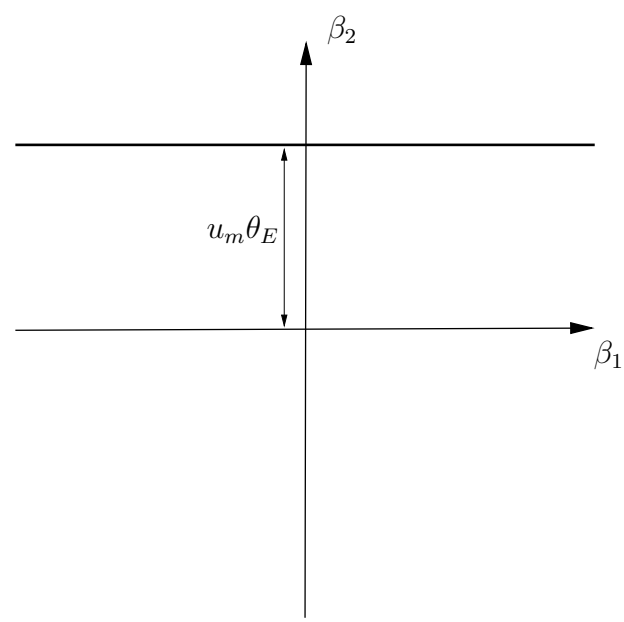
$$\mu \;=\; \left|\mu_+\right| \;+\; \left|\mu_-\right| \;=\; \mu_+ \;-\; \mu_- \;=\; \frac{1}{2}\;+\;\frac{u^2+2}{2u\sqrt{u^2+4}}\;-\;\frac{1}{2}\;+\;\frac{u^2+2}{2u\sqrt{u^2+4}} \;=\; \frac{u^2+2}{u\sqrt{u^2+4}}\,.$$

We evaluate this for a light source that moves in a straight line,

$$\beta_1 \;=\; \theta_E\,\frac{t}{t_E}\,,$$

$$\beta_2 \;=\; \theta_E\,u_m$$

where t_E and u_m are constants. t_E is the time the light source needs for traveling through the angular distance θ_E, while u_m tells how close the light ray comes to the axis; for $u_m = 0$ the light source passes through the axis at time $t = 0$ which means that an Einstein ring is formed.



For such a moving light source, we have

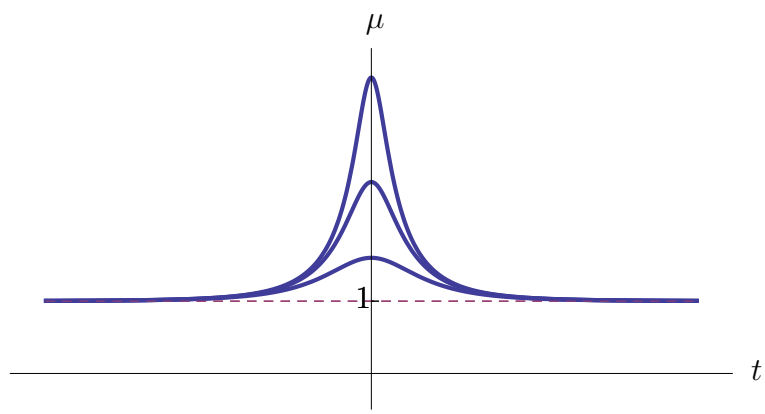
$$u = \frac{\beta}{\theta_E} = \frac{\sqrt{\beta_1^2 + \beta_2^2}}{\theta_E} = \sqrt{u_m^2 + \frac{t^2}{t_E^2}}$$

and the magnification is given, as a function of time t, as

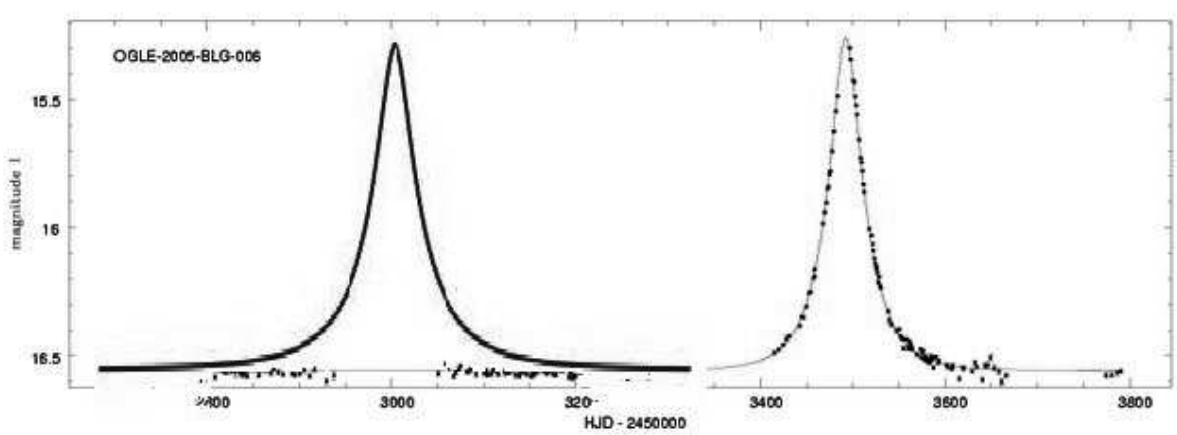
$$\mu = \frac{u^2 + 2}{u\sqrt{u^2 + 4}} = \frac{u_m^2 + \frac{t^2}{t_E^2} + 2}{\sqrt{u_m^2 + \frac{t^2}{t_E^2}}\sqrt{u_m^2 + \frac{t^2}{t_E^2} + 4}}.$$

Note that $\mu \to 1$ tor $t \to \pm\infty$ which means that, far away from the moment of closest approach to the axis, the brightness is that of an unlensed image.

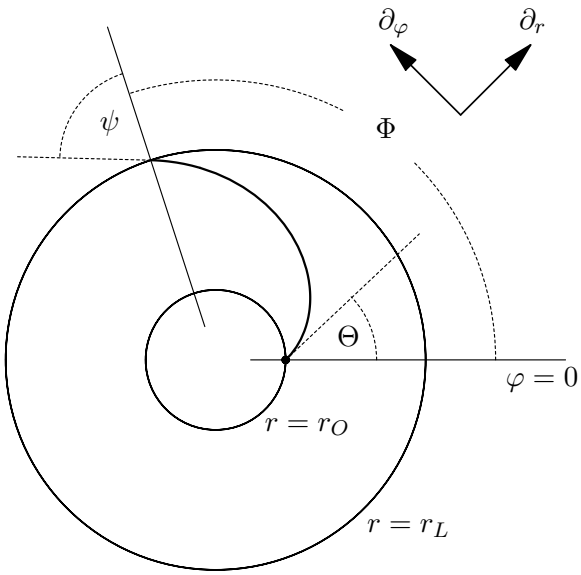
The diagram shows plots of μ as a function of t for three different values of u_m. For $u_m \to 0$ the maximum goes to infinity.



To compare with observations, we assume that over the duration of the microlensing event the travel time of light from the light source to the observer can be considered as a constant. Then our equation for μ as a function of t gives us directly the light curve as predicted by our theoretical model. We can compare this with a light curve that is actually observed. By choosing u_m appropriately, we can fit the maximum of the predicted light curve to the maximum of the observed light curve. By choosing t_E appropriately, we can fit the half-width. These are the two parameters in our model that can be varied. For many observed microlensing events, the observed light curve can be very well described by this simple theoretical model, see the example below. (Obviously, if the light curve has a less symmetric shape a more complicated model of the deflector is necessary.) We can then read the values of t_E and u_m from the observed light curve. However, this information alone does not allow to determine the mass of the deflector. Additional information is needed, e.g. about the distances D_O and D_L.



We have seen that the simplest (symmetric) microlensing light curves can be well explained in the quasi-Newtonian approximation formalism with a point lens model. We will now investigate whether they could also be explained as being produced by a more exotic lens, e.g. a Barriola-Vilenkin monopole. We will use the exact lens map for spherically symmetric and static spacetimes, $\Theta \mapsto \Phi$, which is based on the adjacent diagram, recall p.73.



