Cosmology

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Lectures: Tue 12–14, NW1, N1250
Fri 12–13, NW1, N1250
Tutorials: Fri 13–14, NW1, N1250

Video recordings of the lectures are available at `http://mlecture.uni-bremen.de/ml/`

General Relativity text-books with detailed sections on cosmology

W. Rindler: "Relativity" Oxford UP (2001) Chicago University Pess (1984)
H. Stephani: "Relativity" 3rd edition, Cambridge UP (2004)
L. Ryder: "Introduction to General Relativity" Cambridge UP (2009)

Monographs

V. Mukhanov: "Physical foundations of cosmology" Cambridge UP (2005)
G. Ellis, R. Maartens, M. MacCallum: "Relativistic Cosmology" Cambridge UP (2012)

Living Reviews (`http://relativity.livingreviews.org`)

N. Jackson: The Hubble Constant, lrr-2015-2
S. Carroll: The Cosmological Constant, lrr-2001-1
A. Jones and A. Lasenby: The Cosmic Microwave Background, lrr-1998-11

Contents

1. Historic Introduction

1826 W. Olbers formulates the "Olbers paradox": If we live in a static and eternal universe uniformly filled with stars, then the sky at night must be infinitely bright. The same observation had been made already earlier, by T. Digges ($\approx$ 1580), by J. Kepler (1610) and by J.-P. de Cheseaux (1744).

1915 A. Einstein publishes the field equation of general relativity, in the version without a cosmological constant.

1916 A. Einstein introduces the cosmological constant in order to get static cosmological solutions.

1922/24 A. Friedmann finds the class of homogeneous and isotropic perfect-fluid solutions to Einstein's field equation named after him.

1927 G. Lemaître re-obtains the Friedmann solutions and comes to the conclusion that the universe began with an initial singularity which he called the "primeval atom". In the 1960s, F. Hoyle coins the term "big bang".

1929 E. Hubble discovers the linear distance-redshift relation ("Hubble law") which is usually recognised as observational evidence for an expanding universe.

1936 F. Zwicky postulates the existence of dark matter in order to explain the stability of galaxy clusters.

1941 A. McKellar observes spectral lines from rotational transitions in cyanogen (CN) molecules in the interstellar medium. He comes to the conclusion that the interstellar medium must have a temperature of approximately 2.3 Kelvin. In hindsight, this is the first detection of the cosmic background radiation.

1946-49 G. Gamow and his PhD student R. Alpher develop a theory how hydrogen, helium and the heavier elements were created in the correct proportions after the initial singularity from a state they called "Ylem". As a joke, Gamow puts H. Bethe, who actually was not involved, as a co-author on their paper (Alpher-Bethe-Gamow = $\alpha\beta\gamma$).

1948 R. Alpher and R. Herman predict the cosmic background radiation.

1948 H. Bondi, T. Gold and F. Hoyle invent the steady-state theory in which the universe is expanding, but the matter density remains constant because of a continuous creation of matter. The steady-state theory remains an important rival to the big-bang theory until the detection of the cosmic background radiation is recognised.

1956 W. Rindler introduces in his PhD Thesis the notions of event horizons and particle horizons.

1955-57 E. Leroux, T. Shmaonov and E. Ohm independently observe microwave radiation with the features of the predicted cosmic background radiation. Their observations remain widely unnoticed and are not recognised at the time.

1960-1970 V. Belinsky, I. Khalatnikov and L. Lifshits study general features of cosmological models with an initial singularity.

1964 A. Doroshkevich and I. Novikov make precise predictions for the existence of the cosmic background radiation.

1963-1969 R. Penrose and S. Hawking prove a series of theorems to the effect that the formation of a singularity is generic for solutions to Einstein's field equation where the energy-momentum tensor satisfies certain "energy conditions".

1964 A. Penzias and R. Wilson, when testing a new radio antenna, discover a mysterious isotropic noise. R. Dicke explains to them that they have found the predicted cosmic background radiation. Penzias and Wilson win the Nobel Prize in 1978.

1970-1980 V. Rubin performs a long series of spectral measurements to determine the rotation curves in galaxies. This provides strong evidence for the existence of dark matter.

1980 A. Starobinsky and A. Guth independently introduce the idea of inflation, i.e., that at an early stage the universe was exponentially expanding.

1989-1993 The satellite COBE investigates the cosmic background radiation. It is found that the radiation has a black-body spectrum with a temperature that is almost but not exactly isotropic. For these discoveries J. Mather and G. Smoot win the Nobel Prize in 2006.

1998 By using supernovae of type Ia as standard candles two teams independently find evidence for the fact that the expansion of the universe is accelerating. For the mysterious type of "matter" that causes the accelerated expansion M. Turner coins the word "dark energy". S. Perlmutter, B. Schmidt and A. Riess win the Nobel Prize in 2011 for the discovery of the accelerated expansion.

2001-2010 The satellite WMAP investigates the cosmic background radiation.

2008-2013 The satellite Planck complements these observations.

2. Brief review of general relativity

A general-relativistic spacetime is a pair (M, g) where:

- M is a four-dimensional manifold; local coordinates will be denoted (x^0, x^1, x^2, x^3) and Einstein's summation convention will be used for greek indices $\mu, \nu, \sigma, \ldots = 0, 1, 2, 3$, for lower case latin indices $i, j, k, \ldots = 1, 2, 3$ and for upper case latin indices $A, B, C, \ldots = 1, 2$.

- g is a Lorentzian metric on M, i.e. a covariant second-rank tensor field, $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, that is

  (a) symmetric, $g_{\mu\nu} = g_{\nu\mu}$, and

  (b) non-degenerate with Lorentzian signature, i.e., for any $p \in M$ there are coordinates defined near p such that $g|_p = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

As the metric is non-degenerate, we may introduce contravariant metric components by

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \, .$$

Here and in the following, δ^μ_σ denotes the Kronecker delta, $\delta^\mu_\sigma = 1$ if $\mu = \sigma$ and $\delta^\mu_\sigma = 0$ if $\mu \neq \sigma$. We use $g^{\mu\nu}$ and $g_{\sigma\tau}$ for raising and lowering indices, e.g.

$$g_{\rho\tau} A^\tau = A_\rho \, , \qquad B_{\mu\nu} g^{\nu\tau} = B_\mu{}^\tau \, .$$

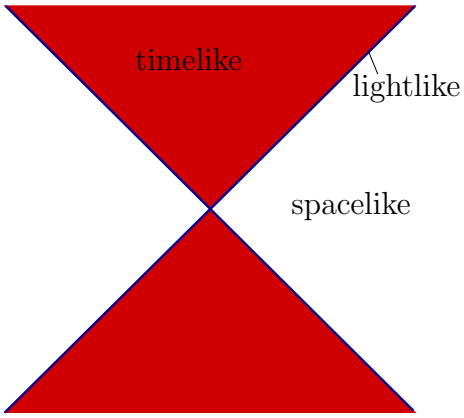
The metric contains all information about the spacetime geometry and thus about the gravitational field. In particular, the metric determines the following.

- The causal structure of spacetime:

  A curve $s \mapsto x(s) = \big(x^0(s), x^1(s), x^2(s), x^3(s)\big)$ is called

  $$\left.\begin{array}{l}\text{spacelike}\\ \text{lightlike}\\ \text{timelike}\end{array}\right\} \iff g_{\mu\nu}\big(x(s)\big)\dot{x}^\mu(s)\dot{x}^\nu(s) \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array}\right.$$

  Timelike curves describe motion at subluminal speed and lightlike curves describe motion at the speed of light. Spacelike curves describe motion at superluminal speed which is forbidden for signals.

  For timelike curves we can choose the parametrisation such that $g_{\mu\nu}\big(x(\tau)\big)\dot{x}^\mu(\tau)\dot{x}^\mu(\tau) = -c^2$. The parameter τ is then called *proper time*.

  The motion of a material continuum, e.g. of a fluid, can be described by a vector field $U = U^\mu \partial_\mu$ with $g_{\mu\nu} U^\mu U^\nu = -c^2$. The integral curves of U are to be interpreted as the worldlines of the fluid elements.

- The geodesics:

  By definition, the geodesics are the solutions to the Euler-Lagrange equations

$$\frac{d}{ds}\frac{\partial \mathcal{L}(x,\dot{x})}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}(x,\dot{x})}{\partial x^\mu} = 0$$

  of the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) = \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \,.$$

  These Euler-Lagrange equations take the form

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}(x)\dot{x}^\nu\dot{x}^\sigma = 0$$

  where

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}\, g^{\mu\tau}\left(\,\partial_\nu g_{\tau\sigma} + \partial_\sigma g_{\tau\nu} - \partial_\tau g_{\nu\sigma}\,\right)$$

  are the socalled Christoffel symbols.

  The Lagrangian $\mathcal{L}(x,\dot{x})$ is constant along a geodesic (see Worksheet 1), so we can speak of timelike, lightlike and spacelike geodesics. Timelike geodesics ($\mathcal{L} < 0$) are to be interpreted as the worldlines of freely falling particles, and lightlike geodesics ($\mathcal{L} = 0$) are to be interpreted as light rays.

  The Christoffel symbols define a *covariant derivative* that takes tensor fields into tensor fields, e.g.

$$\nabla_\nu U^\mu = \partial_\nu U^\mu + \Gamma^\mu{}_{\nu\tau}U^\tau \,,$$
$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\rho{}_{\nu\mu}A_\rho \,.$$

- The curvature.

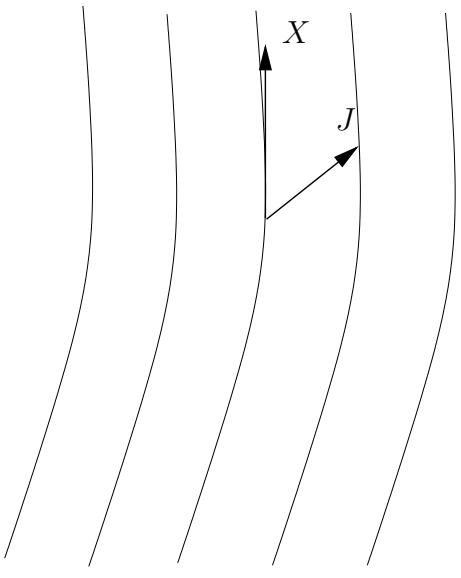
  The Riemannian curvature tensor is defined, in coordinate notation, by

$$R^\tau{}_{\mu\nu\sigma} = \partial_\mu\Gamma^\tau{}_{\nu\sigma} - \partial_\nu\Gamma^\tau{}_{\mu\sigma} + \Gamma^\rho{}_{\nu\sigma}\Gamma^\tau{}_{\mu\rho} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\tau{}_{\nu\rho}\,.$$

  The curvature tensor determines the relative motion of neighbouring geodesics: If $X = X^\mu\partial_\mu$ is a vector field whose integral curves are geodesics, and if $J = J^\nu\partial_\nu$ connects neighbouring integral curves of X (i.e., if the Lie bracket between X and J vanishes), then the *equation of geodesic deviation* or *Jacobi equation* holds:

$$\big(X^\mu\nabla_\mu\big)\big(X^\nu\nabla_\nu\big)J^\sigma = R^\sigma{}_{\mu\nu\rho}X^\mu J^\nu X^\rho \,.$$

  If the integral curves of X are timelike, they can be interpreted as worldlines of freely falling particles. In this case the curvature term in the Jacobi equation gives the *tidal force* produced by the gravitational field.



The curvature tensor satisfies the identities

$$R^{\tau}{}_{\mu\nu\sigma} = -R^{\tau}{}_{\nu\mu\sigma}\,,$$

$$R_{\tau\mu\nu\sigma} = -R_{\sigma\mu\nu\tau}\,,$$

$$R^{\tau}{}_{\mu\nu\sigma} + R^{\tau}{}_{\sigma\mu\nu} + R^{\tau}{}_{\nu\sigma\mu} = 0 \quad (1^{\text{st}} \text{ Bianchi identity})\,,$$

$$\nabla_\rho R^{\tau}{}_{\mu\nu\sigma} + \nabla_\nu R^{\tau}{}_{\rho\mu\sigma} + \nabla_\mu R^{\tau}{}_{\nu\rho\sigma} = 0 \quad (2^{\text{nd}} \text{ Bianchi identity})\,.$$

From the curvature tensor one defines the Ricci tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\sigma\mu\nu}$$

and the Ricci scalar

$$R = R_{\mu\nu} g^{\mu\nu}\,.$$

The Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$. In three dimensions, the curvature tensor is completely determined by the Ricci tensor and the metric tensor. In two dimensions, the curvature tensor is completely determined by the Ricci scalar and the metric tensor.
The spacetime metric is determined, in terms of its sources, by *Einstein's field equation*

$$R_{\mu\nu} \,-\, \frac{R}{2}\, g_{\mu\nu} \,+\, \Lambda\, g_{\mu\nu} \,=\, \kappa\, T_{\mu\nu}\,.$$

The curvature quantity

$$G_{\mu\nu} \,=\, R_{\mu\nu} \,-\, \frac{R}{2}\, g_{\mu\nu}$$

is called the *Einstein tensor field*, Λ is called the *cosmological constant*, and κ is called *Einstein's gravitational constant*.
Based on cosmological observations we believe that we live in a universe with a positive cosmological constant that is of the order of $\Lambda \approx 10^{-52}\mathrm{m}^{-2}$, as we will discuss in detail later.
Einstein's gravitational constant is related to Newton's gravitational constant G according to $\kappa = 8\pi G/c^2$ as follows from the Newtonian limit of Einstein's theory.
The energy-momentum tensor $T_{\mu\nu}$ on the right-hand side of the field equation depends on the matter model that is used for the source of the gravitational field. The most important cases are the following.

- Vacuum: $T_{\mu\nu} = 0$
  Then the field equation simplifies to $R_{\mu\nu} = \Lambda g_{\mu\nu}$, as can be verified by calculating the trace of the field equation and then re-inserting the result into the field equation. The vacuum field equation is a system of ten scalar second-order non-linear partial differential equations for the ten independent metric coefficients $g_{\mu\nu}$. The best known solutions to Einstein's vacuum field equation with $\Lambda = 0$ are the Schwarzschild solution and the Kerr solution. An important solution with $\Lambda > 0$ is the deSitter metric and with $\Lambda < 0$ the anti-deSitter metric. Both will be discussed in this course.

- Electrovacuum: $T_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}$
  In this case Einstein's field equation together with Maxwell's equations gives a system of partial differential equations for the $g_{\mu\nu}$ and the electromagnetic field strength $F_{\mu\nu}$. The best-known electrovacuum solutions without a cosmological constant are the Reissner-Nordström solution (field outside of a charged spherically symmetric static object) and the Kerr-Newman solution (field of a charged and rotating black hole).

- Perfect fluid: $T_{\mu\nu} = \Big(\mu + \frac{p}{c^2}\Big)U_\mu U_\nu + p\, g_{\mu\nu}$
  For solving Einstein's field equation with a perfect-fluid source, one has to specify an equation of state linking the pressure p to the energy density μ. Then Einstein's equation together with the Euler equation

$$\Big(\mu + \frac{p}{c^2}\Big)U^\rho \nabla_\rho U^\sigma + \nabla_\tau\Big(g^{\tau\sigma} + \frac{1}{c^2}U^\tau U^\sigma\Big) = 0$$

  gives a system of partial differential equations for the $g_{\mu\nu}$, the four-velocity U^ρ and the energy density μ. Actually, the Euler equation is a consequence of the energy conservation law $\nabla_\mu T^{\mu\nu} = 0$ which follows from Einstein's field equation. So in the case of a perfect fluid Einstein's field equation determines the equation of motion of the matter source.

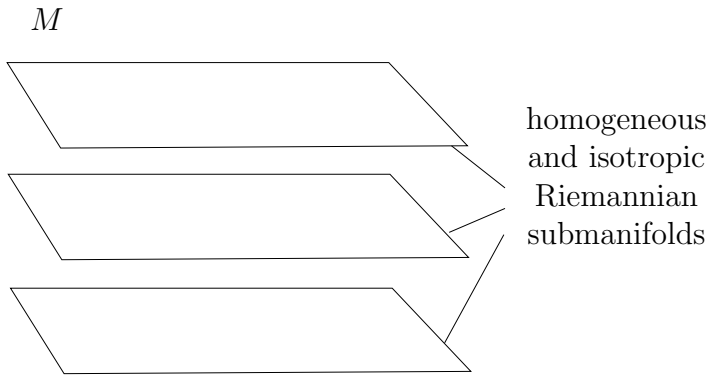
  Perfect fluid solutions without a cosmological constant are of interest as models for the interior of stars. The interior Schwarzschild solution is an example; it describes a spherically symmetric static star with constant density μ. In this course we will intensively study the socalled Friedmann solutions, which are the simplest cosmological models of our universe. They are perfect fluid solutions, possibly with a cosmological constant. If time permits, we will also consider some other cosmological solutions, e.g. the rather pathological Goedel universe (Kurt Goedel's birthday present to Einstein on occasion of his 70th birthday in 1949) which is a dust solution ($p = 0$) with a cosmological constant, and maybe also some cosmological models that are homogeneous but not isotropic (Bianchi models).

2. Homogeneous and isotropic cosmology

In Section 2.1 we give a characterisation of spacetime models that are spatially homogeneous and isotropic and we discuss their properties. This consideration is purely kinematic, i.e., Einstein's field equation is not used. In the subsequent section we will then solve the field equation within the class of spatially homogeneous and isotropic spacetimes, and we will discuss some solutions and their properties in detail.

2.1 Robertson-Walker spacetimes

By definition, a Robertson-Walker spacetime is a general-relativistic spacetime that can be sliced into 3-dimensional spacelike submanifolds that are homogeneous and isotropic, see picture. The general form of such metrics was determined independently by H. P. Robertson and by A. Walker in 1935.



The first step is to determine the geometry of the time slices. The assumption that they are spacelike means that they inherit from the spacetime metric a Riemannian (i.e., positive definite) metric. So the task is to determine all 3-dimensional Riemannian manifolds that are homogeneous and isotropic. Here "homogeneous" means that there are no distinguished points and "isotropic" means that there are no distinguished directions.
We consider a 3-dimensional manifold with a Riemannian metric $g_{ik}dx^i dx^k$ whose curvature tensor we denote by $R^i{}_{jkl}$. (Recall our convention of having latin indices running from 1 to 3.) As $R^i{}_{jk}{}^l = -R^i{}_{kj}{}^l$ and $R^i{}_{jk}{}^l = -R^l{}_{jk}{}^i$, we can define at each point a linear map from the space of antisymmetric second-rank tensors $\Lambda_2 = \left\{\omega_{li}dx^l \otimes dx^i \middle| \omega_{li} = -\omega_{il}\right\}$ onto itself by

$$\Lambda_2 \longrightarrow \Lambda_2$$

$$\omega_{li}dx^l \otimes dx^i \longmapsto \hat{\omega}_{jk}dx^j \otimes dx^k = R^i{}_{jk}{}^l\omega_{li}dx^j \otimes dx^k \,.$$

Owing to the first Bianchi identity, this linear map is symmetric (with respect to the positive definite scalar product induced by the metric), so it has three linearly independent eigenvectors. If the eigenvalues would be different from each other, the eigenvectors would define distinguished directions in the tangent space, in contradiction to the assumption of isotropy. So the three eigenvalues must be equal, i.e, the linear map must be a multiple of the identity map,

$$R^i{}_{jk}{}^l = K\big(\delta^i_k\delta^l_j - \delta^i_j\delta^l_k\big) \qquad (\star)$$

with a scalar factor K. (Note that $\big(\delta^i_k\delta^l_j - \delta^i_j\delta^l_k\big)\omega_{li} = \omega_{jk} - \omega_{kj} = 2\omega_{jk}$, i.e., that the antisymmetrised product of Kroneckers acts, indeed, as the identity operator on antisymmertric tensor fields, up to a factor 2 that was absorbed in the K.) The condition of homogeneity requires K to be a constant. It is common to write $(\star)$ in terms of the covariant components of the curvature tensor. Then the conditions of homogeneity and isotropy require that

$$R_{ijkl} = K\big(g_{ik}g_{lj} - g_{ij}g_{lk}\big)\,, \quad K = \text{constant}.$$

One says that a Riemannian manifold is a "space of constant curvature" if this condition holds. It is interesting to note that in all dimensions $n \neq 2$, and thus in particular in the case $n = 3$ considered here, K is necessarily a constant if the condition of isotropy holds at every point, i.e., the condition of homogeneity need not be required separately. To prove this, it is sufficient to apply the second Bianchi identity to the curvature tensor from $(\star)$ and to contract two pairs of indices away; this results in $(n-2)\nabla_i K = 0$.