We have worked out this lens map for metrics of the form

$$g \,=\, e^{2f(r)} \Big\{ -\, c^2 dt^2 \,+\, S(r)^2 dr^2 \,+\, R(r)^2 \left(d\vartheta^2 \,+\, \sin^2\!\vartheta \, d\varphi^2 \right) \Big\} .$$

The metric of the Barriola-Vilenkin monopole,

$$g \,=\, -\, c^2 dt^2 \,+\, dr^2 \,+\, k^2 \, r^2 \left(d\vartheta^2 \,+\, \sin^2\!\vartheta \, d\varphi^2 \right) ,$$

is of this form with

$$f(r) = 0 \,, \qquad S(r) = 1 \,, \qquad R(r) = k \, r \;.$$

We have derived on p.76 that, for the Barriola-Vilenkin monopole, the lens map $\Theta \mapsto \Phi$ is implicitly given by the equation

$$r_L \sin(\Theta - k\Phi) \,=\, r_O \sin\Theta \;. \qquad \text{(BV1)}$$

For later purpose, we solve this equation for $\cot\Theta$:

$$r_L \sin\Theta \cos\!\big(k\Phi\big) \,-\, r_L \cos\Theta \sin\!\big(k\Phi\big) \,=\, r_O \sin\Theta \;,$$

$$\sin\Theta \Big(r_L \cos\!\big(k\Phi\big) \,-\, r_O \Big) \,=\, r_L \cos\Theta \sin\!\big(k\Phi\big) \;,$$

$$\cot\Theta \,=\, \frac{r_L \cos\!\big(k\Phi\big) \,-\, r_O}{r_L \sin\!\big(k\Phi\big)} \;. \qquad \text{(BV2)}$$

We will also need an expression for $d\Phi/d\Theta$. Differentiation of (BV2) with respect to Θ yields

$$-\,\frac{1}{\sin^2\!\Theta} \,=\, \frac{\Big(-\, r_L \, k \sin^2\!\big(k\Phi\big) \,-\, \big\{ r_L \cos\!\big(k\Phi\big) - r_O \big\} \, k \cos\!\big(k\Phi\big) \Big)}{r_L \sin^2\!\big(k\Phi\big)} \, \frac{d\Phi}{d\Theta} \;,$$

$$\frac{\sin^2\!\Theta \,+\, \cos^2\!\Theta}{\sin^2\!\Theta} \,=\, \frac{k \Big(r_L \,-\, r_O \cos\!\big(k\Phi\big) \Big)}{r_L \sin^2\!\big(k\Phi\big)} \, \frac{d\Phi}{d\Theta} \;.$$

After inserting (BV2) on the left-hand side, this results in

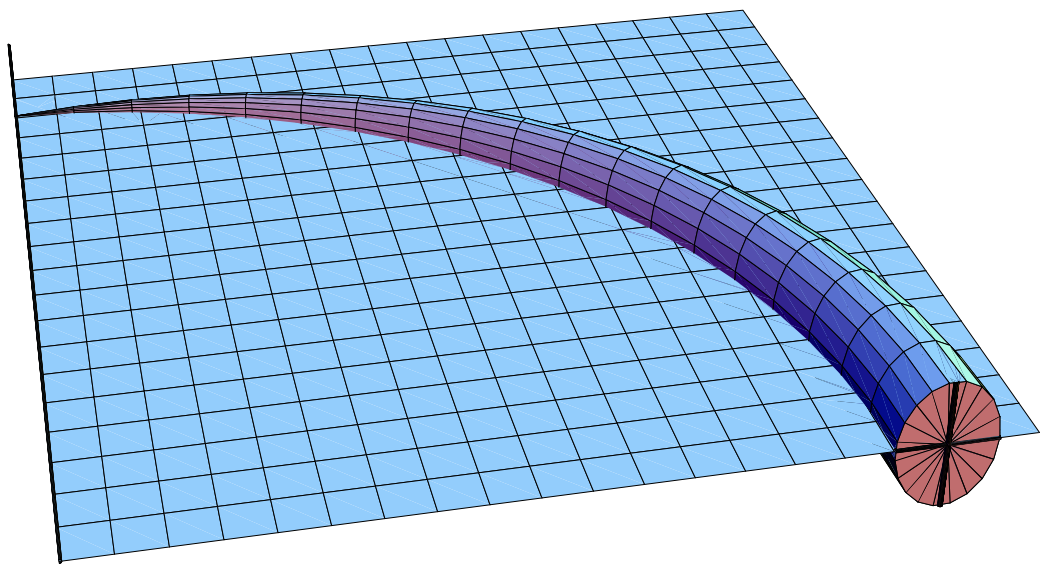
$$1 + \frac{\Big(r_L \cos\big(k\Phi\big) - r_O\Big)^2}{r_L^2 \sin^2\big(k\Phi\big)} = \frac{k\Big(r_L - r_O \cos\big(k\Phi\big)\Big)}{r_L \sin^2\big(k\Phi\big)} \frac{d\Phi}{d\Theta} ,$$

$$k\, r_L \frac{d\Phi}{d\Theta} = \frac{r_L^2 + r_O^2 - 2\, r_L r_O \cos\big(k\Phi\big)}{r_L - r_O \cos\big(k\Phi\big)} . \qquad \text{(BV3)}$$

After these preparations, we will now calculate the luminosity distance D_{lum} which will give us the brightness of images. For the Barriola-Vilenkin metric, the redshift potential $f(r)$ is zero, so the luminosity distance equals the area distance,

$$D_{\text{lum}}^2 = D_{\text{area}}^2 .$$

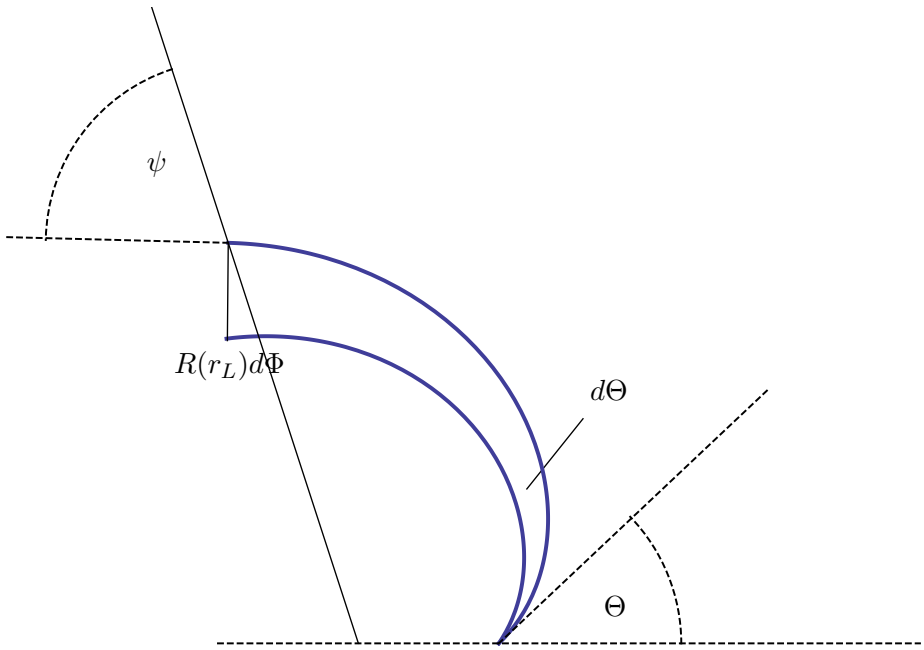
To calculate the area distance we consider, around the ray from the observer to the light source, a light bundle with vertex at the observer. The area distance is the ratio of the cross-sectional area of this bundle at the light source to its opening solid angle. By symmetry, one of the two semi-axes of the cross-section is in the plane of the central ray, the other one is perpendicular to it, as is illustrated in the picture.



We write the area of the cross section as the product of the horizontal semi-axis times the vertical semi-axis. This gives us the area distance as a product

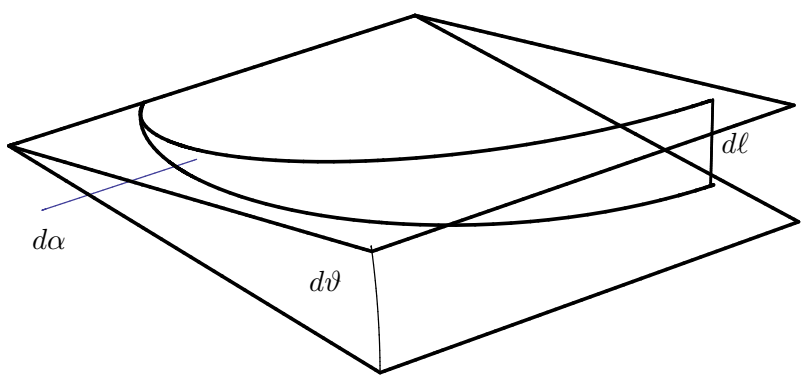
$$D_{\text{area}}^2 = \big|D_+ D_-\big| ,$$

recall Section 2.9.



D_+ is the ratio of the horizontal cross-section to the corresponding opening angle. From the picture we read that

$$D_+ = \frac{\cos\psi\, R(r_L)\, d\Phi}{d\Theta} = \frac{\cos\psi\, k\, r_L\, d\Phi}{d\Theta}$$



D_- is the ratio of the vertical cross-section to the corresponding opening angle. From the picture we read that

$$D_- = \frac{d\ell}{d\alpha} = \frac{R(r_L)\sin\Phi\, \cancel{d\vartheta}}{\sin\Theta\, \cancel{d\vartheta}} = \frac{k\, r_L \sin\Phi\, \cancel{d\vartheta}}{\sin\Theta\, \cancel{d\vartheta}}\,.$$

The formula for D_+ involves the angle ψ. This angle can be expressed in terms of the angle Θ (and the radii r_L and r_O) in the following way.

On p.74 we have shown that the constants of motion E and L satisfy

$$\frac{c^2 L^2}{E^2} \;=\; R(r_O)^2 \sin^2\Theta\,.$$

The same argument, now at the position of the light source, demonstrates that

$$\frac{c^2 L^2}{E^2} \;=\; R(r_L)^2 \sin^2\psi\,.$$

Inserting the resulting expression

$$\cos\psi \;=\; \sqrt{1-\sin^2\psi} \;=\; \sqrt{1\;-\;\frac{R(r_O)^2}{R(r_L)^2}\sin^2\Theta} \;=\; \sqrt{1\;-\;\frac{r_O^2}{r_L^2}\sin^2\Theta}$$

into the formulas for D_+ and D_- results in

$$\big|D_+D_-\big| \;=\; \Big|\, k\,\frac{d\Phi}{d\Theta}\sqrt{r_L^2\;-\;r_O^2\sin^2\Theta}\;\frac{k\,r_L\sin\Phi}{\sin\Theta}\,\Big| \;=$$

$$=\; \Big|\, k^2\,r_L\,\frac{d\Phi}{d\Theta}\sqrt{r_L^2\;-\;r_O^2\;+\;r_L^2\cot^2\Theta}\;\sin\Phi\,\Big|\,.$$

We substitute for $\cot^2\Theta$ from (BV2) and for $d\Phi/d\Theta$ from (BV3).

$$\big|D_+D_-\big| \;=\; \Big|\, k\sqrt{r_L^2\;-\;r_O^2\;+\;\frac{\big(r_L\cos(k\Phi)-r_O\big)^2}{\sin^2(k\Phi)}}\;\frac{\Big(r_L^2\;+\;r_O^2\;-\;2\,r_L\,r_O\cos\big(k\Phi\big)\Big)}{r_L\;-\;r_O\cos\big(k\Phi\big)}\sin\Phi\,\Big| \;=$$

$$=\; \Big|\,\frac{k\sin\Phi}{\sin\big(k\Phi\big)}\,\frac{\sqrt{r_L^2\;+\;r_O^2\cos^2(k\Phi)\;-\;2\,r_L\,r_O\cos(k\Phi)}}{r_L\;-\;r_O\cos\big(k\Phi\big)}\Big(r_L^2\;-\;r_O^2\;+\;r_L^2\cot^2\Theta\Big)\,\Big|\,.$$

This gives us the brightness of images in terms of the energy flux

$$F \;=\; \frac{L}{4\,\pi\,D_{\rm lum}^2} \;=\; \frac{L}{4\,\pi\,\big|D_+D_-\big|} \;=$$

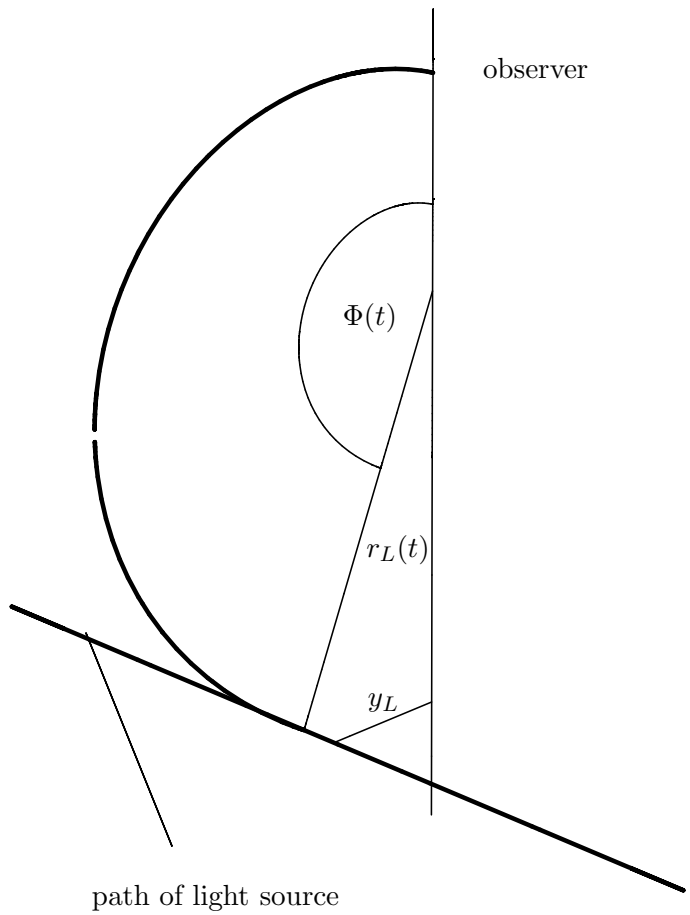
$$=\; \frac{L\,\big|\sin(k\Phi)\big|}{4\,\pi\,k\,\big|\sin\Phi\,\big|\,\Big(r_L^2\;-\;r_O^2\;+\;r_L^2\cot^2\Theta\Big)}\,. \qquad \text{(BVF)}$$

Here L denotes the luminosity of the light source, recall p.58.

We evaluate this formula for a light source moving in a straight line, see diagram,

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v\,t \\ y_L \\ z_L \end{pmatrix} =$$

$$= \begin{pmatrix} r_L(t)\cos\big(\alpha(t)\big)\sin\big(\Phi(t)\big) \\ r_L(t)\sin\big(\alpha(t)\big)\sin\big(\Phi(t)\big) \\ r_L(t)\cos\big(\Phi(t)\big) \end{pmatrix}$$



where v, y_L and z_L are constants. According to the picture, z_L should be negative. v is a measure for the speed of the light source and y_L for its minimum distance from the axis. If $y_L = 0$ the light source passes through the axis at $t = 0$ which means that, at this moment, an Einstein ring is produced.

Solving for $r_L(t)$ and $\Phi(t)$ yields, respectively,

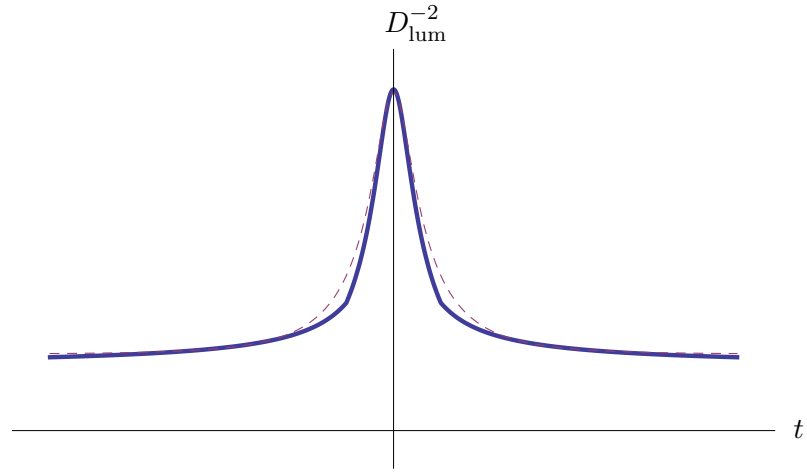
$$r_L(t) \;=\; \sqrt{v^2t^2 + y_L^2 + z_L^2}\,, \qquad \tan\Phi(t) \;=\; \frac{\sqrt{v^2t^2 + yL^2}}{z_L}\,. \qquad \text{(BVM)}$$

We assume that $0.5 < k < 1$ so that we have a double-imaging zone and a single-imaging zone (recall Section 2.14). If y_L is sufficiently small, the light source crosses the double-imaging zone during a time interval $-t_0 < t < t_0$. For t in this interval there are two images, one with $0 \le \Phi_1(t) \le \pi$ and a second one with $\Phi_2(t) = \Phi_1(t) - 2\pi$. For $|t| > t_0$ there is only one image with $0 \le \Phi_1(t) \le \pi$. The brightness of each image is given as a function of time by inserting $r_L(t)$ and $\Phi(t) = \Phi_{1/2}(t)$ from (BVM) into the flux formula (BVF). For times $|t| < t_0$ we add the fluxes from both images together. This gives us the total flux as a function of t which can be plotted.