We have demonstrated that homogeneous and isotropic Riemannian manifolds are necessarily spaces of constant curvature. One can prove that, conversely, the spaces of constant curvature are precisely those Riemannian manifolds for which the conditions of homogeneity and isotropy hold *locally*. More precisely, one can prove the following:

- Any two Riemannian manifolds of constant curvature with the same dimension and the same K are locally isometric. In other words, locally there is only one such geometry for each K. The global structure is not uniquely determined. Spaces of constant curvature that are geodesically complete are called *space forms*. For low dimensions, the space forms have been classified.

- A space of constant curvature is a space with the maximal number of local symmetries. Local symmetries are characterised in terms of *Killing vector fields*, see Worksheet 2. By definition, a vector field $K = K^i\partial_i$ is a Killing vector field if the Lie derivative of the metric in the direction of K vanishes. If K has no zeros, this is equivalent to saying that there is a coordinate system in which $K = \partial_1$ and the metric coefficients are independent of x^1. One can show that the linear combination of two Killing vector fields with *constant* coefficients is again a Killing vector field, i.e., that the Killing vector fields form a vector space over the real numbers. One can further show that the dimension of this vector space for an n-dimensional manifold cannot be bigger than $n(n+1)/2$. So on a 3-dimensional manifold there are at most six linearly independent Killing vector fields. The maximal number is just reached in the case of homogeneity and isotropy where we have 3 translations and 3 rotations.

So the possible time-slices in a Robertson-Walker spacetime are 3-dimensional Riemannian manifolds of constant curvature. As a pre-exercise, to get a better geometric intuition, we consider the 2-dimensional Riemannian manifolds of constant curvature. In this case we use capital indices, taking the values 1 and 2, and we write the curvature tensor as

$$R_{ABCD} = K\big(g_{AC}g_{BD} - g_{AB}g_{CD}\big)\,, \qquad K = \text{constant}.$$

This implies that the Ricci tensor is

$$R_{CD} = R^A{}_{ACD} = K\big(g_{CD} - 2g_{CD}\big) = -Kg_{CD}$$

and the Ricci scalar is

$$R = -2K\,.$$

So with our conventions the Ricci scalar is negative for a space of positive constant curvature and vice versa. We consider the cases $K = 0$, $K > 0$ and $K < 0$ separately.

$K = 0$: The condition $K = 0$ means that $R_{ABCD} = 0$ which is certainly true for the Euclidean plane.

$$g = dx^2 + dy^2\,.$$

If we transform to polar coordinates,

$$x = a\,\chi\cos\varphi\,,$$

$$y = a\,\chi\sin\varphi\,,$$

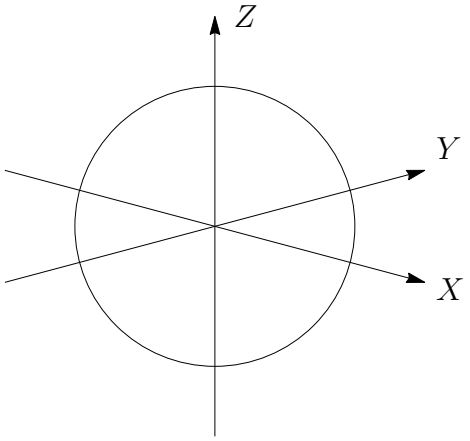
where a is a constant with the dimension of a length and χ is a dimensionless radial coordinate, the metric reads

$$g = a^2\big(d\chi^2 + \chi^2 d\varphi^2\big)\,.$$

All other 2-dimensional Riemannian manifolds of constant curvature $K = 0$ are locally isometric to the Euclidean plane. The space forms, i.e., the geodesically complete cases, can be constructed as quotient manifolds from the plane, i.e., by identifying points on the plane. In addition to the plane itself, there are four of them: The cylinder, the torus, the Moebius strip and the Klein bottle.

$K > 0$: An obvious candidate for a space of constant positive curvature is the sphere of radius a which is defined as a submanifold of Euclidean 3-space by the equation $X^2+Y^2+Z^2 = a^2$. We can parametrise the sphere by angle coordinates (χ, φ) via

$$X = a\cos\varphi\sin\chi\,,$$
$$Y = a\sin\varphi\sin\chi\,,$$
$$Z = a\cos\chi\,.$$



Then we have on the sphere

$$dX = a\cos\varphi\cos\chi\,d\chi - a\sin\varphi\sin\chi\,d\varphi\,,$$
$$dY = a\sin\varphi\cos\chi\,d\chi + a\cos\varphi\sin\chi\,d\varphi\,,$$
$$dZ = -a\sin\chi\,d\chi\,.$$

Inserting these results into the expression $dX^2+dY^2+dZ^2$ demonstrates that the Euclidean 3-metric induces on the sphere the metric

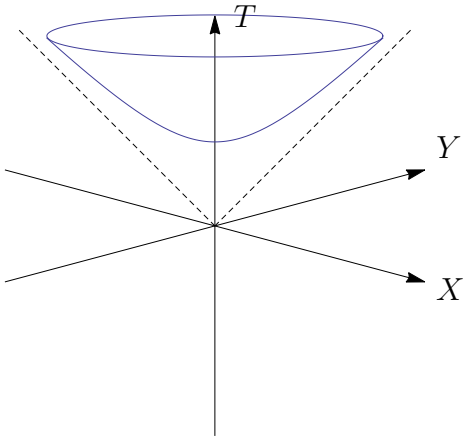
$$g = a^2\big(d\chi^2 + \sin^2\chi\,d\varphi^2\big)\,.$$

By calculating the Christoffel symbols and, thereupon, the Riemann tensor one easily verifies that the condition of constant curvature is indeed satisfied with $K = 1/a^2$.

Again, all other 2-dimensional spaces of constant curvature $K > 0$ are locally isometric to the sphere. The only other space form is 2-dimensional projective space which results from the sphere by identifying antipodal points. 2-dimensional projective space cannot be globally embedded into Euclidean 3-space.

$K < 0$: Guided by the case $K > 0$, one tries a similar construction but now in a way that the curvature comes out as $K = -1/a^2$. This requires to change some signs in the signature of the ambient space and of the embedding formula: The manifold is now given by the equation $X^2 + Y^2 - T^2 = -a^2$ and the ambient space has Minkowskian signature, i.e., the metric is $dX^2 + dY^2 - dT^2$. This defines a hyperboloid which can be parametrised as

$$X = a\cos\varphi\sinh\chi\,,$$
$$Y = a\sin\varphi\sinh\chi\,,$$
$$T = a\cosh\chi\,.$$



Then we have on the hyperboloid

$$dX = a\cos\varphi\cosh\chi\,d\chi - a\sin\varphi\sinh\chi\,d\varphi\,,$$
$$dY = a\sin\varphi\cosh\chi\,d\chi + a\cos\varphi\sinh\chi\,d\varphi\,,$$
$$dZ = a\sinh\chi\,d\chi\,.$$

Inserting these results into the expression $dX^2 + dY^2 - dZ^2$ demonstrates that the 3-dimensional Minkowski metric induces on the hyperboloid the metric

$$g = a^2\big(d\chi^2 + \sinh^2\chi\, d\varphi^2\big)\,.$$

Again, the Christoffel symbols and the components of the curvature tensor are readily calculated and one finds that, indeed, this is a space of constant curvature with $K = -1/a^2$.

The space of constant negative curvature is known as *hyperbolic space*, as *Lobachevsky space* or as *Lobachevsky-Bolyai* space, named after Nikolai Lobachevsky and Jánós Bolyai who independently discovered this geometry in the 1820s. In hyperbolic space, the sum of the angles in a geodesic triangle is smaller than π while on a sphere it is bigger. Hyperbolic geometry satisfies all axioms of Euclid, i.e., all axioms of flat geometry, with the exception of the parallel axiom.

In addition to the isometric embedding into 3-dimensional Minkowski space, hyperbolic space can be represented in various different ways. However, it cannot be isometrically embedded into 3-dimensional Euclidean space. (A non-global embedding is possible in the form of a surface of revolution generated by a *tractrix*.)

Other space forms of negative curvature can be constructed as quotient manifolds from the hyperboloid in Minkowski spacetime.

We summarise our results in the following way: A 2-dimensional Riemannian manifold of constant curvature K is given by the metric

$$g = a^2\big(d\chi^2 + \eta(\chi)^2 d\varphi^2\big)\,, \qquad \eta(\chi) = \begin{cases} \sin\chi & \text{for } K = \dfrac{1}{a^2} > 0\,, \\ \chi & \text{for } K = 0\,, \\ \sinh\chi & \text{for } K = -\dfrac{1}{a^2} < 0\,. \end{cases}$$

The transition from 2 to 3 dimensions simply requires the circle parametrised by φ to be replaced with a sphere parametrised by standard spherical coordinates (ϑ, φ), i.e., a 3-dimensional Riemannian manifold of constant curvature K is given by the metric

$$g = a^2\big(d\chi^2 + \eta(\chi)^2\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\}\big)\,, \qquad \eta(\chi) = \begin{cases} \sin\chi & \text{for } K = \dfrac{1}{a^2} > 0\,, \\ \chi & \text{for } K = 0\,, \\ \sinh\chi & \text{for } K = -\dfrac{1}{a^2} < 0\,. \end{cases}$$

One cannot visualise a 3-dimensional space of constant curvature, except in the case $K = 0$, but one can visualise the equatorial section $\vartheta = \pi/2$ which is a 2-dimensional space of constant curvature.

There are two alternative coordinate representations of spaces of constant curvature. Firstly, we can make a coordinate transformation $(\chi, \vartheta, \varphi) \mapsto (\eta, \vartheta, \varphi)$ with $\eta(\chi)$ from above. For $K > 0$ we find

$$\eta = \sin\chi\,, \quad d\eta = \cos\chi\, d\chi = \sqrt{1-\eta^2}\, d\chi\,,$$

for $K = 0$ we simply have

$$\eta = \chi\,, \quad d\eta = d\chi$$

and for $K < 0$

$$\eta = \sinh\chi\,, \quad d\eta = \cosh\chi\, d\chi = \sqrt{1+\eta^2}\, d\chi.$$

The three cases can be written in a unified way as

$$d\chi = \frac{d\eta}{\sqrt{1-k\,\eta^2}} \qquad \text{where} \begin{cases} k=1 & \text{if } K>0\,, \\ k=0 & \text{if } K=0\,, \\ k=-1 & \text{if } K<0\,, \end{cases}$$

so the metric becomes

$$g = a^2\left(\frac{d\eta^2}{1-k\eta^2} + \eta^2\big\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big\}\right)$$

where $k = 1$, 0 or -1.

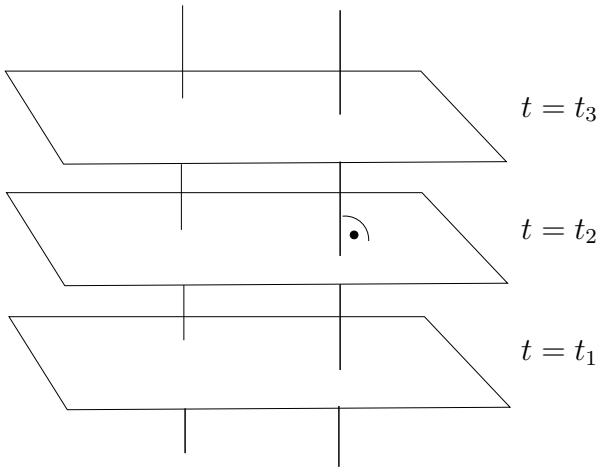
Secondly, another coordinate transformation $(\eta, \vartheta, \varphi) \mapsto (\rho, \vartheta\,\varphi)$ can be found such that the metric takes the form

$$g = \frac{a^2\Big(d\rho^2 + \rho^2\big\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big\}\Big)}{\Big(1+\dfrac{k}{4}\,\rho^2\Big)^2}\,.$$

This transformation and the geometric meaning of the coordinate ρ will be discussed in Worksheet 2. All three coordinate representations of spaces of constant curvature are frequently used for applications in cosmology.

Now we know the 3-dimensional homogeneous and isotropic Riemannian manifolds. This was the first step for constructing Robertson-Walker spacetimes. The second step is adding the time dimension.

As we assume that the spacetime admits a slicing into 3-dimensional Riemannian submanifolds of constant curvature, there is a distinguished timelike direction at each point, viz., the direction perpendicular to the slices. These timelike directions have as their integral curves a distinguished family of observer worldlines. Along these worldlines we can use proper time as a parametrisation. Homogeneity requires that the proper time that elapses between any two fixed sclices is the same along all worldlines perpendicular to the slices. Hence, the proper time parametrisatiom defines a time coordinate t on the spacetime such that the slices become submanifolds $\{t = \text{constant}\}$. Then the metric reads



$$g = -c^2dt^2 + a(t)^2\big(d\chi^2 + \eta(\chi)^2\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\}\big) =$$

$$= -c^2dt^2 + a(t)^2\left(\frac{d\eta^2}{1-k\eta^2} + \eta^2\big\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big\}\right) =$$

$$= -c^2dt^2 + \frac{a(t)^2\Big(d\rho^2 + \rho^2\big\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big\}\Big)}{\Big(1 + \dfrac{k}{4}\,\rho^2\Big)^2}$$

which is the general form of a Robertson-Walker metric. We have no mixed metric components, $g_{0i} = 0$, because the t-lines are perpendicular to the $\{t = \text{constant}\}$-sclices. The coefficient in front of dt^2 must be $-c^2$ because t is supposed to be proper time on the worldlines perpendicular to the slices. And the spatial part is a metric of constant curvature with a scale factor that may depend on t, indicating that the universe may expand or contract.

A Robertson-Walker universe is locally (but not globally) fixed by the curvature parameter k and by the scale factor $a(t)$. There is a freedom in choosing the topology. We say that for $k = 1$ the *natural topology* is a 3-sphere while for $k = 0$ and $k = -1$ it is $\mathbb{R}^3$. However, we are free to form quotient manifolds, e.g., to consider a Robertson-Walker universe with $k = 0$ that has a toroidal spatial topology. Therefore it is misleading to call the universes with $k > 0$ "closed" and the ones with $k \leq 0$ "open". By the same token, the time slices of a Robertson-Walker universe may have a finite volume (and be geodesically complete) even if $k = 0$ or $k = -1$.

If the scale factor takes the value 0, the metric degenerates. In such cases the function a has to be restricted to a maximal interval on which $a(t) \neq 0$. As only $a(t)^2$ matters, we may choose $a(t) > 0$ without loss of generality. So a is a map of the form

$$a :]\, t_i, t_f\, [\longrightarrow]\, 0, \infty\, [$$

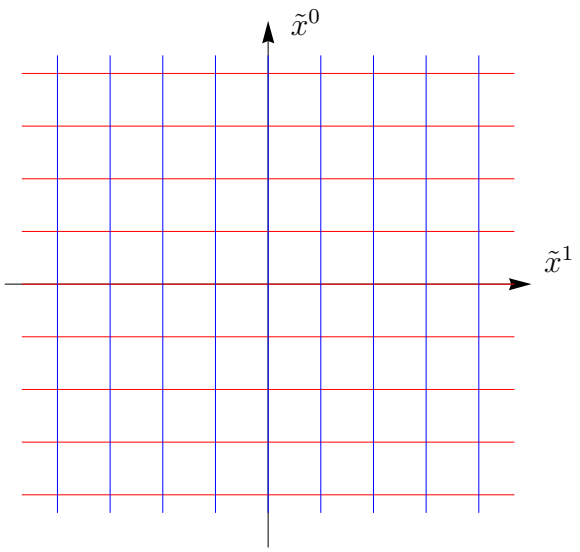
$$t \longmapsto a(t)\,.$$

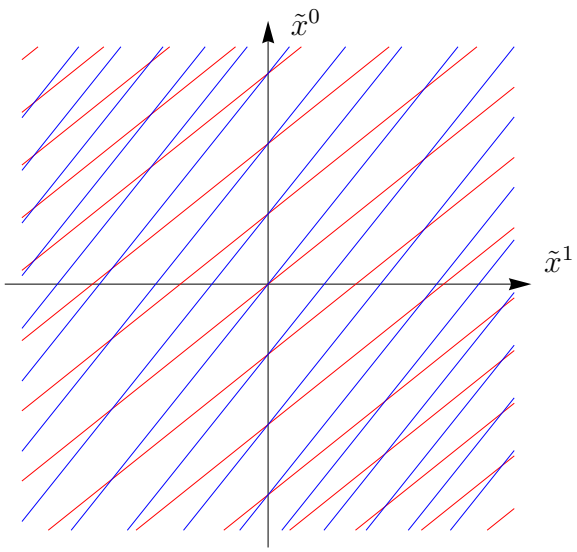
Here either $t_i = -\infty$ or t_i is a finite value with $a(t) \to 0$ for $t \to t_i$ and, analogously, either $t_f = \infty$ or t_f is a finite value with $a(t) \to 0$ for $t \to t_f$. The index i stands for "initial" and the index f stands for "final".

Models with $t_i \neq -\infty$ are called *big bang* models while models with $t_f \neq \infty$ are called *big crunch* models. Models with $a(t) = \text{constant}$ are called *steady state* models.

The t-lines, i.e., the timelike curves perpendicular to the slices of constant curvature, are called the worldlines of the *standard observers*. In particular in expanding Robertson-Walker models it is also common to call the flow of the vector field ∂_t the *Hubble flow*.

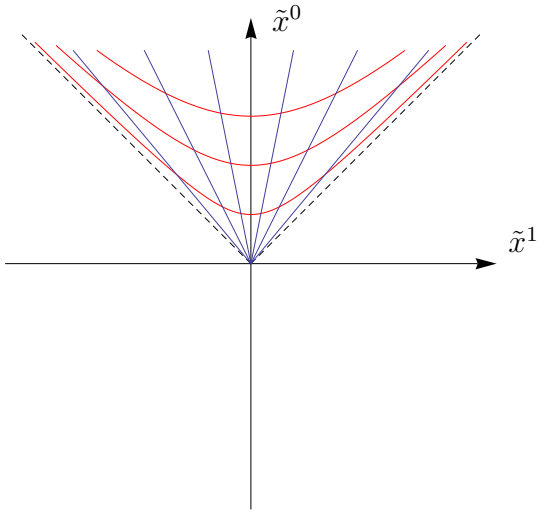
A (trivial) example of a Robertson-Walker spacetime is Minkowski spacetime in inertial coordinates. In this case $k = 0$ and $a = \text{constant}$. Of course, from one such slicing of Minkowski spacetime into flat spacelike hyperplanes we can change to another one by a Lorentz boost, see the picture on the next page. So this example demonstrates that the family of standard observers in a Robertson-Walker spacetime is not necessarily unique.





Another (less trivial) example of a Robertson-Walker universe is the *Milne model* which was suggested, as a special-relativistic model of our universe, by E. Milne in 1935. It has hyperbolic spatial geometry, $k = -1$, and a scale factor $a(t) = ct$.

The Milne model can be isometrically embedded into Minkowski spacetime where it covers the future light-cone of an event, see picture on the right. Therefore, it is a vacuum solution of Einstein's field equation with $\Lambda = 0$ and thus not a realistic model of our universe. The slices $t =$ constant are 3-dimensional hyperboloids (red) and the worldlines of the standard observers are straight lines (blue). This model starts at $t_i = 0$ with a big bang in the sense that at this time all standard observers were compressed into one point, but this is of course neither a curvature singularity nor a point of infinite matter density (if Einstein's field equation is considered).



The geodesics in a Robertson-Walker spacetime are the solutions to the Euler-Lagrange equations

$$\frac{d}{ds}\Big(\frac{\partial\mathcal{L}\big(x,\dot{x}\big)}{\partial\dot{x}^\mu}\Big) = \frac{\partial\mathcal{L}\big(x,\dot{x}\big)}{\partial x^\mu}$$

with the Lagrangian

$$\mathcal{L}\big(x,\dot{x}\big) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\Big(-c^2\dot{t}^2 + a(t)^2\Big\{\dot{\chi}^2 + \eta(\chi)^2\big(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\big)\Big\}\Big)$$

where the overdot means derivative with respect to the affine parameter s. The four components of the Euler-Lagrange equation read

$$\frac{d}{ds}\Big(-c^2\dot{t}\Big) = a(t)\frac{da(t)}{dt}\Big\{\dot{\chi}^2 + \eta(\chi)^2\big(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\big)\Big\}\,, \qquad \text{(G1)}$$

$$\frac{d}{ds}\Big(a(t)^2\dot{\chi}\Big) = a(t)^2\eta(\chi)\frac{d\eta(\chi)}{d\chi}\big(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\big)\,, \qquad \text{(G2)}$$

$$\frac{d}{ds}\Big(a(t)^2\eta(\chi)^2\dot\vartheta\Big) = a(t)^2\eta(\chi)^2\sin\vartheta\cos\vartheta\,\dot\varphi^2\,, \qquad \text{(G3)}$$

$$\frac{d}{ds}\Big(a(t)^2\eta(\chi)^2\sin^2\vartheta\dot\varphi\Big) = 0\,. \qquad \text{(G4)}$$

From (G3) and (G4) we read that, if we choose initial conditions $\dot\vartheta(s_o) = 0$ and $\dot\varphi(s_o) = 0$, then the solution satisfies $\dot\vartheta(s) = 0$ and $\dot\varphi(s) = 0$ for all s. In other words, a geodesic remains radial if it starts in the radial direction. Of course, this is an obvious consequence of the isotropy. Moreover, the spacetime is spatially homogeneous. Therefore, if we want to know all the geodesics issuing from a certain event, we may choose the coordinate system such that this event is on the worldline $\chi = 0$, i.e., at the spatial origin of the coordinate system. Combining these two observations tells us that it is sufficient to consider radial geodesics, $\dot\vartheta$ and $\dot\varphi = 0$, with the initial condition $\big(t(s_o) = t_o, \chi(s_o) = 0\big)$. Then (G3) and (G4) are automatically satisfied. We analyse the remaining equations for the case of timelike and lightlike geodesics separately.