Note that t is the time at the light source. Therefore, our construction gives us the observed light curve only if the travel time of the light rays can be considered as constant during the whole microlensing event.

Also, for a moving source D_{lum} and D_{area} differ, strictly speaking, by a redshift factor; this, however, can be ignored as long as the speed v of the light source is small in comparison to the speed of light.

The diagram shows the resulting light curve for a Barriola-Vilenkin monopole with $k = 0.99999915$. We have chosen $z_L = -0.6\, r_O$ and $y_L = 0.6 \times 10^{-6} r_O$. The dotted curve shows, for the sake of comparison, the light curve of a point lens in the quasi-Newtonian approximation, with the two free parameters u_m and Θ_E fitted such that the two curves coincide as well as possible near their maxima. One clearly sees qualitative differences: The Barriola-Vilenkin monopole produces a light curve that has a characteristic discontinuity in the derivative where the light source crosses the boundary of the double-imaging region.



3.2 Arcs and rings

In this section we are going to discuss how strong deformation effects, such as the formation of giant arcs and Einstein rings, can be theoretically described. For images of examples see p.12/13/14. We begin with the simplest mathematical model of a deflector, i.e., a point lens in the quasi-Newtonian approximation. We will see what sort of arcs such a simple model can explain, and we will then discuss how the model can be modified to explain more complicated shapes.

For a point lens, the lens equation reads

$$\vec{\beta} = \vec{\theta} - \frac{\theta_E^2}{\theta^2}\,\vec{\theta}$$

with

$$\theta_E := \sqrt{4GMD_L}c^2(D_L + D_O)D_O\,,$$

recall p.78/79. The lens map is a map, $\vec{\theta} \to \vec{\beta}$, from the deflector plane to the source plane. We want to consider an extended light source with a circular contour, i.e., a circle in the source plane, and we want to calculate the pre-images of this circle in the deflector plane. (There are two such pre-images, because a point lens produces double-imaging.) The shapes of these pre-images give us the apparent shapes of the light source in the observer's sky.

For each non-zero $\vec{\beta}$, the pre-images can be found in the following way. We write $\vec{\beta} = \beta\,\vec{e}$ with a unit vector $\vec{e}$ and a scalar $\beta > 0$. From the lens equation we read that then $\vec{\theta}$ must be of the form $\vec{\theta} = \theta\,\vec{e}$ where θ may be positive or negative. The lens equation becomes a scalar equation,

$$\beta = \theta - \frac{\theta_E^2}{\theta} \qquad \Longleftrightarrow \qquad \theta^2 - \beta\,\theta - \theta_E^2 = 0\,.$$

This quadratic equation for θ has two solutions,

$$\theta_\pm = \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} + \theta_E^2}\,.$$

Now we consider an extended light source with a circular contour given by

$$\vec{\beta} = \begin{pmatrix} \delta + \varepsilon \cos\varphi \\ \varepsilon \sin\varphi \end{pmatrix}, \qquad \varphi \in [0, 2\pi] .$$

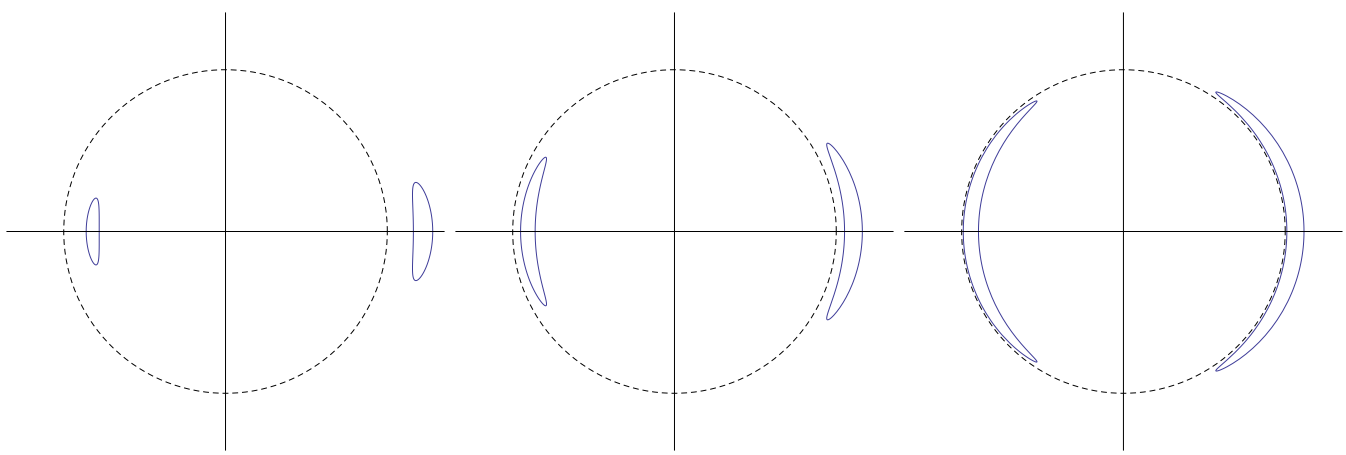
δ is the (angular) distance of the centre of the circle from the axis and ε is the (angular) radius of the circle. The two pre-images of this circle are given by

$$\vec{\theta}_\pm = \theta_\pm \, \vec{e} = \left(\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} + \theta_E^2} \right) \frac{\vec{\beta}}{\beta} =$$

$$= \frac{1}{2} \left(1 \pm \sqrt{1 + \frac{4\,\theta_E^2}{\beta^2}} \right) \vec{\beta} =$$

$$= \frac{1}{2} \left(1 \pm \sqrt{1 + \frac{4\,\theta_E^2}{\delta^2 + 2\,\delta\,\varepsilon\cos\varphi + \varepsilon^2}} \right) \begin{pmatrix} \delta + \varepsilon \sin\varphi \\ \varepsilon \cos\varphi \end{pmatrix} .$$

This gives us the contours of the two images in the sky, parametrised by the angle φ which ranges from 0 to 2π. The diagrams show plots of these parametrised curves, for a fixed value of ε and three different values of δ. We have chosen $\varepsilon = 0.1\,\theta_E$ and, from left to right, $\delta = 0.4\,\theta_E$, $\delta = 0.2\,\theta_E$ and $\delta = 0.12\,\theta_E$. The dashed line is the Einstein ring, i.e., the circle $\left|\vec{\theta}\right| = \theta_E$ which is the pre-image under the lens map of a point source precisely on the axis.



For all values of δ there is a pair of arcs. If the light source approaches the axis, $\delta \to 0$, the arcs become longer and longer and approach the Einstein ring. Note that there is a mirror symmetry with respect to the Einstein ring. This is a characteristic feature not only of the point lens but of all rotationally symmetric deflectors. Some arcs which are observed show this mirror symmetry, but others do not. In the latter case we need a deflector model that is not rotationally symmetric.

We will now determine the shape of arcs that are produced by lenses without rotational symmetry. We will restrict to the case that the deviation from rotational symmetry can be treated as a small perturbation.

As before, we assume that the quasi-Newtonian approximation is valid, so we start out from the quasi-Newtonian lens equation

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{(D_L + D_O)}{D_O} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - D_L \begin{pmatrix} \hat{\alpha}_1(\vec{\xi}) \\ \hat{\alpha}_2(\vec{\xi}) \end{pmatrix}$$

where the bending angle $\vec{\hat{\alpha}}$ is the gradient of the lensing potential,

$$\begin{pmatrix} \hat{\alpha}_1(\vec{\xi}) \\ \hat{\alpha}_2(\vec{\xi}) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V(\vec{\xi})}{\partial \xi_1} \\ \dfrac{\partial V(\vec{\xi})}{\partial \xi_2} \end{pmatrix}, \qquad V(\vec{\xi}) = \frac{4\,G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}\,') \ln\left|\vec{\xi}\,' - \vec{\xi}\right| d^2\,\vec{\xi}\,' .$$

We want to consider the case that V consists of a rotationally symmetric part and a small perturbation, i.e., that

$$V(\vec{\xi}) = V_0(|\vec{\xi}\,|) + \varepsilon\, V_1(\vec{\xi})$$

where ε is so small that, in the following, we may linearise all equations with respect to ε.

We divide the lens equation by $(D_L + D_O)$.

$$\frac{1}{(D_L + D_O)} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{D_O} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - \frac{D_L}{(D_L + D_O)} \left\{ \frac{V_0'(|\vec{\xi}\,|)}{|\vec{\xi}\,|} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \varepsilon \begin{pmatrix} \dfrac{\partial V_1(\vec{\xi})}{\partial \xi_1} \\ \dfrac{\partial V_1(\vec{\xi})}{\partial \xi_2} \end{pmatrix} \right\}$$

Switching to dimensionless (angular) variables

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{(D_L + D_O)} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{D_O} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

gives us the lens equation in the form

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \frac{D_L}{(D_L + D_O)} \left\{ \frac{V_0'(D_O\,|\vec{\theta}\,|)}{|\vec{\theta}\,|} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \frac{\varepsilon}{D_O} \begin{pmatrix} \dfrac{\partial V_1(D_O\,\vec{\theta}\,)}{\partial \theta_1} \\ \dfrac{\partial V_1(D_O\,\vec{\theta}\,)}{\partial \theta_2} \end{pmatrix} \right\}.$$

We introduce the abbreviations

$$s(|\vec{\theta}\,|) = 1 - \frac{D_L\, V_0'(D_O\,|\vec{\theta}\,|)}{(D_L + D_O)\,|\vec{\theta}\,|}, \qquad \psi(\vec{\theta}\,) = \frac{D_L\, V_1(D_O\,\vec{\theta}\,)}{D_O\,(D_L + D_O)}.$$

Then the lens equation can be written more concisely as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = s\big(|\vec{\theta}\,|\big) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \varepsilon \begin{pmatrix} \dfrac{\partial\psi(\vec{\theta}\,)}{\partial\theta_1} \\ \dfrac{\partial\psi(\vec{\theta}\,)}{\partial\theta_2} \end{pmatrix} .$$

We first consider the unperturbed system

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = s\big(|\vec{\theta}\,|\big) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} .$$

For a light source on the axis, $\beta_1 = \beta_2 = 0$, there is an Einstein ring of angular radius $\big|\vec{\theta}\,\big| = \theta_E$, where θ_E is determined by the equation $s(\theta_E) = 0$. (The latter equation may have several solutions; we pick one of them for the following consideration.) The Einstein ring $\big|\vec{\theta}\,\big| = \theta_E$ is mathematically characterised as the set in the deflector plane where the magnification μ becomes infinite. Quite generally, this set is called the *critical curve* of the lens map, and its image under the lens map is called the *caustic*. So in the unperturbed case the critical curve is a circle, $\big|\vec{\theta}\,\big| = \theta_E$, and the caustic is a point, $\vec{\beta} = \vec{0}$. We now ask what happens to the critical curve and to the caustic if the perturbation is switched on.

To determine the critical curve, we have to find all $\vec{\theta}$ where

$$\mu^{-1} = \det\Big(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\Big) = 0 .$$

It is convenient to introduce polar coordinates (r, φ) in the deflector plane,

$$\theta_1 = r\cos\varphi , \qquad \theta_2 = r\sin\varphi .$$

By the chain rule, the partial derivatives satisfy

$$\frac{\partial\psi}{\partial\theta_1} = \cos\varphi\,\frac{\partial\psi}{\partial r} - \frac{1}{r}\sin\varphi\,\frac{\partial\psi}{\partial\varphi} , \qquad \frac{\partial\psi}{\partial\theta_2} = \sin\varphi\,\frac{\partial\psi}{\partial r} + \frac{1}{r}\cos\varphi\,\frac{\partial\psi}{\partial\varphi} .$$

Then the lens map reads

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = s(r) \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} - \varepsilon \begin{pmatrix} \cos\varphi\,\dfrac{\partial\psi}{\partial r} - \dfrac{1}{r}\sin\varphi\,\dfrac{\partial\psi}{\partial\varphi} \\ \sin\varphi\,\dfrac{\partial\psi}{\partial r} + \dfrac{1}{r}\cos\varphi\,\dfrac{\partial\psi}{\partial\varphi} \end{pmatrix} .$$

With the abbreviation

$$f(r) = s(r)\,r$$

this can be rewritten as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \Big(f(r) - \varepsilon\,\frac{\partial\psi}{\partial r} \Big) \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} - \frac{\varepsilon}{r}\,\frac{\partial\psi}{\partial\varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} .$$

The equation $\mu^{-1} = 0$ holds at all points where the partial derivatives $\partial\vec{\beta}/\partial\theta_1$ and $\partial\vec{\beta}/\partial\theta_2$ are linearly dependent. On the other hand, the transformation from Cartesian to polar coordinates, $(\theta_1, \theta_2) \mapsto (r, \varphi)$, is regular everywhere except at $r = 0$. (As we want to consider a small neighbourhood of the Einstein ring $r = \theta_E \neq 0$, the coordinate singularity at $r = 0$ is irrelevant for us.) Therefore, the equation $\mu^{-1} = 0$ is equivalent to the condition of $\partial\vec{\beta}/\partial r$ and $\partial\vec{\beta}/\partial\varphi$ being linearly dependent. We now calculate these partial derivatives.

$$\frac{\partial}{\partial r}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \Big(f'(r) - \varepsilon\,\frac{\partial^2\psi}{\partial r^2}\Big)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} - \varepsilon\Big\{\frac{1}{r}\,\frac{\partial^2\psi}{\partial\varphi\partial r} - \frac{1}{r^2}\,\frac{\partial\psi}{\partial\varphi}\Big\}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}.$$

$$\frac{\partial}{\partial\varphi}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = -\varepsilon\frac{\partial^2\psi}{\partial\varphi\partial r}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} + \Big(f(r) - \varepsilon\,\frac{\partial\psi}{\partial r}\Big)\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$-\frac{\varepsilon}{r}\,\frac{\partial^2\psi}{\partial\varphi^2}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} - \frac{\varepsilon}{r}\,\frac{\partial\psi}{\partial\varphi}\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\end{pmatrix} =$$

$$= \varepsilon\Big(\frac{1}{r}\,\frac{\partial\psi}{\partial\varphi} - \frac{\partial^2\psi}{\partial\varphi\partial r}\Big)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} + \Big(f(r) - \varepsilon\,\frac{\partial\psi}{\partial r} - \frac{\varepsilon}{r}\,\frac{\partial^2\psi}{\partial\varphi^2}\Big)\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}.$$

These two vectors are linearly dependent if and only if

$$\frac{f'(r) - \varepsilon\,\dfrac{\partial^2\psi}{\partial r^2}}{\varepsilon\Big(\dfrac{1}{r}\,\dfrac{\partial\psi}{\partial\varphi} - \dfrac{\partial^2\psi}{\partial\varphi\partial r}\Big)} = \frac{\varepsilon\Big(\dfrac{1}{r^2}\,\dfrac{\partial\psi}{\partial\varphi} - \dfrac{1}{r}\,\dfrac{\partial^2\psi}{\partial\varphi\partial r}\Big)}{f(r) - \varepsilon\Big(\dfrac{\partial\psi}{\partial r} + \dfrac{1}{r}\,\dfrac{\partial^2\psi}{\partial\varphi^2}\Big)}.$$

We evaluate this equation up to linear order with respect to ε.

$$f'(r)\,f(r) - \varepsilon\Big\{f(r)\,\frac{\partial^2\psi}{\partial r^2} + f'(r)\Big(\frac{\partial\psi}{\partial r} + \frac{1}{r}\,\frac{\partial^2\psi}{\partial\varphi^2}\Big)\Big\} = O\big(\varepsilon^2\big).$$

This equation gives the critical curve in implicit form, i.e., in the form of an equation $\Phi(r, \varphi) = 0$. For the unperturbed system, the critical curve is the Einstein ring $r = \theta_E$. For the perturbed system, the critical curve is, thus, given by an equation of the form

$$r = \theta_E + \varepsilon\,\delta(\varphi) + O\big(\varepsilon^2\big)$$

with a function $\delta(\varphi)$ to be determined. By Taylor expansion,

$$f(r) = f'(\theta_E)\,\varepsilon\,\delta(\varphi) + O\big(\varepsilon^2\big), \qquad f'(r) = f'(\theta_E) + O(\varepsilon),$$

where we have used that $f(\theta_E) = s(\theta_E)\,\theta_E = 0$.