(a) Timelike geodesics: Then we can choose proper time as the affine parameter, $s = \tau$, hence

$$-c^2\dot t^2 + a(t)^2\dot\chi^2 = -c^2\,.$$

As $\dot\vartheta = 0$ and $\dot\varphi = 0$, equation (G2) requires

$$\frac{d}{d\tau}\Big(a(t)^2\dot\chi\Big) = 0\,, \qquad a(t)^2\dot\chi = A\,.$$

The constant of motion A determines the initial velocity with respect to the standard observers. Equation (G1) will not be needed in the following because it yields no additional information. Inserting the second equation into the first results in

$$-c^2\frac{\dot t^2}{\dot\chi^2} + a(t)^2 = \frac{-c^2a(t)^4}{A^2}\,,$$

$$c^2\frac{\dot t^2}{\dot\chi^2} = a(t)^2\Big(1 + \frac{c^2a(t)^2}{A^2}\Big)\,,$$

$$A^2c^2\left(\frac{dt}{d\chi}\right)^2 = a(t)^2\Big(A^2 + c^2a(t)^2\Big)\,,$$

$$Ac\frac{dt}{d\chi} = \pm a(t)\sqrt{A^2 + c^2a(t)^2}\,,$$

$$d\chi = \frac{\pm Ac\,dt}{a(t)\sqrt{A^2 + c^2a(t)^2}}\,.$$

If we integrate this equation with the initial condition $\big(t(\tau_o) = t_o, \chi(\tau_o) = 0\big)$, we find

$$\chi = \pm\int_{t_o}^{t}\frac{A\,c\,d\tilde t}{a\big(\tilde t\,\big)\sqrt{A^2 + c^2a(\tilde t)^2}}\,.$$

Note that χ takes only positive values.Without loss of generality we may assume that $A \geq 0$. Then we have to choose the plus sign for times $t > t_o$ and the minus sign for times $t < t_o$. For $A = 0$ one gets the t-line which we have chosen as the spatial origin.

As we can choose any t-line, we have thus proven that all the standard observers are freely falling, i.e., that they stay on their worldlines without a thrust. Of course, this is an obvious consequence of the isotropy: If the standard observers were non-geodesic, they had a non-vanishing 4-acceleration and this 4-acceleration would distinguish a spatial direction.

Note that in a spatially compact universe χ cannot take all positive values. This happens, e.g., in a Robertson-Walker universe with $k = 1$ and the spatial topology of a 3-sphere, where χ is restricted to values between 0 and π. In this case a timelike geodesic with $A \neq 0$ may return to the same point in space (i.e., to the same standard observer) from where it started. An example of this kind will be treated in Worksheet 3.

(b) Lightlike geodesics: A lightlike radial curve must satisfy

$$-c^2\dot{t}^2 + a(t)^2\dot{\chi}^2 = 0\,.$$

This equation alone already determines the paths of lightlike geodesics, i.e., we need neither (G1) nor (G2). The reason is, again, obvious from the isotropy: Any radial lightlike curve is necessarily a geodesic (up to parametrisation). We find

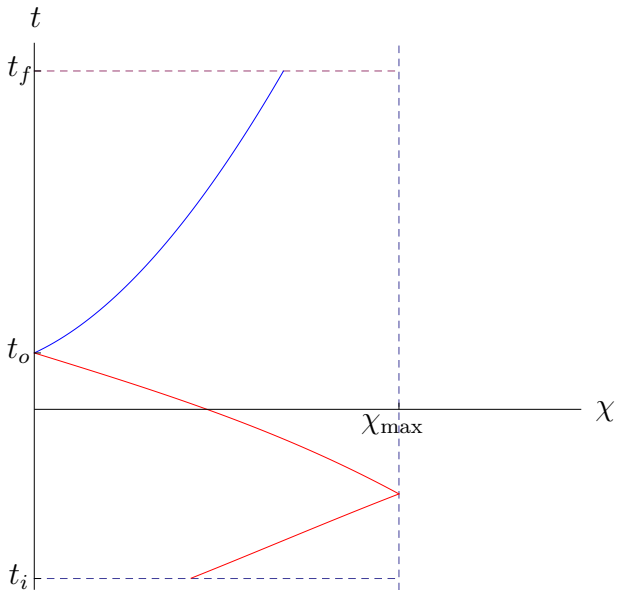
$$c^2\Big(\frac{dt}{d\chi}\Big)^2 = c^2\frac{\dot{t}^2}{\dot{\chi}^2} = a(t)^2\,, \qquad d\chi = \frac{\pm\, c\, dt}{a(t)}\,,$$

and, upon integration with the chosen initial condition,

$$\chi = \pm\int_{t_o}^{t}\frac{c\,d\tilde{t}}{a\big(\tilde{t}\,\big)}\,.$$

As before, the plus sign must be chosen for $t > t_o$ and the minus sign for $t < t_o$.

The picture illustrates radial lightlike geodesics that issue into the future (blue) and radial lightlike geodesics that issue into the past (red) for a case where χ is restricted to an interval $0 < \chi < \chi_{\max}$ (for a spherical universe $\chi_{\max} = \pi$) and where the scale factor restricts the t coordinate to a finite interval $t_i < t < t_f$. Note that every point in this diagram represents a sphere because the ϑ and φ coorrdinates are not shown. In a non-spherical but spatially compact universe $\chi_{\max}$ may depend on ϑ and φ; this happens, e.g., in a Robertson-Walker universe with $k = 0$ and a toroidal topology.



We will now discuss three observable features of Robertson-Walker universes all of which are related to lightlike geodesics: The redshift, the horizons and various distance measures.

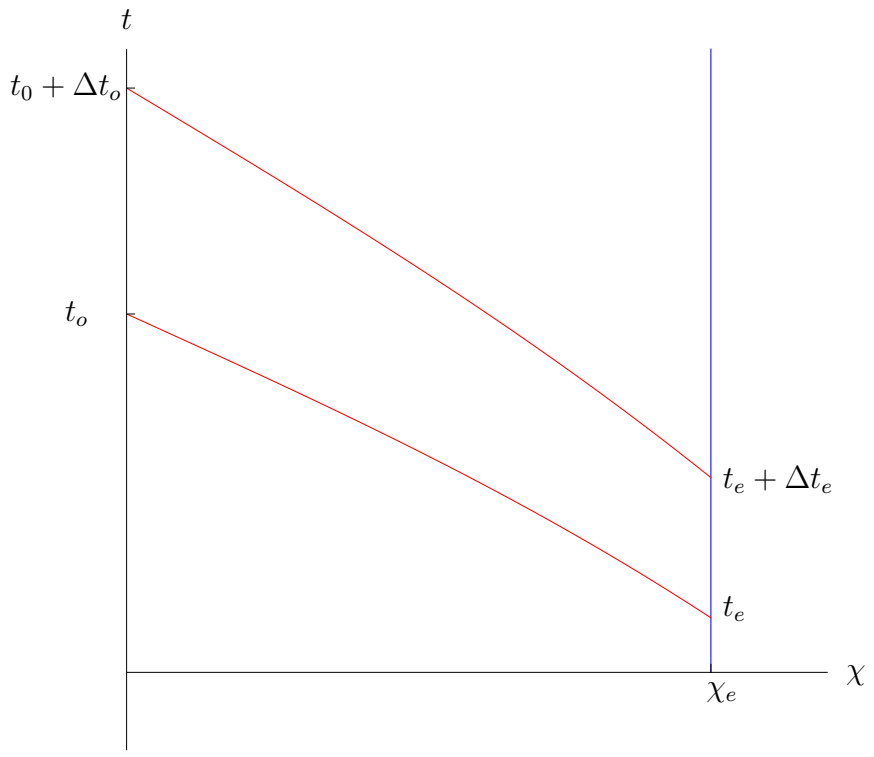
(i) The redshift: The redshift can be defined, for any pair of worldlines parametrised by proper time in any spacetime model, in the following way. Assume that from one of the worldlines light rays are emitted at proper times labelled t_e and that they are received on the other worldline at proper times labelled t_o. The indices e and o stand for "emitter" and "observer", respectively. Then we can calculate the frequency ratio

$$\frac{dt_o}{dt_e} = \lim_{\Delta t_e \to 0} \frac{\Delta t_o}{\Delta t_e} = \frac{\omega_e}{\omega_o} = \frac{\lambda_o}{\lambda_e}$$

where we have used that a process with period Δt_e has frequency $\omega_e = 2\pi/\Delta t_e$ and similarly for Δt_o and ω_o. Also, for light propagating in vacuo we can use the dispersion relation $\omega\lambda = c$ to convert a frequency ω into a wave-length λ. The limit $\Delta t_e \to 0$ is necessary to make the result unique. Astronomers define the redshift z as the change of wave-length divided by the emitted wave-length, i.e.

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{dt_o}{dt_e} - 1\,.$$

The figure illustrates this situation for the case that both the emitter and the observer are standard observers in a Robertson-Walker spacetime. Without loss of generality, we choose the coordinate system such that the observer is at the origin, $\chi_o = 0$, while the emitter is at a certain radius coordinate χ_e. Here it is important to recall that the time coordinate t gives proper time for the standard observers, i.e., $\omega_e = 2\pi/\Delta t_e$ and $\omega_o = 2\pi/\Delta t_o$ are, indeed, the frequencies as measured with standard clocks.



Differentiating the equation for lightlike geodesics

$$\chi_e = -\int_{t_o}^{t_e} \frac{c\,dt}{a(t)} = \int_{t_e}^{t_o} \frac{c\,dt}{a(t)}$$

with respect to t_e yields

$$0 = \frac{c}{a(t_o)}\frac{dt_o}{dt_e} - \frac{c}{a(t_e)}\,, \qquad \frac{dt_o}{dt_e} = \frac{a(t_o)}{a(t_e)}\,,$$

hence

$$z = \frac{a(t_o)}{a(t_e)} - 1\,.$$

This is the redshift law for standard observers in a Robertson-Walker universe. If the observer and/or the emitter is not a standard observer, one has to apply additional Doppler factors that are determined by the velocity relative to a standard observer.

With a Taylor expansion

$$a(t_e) = a(t_o) + \frac{da}{dt}\big(t_o\big)(t_e - t_o) + \frac{1}{2}\frac{d^2a}{dt^2}\big(t_o\big)(t_e - t_o)^2 + \ldots$$

we find

$$z = \frac{a(t_o) - a(t_e)}{a(t_e)} = \frac{-\frac{da}{dt}\big(t_o\big)(t_e - t_o) - \frac{1}{2}\frac{d^2a}{dt^2}\big(t_o\big)(t_e - t_o)^2 + \ldots}{a(t_o) + \frac{da}{dt}\big(t_o\big)(t_e - t_o) + \ldots} =$$

$$= \frac{\frac{da}{dt}\big(t_o\big)(t_o - t_e) - \frac{1}{2}\frac{d^2a}{dt^2}\big(t_o\big)(t_0 - t_e)^2 + \ldots}{a(t_o)\Big(1 - \frac{da}{dt}\big(t_o\big)\frac{(t_o - t_e)}{a(t_o)} + \ldots\Big)} =$$

$$= \Big(\frac{da}{dt}\big(t_o\big)(t_o - t_e) - \frac{1}{2}\frac{d^2a}{dt^2}\big(t_o\big)(t_0 - t_e)^2 + \ldots\Big)\frac{1}{a(t_o)}\Big(1 + \frac{da}{dt}\big(t_o\big)\frac{(t_o - t_e)}{a(t_o)} + \ldots\Big) =$$

$$= \frac{1}{a(t_o)}\frac{da}{dt}\big(t_o\big)(t_o - t_e) + \left(\frac{1}{a(t_o)^2}\Big(\frac{da}{dt}\big(t_o\big)\Big)^2 - \frac{1}{2a(t_o)}\frac{d^2a}{dt^2}\big(t_o\big)\right)\big(t_o - t_e\big)^2 + \ldots$$

It is common to define the *Hubble constant*

$$H(t_o) := \frac{1}{a(t_o)}\frac{da}{dt}\big(t_o\big)$$

and the *deceleration parameter*

$$q(t_o) := \frac{-a(t_o)}{\Big(\frac{da}{dt}\big(t_o\big)\Big)^2}\frac{d^2a}{dt^2}\big(t_o\big)\,.$$

Then the expression for the redshift becomes

$$z = H(t_o)(t_o - t_e) + H(t_o)^2\Big(1 + \frac{q(t_o)}{2}\Big)(t_o - t_e)^2 + \ldots$$

The travel time $t_o - t_e$ can be viewed as a measure for the distance (if multiplied with c, for dimensional reasons). We will discuss other distance measures below in this section and we will see that all of them coincide up to first order in $t_o - t_e$. So we may say that the relation between distance and redshift is unambiguously determined by the Hubble constant to within a linear approximation. Note that the Hubble "constant" and the deceleration parameter depend on t_o. We will discuss later in detail that we believe to live in a universe where now (at time t_o) the Hubble constant is positive and the deceleration parameter is negative, i.e., in a universe that is expanding and where the expansion rate is even increasing.

(ii) Horizons: There are two types of horizons in a Robertson-Walker universe, known as "particle horizons" and "event horizons". Here the word "particle" is used as synonymous with "standard observer". These notions are defined in the following way.

Fix an event p_o. Then the *particle horizon* of p_o separates particles that can be seen at p_o from those that cannot.

Fix a particle (i.e., a standard observer) P_o. Then the *event horizon* of P_o separates events that can be seen by P_o from those that cannot.

Here p_o is a point in the spacetime while P_o is a worldline. Always keep in mind that events have particle horizons whereas particles have event horizons.

It is now our goal to give a mathematical criterion for the existence or non-existence of horizons. To that end we perform a coordinate transformation where only the time coordinate is transformed, $(t, \chi, \vartheta, \varphi) \mapsto (T, \chi, \vartheta, \varphi)$, defined by

$$T = \int_{t_o}^{t} \frac{d\tilde{t}}{a\big(\tilde{t}\,\big)}$$

where t_o is a constant that can be chosen at will, except for the restriction $t_i < t_o < t_f$. Then t_o corresponds to $T_o = 0$ while t_i and t_f correspond to

$$T_i = \int_{t_o}^{t_i} \frac{d\tilde{t}}{a\big(\tilde{t}\big)} = -\int_{t_i}^{t_o} \frac{d\tilde{t}}{a\big(\tilde{t}\,\big)} < 0$$

and

$$T_f = \int_{t_o}^{t_f} \frac{d\tilde{t}}{a\big(\tilde{t}\,\big)} > 0\,,$$

respectively. If we express the scale factor as a function of T,

$$a(t) = \hat{a}(T)\,,$$

the metric reads

$$g = \hat{a}(T)^2\Big(-c^2 dT^2 + d\chi^2 + \eta(\chi)^2\big\{d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big\}\Big)\,.$$

Multiplying a metric with a positive function is called a *conformal transformation*. We see that T is proper time along the worldlines of the standard observers with respect to

the conformally transformed metric $\hat{a}(T)^{-2}g$; therefore, T is called the *conformal time*. We see that the coefficients of the metric $\hat{a}(T)^{-2}g$ are independent of T, i.e., that this metric is static. The given representation thus shows that every Robertson-Walker metric is *conformally static*, i.e., conformal to a static metric.

If we use the conformal time coordinate T instead of t, the equation for lightlike radial geodesics,

$$\chi = \pm \int_{t_o}^{t} \frac{c\, d\tilde{t}}{a(\tilde{t})}\,,$$

reads

$$\chi = \pm\, c\, T\,,$$

i.e., in a (χ, cT)-diagram the radial light rays are represented by lines under 45 degrees. Note that cT is a dimensionless time coordinate, just as χ is a dimensionless radius coordinate.

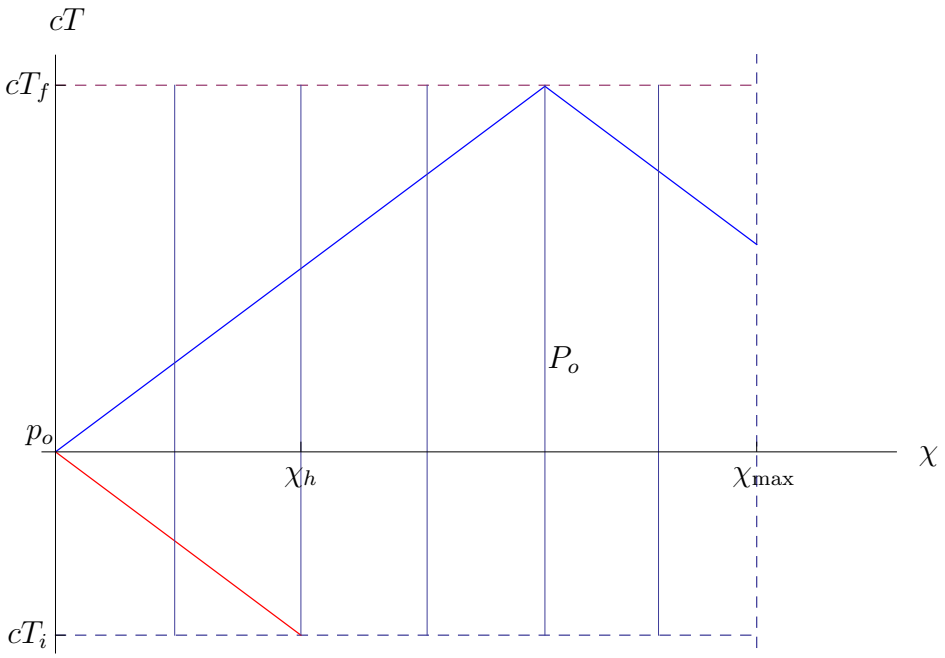
It is now obvious that the following existence criteria for horizons are true, see the figure below.

Particle horizons exist if and only if $c\,|T_i| < \chi_{\max}$.

Event horizons exist if and only if $c\,T_f < \chi_{\max}$.

In particular, horizons do not exist if $T_i = -\infty$ and $T_f = \infty$, i.e., if the function $T \mapsto \hat{a}(T)$ is defined on all of $\mathbb{R}$.

The picture shows the situation for a case where T_i, T_f and $\chi_{\max}$ are finite. The past light-cone of the event p_o is shown in red, so the particle horizon of p_o is situated at the radius coordinate χ_h. The event horizon of the particle P_o is shown in blue.



(iii) Distance measures

There are various ways of assigning a distance to a pair of standard observers. We will discuss several of them, interpreting one of the standard observers as the emitter of light and the other as the observer. As the scale factor depends on time, each of the distance measures has to be viewed as a function of the observation time t_o. We choose the observer as the origin of the spatial coordinate system unless otherwise stated. Astronomers prefer to use the redshift z as an independent variable because it is directly measurable (if spectral lines can be identified). Following this practice, we write each of the distance measures as a power series in terms of z ("Kristian-Sachs series"). We will see that all distance measures coincide to within linear order.

– Distance by travel time of light

If a light ray starts at time t_e at the emitter and arrives at time t_o at the observer, we can use the expression

$$D_T = c(t_o - t_e)$$

as a measure for the distance between emitter and observer. D_T is not a directly measurable quantity because t_e is not known. However, in some cases t_e and thus D_T can be estimated.

With the series expansion for the redshift from p.18,

$$z = H(t_o)(t_o - t_e) + H(t_o)^2\Big(1 + \frac{q(t_o)}{2}\Big)(t_o - t_e)^2 + \dots\,,$$

we can express D_T as a power series in terms of z. For that purpose, we write

$$t_o - t_e = \alpha z + \beta z^2 + \dots$$

and insert this expression into the series expansion for the redshift,

$$z = H(t_o)\Big(\alpha\, z + \beta\, z^2\Big) + H(t_o)^2\Big(1 + \frac{q(t_o)}{2}\Big)\alpha^2 z^2 + \dots\, =$$

$$=\; H(t_o)\,\alpha\, z + \Big\{H(t_o)\beta + H(t_o)^2\Big(1 + \frac{q(t_o)}{2}\Big)\alpha^2\Big\}z^2 + \dots$$

Comparison of coefficients yields

$$1 = H(t_o)\alpha\,, \qquad 0 = H(t_o)\Big\{\beta + H(t_o)\Big(1 + \frac{q(t_o)}{2}\Big)\alpha^2\Big\}$$

hence

$$\alpha = \frac{1}{H(t_o)}\,, \qquad \beta = -\frac{1}{H(t_o)}\Big(1 + \frac{q(t_o)}{2}\Big)\,.$$

This gives the expansion for $D_T = c(t_o - t_e)$ as a power series in terms of z,

$$D_T = \frac{c\,z}{H(t_o)} - \Big(1 + \frac{q(t_o)}{2}\Big)\frac{c\,z^2}{H(t_o)} + \dots$$

– Proper distance

From a mathematical point of view, the most natural way of measuring the distance between two standard observers (emitter and observer) at time t_o is the proper length of the radial line that connects them in the hypersurface $t = t_o$. From the form of the metric

$$g = -c^2 dt^2 + a(t)^2 \Big(d\chi^2 + \eta(\chi)^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \Big) \,,$$

we read that along a radial line ($t = t_o$, ϑ = constant and φ = constant) proper length ℓ is given by

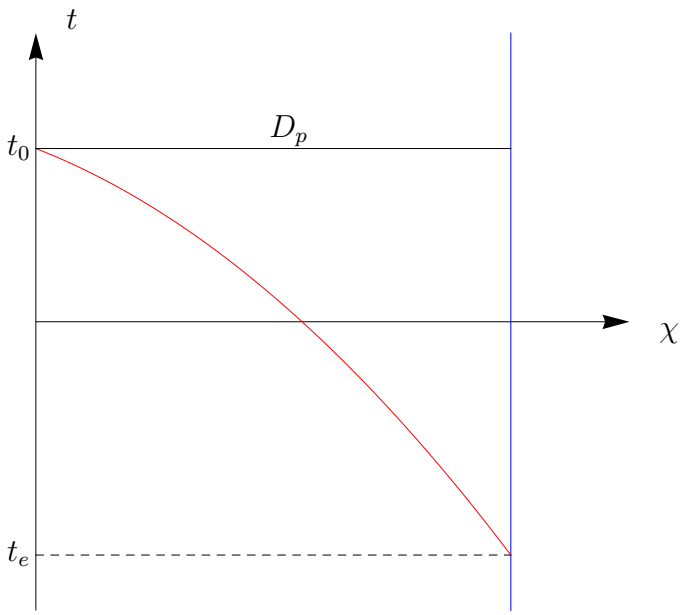
$$d\ell^2 = a(t_o)^2 d\chi^2 \,,$$

so the proper distance between an emitter at $\chi_e = \chi$ and an observer at $\chi_o = 0$ is

$$D_p = \int_0^\chi a(t_o) d\chi = a(t_o)\chi \,.$$

From this expression we find a new version of the Hubble law,

$$\frac{dD_p}{dt}(t_o) = \frac{da}{dt}(t_o)\chi = \frac{da}{dt}(t_o)\frac{D_p}{a(t_o)} = H(t_o) D_p \,.$$



We may interpret this formula as saying that "the radial velocity of a distant source is proportional to its distance, with the Hubble constant as the factor of proportionality". This formula is exact, i.e., not only valid to within a linear approximation. In this sense, it is more universal than the linear relation between travel time and redshift which is true only if terms of higher-order in the travel time are neglected. On the other hand, the "radial velocity" and the "distance" in this formula are purely theoretical quantities that are not related to observations: The proper distance D_p is based on connecting two events in a hypersurface $t = t_o$, i.e., at equal times; no signal can realise this connecting line. Moreover, the "radial velocity" dD_p/dt does *not* give the velocity of the emitter relative to its neighbourhood but rather the change of a mathematically defined distance between emitter and observer. Therefore, it is no contradiction to the rules of relativity that this "radial velocity" may very well be bigger than c. In view of the fact that the Hubble constant relates a distance to a velocity it is usually given in units of (km/s)/Mpc. Of course, this is the same as an inverse time.

As in the case of the distance by travel time of light, we may write D_p as a series in terms of the redshift z. In this case we need the Taylor expansion of χ using the equation for a radial light ray from p. 16,

$$\chi = \int_{t_e}^{t_o} \frac{c\,dt}{a(t)} = \int_{t_e}^{t_0} \frac{c\,dt}{a(t_o) + \frac{da}{dt}\big(t_o\big)(t - t_o) + \ldots} =$$

$$= \int_{t_e}^{t_0} \frac{c\,dt}{a(t_o)\Big(1 + H(t_o)(t - t_o) + \ldots\Big)} =$$

$$= \frac{c}{a(t_o)} \int_{t_e}^{t_o} \Big(1 - H(t_o)(t - t_o) + \ldots\Big) dt =$$

$$= \frac{c}{a(t_o)} \Big\{ t_o - t_e - H(t_o)\Big(\frac{t_o^2}{2} - \frac{t_e^2}{2} - t_o\big(t_o - t_e\big)\Big) + \ldots \Big\} =$$

$$= \frac{c}{a(t_o)} \big(t_o - t_e\big) + \frac{c}{2a(t_o)} H(t_o)(t_o - t_e)^2 + \ldots$$

Inserting the Taylor expansion of $t_o - t_e$ in terms of z yields

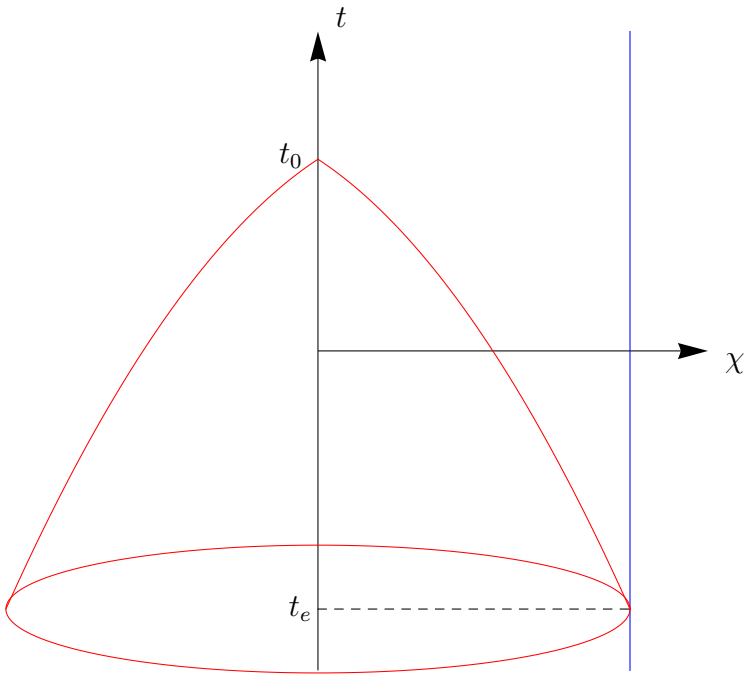
$$D_p = a(t_o)\chi = c\Big\{\frac{z}{H(t_o)} - \Big(1 + \frac{q(t_o)}{2}\Big)\frac{z^2}{H(t_o)}\Big\} + \frac{c\,z^2}{2\,H(t_o)} + \cdots =$$

$$= \frac{c\,z}{H(t_o)} - \frac{c}{2H(t_o)}\Big(1 + q(t_o)\Big)z^2 + \ldots$$

– Area distance (=angular diameter distance)

In Newtonian physics, which is based on the assumption that Euclidean geometry holds in our 3-space, the apparent size of an object is inverse proportional to the square of its distance. If we have "standard rulers" (i.e., objects whose true size we know) at our disposal, we can determine their distance directly from measuring their apparent size in the sky. In a curved geometry, we can *define* a distance measure in such a way that this relation still holds. This can be done either by comparing the true cross-sectional area of the object to the solid angle it suspends in the sky, or by comparing the true length of a particular diameter of the object to the angle this diameter suspends in the sky. The first distance measure is known as the "area distance" and the second as the "angular diameter distance". In a Robertson-Walker universe the two notions coincide because of the isotropy. Moreover, isotropy implies that it suffices to consider the area of spheres about the observer. So we choose the observer as the spatial origin, as before, and we consider the past light-cone of an observation event at time t_o. For any earlier time t_e, the intersection of this light-cone with the hypersurface $t = t_e$ is a sphere of coordinate radius

$$\chi = \int_{t_e}^{t_o} \frac{c\,dt}{a(t)}\,.$$



From the metric we read that the area of this sphere is $4\pi a(t_e)^2\eta(\chi)^2$. The area distance D_A is *defined* by equating this expression to the Euclidean expression for the area of a sphere,

$$4\pi a(t_e)^2\eta(\chi)^2 = 4\pi D_A^2\,,$$

hence

$$D_A = a(t_e)\eta(\chi)\,.$$

As $\eta(\chi) = \sin\chi$, $\eta(\chi) = \chi$ or $\eta(\chi) = \sinh\chi$, we have in any case $\eta(\chi) = \chi + O(\chi^3)$, so we may write

$$D_A = a(t_e)\Big(\chi + O(\chi^3)\Big)\,.$$

With the Taylor series for χ from above this can be rewritten as

$$\begin{aligned} D_A &= \frac{a(t_e)}{a(t_o)}\,a(t_o)\,\left(\frac{c}{a(t_o)}\,(t_o - t_e) \;+\; \frac{c\,H(t_o)}{2a(t_o)}\,(t_o - t_e)^2 + \dots\right) \\ &= \frac{1}{(1+z)}\,\left(D_T \;+\; \frac{H(t_o)}{2\,c}\,D_T^2 + \dots\right) \end{aligned}$$

Inserting the Taylor series for D_T yields

$$\begin{aligned} D_A &= \big(1 - z + \dots\big)\,\left(\Big\{\frac{c\,z}{H(t_o)} - \Big(1 + \frac{q(t_o)}{2}\Big)\frac{c\,z^2}{H(t_o)} + \dots\Big\} + \frac{H(t_o)}{2}\,\frac{c\,z^2}{H(t_o)^2} + \dots\right) \\ &= \frac{c\,z}{H(t_o)} - \big(3 + q(t_o)\big)\,\frac{c\,z^2}{2\,H(t_o)} + \dots \end{aligned}$$

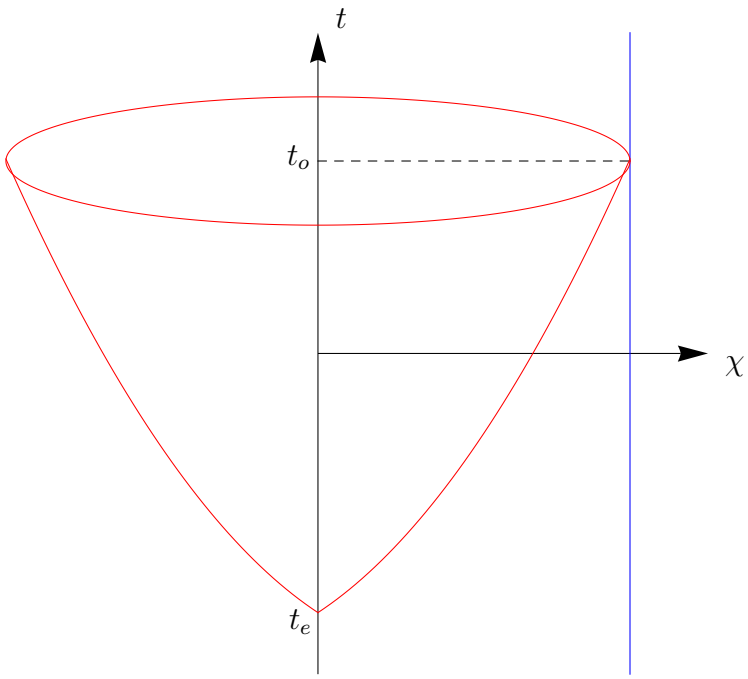
If we had standard rulers available distributed over the universe, we could measure D_A and z for each of them and then determine $H(t_o)$ and $q(t_o)$ from this formula. Actually, we have better "standard candles" than "standard rulers", i.e., it is more promising to consider the luminosity of a light source rather than its size, see the next item.

– Luminosity distance

In Newtonian physics, not only the apparent size but also the apparent luminosity of a light source falls off with the square of the distance. So if we have "standard candles" (i.e., objects whose true luminosity we know) at our disposal, we can determine their distance directly from measuring their apparent luminosity. The apparent luminosity is given by the energy flux arriving at the observer. Astronomers use a logarithmic scale and express the energy flux in *magnitudes*.

In analogy to the area distance, which was defined in a way that it is related to the true size of a light source by the same formula as in the Newtonian (i.e., Euclidean) case, we can define a luminosity distance in a way that it is related to the true luminosity by the Newtonian formula. In a Robertson-Walker universe, we may again take advantage of the isotropy and consider the future light-cone of an emission event at time t_e. In this case it is convenient to place the emitter in the spatial origin, $\chi_e = 0$, and to have the observer at a radius coordinate $\chi_o = \chi$. Then for any observation time $t_o > t_e$, the intersection of the considered future light-cone with the hypersurface $t = t_o$ has area $4\pi a(t_o)^2 \eta(\chi)^2$. We assume that the emitter sends photon isotropically into all spatial directions. If the apparent luminosity were measured in terms of a number flux of photons, the desired distance measure would be given by the equation

$$4\pi a(t_o)^2 \eta(\chi)^2 = 4\pi \tilde{D}_L^2 \,.$$



One calls $\tilde{D}_L$ the *corrected luminosity distance*. Actually, the apparent luminosity is not given by the number flux of photons but rather by the energy flux. The latter differs from the first by a redshift factor, because the energy of a photon undergoes a redshift on its way from the emitter to the observer. Therefore, one defines the (uncorrected) *luminosity distance* D_L by the equation

$$D_L = (1+z)\tilde{D}_L = a(t_o)(1+z)\eta(\chi) \,.$$

This is related to the energy flux F at the observer by the formula

$$F = \frac{L}{4\pi D_L^2}$$

where L is the true luminosity of the source. (One usually considers the luminosity integrated over all frequencies which is called the *bolometric luminosity*.) Astronomers use a logarithmic scale, as all human senses respond logarithmically to a physical stimulus ("Weber-Fechner law") and define the (apparent) *magnitude* m of a light source such that

$$m \;=\; -\,2.5\log_{10}(L) \;+\; 2.5\log_{10}\big(D_L^2\big) \;+\; m_0$$

where m_0 is a constant. Note that the luminosity distance can be rewritten as

$$D_L = \frac{a(t_o)}{a(t_e)}\, a(t_e)\,(1+z)\,\eta(\chi) = (1+z)^2 a(t_e)\,\eta(\chi)\,.$$

Comparison with the formula for the area distance demonstrates that

$$D_L = (1+z)^2 D_A\,.$$

This relation between luminosity distance and area distance is obvious in a Robertson-Walker universe. Actually, it is true in *any* spacetime, but the general proof is rather involved and will not be given here. It is based on the so-called *reciprocity theorem* for light bundles which was proven by Etherington in 1933.
With the relation between luminosity distance and area distance at hand, we can now easily write the luminosity distance as a power series in terms of z,

$$D_L = \big(1+2z+z^2\big)\Big(\frac{c\,z}{H(t_o)} - \big(3+q(t_o)\big)\,\frac{c\,z^2}{2\,H(t_o)} + \ldots\Big) =$$

$$= \frac{c\,z}{H(t_o)} + \big(1-q(t_o)\big)\,\frac{c\,z^2}{2\,H(t_o)} + \ldots$$

If we have standard candles distributed in the universe, we can measure their luminosity distance and their redshift and determine $H(t_o)$ and $q(t_o)$ from the last equation. The best standard candles we have to date are supernovae of type Ia. In the next chapter we will discuss in detail how they have been used to determine $q(t_o)$ in the late 1990s. The surprising result was that $q(t_o) < 0$, i.e., that the expansion of our universe is accelerating. The present value of the Hubble constant is $H(t_o) = \big(67.8 \pm 0.77\big)$(km/s)/Mpc. This, however, was not determined from the relation between luminosity distance and redshift, which yields a considerably lower accuracy, but rather from the cosmic background radiation, see below.

For each of the distance measures we have discussed the series expansion in terms of z is true in *any* Robertson-Walker universe. The results are purely kinematical in the sense that Einstein's field equation has not been used. Also note that our formulas, which include terms up to the second order, are independent of k, i.e., they hold for universes of $k = 1$, $k = 0$ and $k = -1$. The third-order terms, however, do depend on k, at least for D_A and D_L, because then the $O(\chi^3)$ term in $\eta(\chi) = \chi + O(\chi^3)$ has to be taken into account. To date the third-order terms are beyond the reach of observations.

2.2 Friedmann solutions

In this section we will study those Roberston-Walker spacetimes that satisfy Einstein's field equation with a perfect fluid source. We will see that *any* Robertson-Walker spacetime can be viewed as such a solution if we do not impose conditions on the density or the pressure. However, in view of applications to cosmology it is of particular interest to study solutions where certain properties of the density or the pressure have been specified. This is what one calls the Friedmann solutions.

It is convenient to use Robertson-Walker spectimes in the coordinate representation

$$g = -c^2 dt^2 + a(t)^2 \left(\frac{d\eta^2}{1 - k\,\eta^2} + \eta^2 \Big(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \Big) \right) .$$

Writing a prime for the derivative of the scale factor with respect to the time coordinate t, one finds that the components of the Ricci tensor are

$$R_{tt} = -\frac{3}{a(t)}\, a''(t)\,,$$

$$R_{\eta\eta} = \frac{1}{(1 - k\,\eta^2)} \left(2\,k + \frac{2}{c^2} a'(t)^2 + \frac{a(t)}{c^2}\, a''(t) \right) ,$$

$$R_{\varphi\varphi} = \sin^2\vartheta \, R_{\vartheta\vartheta} = \eta^2 \big(1 - k\,\eta^2\big) R_{\eta\eta}\,,$$

and $R_{\mu\nu} = 0$ for $\mu \neq \nu$. As a consequence, the Ricci scalar reads

$$R = \frac{6}{c^2 a(t)^2} \Big(c^2 k + a'(t)^2 + a(t) a''(t) \Big) .$$

We want to solve Einstein's field equation with an energy-momentum tensor of a perfect fluid,

$$T_{\mu\nu} = \Big(\mu + \frac{p}{c^2} \Big) U_\mu U_\nu + p\, g_{\mu\nu}$$

where the fluid is supposed to be at rest with respect to the standard observers. The latter condition requires, in combination with the normalisation condition $U^\mu U_\mu = -c^2$, that

$$U^\mu = \delta^\mu_t\,, \quad U_\nu = g_{\nu t} = -c^2 \delta^t_\nu\,.$$

We interpret U^μ as the 4-velocity of the mean flow of galaxies.

For the components of the energy-momentum tensor we find

$$T_{tt} = \Big(\mu + \frac{p}{c^2} \Big) (g_{tt})^2 + p\, g_{tt} = \mu\, c^4\,,$$

$$T_{\eta\eta} = p g_{\eta\eta} = \frac{p a(t)^2}{1 - k\,\eta^2}\,,$$

$$T_{\varphi\varphi} = \sin^2\vartheta \, T_{\vartheta\vartheta} = \eta^2 \left(\big(1 - k\,\eta^2\big) \sin^2\vartheta \, T_{\eta\eta}\,. \right.$$

For $\mu \neq \nu$ we have $T_{\mu\nu} = 0$.

Einstein's field equation

$$R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

gives two independent component equations. The tt component yields

$$-\frac{3}{a(t)}\, a''(t) + \frac{3\, c^2}{c^2 a(t)^2}\left(c^2 k + a'(t)^2 + a(t)a''(t)\right) - \Lambda\, c^2 = \kappa\mu\, c^4\,,$$

$$\frac{3c^2 k}{a(t)^2} + \frac{3}{a(t)^2}\, a'(t)^2 - \Lambda\, c^2 = \kappa\mu\, c^4\,, \qquad \text{(F1)}$$

and the $\eta\eta$ component yields

$$\frac{1}{(1-k\,\eta^2)}\left(2\,k + \frac{2}{c^2}a'(t)^2 + \frac{a(t)}{c^2}\, a''(t)\right) - \frac{3a(t)^2}{c^2 a(t)^2(1-k\eta^2)}\left(c^2 k + a'(t)^2 + a(t)\, a''(t)\right)$$

$$+\frac{\Lambda a(t)^2}{1-k\eta^2} = \frac{\kappa\, p\, a(t)^2}{1-k\,\eta^2}\,,$$

$$-\,k - \frac{1}{c^2}\, a'(t)^2 - \,2\,\frac{a(t)}{c^2}\, a''(t) + \Lambda a(t)^2 = \kappa\, p\, a(t)^2\,. \qquad \text{(F2)}$$

The $\varphi\varphi$ and $\vartheta\vartheta$ components give again (F2) and the $\mu\nu$ components with $\mu \neq \nu$ just give the identity $0 = 0$.