Inserting the Taylor expansions of $f(r)$ and $f'(r)$ into the expression for the critical curve yields

$$f'(\theta_E)^2\,\varepsilon\,\delta(\varphi) \,-\, \varepsilon\, f'(\theta_E)\left\{\frac{\partial\psi}{\partial r} \,+\, \frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}\right\}\Big|_{r=\theta_E} \,=\, O\big(\varepsilon^2\big)\,,$$

and, comparing leading-order terms,

$$f'(\theta_E)\,\delta(\varphi) \,=\, \left\{\frac{\partial\psi}{\partial r} \,+\, \frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}\right\}\Big|_{r=\theta_E}\,.$$

With $\delta(\varphi)$ determined by this equation, the critical curve is given by

$$r \,=\, \theta_E \,+\, \frac{\varepsilon}{f'(\theta_E)}\left\{\frac{\partial\psi}{\partial r} \,+\, \frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}\right\}\Big|_{r=\theta_E}$$

where we have neglected terms of second or higher order with respect to ε.

The caustic is the image of the critical curve under the lens map, so it is given by the equation

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \,=\, \Big(f(r) \,-\, \varepsilon\,\frac{\partial\psi}{\partial r}\Big)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \,-\, \frac{\varepsilon}{r}\frac{\partial\psi}{\partial\varphi}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} \,+\, O\big(\varepsilon^2\big) \,=$$

$$=\, \Big(f'(\theta_E)\,\varepsilon\,\delta(\varphi) \,-\, \varepsilon\,\frac{\partial\psi}{\partial r}\Big|_{r=\theta_E}\Big)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \,-\, \frac{\varepsilon}{r}\frac{\partial\psi}{\partial\varphi}\Big|_{r=\theta_E}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} \,+\, O\big(\varepsilon^2\big)\,.$$

If we neglect second and higher order terms with respect to ε, we find

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \,=\, \varepsilon\left\{\Big(\cancel{\frac{\partial\psi}{\partial r}} \,+\, \frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2} \,-\, \cancel{\frac{\partial\psi}{\partial r}}\Big)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \,-\, \frac{\varepsilon}{r}\frac{\partial\psi}{\partial\varphi}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}\right\}\Bigg|_{r=\theta_E}\,.$$

This equation gives us the caustic as a parametrised curve, $\varphi \mapsto \vec{\beta}$, in the source plane. Is this curve smooth, i.e., is the tangent vector everywhere non-zero? We calculate

$$\frac{d}{d\varphi}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \,=\, \varepsilon\left\{\frac{1}{r}\frac{\partial^3\psi}{\partial\varphi^3}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \,+\, \cancel{\frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}\right.$$
$$\left. -\,\cancel{\frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2}}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} \,-\, \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\end{pmatrix}\right\}\Big|_{r=\theta_E}\,,$$

hence

$$\frac{d}{d\varphi}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \,=\, \frac{\varepsilon}{\theta_E}\,\frac{dF(\varphi)}{d\varphi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$$

with

$$F(\varphi) \,=\, \left\{\frac{\partial^2\psi}{\partial\varphi^2} \,+\, \psi\right\}\Big|_{r=\theta_E}\,.$$

Now assume that $dF(\varphi)/d\varphi$ is either strictly positive or strictly negative for all φ; then

$$\int_0^{2\pi} \frac{dF(\varphi)}{d\varphi} d\varphi \neq 0 \, .$$

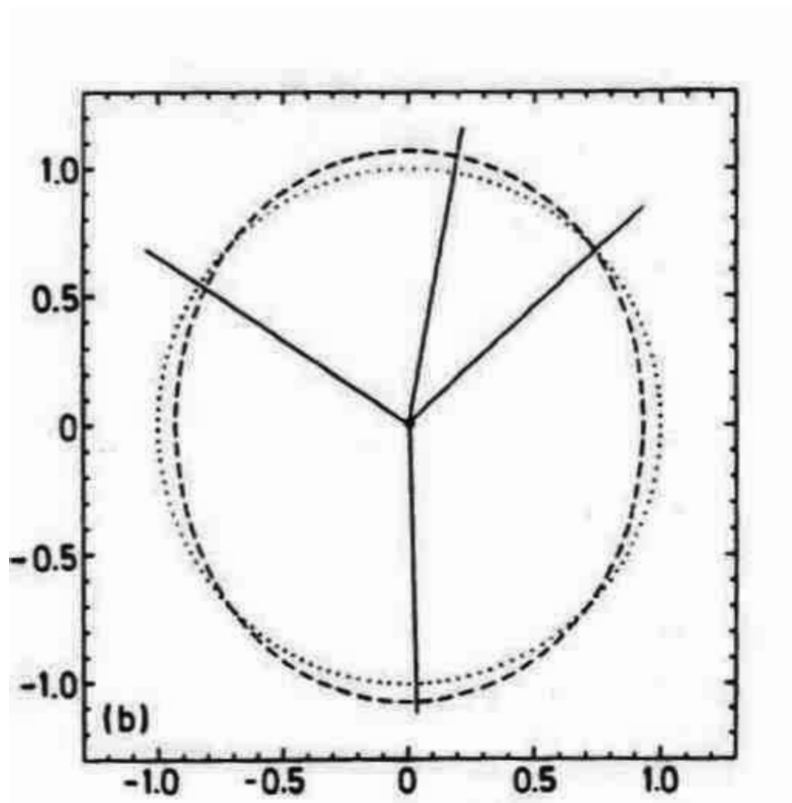
On the other hand, $\varphi = 0$ is the same point as $\varphi = 2\pi$, hence

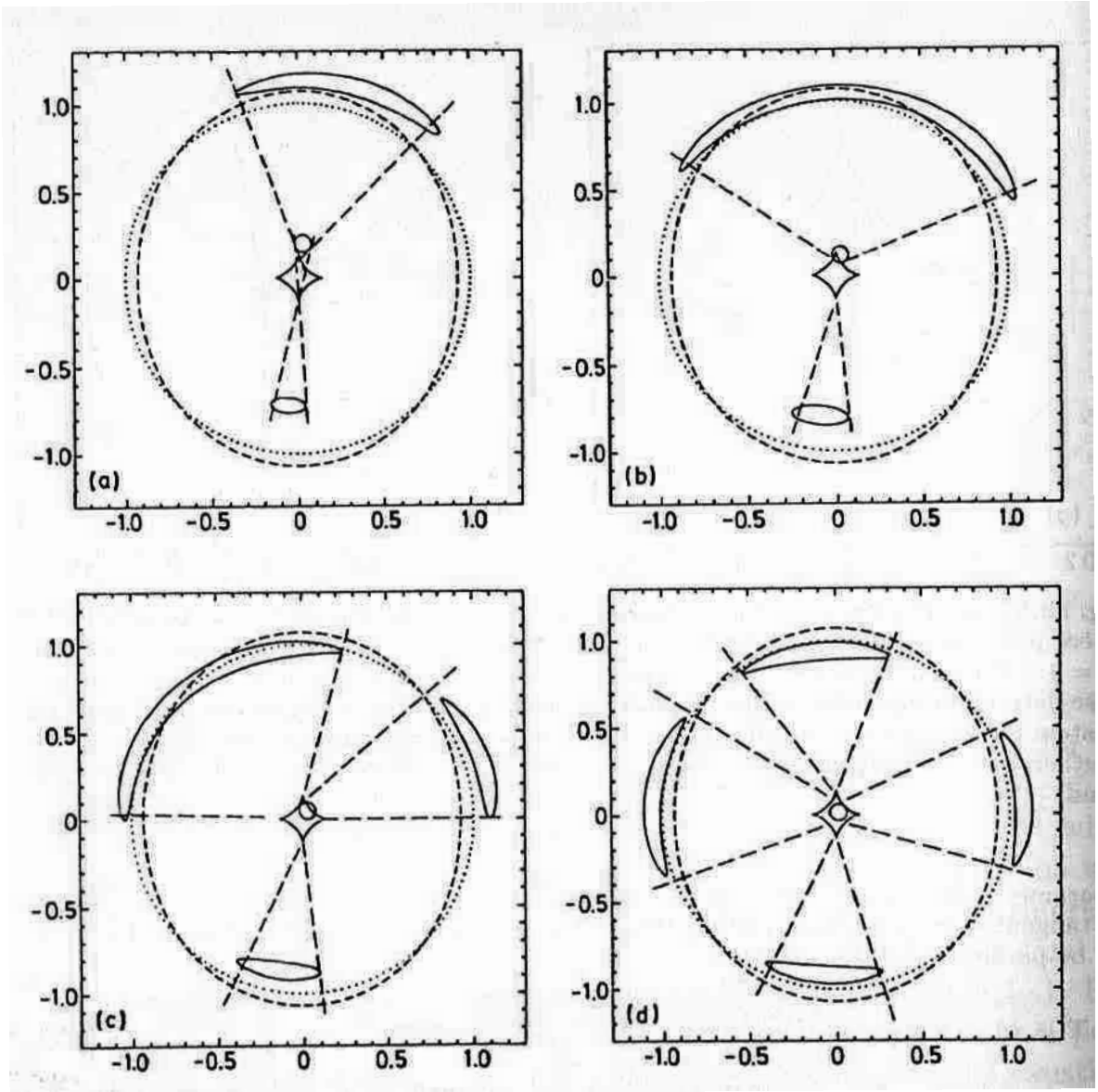
$$\int_0^{2\pi} \frac{dF(\varphi)}{d\varphi} d\varphi = F(2\pi) - F(0) = 0 \, .$$