(F1) and (F2) are known as the (generalised) *Friedmann equations*. As the left-hand sides are functions of t only, these equations require that μ and p be also functions of t only. Moreover, we read from (F1) and (F2) that $a(t)$, k and Λ can be chosen arbitrarily and then $\mu(t)$ and $p(t)$ are uniquely determined. In this sense, *any* Robertson-Walker spacetime solves Einstein's field equation with a perfect fluid source that is at rest with respect to the standard observers. However, it is *not* guaranteed that pressure and density are non-negative and related by an equation of state, i.e., by a relation of the form $F(\mu, p) = 0$ that does not depend explicitly on time. So what one has to discuss is the question of which Robertson-Walker spacetimes are *physically reasonable* perfect fluid solutions.

(a) Vacuum solutions
Although our real universe is certainly not a vacuum spacetime, vacuum solutions to the Friedmann equations are of some interest in cosmology as limiting cases. For vacuum, $\mu = 0$ and $p = 0$, the Friedmann equations reduce to

$$c^2 k + a'(t)^2 - \,\frac{\Lambda}{3}\, c^2 a(t)^2 = 0\,, \qquad \text{(F1}')$$

$$-c^2 k - a'(t)^2 - \,2\, a(t)\, a''(t) + \Lambda\, c^2\, a(t)^2 = 0\,. \qquad \text{(F2}')$$

We will first show that in the vacuum case the second Friedmann equation is redundant.

Claim: (F2′) is a consequence of (F1′).

Proof: Differentiation of (F1′) yields

$$2\,a'(t)\,a''(t) - \frac{\Lambda}{3}\,c^2 2\,a(t)\,a'(t) = 0\,.$$

Multiplication with $a(t)/a'(t)$ yields

$$2\,a(t)\,a''(t) = \frac{\Lambda}{3}\,c^2 2\,a(t)^2\,.$$

Using again (F1′), this can be rewritten as

$$2\,a(t)\,a''(t) + a'(t)^2 = \frac{\Lambda}{3}\,c^2 2\,a(t)^2 - c^2 k + \frac{\Lambda}{3}\,c^2 a(t)^2$$

which is just equation (F2′).

Therefore, for the following discussion we have to consider only the first Friedmann equation (F1′). We solve this equation for all possible values of Λ and k.

(i) $\Lambda = 0$

- $k = 1$
  In this case the Friedmann equation reads
  $$c^2 + a'(t)^2 = 0$$
  which cannot be satisfied by any real function a.
- $k = 0$
  The Friedmann equation
  $$a'(t)^2 = 0$$
  requires a to be a constant, $a(t) = a_0$. In this case the metric reads
  $$g = -c^2 + a_0^2\Big(d\chi^2 + \chi^2\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)\Big).$$
  A coordinate transformation $(t, \chi, \vartheta, \varphi) \mapsto (t, r = a_o\chi, \vartheta, \varphi)$ shows that this is just the Minkowski metric,
  $$g = -c^2 + dr^2 + r^2\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)\,.$$
  The time slices are the usual flat Euclidean spaces of an inertial system, represented in spherical polar coordinates.
- $k = -1$
  Now we have to solve the differential equation
  $$-c^2 + a'(t)^2 = 0$$
  which yields
  $$\frac{da}{dt} = \pm c\,, \quad a(t) = \pm c\,(t - t_0)\,.$$

As we are free to shift the zero point on the time axis, we may choose $t_0 = 0$. Moreover, we only consider the plus sign because the minus sign gives the same spacetime just with the time reversed. The scale factor $a(t) = c\,t$ is then defined on the interval $]\,t_i = 0\,,\,t_f = \infty\,[$. The metric reads

$$g = -c^2dt^2 + c^2t^2\Big(d\chi^2 + \sinh^2\chi\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)\Big)$$

which is *Milne's universe* that was already mentioned. Milne's universe is just part of Minkowski spacetime, with a slicing into hyperboloids, see the picture on p.14. The transformation to standard Minkowski coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ is given by

$$\begin{aligned}
\tilde{x}^0 &= c\,t\cosh\chi\,,\\
\tilde{x}^1 &= c\,t\sinh\chi\cos\varphi\sin\vartheta\,,\\
\tilde{x}^2 &= c\,t\sinh\chi\sin\varphi\sin\vartheta\,,\\
\tilde{x}^3 &= c\,t\sinh\chi\cos\vartheta\,.
\end{aligned}$$

(ii) $\Lambda > 0$

- $k = 1$

  In this case the Friedmann equation reads

$$c^2 + a'(t)^2 - \frac{\Lambda}{3}\,c^2a(t)^2 = 0\,,$$

$$\frac{da}{dt} = \pm\,c\,\sqrt{\frac{\Lambda}{3}\,a^2 - 1}\,.$$

  As Λ is positive, we may substitute

$$\sqrt{\frac{\Lambda}{3}}\,a = \cosh u\,,$$

$$\sqrt{\frac{\Lambda}{3}}\,da = \sinh u\,du\,,$$

  hence

$$\sqrt{\frac{3}{\Lambda}}\,\frac{\cancel{\sinh u}\,du}{\cancel{\sqrt{\cosh^2 u - 1}}} = \pm\,c\,dt\,.$$

  This results in

$$u = \pm\,\sqrt{\frac{\Lambda}{3}}\,c\,(t - t_0)\,,$$

$$a(t) = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\,c\,(t - t_0)\right)\,.$$

  Without loss of generality we choose $t_0 = 0$. The scale factor is defined for all real values of t. It decreases from $t = -\infty$ to a minimum value at $t = 0$ and then increases again to $+\infty$. The metric reads

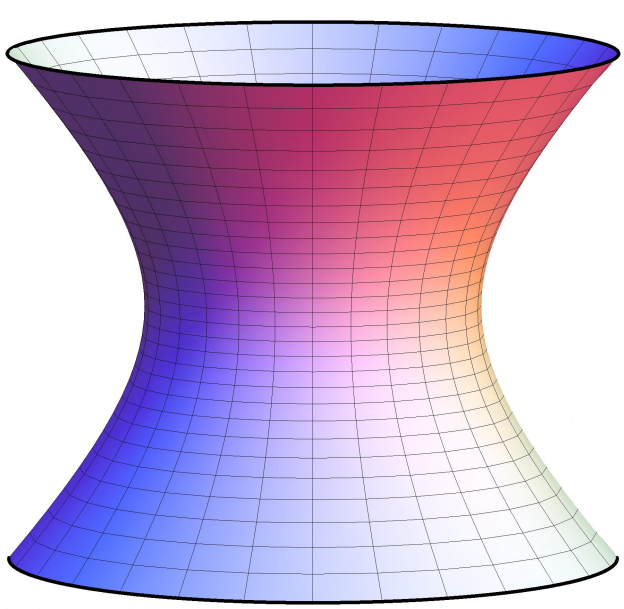
$$g = -c^2dt^2 + \frac{3}{\Lambda}\cosh^2\Big(\sqrt{\frac{\Lambda}{3}}\,c\,t\Big)\Big(d\chi^2 + \sin^2\chi\big(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big)\Big)\,.$$

This spacetime is known as the *deSitter universe*. It was found by Durtch astronomer Willem deSitter in 1917. The deSitter universe can be isometrically embedded as the hyperboloid

$$X^2 + Y^2 + Z^2 + W^2 - V^2 = \frac{3}{\Lambda}$$

into 5-dimensional Minkowski space,

$$g^{(5)} = dX^2 + dY^2 + dZ^2 + dW^2 - dV^2 \,.$$



In terms of our coordinates $(t, \chi, \vartheta, \varphi)$, the embedding is given by the map

$$X = \sqrt{\frac{3}{\Lambda}} \cosh\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sin\chi \cos\varphi \sin\vartheta \,,$$

$$Y = \sqrt{\frac{3}{\Lambda}} \cosh\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sin\chi \sin\varphi \sin\vartheta \,,$$

$$Z = \sqrt{\frac{3}{\Lambda}} \cosh\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sin\chi \cos\vartheta \,,$$

$$W = \sqrt{\frac{3}{\Lambda}} \cosh\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \cos\chi \,,$$

$$V = \sqrt{\frac{3}{\Lambda}} \sinh\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \,,$$

Our representation gives the deSitter universe with a slicing into hyperspaces $t =$ constant that are 3-spheres S^3, so the topology of the spacetime is $S^3 \times \mathbb{R}$. In the picture the 3-spheres are represented by circles which are given by intersecting the hyperboloid with horizontal planes.

We will briefly check if there are horizons in the deSitter universe with its sclicing into 3-spheres. To that end we have to consider the conformal time T which is defined by

$$c\,dT = \frac{c\,dt}{a(t)} = c\sqrt{\frac{\Lambda}{3}}\,\frac{dt}{\cosh\Big(\sqrt{\frac{\Lambda}{3}}\,c\,t\Big)}\,.$$

Integration yields

$$c\,T = \arcsin\left(\tanh\Big(\sqrt{\frac{\Lambda}{3}}\,c\,t\Big)\right) - \frac{\pi}{2}$$

where we have chosen the integration constant, in disagreement with the convention introduced on p.19, as $-\pi/2$ because this is usual. The conformal time T is then related to t by

$$\cos\big(c\,T\big) = \tanh\Big(\sqrt{\frac{\Lambda}{3}}\,c\,t\Big)\,.$$

If t runs over its domain from $-\infty$ to ∞, the dimensionless conformal time parameter cT runs from $-\pi$ to 0. For every event in the spacetime, $cT - cT_i = cT + \pi$ is smaller than $\chi_{\max} = \pi$, so there are particle horizons. The part of the 3-sphere that is visible becomes bigger and bigger for $t \to \infty$ and the antipodal point is the only point that comes never into view. Similarly, $cT_f - cT = |cT|$ is smaller than $\chi_{\max}$, so there are event horizons.

The deSitter universe is of great relevance for cosmology. An interesting class of matter solutions asymptotically approach the deSitter universe. Moreover, it made its appearance in the steady-state model (which now is usually considered only of historic interest). We will come back to the deSitter universe when discussing the theory of inflation and the idea of dark energy.

- $k = 0$
  Now we have to solve the differential equation

$$a'(t)^2 - \frac{\Lambda}{3}\,c^2 a(t)^2 = 0$$

with a positive Λ, hence

$$\frac{da}{a} = \pm\,c\sqrt{\frac{\Lambda}{3}}\,dt\,,$$

$$\ln a - \ln a_0 = \pm\,c\sqrt{\frac{\Lambda}{3}}\,t$$

with a positive integration constant a_0 which is usually chosen as $a_0 = \sqrt{3/\Lambda}$. We only consider the solution with the plus sign. The solution with the minus sign gives the same universe with the time direction reversed. The scale factor

$$a(t) = \sqrt{\frac{3}{\Lambda}}\exp\Big(c\sqrt{\frac{\Lambda}{3}}\,t\Big)$$

expands monotonically from $t = -\infty$ to $t = +\infty$. The metric reads

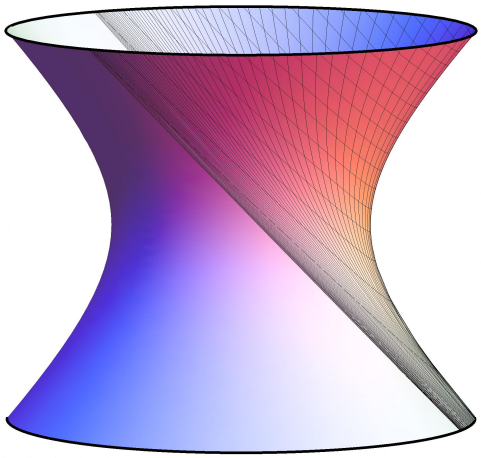
$$g = -c^2 dt^2 + \frac{3}{\Lambda} \exp\Big(\sqrt{\frac{\Lambda}{3}}\, 2\, c\, t\Big) \Big(d\chi^2 + \chi^2 \big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\Big)\,.$$

This is again the deSitter universe, but this time only half of it and with a slicing into *flat* 3-spaces. We see that a different choice of the integration constant a_0 could be compensated for by a rescaling of χ.

Note that this particular scale factor gives a "Hubble constant" that is really a constant, i.e., independent of time,

$$\frac{a'(t)}{a(t)} = c\sqrt{\frac{\Lambda}{3}}\,,$$

so the universe expands at a constant rate.



The embedding into the full deSitter universe is shown in the picture. The 3-dimensional flat slices $t =$ constant are represented as lines that come about as the sections of the hyperboloid with planes under 45 degress. The boundary of the region covered corresponds to $t = -\infty$. The embedding is given by the equations

$$X = \sqrt{\frac{3}{\Lambda}} \exp\Big(\sqrt{\frac{\Lambda}{3}} ct\Big)\, \chi \cos\varphi \sin\vartheta\,,$$

$$Y = \sqrt{\frac{3}{\Lambda}} \exp\Big(\sqrt{\frac{\Lambda}{3}} ct\Big)\, \chi \cos\varphi \sin\vartheta\,,$$

$$Z = \sqrt{\frac{3}{\Lambda}} \exp\Big(\sqrt{\frac{\Lambda}{3}} ct\Big)\, \chi \cos\vartheta\,,$$

$$W = \sqrt{\frac{3}{\Lambda}} \cosh\Big(\sqrt{\frac{\Lambda}{3}} ct\Big) - \sqrt{\frac{3}{\Lambda}}\, \frac{\chi^2}{2} \exp\Big(\sqrt{\frac{\Lambda}{3}} ct\Big)\,,$$

$$V = \sqrt{\frac{3}{\Lambda}} \sinh\Big(\sqrt{\frac{\Lambda}{3}} ct\Big) + \sqrt{\frac{3}{\Lambda}}\, \frac{\chi^2}{2} \exp\Big(\sqrt{\frac{\Lambda}{3}} ct\Big)\,.$$

- $k = -1$
  In this case the Friedmann equation reads

$$-c^2 + a'(t)^2 - \frac{\Lambda}{3} c^2 a(t)^2 = 0\,,$$

$$\frac{da}{dt} = \pm c \sqrt{\frac{\Lambda}{3} a^2 + 1}\,.$$

  As Λ is positive, we may substitute

$$\sqrt{\frac{\Lambda}{3}}\, a = \sinh u\,,$$

$$\sqrt{\frac{\Lambda}{3}}\, da = \cosh u\, du\,,$$

  hence

$$\sqrt{\frac{3}{\Lambda}} \frac{\cancel{\cosh u}\, du}{\cancel{\sqrt{\sinh^2 u + 1}}} = \pm c\, dt\,.$$

  This results in

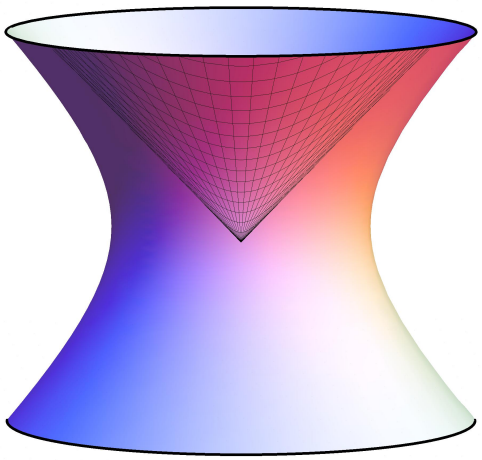
$$u = \pm \sqrt{\frac{\Lambda}{3}}\, c(t - t_0)\,,$$

$$a(t) = \pm \sqrt{\frac{3}{\Lambda}} \sinh\Big(\sqrt{\frac{\Lambda}{3}}\, c(t - t_0)\Big)\,.$$

  We choose $t_0 = 0$ and the plus sign. (Again, the minus sign gives a time-reversed version of the same spacetime.) The scale factor is defined for $0 < t < \infty$ and increases monotonically. The metric reads

$$g = -c^2 dt^2 + \frac{3}{\Lambda} \sinh^2\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big)\Big(d\chi^2 + \sinh^2\chi\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\Big)\,.$$

  This is again part of the deSitter universe, this time with a slicing into hyperbolic spaces. The part covered by the slicing sits in the entire deSitter universe in a similar fashion as the Milne universe sits in the entire Minkowski spacetime, see the picture.



The embedding into the hyperboloid is given by the equations

$$X = \sqrt{\frac{3}{\Lambda}}\sinh\Big(\sqrt{\frac{\Lambda}{3}}ct\Big)\sinh\chi\cos\varphi\sin\vartheta\,,$$

$$Y = \sqrt{\frac{3}{\Lambda}}\sinh\Big(\sqrt{\frac{\Lambda}{3}}ct\Big)\sinh\chi\sin\varphi\sin\vartheta\,,$$

$$Z = \sqrt{\frac{3}{\Lambda}}\sinh\Big(\sqrt{\frac{\Lambda}{3}}ct\Big)\sinh\chi\cos\vartheta\,,$$

$$W = \sqrt{\frac{3}{\Lambda}}\cosh\Big(\sqrt{\frac{\Lambda}{3}}ct\Big)\,,$$

$$V = \sqrt{\frac{3}{\Lambda}}\sinh\Big(\sqrt{\frac{\Lambda}{3}}ct\Big)\cosh\chi\,,$$

(iii) $\Lambda < 0$

- $k = 1$
  In this case there is no solution because the differential equation

  $$c^2 + a'(t)^2 - \frac{\Lambda}{3}c^2a(t)^2 = 0\,,$$

  with a negative Λ cannot be satisfied by a real function a.
- $k = 0$
  Again, there is no solution because

  $$a'(t)^2 - \frac{\Lambda}{3}c^2a(t)^2 = 0\,,$$

  cannot hold for a real a if Λ is negative.
- $k = -1$
  In this case the Friedmann equation

  $$-c^2 + a'(t)^2 - \frac{\Lambda}{3}c^2a(t)^2 = 0$$

  requires

  $$\frac{da}{dt} = \pm c\sqrt{1 + \frac{\Lambda}{3}a^2}\,.$$

  As Λ is negative, we may substitute

  $$\sqrt{-\frac{\Lambda}{3}}\,a = \sin u\,,$$

  $$\sqrt{\frac{-\Lambda}{3}}\,da = \cos u\,du$$

  hence

  $$\sqrt{-\frac{3}{\Lambda}}\,\frac{\cancel{\cos u}\,du}{\cancel{\sqrt{1-\sin^2 u}}} = \pm c\,dt\,.$$

This results in

$$u - u_0 = \pm \sqrt{\frac{\Lambda}{3}}\, c\, t$$

with an integration constant u_0. If we choose $u_0 = 0$ and the plus sign, the scale factor

$$a(t) = \sqrt{-\frac{3}{\Lambda}} \sin\Big(\sqrt{\frac{-\Lambda}{3}}\, c\, t\Big)$$

is defined on the time interval $0 < t < \big(-3/\Lambda\big)^{-1/2}\pi/c$. Here the other choice of the sign (and another choice of the integration constant) gives the same behaviour because the situation is time symmetric. The universe starts with a "big bang" and ends in a "big crunch". The metric reads

$$g = -c^2 dt^2 - \frac{3}{\Lambda} \sin^2\Big(\sqrt{-\frac{\Lambda}{3}}\, c\, t\Big)\Big(d\chi^2 + \sinh^2\chi\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\Big)\,.$$

This spacetime is part of the socalled *anti-deSitter universe*. The full anti-deSitter universe is the isometrically embedded hyperboloid

$$X^2 + Y^2 + Z^2 - W^2 - V^2 = -\frac{3}{\Lambda}$$

in the 5-dimensional pseudo-Euclidean space with metric

$$g^{(5)} = dX^2 + dY^2 + dZ^2 - dW^2 - dV^2\,.$$

The full anti-deSitter universe has the topology of $\mathbb{R}^3 \times S^1$ where S^1 is a 1-dimensional sphere, i.e., a circle. The cyclic dimension is timelike, i.e., in the anti-deSitter universe there are closed timelike curves through each point. One can remove this unwanted feature by considering the universal covering space.

The slicing into hyperbolic spaces covers only part of the anti-deSitter universe, see the picture on the next page. The embedding into the hyperboloid is given by the equations

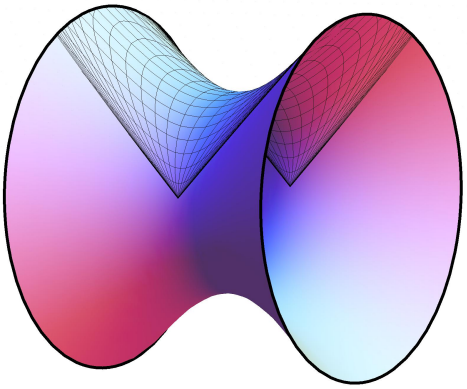
$$X = \sqrt{\frac{3}{\Lambda}} \sin\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sinh\chi \cos\varphi \sin\vartheta\,,$$

$$Y = \sqrt{\frac{3}{\Lambda}} \sin\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sinh\chi \sin\varphi \sin\vartheta\,,$$

$$Z = \sqrt{\frac{3}{\Lambda}} \sin\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \sinh\chi \cos\vartheta\,,$$

$$W = \sqrt{\frac{3}{\Lambda}} \cos\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big)\,,$$

$$V = \sqrt{\frac{3}{\Lambda}} \sin\Big(\sqrt{\frac{\Lambda}{3}}\, c\, t\Big) \cosh\chi\,.$$



Anti-deSitter spacetime is not considered as a realistic model of our universe, not even in the sense of a limit. However, anti-deSitter (AdS) spaces of various dimension play an important role in string theory, in particular as mathematical tools for calculations in conformal field theories (CFT). This approach is known as the AdS-CFT correspondence.

We summarise our results on vacuum solutions to the Friedmann equations. We have seen that the only solutions are Minkowski spacetime ($\Lambda = 0$), deSitter spacetime ($\Lambda > 0$) and anti-deSitter spacetime ($\Lambda < 0$). Minkowski spacetime can be viewed as a Robertson-Walker spacetime in two different ways, with slices of curvature $k = 0$ or $k = -1$. For the deSitter spacetime all three kinds of slicings, $k = 1$, $k = 0$ and $k = -1$, are possible, whereas for the anti-deSitter spacetime it only works with $k = -1$. The slicings with $k = -1$ have an initial singularity in the sense that the standard observers are compressed into one point, but this is of course not a curvature singularity. To put this another way, it is a singularity of the sclicing and not of the spacetime.

Minkowski, deSitter and anti-deSitter spacetime have constant curvature $\Lambda/3$, i.e., the curvature tensor satisfies

$$R_{\mu\nu\sigma\tau} = \frac{\Lambda}{3}\Big(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\nu}g_{\sigma\tau}\Big)$$

as can be verified in any of the given coordinate representations. Also, they are the only spacetimes with ten linearly independent Killing vector fields which is the maximal number in a pseudo-Riemannian manifold of dimension 4. (Here "linearly independent" refers to linear combinations with *constant* coefficients; of course, if we allow for coefficients that depend on the foot-point there cannot be more than four linearly independent vector fields.) In the Minkowski case, $\Lambda = 0$, these ten Killing vector fields generate the Poincaré group, i.e., the 4 translations, the 3 spatial rotations and the 3 Lorentz boosts. The corresponding symmetry groups for $\Lambda > 0$ and $\Lambda < 0$ are known as the *deSitter group* and the *anti-deSitter group*, respectively.

We have seen that Minkowski spacetime with the flat slicing is the only vacuum solution to the Friedmann equations with a constant scale factor. This does not mean that the deSitter spacetime and the anti-deSitter spacetime are not static: Actually, they do admit a timelike Killing vector field that is perpendicular to spacelike slices, but these sclices are not spaces of constant curvature, so in this static representation the metric does not have the Robertson-Walker form. We will discuss these static representations in Worksheet 5.

(b) Dust solutions

We now consider dust solutions to the Friedmann equations, i.e., solutions with $p = 0$. A dust is a good mathematical model for the ordinary (baryonic) matter in galaxies and also for "cold" dark matter. (Quite generally, the terms "cold fluid" and "dust" are synonymous, meaning a perfect fluid with vanishing pressure.)

We have to solve the equations

$$\frac{3c^2k}{a(t)^2} + \frac{3}{a(t)^2}\, a'(t)^2 - \Lambda\, c^2 = \kappa\, c^4\, \mu(t)\,, \qquad \text{(F1)}$$

$$-\,k - \frac{1}{c^2}\, a'(t)^2 - \,2\,\frac{a(t)}{c^2}\, a''(t) + \Lambda a(t)^2 = 0\,. \qquad \text{(F2}')$$

We require $\mu(t) > 0$ throughout.

We first look for static solutions, $a(t) = a_0 = \text{constant}$. Then (F1) and (F2$'$) require

$$\frac{3c^2k}{a_0^2} - \Lambda\, c^2 = \kappa\, c^4\, \mu(t)\,,$$

$$-\,k + \Lambda a_0^2 = 0\,.$$

Of course, the density must be constant, $\mu(t) = \mu_0$. Solving for Λ and μ_0 yields

$$\Lambda = \frac{k}{a_0}\,, \quad \mu_0 = \frac{2\,k}{\kappa\, c^2 a_0^2}\,.$$

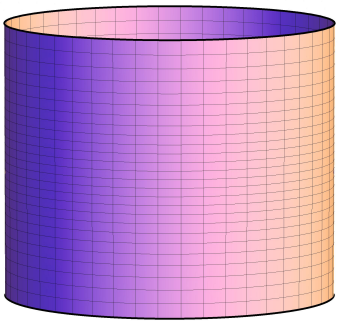
As we assume that the density is positive, the second equation implies $k = 1$, so the first one requires $\Lambda > 0$. We summarise this important result in the following way:

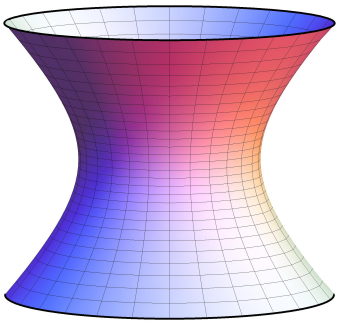
> If we consider Einstein's field equation without a cosmological constant, there is no static solution to the Friedmann equations for a dust of positive mass density.

It was this observation that led Einstein to introducing the cosmological constant in 1917. With a positive cosmological constant, a static dust solution does exist. It has $k = 1$, so the natural spatial topology is that of a 3-sphere. The metric reads

$$g = -c^2dt^2 + a_0^2\Big(d\chi^2 + \sin^2\chi\big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big)\Big)\,.$$

This is *Einstein's static universe*, also known as *Einstein's cylinder spacetime*, refering to the fact that the natural topology of this spacetime is $S^3 \times \mathbb{R}$. Einstein advertised this spacetime as a viable mathematical model of our universe in 1917. In the same year, deSitter introduced the universe named after him. Until Friedmann's work in 1922/23, these two spacetimes were the only cosmological models that were discussed on the basis of general relativity. We compare them on the next page, where we consider the "natural" (global) slicing of the deSitter universe.





	Einstein's static universe	deSitter universe
topology	$S^3 \times \mathbb{R}$	$S^3 \times \mathbb{R}$
time dependence	static	contracting, then expanding
Lambda term	$\Lambda > 0$	$\Lambda > 0$
matter	$\mu > 0$, $p = 0$	$\mu = 0$, $p = 0$

Recall that a positive cosmological constant has a repellent effect. In Einstein's static universe this repellent effect is balanced by the gravitational attraction of the dust. In the deSitter universe the cosmological constant delecerates the initial contraction and then causes a re-expansion.

Having found all static solutions to the equations (F1) and (F2′), we assume $a'(t) \neq 0$ from now on.

Claim: (F1) and (F2') imply the conservation law

$$\mu(t)a(t)^3 = \text{constant}\,.$$

Proof: We write (F1) in the form

$$\frac{\kappa}{3}\,c^2\,\mu(t)\,a(t)^3 = k\,a(t) + \frac{a(t)}{c^2}\,a'(t)^2 - \frac{\Lambda}{3}\,a(t)^3$$

and differentiate with respect to t:

$$\frac{\kappa}{3}\,c^2\,\frac{d\big(\mu(t)\,a(t)^3\big)}{dt} = k\,a'(t) + \frac{a'(t)^3}{c^2} + \frac{2}{c^2}\,a(t)\,a'(t)\,a''(t) - \Lambda\,a(t)^2 a'(t)$$

$$= a'(t)\Big(k + \frac{a'(t)^2}{c^2} + \frac{2}{c^2}\,a(t)\,a''(t) - \Lambda\,a(t)^2\Big)\,.$$

By (F2′), this is indeed zero.

Of course, this result just establishes the conservation of mass. (For a dust, the only form of rest energy is the rest mass of the dust particles.) As the equation $\nabla^\mu T_{\mu\nu} = 0$ is a consequence of Einstein's field equation, it should not come as a surprise that this conservation law is implied by the Friedmann equations.

Hence, the most convenient way of solving the Friedmann equations for dust is by determining $a(t)$ from the differential equation

$$k\,a(t) \;+\; \frac{a(t)}{c^2}\,a'(t)^2 \;-\; \frac{\Lambda}{3}\,a(t)^3 = a_0\,, \qquad \text{(F1}''\text{)}$$

with a positive constant a_0. Once a solution $a(t)$ has been found, the corresponding density $\mu(t)$ is determined by the equation

$$\frac{\kappa}{3}\,c^2\,\mu(t)\,a(t)^3 = a_0\,.$$

The above proof of the conservation law demonstrates that any solution of (F1″) automatically satisfies (F2′). Note, however, that this requires dividing by $a'(t)$, so this method excludes the static solutions.

We will now solve (F1″) for all values of k with $\Lambda = 0$. Thereafter, we will discuss how a non-vanishing Λ influences these solutions.

(i) $\Lambda = 0$
We have to solve

$$k\,a + \frac{a}{c^2}\left(\frac{da}{dt}\right)^2 = a_0\,.$$

To that end, it is convenient to introduce the conformal time,

$$dT = \frac{dt}{a}\,,$$

hence

$$\frac{da}{dt} = \frac{da}{dT}\,\frac{dT}{dt} = \frac{da}{dT}\,\frac{1}{a}\,.$$

Then the differential equation takes the form

$$k\,a + \frac{\cancel{a}}{c^2}\,\frac{1}{\cancel{a^2}}\left(\frac{da}{dT}\right)^2 = a_0\,,$$

$$\left(\frac{da}{dT}\right)^2 = c^2\big(a_0 a - k\,a^2\big)\,,$$

$$\frac{da}{\sqrt{a_0\,a - k\,a^2}} = \pm\,c\,dT\,.$$

- $k = 1$
  Then we have to integrate

$$\frac{2\,da}{a_0\sqrt{1 - \left(1 - \dfrac{2\,a}{a_0}\right)^2}} = \pm\,c\,dT$$

which can be done with the substitution

$$1 \;-\; \frac{2\,a}{a_0} \;=\; \cos u\,, \quad \frac{2\,da}{a_0} \;=\; \sin u\,du\,.$$

The resulting integral reads

$$\int \frac{\cancel{\sin u}\, du}{\cancel{\sqrt{1-\sin^2 u}}} = \pm c \int dT\,,$$

$$u \;=\; \mp c\,\big(T - T_i\big)$$

with an integration constant T_i. Choosing $T_i = 0$, this yields

$$\arccos\Big(1 \,-\, \frac{2a}{a_0}\Big) \;=\; \mp c\,T\,,$$

$$1 \,-\, \frac{2a}{a_0} \;=\; \cos\big(cT\big)\,,$$

$$a = \frac{a_0}{2}\Big(1 - \cos\big(cT\big)\Big).$$

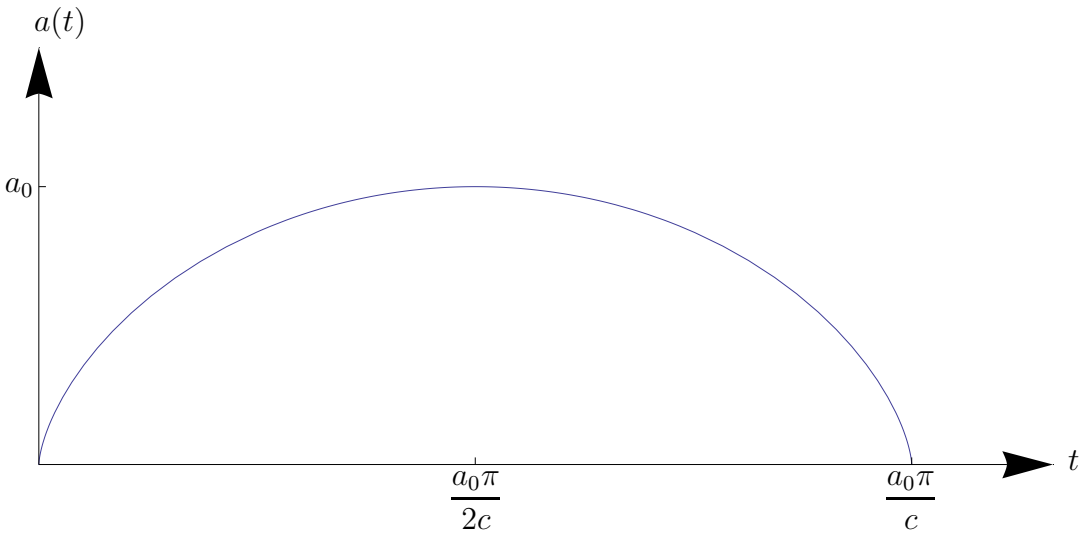
With this result, the relation between t and T reads

$$c\,dt = c\,a\,dT = \frac{c\,a_0}{2}\Big(1 - \cos\big(cT\big)\Big)dT\,,$$

and, upon choosing the integration constant appropriately,

$$c\,t \;=\; \frac{a_0}{2}\Big(c\,T - \sin\big(cT\big)\Big)\,.$$

We have thus found the graph of the function $t \mapsto a$ in parametric form, with T as the curve parameter. The resulting curve is a cycloid. (A cycloid is the curve traced by a point on the rim of a wheel that is rolling on a horizontal surface along a straight line.) The universe begins with a big bang at $t = 0$, reaches a maximal extension at $t = a_0\pi/(2c)$ and ends in a big crunch at $t = a_0\pi/c$.



- $k = 0$
  In this case it is not actually necessary to consider the conformal time, but in view of consistency with the other cases we proceed analogously. Integrating

$$\frac{1}{\sqrt{a_0}} \int \frac{da}{\sqrt{a}} = \pm\, c \int dT$$

  yields

$$\frac{2}{\sqrt{a_0}} \sqrt{a} = \pm\, c \left(T - T_i\right) .$$

  We choose again $T_i = 0$. Then

$$\frac{4\, a}{a_0} = c^2\, T^2 .$$

  The relation between t and T reads

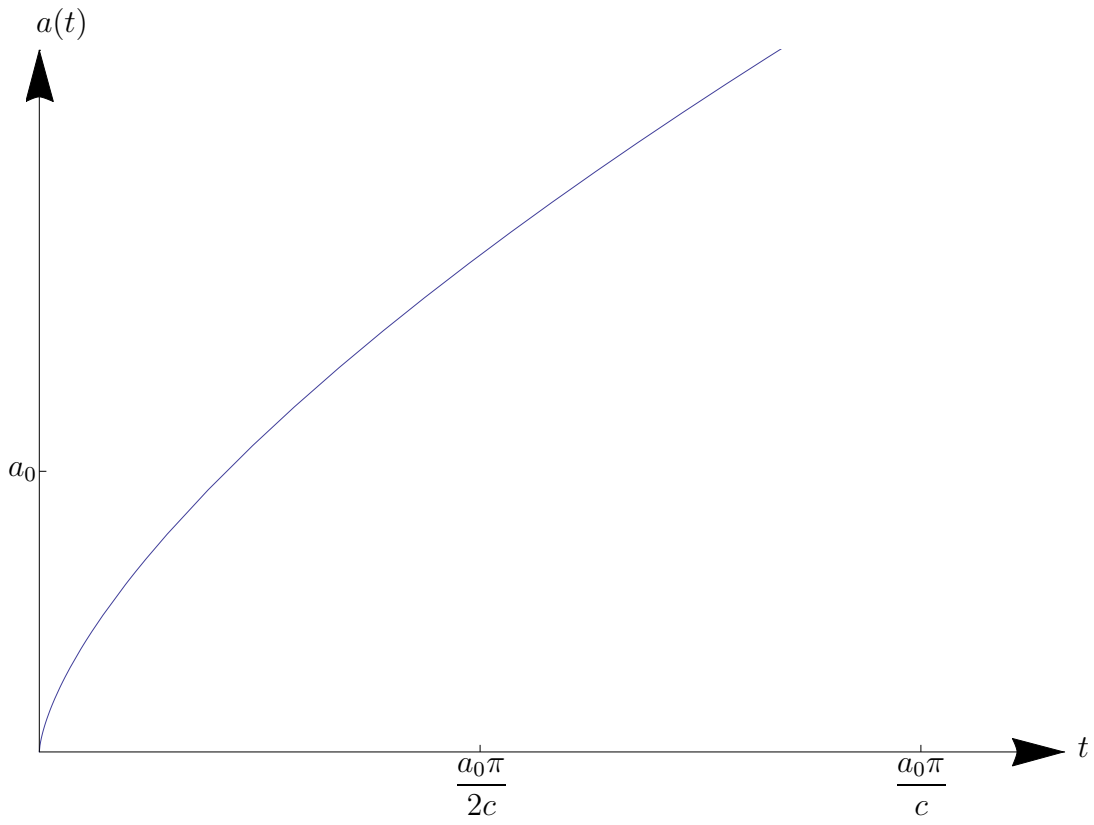
$$dt = a\, dT \;=\; \frac{a_0}{4}\, c^2\, T^2\, dT ,$$

  and upon integration

$$t \;=\; \frac{a_0}{4}\, c^2\, \frac{T^3}{3} ;$$

  here we have chosen the initial condition such that $t = 0$ corresponds to $T = 0$. Hence

$$a(t) \;=\; \frac{a_0}{4}\, c^2\, T^2 \;=\; \frac{a_0}{4}\, c^2 \left(\frac{12\, t}{a_0\, c^2}\right)^{2/3} \;=\; \left(\frac{9}{4}\, a_0\, c^2\right)^{1/3} t^{2/3} .$$



The metric reads

$$g = -c^2 dt^2 + \Big(\frac{9}{4}\, a_0\, c^2\Big)^{2/3} t^{4/3} \Big(d\chi^2 + \chi^2 \big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \big) \Big) \,.$$

This is the *Einstein-deSitter universe*. It is a dust-filled, forever expanding spacetime model with flat spatial sections, so the natural topology of this spacetime is $\mathbb{R}^4$. After the expansion of our universe had been widely accepted around 1930, Einstein and deSitter advertised this special solution to the Friedmann equations as the most promising cosmological world model in a joint paper.