So our hypothesis must be false. This leaves us two possibilities. The first possibility is that $dF(\varphi)/d\varphi$ is identically zero; this happens if the perturbation does not break the rotational symmetry, so the caustic is still a point. Such trivial perturbations are of no interest for our present discussion. The second possibility is that there is a point $\varphi = \varphi_0$ where $dF(\varphi)/d\varphi$ changes its sign. (Actually, because of the 2π-periodicity of φ, it is clear that then there must be at least *two* such points: One where $dF(\varphi)/d\varphi$ changes from positive to negative values and another one where it changes back from negative to positive values. A more careful analysis shows that, in the case at hand, the function $dF(\varphi)/d\varphi$ must have at least four zeros.) At such a point the tangent vector to the caustic turns backwards which means that the caustic has a cusp. We have thus proven that the caustic *must* have a cusp, for any non-trivial perturbation. (Actually, it must have at least four cusps.)

The following pictures show an example, taken from P. Schneider, J. Ehlers and E. Falco [*Gravitational Lenses*, Springer (1992)]. Here the unperturbed system is a singular isothermal sphere (cf. Worksheet 10), and the perturbation is of quadrupole symmetry (i.e., it preserves the reflection symmetry with respect to both axes). This is a reasonable model for a galaxy.

The picture on the right shows the Einstein ring of the unperturbed system (dotted) and the critical curve of the perturbed system (dashed). The pictures on the next page show the caustic (diamond shaped curve with four cusps) in the source plane and, for four different positions of an extended source with circular contour, the resulting arcs in the deflector plane. In contrast to the rotationally symmetric case, it is now possible to have more than two arcs, and they need not be symmetric with respect to the origin.





3.3 Weak lensing

If a light source is not close to the caustic, then its image is only slightly distorted. Even in this case it is possible to get information on the surface mass density Σ of the deflector with statistical methods. The idea is to assume that, in the case of no deflection by intervening masses, the ellipticities of background galaxies would be random. Any deviation from randomness would then be interpreted as a gravitational lens effect. A rather sophisticated mathematical formalism has been evaluated for numerically calculating surface mass densities from such statistical observations. This goes under the name of *weak lensing*. The method of weak lensing has been used, in particular, for determining the surface mass density in galaxy clusters; for pictures see p.16. In the following we give a rough outline of the general method.

Again, we start out from the quasi-Newtonian lens equation

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{(D_L + D_O)}{D_O} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - D_L \begin{pmatrix} \hat{\alpha}_1(\vec{\xi}) \\ \hat{\alpha}_2(\vec{\xi}) \end{pmatrix}$$

where the bending angle $\vec{\hat{\alpha}}$ is the gradient of the lensing potential,

$$\begin{pmatrix} \hat{\alpha}_1(\vec{\xi}) \\ \hat{\alpha}_2(\vec{\xi}) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V(\vec{\xi})}{\partial \xi_1} \\ \dfrac{\partial V(\vec{\xi})}{\partial \xi_2} \end{pmatrix}, \qquad V(\vec{\xi}) = \frac{4\,G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}\,') \ln\left|\vec{\xi}\,' - \vec{\xi}\right| d^2\vec{\xi}\,' .$$

It is our goal to get information on the surface mass density Σ from statistical observations of the ellipticities of background galaxies. For this purpose, it is convenient to stick with the dimensional form of the lens equation, so we will not switch to the dimensionless (angular) variables $\vec{\beta}$ and $\vec{\theta}$.

We calculate the Jacobi matrix of the lens map $\vec{\xi} \mapsto \vec{\eta}$,

$$\frac{\partial \vec{\eta}}{\partial \vec{\theta}} = \frac{(D_L + D_O)}{D_O} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{D_L D_O}{(D_L + D_O)} \begin{pmatrix} \dfrac{\partial^2 V}{\partial \xi_1^2} & \dfrac{\partial^2 V}{\partial \xi_1 \partial \xi_2} \\ \dfrac{\partial^2 V}{\partial \xi_1 \partial \xi_2} & \dfrac{\partial^2 V}{\partial \xi_2^2} \end{pmatrix} \right\} .$$

We define

$$\kappa(\vec{\xi}) = \frac{D_L D_O}{2(D_L + D_O)} \left(\frac{\partial^2 V}{\partial \xi_1^2} + \frac{\partial^2 V}{\partial \xi_2^2} \right) ,$$

$$\gamma_1(\vec{\xi}) = \frac{D_L D_O}{2(D_L + D_O)} \left(\frac{\partial^2 V}{\partial \xi_1^2} - \frac{\partial^2 V}{\partial \xi_2^2} \right) ,$$

$$\gamma_2(\vec{\xi}) = \frac{D_L D_O}{(D_L + D_O)} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} .$$

Then the Jacobi matrix reads

$$\frac{\partial \vec{\eta}}{\partial \vec{\theta}} = \frac{(D_L + D_O)}{D_O} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \right\} =$$

$$= \frac{(D_L + D_O)}{D_O} \left\{ \begin{pmatrix} 1 - \kappa & 0 \\ 0 & 1 - \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \right\} .$$

κ is a measure of the expansion of a bundle with vertex at the observer, and (γ_1, γ_2) is a measure of its shear. In this sense, the quantities κ and (γ_1, γ_2) are the quasi-Newtonian analogues of the quantities θ and (σ_1, σ_2) for light bundles in the full formalism, recall p.48.

With the complex shear

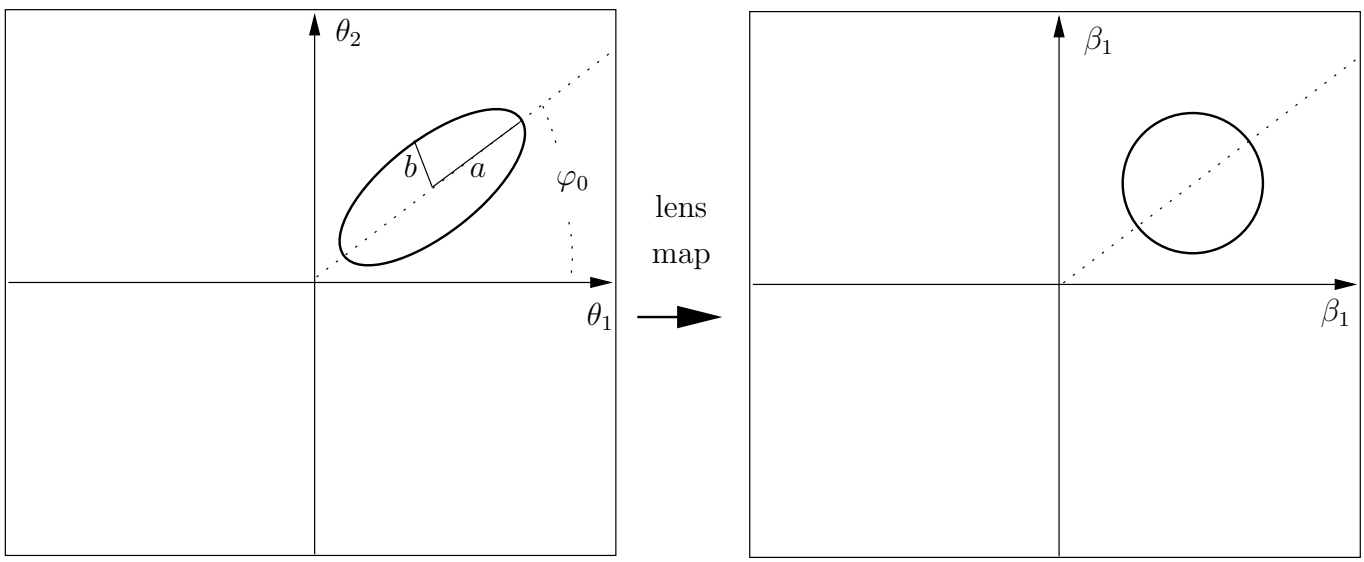
$$\gamma \,=\, \gamma_1 \,+\, i\,\gamma_2 \,=\, |\gamma|\, e^{2i\varphi_0}$$