- $k = -1$

The mathematics is quite similar to the case $k = 1$. We have to integrate

$$\frac{2\, da}{a_0 \sqrt{\Big(\frac{2\, a}{a_0} + 1\Big)^2 - 1}} = \pm\, c\, dT$$

which can be done with the substitution

$$\frac{2\, a}{a_0} + 1 \;=\; \cosh u \,, \quad \frac{2\, da}{a_0} \;=\; -\sinh u\, du \,.$$

Then the integral reads

$$\int \frac{\cancel{\sinh u}\, du}{\cancel{\sqrt{\cosh^2 u - 1}}} = \pm\, c \int dT \,,$$

$$u \;=\; \mp\, c \big(T - T_i \big)$$

with an integration constant T_i. Choosing $T_i = 0$, this yields

$$\operatorname{arcosh}\Big(\frac{2a}{a_0} + 1\Big) \;=\; \mp c\, T \,,$$

$$\frac{2a}{a_0} + 1 = \cosh\big(cT\big) \,,$$

$$a = \frac{a_0}{2}\Big(\cosh\big(cT\big) - 1\Big).$$

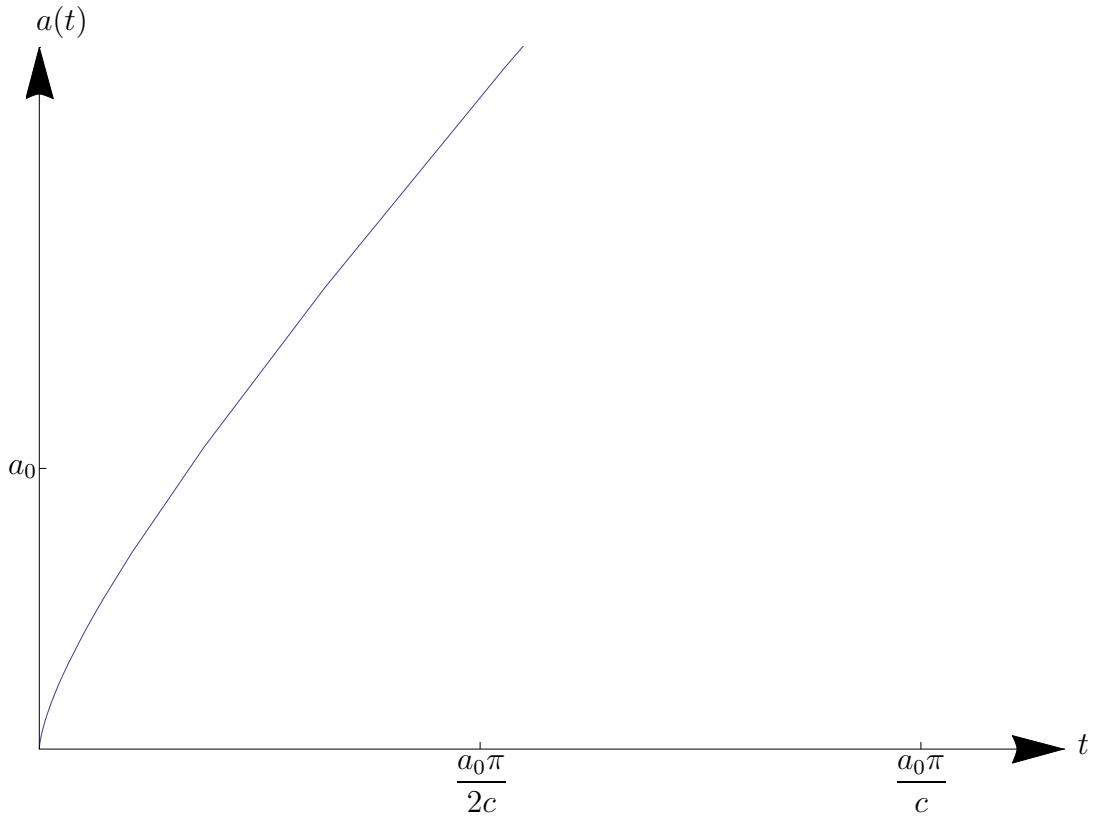
With this result, the relation between t and T reads

$$c\, dt = c\, a\, dT = \frac{c\, a_0}{2}\Big(\cosh\big(cT\big) - 1\Big) dT \,,$$

and, upon choosing the integration constant appropriately,

$$c\, t \;=\; \frac{a_0}{2}\Big(\sinh\big(cT\big) - c\, T\Big) \,.$$

Again, we have found the graph of the function $t \mapsto a$ in parametric form, with T as the curve parameter. The resulting curve is the hyperbolic analogue of a cycloid, sometimes called a "hyperbolic cycloid". The universe begins with a big bang at $t = 0$ and expands forever.



We see that, for $k = 1$, $k = 0$ and $k = -1$, the dust universe without a cosmological constant is always *decelerating*, i.e., $q(t_o) > 0$ for all t_o. For $k = -1$, the initial "explosion" is strong enough to make the universe expand forever. For $k = 1$, however, the self-gravitating dust is dense enough to make the universe re-collapse into a big crunch. The case $k = 0$ is the critical case where "the turning point is at infinity", i.e., the universe just makes it to avoid re-collapse. This borderline case can be characterised by a critical density in the following way. At any chosen time t_o, the first Friedmann equation without a cosmological constant can be written as an equality between densities,

$$\frac{3c^2k}{\kappa c^4 a(t_o)^2} = \mu(t_o) - \frac{3\,a'(t_o)^2}{\kappa c^4 a(t_o)^2}\,.$$

The sign of the left-hand side is determined by k. If we define a critical density (which depends on t_o) by

$$\mu_c(t_o) := \frac{3\,a'(t_o)^2}{\kappa c^4 a(t_o)^2} = \frac{3\,H(t_o)^2}{\kappa c^4}\,,$$

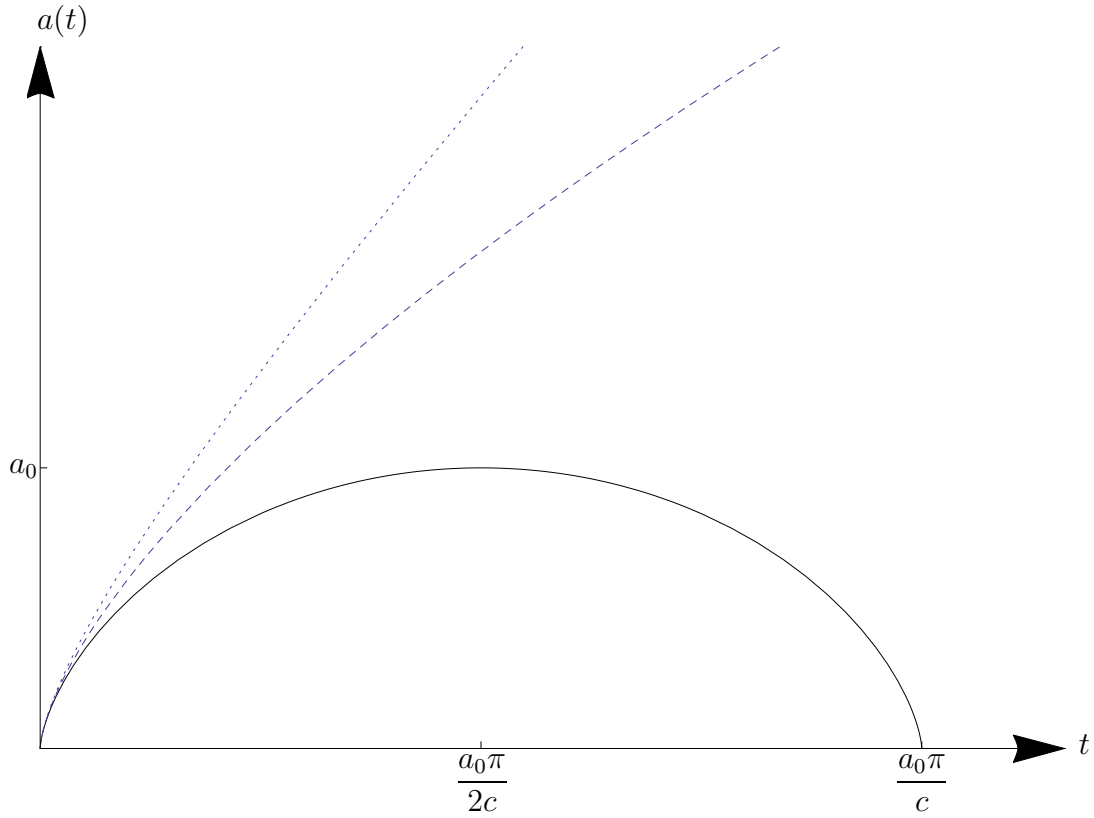
we see that

$$\begin{aligned} \mu(t_o) &> \mu_c(t_o) \quad \text{if } k = 1\,, \\ \mu(t_o) &= \mu_c(t_o) \quad \text{if } k = 0\,, \\ \mu(t_o) &< \mu_c(t_o) \quad \text{if } k = -1\,. \end{aligned}$$

One often uses the *density parameter*

$$\Omega_m(t_o) = \frac{\mu(t_o)}{\mu_c(t_o)}$$

where the index m (for "matter") is meant as a reminder that we are considering a dust universe here. The picture shows the behaviour of the scale factor for the three cases in one diagram: Solid for overcritical density, dashed for critical density and dotted for undercritical density.



For the dust universes without a cosmological constant W. Mattig found in 1957 an *exact* distance-redshift relation. We will derive this Mattig formula now. As a preparation, we first establish three auxiliary equations which we denote (E1), (E2) and (E3).

For the following calculation we assume that $a'(t) > 0$. If $k = -1$ or $k = 0$, this is true for all t; if $k = 1$, however, this assumption restricts the validity to times $0 < t < a_0\pi/(2c)$. From the differential equation

$$k\,c^2 \,+\, a'(t)^2 \,=\, \frac{c^2 a_0}{a(t)}$$

we have

$$a'(t) \,=\, c\,\sqrt{\frac{a_0}{a(t)} - k}\,, \qquad \text{(E1)}$$

and, on the other hand, by differentiation

$$2\,a'(t)\,a''(t) \,=\, -\,\frac{c^2 a_0}{a(t)^2}\,a'(t)\,.$$

As we have excluded the static case, we have $a'(t) \neq 0$ for almost all t, so we may divide by $a'(t)$ and get

$$2\,a''(t) \;=\; -\,\frac{c^2 a_0}{a(t)^2}\,.$$

Multiplication by $a(t)/a'(t)^2$ yields

$$\frac{2\,a(t)\,a''(t)}{a'(t)^2} \;=\; -\,\frac{c^2 a_0}{a(t)\,a'(t)^2}$$

which, by (E1), can be rewritten as

$$\frac{2\,a(t)\,a''(t)}{a'(t)^2} \;=\; -\,\frac{a_0}{a(t)\Big(\dfrac{a_0}{a(t)} - k\Big)}\,.$$

Evaluation at time t_o gives the deceleration parameter at this time,

$$2\,q(t_0) \;=\; \frac{a_0}{a_0 - k\,a(t_o)}\,,$$

hence

$$k\,\frac{a(t_o)}{a_0} \;=\; 1 \;-\; \frac{1}{2\,q(t_o)}\,. \qquad \text{(E2)}$$

The third auxiliary equation follows from inserting (E1) into the equation that defines the Hubble constant,

$$H(t_o) = \frac{a'(t_o)}{a(t_o)} \;=\; \frac{c}{a(t_o)}\,\sqrt{\frac{a_0}{a(t_o)} - k} \;=\; \frac{c\,\sqrt{k}}{a(t_o)}\,\sqrt{\frac{a_0}{k\,a(t_o)} - 1}\,.$$

For $k = -1$, both square roots are purely imaginary, so the right-hand side is real. For $k = 0$, the right-hand side is to be understood in the sense of a limiting procedure. With (E2), we find from the last equation

$$H(t_o) = \frac{c\,\sqrt{k}}{a(t_o)}\,\sqrt{\frac{1}{1 - \dfrac{1}{2q(t_o)}} - 1} = \frac{c\,\sqrt{k}}{a(t_o)}\,\sqrt{\frac{\cancel{1} - \cancel{1} + \dfrac{1}{2q(t_o)}}{1 - \dfrac{1}{2q(t_o)}}}$$

$$= \frac{c\,\sqrt{k}}{a(t_o)\,\sqrt{2q(t_o) - 1}}\,,$$

hence

$$a(t_o) = \frac{c\,\sqrt{k}}{H(t_o)\,\sqrt{2\,q(t_o) - 1}}\,. \qquad \text{(E3)}$$

It is now our goal to determine the luminosity distance D_L as a function of the redshift z. Recall from p.25 that, for light from an emission event at time t_e to an observation event at time t_o, the luminosity distance is given by the equation

$$D_L \;=\; a(t_o)\,(1+z)\,\eta(\chi)$$

where

$$\eta(\chi) \;=\; \frac{\sin\bigl(\sqrt{k}\,\chi\bigr)}{\sqrt{k}} \;=\; \begin{cases} \sin\chi & \text{if } k=1\,, \\ \chi & \text{if } k=0\,, \\ \sinh\chi & \text{if } k=-1\,. \end{cases}$$

Keeping the observation event t_0 fixed, we shall determine $\eta(\chi)$ as a function of z. As we want to use (E1) and the equations derived from it, we have to assume that $t_o < a_0\pi/(2c)$ if $k=1$. From

$$\chi = \int_{t_e}^{t_o} \frac{c\,dt}{a} \;=\; \int_{a(t_e)}^{a(t_o)} \frac{c\,dt\,da}{a\,da}$$

we find, with (E1),

$$\chi = \int_{a(t_e)}^{a(t_o)} \frac{\not{c}\,da}{a\,\not{c}\,\sqrt{\dfrac{a_0}{a}-k}} = \int_{a(t_e)}^{a(t_o)} \frac{da}{\sqrt{a_o\,a-k\,a^2}}\,.$$

This is an elementary integral,

$$\chi = \frac{1}{\sqrt{k}}\arcsin\Bigl(\frac{2\,k\,a}{a_o}-1\Bigr)\Bigr|_{a(t_e)}^{a(t_o)}\,,$$

$$\sqrt{k}\,\chi \;=\; \arcsin\Bigl(\frac{2\,k\,a(t_o)}{a_o}-1\Bigr) - \arcsin\Bigl(\frac{2\,k\,a(t_e)}{a_o}-1\Bigr)$$

$$= \underbrace{\arcsin\Bigl(\frac{2\,k\,a(t_o)}{a_o}-1\Bigr)}_{=\alpha} \;-\; \underbrace{\arcsin\Bigl(\frac{2\,k\,a(t_o)}{a_o(1+z)}-1\Bigr)}_{=\beta}\,.$$

Hence

$$D_L = a(t_o)(1+z)\,\frac{\sin\bigl(\sqrt{k}\,\chi\bigr)}{\sqrt{k}} = a(t_o)(1+z)\,\frac{\sin\bigl(\alpha-\beta\bigr)}{\sqrt{k}}$$

$$= \frac{a(t_o)}{\sqrt{k}}\,(1+z)\,\Bigl(\sin\alpha\cos\beta-\sin\beta\cos\alpha\Bigr)$$

$$= \frac{a(t_o)}{\sqrt{k}}\,(1+z)\,\Bigl(\sin\alpha\,\sqrt{1-\sin^2\beta}-\sin\beta\,\sqrt{1-\sin^2\alpha}\,\Bigr)\,.$$

Inserting the expressions for α and β yields

$$
\begin{aligned}
D_L = \frac{a(t_o)}{\sqrt{k}}\,(1+z)\,\Bigg\{ &\Big(\frac{2\,k\,a(t_o)}{a_0}-1\Big)\sqrt{1-\Big(\frac{2\,k\,a(t_o)}{a_0(1+z)}-1\Big)^2} \\
&-\Big(\frac{2\,k\,a(t_o)}{a_0(1+z)}-1\Big)\sqrt{1-\Big(\frac{2\,k\,a(t_o)}{a_0}-1\Big)^2}\;\Bigg\} \\
= \frac{a(t_o)}{\sqrt{k}}\,\Bigg\{ &\Big(\frac{2\,k\,a(t_o)}{a_0}-1\Big)\sqrt{(1+z)^2-\Big(\frac{2\,k\,a(t_o)}{a_0}-1-z\Big)^2} \\
&-\Big(\frac{2\,k\,a(t_o)}{a_0}-1-z\Big)\sqrt{1-\Big(\frac{2\,k\,a(t_o)}{a_0}-1\Big)^2}\;\Bigg\}\,.
\end{aligned}
$$

With (E2) this can be rewritten as

$$
\begin{aligned}
D_L = \frac{a(t_o)}{\sqrt{k}}\,\Bigg\{ &\Big(1-\frac{1}{q(t_o)}\Big)\sqrt{(1+z)^2-\Big(1-z-\frac{1}{q(t_o)}\Big)^2} \\
&-\Big(1-z-\frac{1}{q(t_o)}\Big)\sqrt{1-\Big(1-\frac{1}{q(t_o)}\Big)^2}\;\Bigg\} \\
= \frac{a(t_o)}{\sqrt{k}\,q(t_o)^2}\,\Bigg\{ &\Big(q(t_o)-1\Big)\sqrt{q(t_o)^2(1+z)^2-\Big(q(t_o)(1-z)-1\Big)^2} \\
&-\Big(q(t_o)-q(t_o)z-1\Big)\sqrt{q(t_o)^2-\Big(q(t_o)-1\Big)^2}\;\Bigg\} \\
= \frac{a(t_o)}{\sqrt{k}\,q(t_o)^2}\,\Big\{ &\Big(q(t_o)-1\Big)\sqrt{q(t_o)^2 4\,z+2\,q(t_o)(1-z)-1} \\
&-\Big(q(t_o)-q(t_o)z-1\Big)\sqrt{2\,q(t_o)-1}\;\Big\} \\
= \frac{a(t_o)\,\sqrt{2\,q(t_o)-1}}{\sqrt{k}\,q(t_o)^2}\,\Big\{ &\Big(q(t_o)-1\Big)\sqrt{2\,q(t_o)\,z+1}\;-\;q(t_o)+q(t_o)z+1\;\Big\}\,.
\end{aligned}
$$

Finally, with (E3), we get the Mattig formula

$$
D_L = \frac{c}{H(t_o)\,q(t_o)^2}\,\Big\{q(t_o)\,z+\Big(q(t_o)-1\Big)\Big(\sqrt{2\,q(t_o)\,z+1}\;-\;1\Big)\Big\}\,.
$$

This relation holds for all three cases, $k=-1$, $k=0$ and $k=1$. However, in the case $k=1$ our derivation is valid only for observation times $0<t_o<a_0\pi/(2c)$. For later observation times the relation between z and D_L is no longer one-to-one: In can be shown that then there are two values of D_L corresponding to the same value of z and only one of them is given by the Mattig formula.

(ii) $\Lambda \neq 0$

The Friedmann equation (F1″) can be solved in the case $\Lambda \neq 0$ with the same method as in the case $\Lambda \neq 0$, by introducing the conformal time. In this case one finds the parametric relation between the scale factor and conformal time given by an elliptic integral. We will not work this out here but restrict to a qualitative analysis.

To that end we write (F1″) in the form

$$\left(\frac{da}{dt}\right)^2 - \frac{a_0\, c^2}{a} - \frac{\Lambda}{3}\, c^2\, a^2 \;=\; -\,k\, c^2\,.$$

This equation has the form of the energy-conservation law of classical mechanics for a particle moving in one spatial dimension,

$$\left(\frac{da}{dt}\right)^2 + V(a) \;=\; E\,.$$

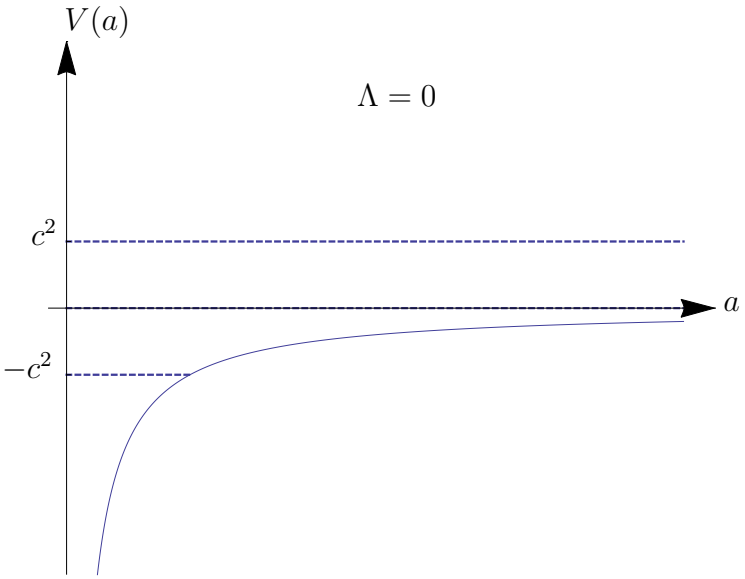
In this analogy, we have to identify a with the position coordinate of the particle, usually denoted x, and

$$V(a) \;=\; -\,\frac{a_0\, c^2}{a} - \frac{\Lambda}{3}\, c^2\, a^2\,,$$

$$E \;=\; -\,k\, c^2\,.$$

As the "kinetic energy" $(da/dt)^2$ cannot be negative, we must have $V(a) \leq E$. Hence, for each value of E (i.e., for $k = 1$, $k = 0$ and $k = -1$) the accessible range of a is given by that part of the line $V(a) = E$ that lies above the graph of the potential. Points where $V(a) = E$ are turning points where da/dt is zero.

For $\Lambda = 0$ we can read from the diagram the following behaviour. If $k = -1$ (i.e., $E = c^2$), the universe starts with a big bang at $a = 0$ and extends forever up to infinity. For $k = 1$ (i.e., $E = -c^2$) it starts with a big bang, reaches a maximum value where $da/dt = 0$ and then recollapses towards a big crunch. $k = 0$ is the borderline case where the turning point is at infinity, i.e., the universe just makes it to expand forever. These observations reproduce, in a qualitative fashion, what we have found with the exact analytical solutions for the case $\Lambda = 0$, see the diagram on p.45.