the Jacobi matrix of the lens map reads

$$\frac{\partial\vec{\eta}}{\partial\vec{\theta}} \,=\, \frac{(D_L + D_O)}{D_O} \left\{ \begin{pmatrix} 1 \,-\, \kappa & 0 \\ 0 & 1 \,-\, \kappa \end{pmatrix} - \begin{pmatrix} |\gamma|\cos(2\varphi_0) & |\gamma|\sin(2\varphi_0) \\ |\gamma|\sin(2\varphi_0) & -\,|\gamma|\cos(2\varphi_0) \end{pmatrix} \right\} =$$

$$= \frac{(D_L + D_O)}{D_O} \left\{ \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} 1 \,-\, \kappa \,-\, |\gamma| & 0 \\ 0 & 1 \,-\, \kappa \,+\, |\gamma| \end{pmatrix} \begin{pmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \right\} .$$

The last equality sign can be easily verified by multiplying out the expression on the right-hand side. From this last expression we read that the Jacobi matrix (i.e., the linearised lens map) consists of a rotation by $-\varphi_0$, a stretching of the axes by $1 \,-\, \kappa \,-\, |\gamma|$ and $1 \,-\, \kappa \,+\, |\gamma|$ respectively, and a rotation by φ_0, see diagram.



This means that an ellipse with semi-axes a and b is mapped onto a circle if

$$\frac{b}{a} \,=\, \frac{1 \,-\, \kappa \,-\, |\gamma|}{1 \,-\, \kappa \,+\, |\gamma|} \,.$$

In the quasi-Newtonian approximation formalism it is usual to use, as a measure for the ellipticity, the complex quantity

$$\varepsilon \,=\, \frac{a-b}{a+b}\, e^{2i\varphi_0} \,=\, \frac{1 \,-\, \dfrac{b}{a}}{1 \,+\, \dfrac{b}{a}}\, e^{2i\varphi_0} \,=\, \frac{1 \,-\, \dfrac{1 \,-\, \kappa \,-\, |\gamma|}{1 \,-\, \kappa \,+\, |\gamma|}}{1 \,+\, \dfrac{1 \,-\, \kappa \,-\, |\gamma|}{1 \,-\, \kappa \,+\, |\gamma|}}\, e^{2i\varphi_0} \,=$$

$$= \frac{\cancel{1} \,-\, \cancel{\kappa} \,+\, |\gamma| \,, -\cancel{1} \,+\, \cancel{\kappa} \,+\, |\gamma|}{1 \,-\, \kappa \,+\, \cancel{|\gamma|} \,+\, 1 \,-\, \kappa \,-\, \cancel{|\gamma|}}\, e^{2i\varphi_0} \,=\, \frac{\cancel{2}\,|\gamma|}{\cancel{2}\,(\,1 \,-\, \kappa\,)}\, e^{2i\varphi_0} \,=\, \frac{\gamma}{(\,1 \,-\, \kappa\,)} \,.$$

The method of weak lensing is based on the hypothesis that intrinsic ellipticities are distributed randomly. Then $\varepsilon = \gamma/(1-\kappa)$ can be measured by subdividing the field of view into sufficiently large subfields and averaging over the background galaxies in each subfield. What information on the surface mass density Σ can be deduced from these measured values of $\varepsilon = \gamma/(1-\kappa)$? Recall that κ and γ have been defined as second-order derivatives of the lensing potential, and thus as integrals over Σ,

$$\kappa(\vec{\xi}) = \frac{D_L D_O}{2(D_L + D_O)} \left(\frac{\partial^2 V}{\partial \xi_1^2} + \frac{\partial^2 V}{\partial \xi_2^2} \right) =$$

$$= \frac{D_L D_O}{2(D_L + D_O)} \frac{4G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}\,') \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \right) \ln\left| \vec{\xi}\,' - \vec{\xi} \right| d^2 \vec{\xi}\,' ,$$

$$\gamma(\vec{\xi}) = \frac{D_L D_O}{2(D_L + D_O)} \left(\frac{\partial^2 V}{\partial \xi_1^2} - \frac{\partial^2 V}{\partial \xi_2^2} + 2\,i \frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} \right) =$$

$$= \frac{D_L D_O}{2(D_L + D_O)} \frac{4G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}\,') \left(\frac{\partial^2}{\partial \xi_1^2} - \frac{\partial^2}{\partial \xi_2^2} + 2\,i \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \right) \ln\left| \vec{\xi}\,' - \vec{\xi} \right| d^2 \vec{\xi}\,' .$$

In the case of κ, we can use that $\ln\left| \vec{\xi}\,' - \vec{\xi} \right|$ is the Green function of the two-dimensional Laplacian,

$$\left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \right) \ln\left| \vec{\xi}\,' - \vec{\xi} \right| = 2\,\pi\,\delta(\vec{\xi}\,' - \vec{\xi}) ,$$

where δ is the Dirac-delta distribution, hence

$$\kappa(\vec{\xi}) = \frac{4\,\pi\,G\,D_L\,D_O}{c^2\,(D_L + D_O)} \Sigma(\vec{\xi}) .$$

In the case of γ, the integration cannot be carried through explicitly, but the integral can be simplified to

$$\gamma(\vec{\xi}) = \frac{-4\,G\,D_L\,D_O}{c^2\,(D_L + D_O)} \int_{\mathbb{R}^2} \frac{\Sigma(\vec{\xi}\,')\; d^2 \vec{\xi}\,'}{\left(\xi_1 - \xi_1' - i\,(\xi_2 - \xi_2') \right)^2} .$$

If $\varepsilon(\vec{\xi})$ has been determined by observations, the equation $1 - \kappa = \varepsilon\,\gamma$ becomes an integral equation for the unknown function $\kappa(\vec{\xi})$, namely

$$1 - \kappa(\vec{\xi}) = -\varepsilon(\vec{\xi}) \int_{\mathbb{R}^2} \frac{2\,\kappa(\vec{\xi}\,')\; d^2 \vec{\xi}\,'}{\left(\xi_1 - \xi_1' - i\,(\xi_2 - \xi_2') \right)^2} .$$

This equation can be solved iteratively: Start with a guess for κ as your zeroth order approximation. Insert it on the right-hand side and calculate what the left-hand side gives for κ. Use this as your first order approximation, feed it into the right-hand side and calculate what the left-hand side now gives for κ. Use this as your second order approximation, and so on.

This gives the basic idea of how the blue clouds in the pictures on p.16 have been determined. Note that it is primarily κ, not Σ, that is determined. These two quantities differ by a constant factor. Calculation of the surface mass density Σ from the dimensionless quantity κ requires that the factor $D_L D_O/(D_L + D_O)$ is known (or can be reasonably guessed). Also note that there are many technical problems when this method is worked out in practice. Most importantly, every telescope shows ideal points as fuzzy discs of a certain finite diameter. (This is caused by aberrations of the telescope optic, in addition to the theoretical limits placed on the resolving power by diffraction.) As a consequence, there is a systematic error in the measurement of ellipticities, making all ellipses more circular than they actually are. Typically, this systematic error is bigger than the effect on the ellipticities caused by weak lensing! Therefore, any weak lensing observation requires a careful measurement of how ideal points are mapped by the telescope used. This deformation effect varies over the field of view of the telecope, and it is known as the *point-spread function*.

This last chapter on applications to astrophysics was largely based on the quasi-Newtonian approximation formalism. As a matter of fact, this approximation formalism can satisfactorily explain all lensing observations up to now. However, with the further improvement of telescopes, in particular radio telescopes, it can be expected that lensing effects beyond the quasi-Newtonian approximation formalism will be observed soon. In particular, we expect that the shadow of the black hole at the centre of our galaxy will be seen within a few years. The very notion of this shadow cannot be understood on the basis of approximations assuming that bending angles are small. So we will really need the full formalism of General Relativity for understanding these features. The mathematical methods developed in Chapter 2 of this course provide the necessary tools for investigating this next generation of lensing observations.