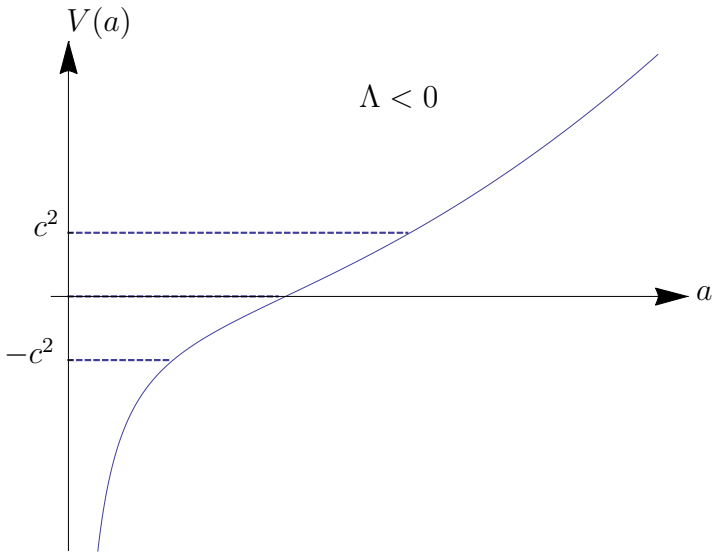
If $\Lambda < 0$, the potential $V(a)$ monotonically increases from $-\infty$ to ∞. As a consequence, the universe is *always* recollapsing, for $k = -1$, $k = 0$ and $k = 1$. The maximal value $a_{\max}$ of the scale factor is determined by the equation

$$V(a_{\max}) \;=\; -\,k\,c^2\,.$$

This is a cubic equation,

$$\frac{\Lambda}{3}\,c^2\,a_{\max}^3 - k\,c^2\,a_{\max} + a_0\,c^2 \;=\; 0\,,$$

which has, indeed, precisely one real and positive solution $a_{\max}$ if $\Lambda < 0$.



The case $\Lambda > 0$ is more subtle. Then the potential increases from $-\infty$ to a maximum at a certain value a_M and decreases again to $-\infty$. The behaviour of the universe with $k = 1$ depends on whether the maximum value $V(a_M)$ is smaller than, equal to or bigger than $-c^2$. The value a_M is determined by

$$V'(a_M) \;=\; \frac{a_0\,c^2}{a_M^2} - \frac{2\,\Lambda}{3}\,c^2\,a_M \;=\; 0\,, \quad a_M \;=\; \Big(\frac{3\,a_0}{2\,\Lambda}\Big)^{1/3}\,.$$

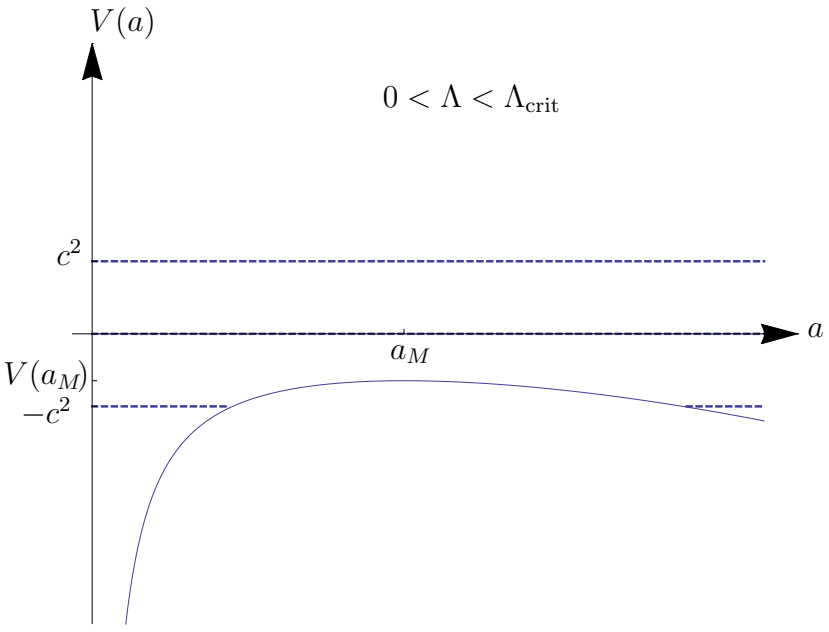
At this maximum, the potential takes the value

$$\begin{aligned} V(a_M) \;&=\; -\,a_0\,c^2\,\Big(\frac{2\,\Lambda}{3\,a_0}\Big)^{1/3} - \frac{\Lambda}{3}\,c^2\,\Big(\frac{3\,a_0}{2\,\Lambda}\Big)^{2/3} \\ &=\; -\,a_0^{2/3}\,c^2\,\Big(\frac{2\,\Lambda}{3}\Big)^{1/3}\Big(1 + \frac{1}{2}\Big) \;=\; -\,\Big(\frac{3\,a_0}{2}\Big)^{2/3}\,c^2\,\Lambda^{1/3}\,. \end{aligned}$$

We consider first the case that $V(a_M) > -c^2$, i.e.

$$\Lambda \;<\; \frac{4}{9\,a_0^2} \;=\; \Lambda_{\rm crit}\,.$$

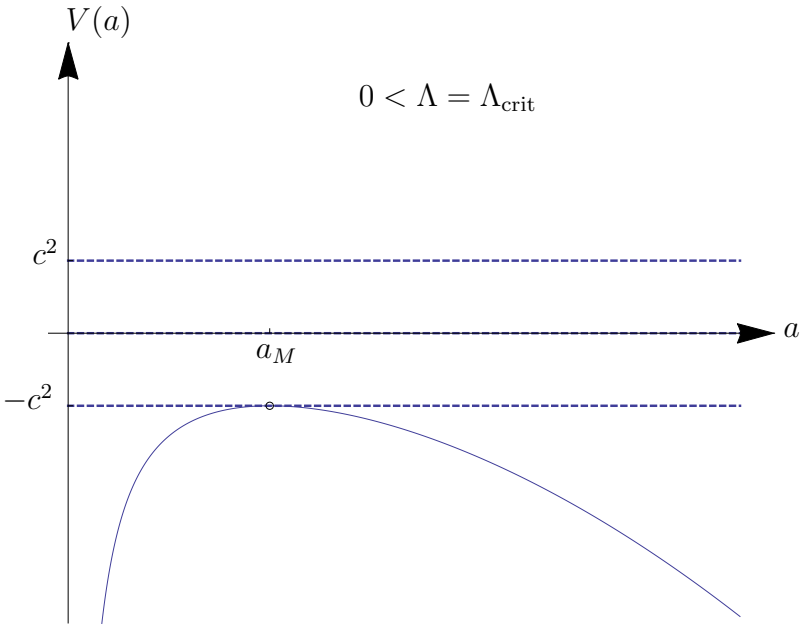
If $k = -1$ or $k - 0$, the universe begins with a big bang and is expanding forever. In the case $k = 1$ there are two universes: One is starting with a big bang, reaches a maximum scale factor, and is then recollapsing. The other comes in from infinity, reaches a minimum scale factor and is then re-expanding to infinity.



If $V(a_M) = -c^2$, i.e.

$$\Lambda = \frac{4}{9\,a_0^2} = \Lambda_{\text{crit}}\,,$$

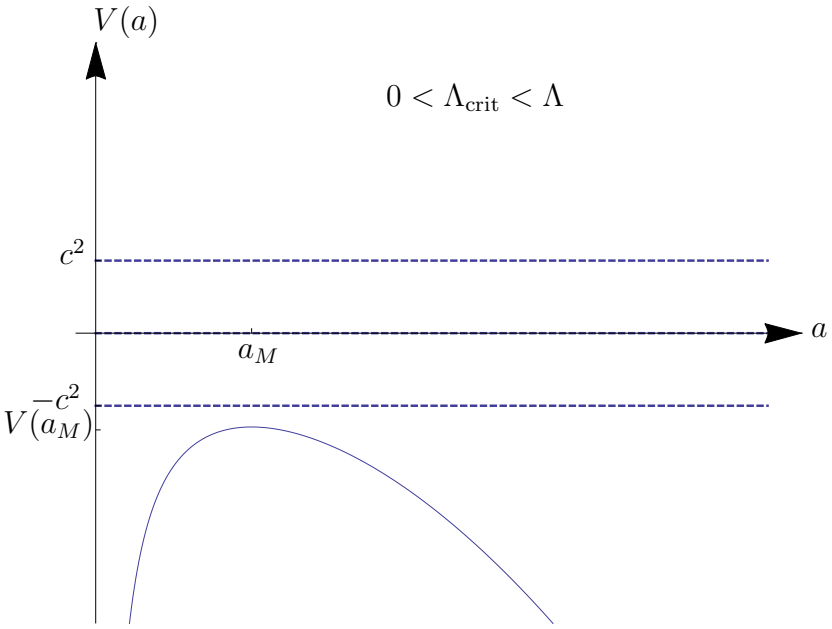
for $k = -1$ or $k = 0$ the universe begins with a big bang and is expanding forever. In the case $k = 1$ there are two universes: One is starting with a big bang, then the scale factor is increasing and approaches the finite value a_M asymptotically from below. The other comes in from infinity, then the scale factor decreases monotonically and approaches the value a_M asymptotically from above.



If $V(a_M) < -c^2$, i.e.

$$\Lambda > \frac{4}{9\,a_0^2} = \Lambda_{\text{crit}}\,,$$

for all three cases $k = -1$, $k = 0$ and $k = 1$ the universe begins with a big bang and is expanding forever. From the differential equation we read that a non-zero cosmological constant always dominates the dynamical behaviour for big a. If $\Lambda > 0$, the spacetime is similar to deSitter for big a.



This is one of the reasons why the deSitter universe is relevant: Dust solutions with a positive cosmological constant asymptotically approach deSitter for $a \to \infty$.

(c) Perfect fluid solutions with pressure

We will now consider the Friedmann equations in its generality, with a non-vanishing density μ and a non-vanishing pressure p,

$$\frac{3c^2k}{a(t)^2} + \frac{3}{a(t)^2}\,a'(t)^2 - \Lambda\,c^2 = \kappa\,c^4\,\mu\,, \qquad \text{(F1)}$$

$$-k - \frac{1}{c^2}\,a'(t)^2 - 2\,\frac{a(t)}{c^2}\,a''(t) + \Lambda a(t)^2 = \kappa\,p\,a(t)^2\,. \qquad \text{(F2)}$$

As long as we do not make any assumptions about μ or p, there is no equation to be solved: Given any k and any $a(t)$, one is even free to choose Λ at will and then (F1) and (F2) determine the density and the pressure. The equations are invariant under transformations

$$\Lambda \mapsto \Lambda + \Lambda_0\,, \quad \mu(t) \mapsto \mu(t) - \frac{\Lambda_0}{\kappa c^2}\,, \quad p(t) \mapsto p(t) + \frac{\Lambda_0}{\kappa}\,.$$

We can utilise this fact for re-interpreting a solution with a cosmological constant as a solution without a cosmological constant, by choosing $\Lambda_0 = -\Lambda$. E.g. we know that the deSitter universe is a solution with $\Lambda > 0$, $\mu = 0$ and $p = 0$. We may re-interpret it as a solution with $\Lambda = 0$, a constant density $\mu > 0$ and a constant pressure $p = -c^2\mu < 0$, see Worksheet 5, Problem2.

If we make special assumptions on μ and p, e.g. if we assume that p is related to μ by an equation of state, then (F1) and (F2) become a system of equations for $a(t)$ and $\mu(t)$. Before solving this system for special cases, we consider the static solutions, i.e., we consider the Friedmann equations for the case that $a(t) = a_0 = \text{constant}$:

$$\frac{3k}{a_0^2} - \Lambda = \kappa\, c^2\, \mu\,,$$

$$-\frac{k}{a_0^2} + \Lambda = \kappa\, p\,.$$

We have seen before that, with $\mu \geq 0$ and $p = 0$, static solutions exist only for the cases $k = 1$ and $k = 0$. Now, with a pressure, we also have solutions with $k = -1$; however, as

$$\kappa\big(\mu\, c^2 + p\big) = \frac{2\, k}{a_0^2}\,,$$

we see that in this case the pressure (or the density) has to be negative. We summarise the static solutions:

- $k = 1$: This is Einstein's static universe,

  $$g = -c^2\, dt^2 + a_0^2\Big(d\chi^2 + \sin^2\chi\, d\Omega^2\Big)\,.$$

  It is a vacuum solution with a positive cosmological constant ($\Lambda > 0$, $\mu = 0$ and $p = 0$). Again, we are free to re-interpret it, e.g. as a solution without a cosmological constant but with a positive density and a negative pressure.

- $k = 0$: This is Minkowski spacetime with the natural slicing associated with inertial systems,

  $$g = -c^2\, dt^2 + a_0^2\Big(d\chi^2 + \chi^2 d\Omega^2\Big)\,.$$

  It is a vacuum solution without a cosmological constant ($\Lambda = 0$, $\mu = 0$ and $p = 0$). Note, however, that we are free to re-interpret it as a solution with a cosmological constant and a funny matter content.

- $k = -1$: This spacetime is known as static hyperbolic spacetime,

$$g \;=\; -\,c^2\,dt^2 \;+\; a_0^2\Big(d\chi^2 + \sinh^2\chi\,d\Omega^2\Big)\,.$$

  We are free to choose the cosmological constant at will. Whatever choice we make, the pressure comes out negative if we want to have the density non-negative.

Having the static solutions out of the way, we try to reduce the Friedmann equations to one first-order equation, as we did for a dust, by utilising a conservation law. From (F1) we find that

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\kappa}{3}\,\mu\,c^4\,a^3\Big) \;&=\; \frac{d}{dt}\left(c^2\,k\,a \;+\; \Big(\frac{da}{dt}\Big)^2\,a \;-\; \frac{\Lambda}{3}\,c^2\,a^3\right)\\ &=\; c^2\,k\,\frac{da}{dt} \;+\; \Big(\frac{da}{dt}\Big)^3 \;+\; 2\,\frac{da}{dt}\,\frac{d^2a}{dt^2}\,a \;-\; \Lambda\,c^2\,a^2\,\frac{da}{dt}\\ &=\; \frac{da}{dt}\left(c^2\,k \;+\; \Big(\frac{da}{dt}\Big)^2 \;+\; 2\,\frac{d^2a}{dt^2}\,a \;-\; \Lambda\,c^2\,a^2\right)\,.\end{aligned}$$

By (F2), this can be rewritten as

$$\frac{d}{dt}\Big(\frac{\kappa}{3}\,\mu\,c^4\,a^3\Big) \;=\; -\,c^2\,\kappa\,p\;a^2\,\frac{da}{dt}\,,$$

hence

$$\frac{d}{dt}\Big(\mu\,c^2\,a^3\Big) \;=\; -\,p\,\frac{d}{dt}a^3\,. \qquad (C1)$$

This is the first law of thermodynamics for a volume element, $dU = T\,dS \;-\; p\,dV$, for the case of an isentropic process, $dS = 0$. Isentropic means that there is no heat transfer between the volume element and its neighbourhood; this assumption is implicitly included by requiring that the energy-momentum tensor has the form of a perfect fluid. Although we have energy conservation in the sense that no energy is produced, $\nabla_\mu T^{\mu\nu} = 0$, the energy in a comoving volume is *not* preserved because the pressure is doing work.

As long as μ and p are unrelated, the energy balance law (C1) cannot be further specified. In particular, we cannot rewrite (C1) in the form $d(\dots)/dt = 0$. However, if we assume an equation of state, then this is possible. We consider here only the special kind of an equation of state where the pressure is directly proportional to the energy density,

$$p(t) \;=\; w\,c^2\,\mu(t)\,, \qquad w \;=\; \text{constant}\,.$$

The energy-momentum tensor is then of the form

$$T_{\rho\sigma} \;=\; \Big(\mu \;+\; \frac{p}{c^2}\Big)\;U_\rho\,U_\sigma \;+\; p\,g_{\rho\sigma} \;=\; \mu\,\Big(\big(1 \;+\; w\big)\,U_\rho\,U_\sigma \;+\; w\,c^2\,g_{\rho\sigma}\Big)\,.$$

The energy balance law (C1) specifies to

$$\frac{d}{dt}\Big(\mu\,a^3\Big) \;=\; -\,w\,\mu\,\frac{d}{dt}a^3\,,$$

$$\frac{d\mu}{dt}\,a^3 \,+\, \mu\,\frac{da^3}{dt} \;=\; -\,w\,\mu\,\frac{da^3}{dt}$$

$$\frac{d\mu}{\mu} \,+\, \big(1\,+\,w\big)\,\frac{da^3}{a^3} \;=\; 0,$$

$$\ln\big(\mu\big) \,+\, \big(1\,+\,w\big)\,\ln\big(a^3\big) \;=\; C_0$$

with an integration constant C_0, hence

$$\mu(t)\,a(t)^{3(1+w)} \;=\; \text{constant}\,. \qquad (C2)$$

Three cases are of particular interest in view of applications:

(i) $w = 0$: This is the dust case we have already considered,

$$p \;=\; 0\,,$$

$$T_{\rho\sigma} \;=\; \mu\,U_\rho\,U_\sigma\,.$$

The conservation law (C2) reads

$$\mu(t)\,a(t)^3 \;=\; \text{constant}$$

which is just the statement that the mass in a comoving volume is constant.

(ii) $w = -1$: This is a perfect fluid mimicking a cosmological constant,

$$p \;=\; -\,c^2\,\mu\,,$$

$$T_{\rho\sigma} \;=\; -\,c^2\,\mu\,g_{\rho\sigma}\,.$$

The conservation law (C2) requires the density to be constant,

$$\mu(t) \;=\; \text{constant}\,,$$

so the energy-momentum tensor has, indeed, the form of a cosmological term if we shift it to the left-hand side of the field equation.

(iii) $w = 1/3$: This is an important case we have not yet treated so far. It describes a perfect fluid that models radiation in terms of a "photon gas":

$$p \;=\; \frac{1}{3}\,c^2\,\mu\,,$$

$$T_{\rho\sigma} \;=\; \frac{\mu}{3}\Big(4\,U_\rho\,U_\sigma \,+\, c^2\,g_{\rho\sigma}\Big)\,.$$

The trace of the energy-momentum tensor vanishes,

$$T_\rho{}^\rho \;=\; \frac{\mu}{3}\Big(4\,U_\rho\,U^\rho \,+\, c^2\,\delta^\rho_\rho\Big) \;=\; \frac{\mu}{3}\Big(\,-\,4\,c^2 \,+\, 4\,c^2\Big) \;=\; 0\,.$$

The conservation law (C2) requires

$$\mu(t)\,a(t)^4 \;=\; \text{constant}\,.$$

A rigourous justification of the statement that such a perfect fluid describes radiation would require a derivation from kinetic theory. We cannot do this here, but we will give two arguments indicating that the statement is true. The first argument builds upon the fact that the trace of the energy-momentum tensor vanishes: The trace $T_\rho{}^\rho$ of the energy-momentum tensor is a scalar, invariant under coordinate transformations. From the field equation we read that $\kappa\, T_\rho{}^\rho$ has the dimension 1/length^2, so an energy-momentum tensor with non-vanishing trace defines a length scale. For a gas that consists of particles with a certain non-zero rest-mass m, this length scale is determined by the Schwarzschild radius associated with m. If the rest mass of the particles is zero, as it is for photons, such a length scale does not exist which means that the energy-momentum tensor must be trace-free. The second argument is based on the conservation law. For a dust that consists of massive particles, the rest mass of each particle remains constant, so the energy density falls off with the volume, $\mu \sim a^{-3}$. For a photon, the energy changes according to the redshift law, $1+z = a(t_o)/a(t_e)$, so the energy density gets a fourth factor of $1/a$ such that $\mu \sim a^{-4}$.

With the conservation law (C2) we may rewrite (F1) as

$$k\, a(t)^{1+3w} \,+\, \frac{1}{c^2}\, a'(t)^2\, a(t)^{1+3w} \,-\, \frac{\Lambda}{3}\, a(t)^{3(1+w)} \,=\, \frac{\kappa}{3}\, c^2\, \mu(t)\, a(t)^{3(1+w)} \,=:\, a_0^{1+3w}$$

where the constant a_0 has the dimension of a length as can be read from comparing with the left-hand side. As for the dust case, we do not have to consider eq. (F2) separately because it is automatically satisfied as long as $a' \neq 0$. We have now three parameters at our disposal: The discrete parameter k which takes the value -1, 0 or 1, and the two parameters Λ and w which may take any real values. Here we will consider only the special case that

$$k = 0\,, \quad \Lambda = 0$$

to concentrate on the influence of the pressure. Then the Friedmann equation simplifies to

$$\frac{1}{c^2}\, a'(t)^2\, a(t)^{1+3w} \,=\, a_0^{1+3w}\,,$$

$$a^{(1+3w)/2}\, \frac{da}{dt} \,=\, \pm\, c\, a_0^{(1+3w)/2}\,.$$

We will solve this differential equation for the three cases which are of particular interest in view of applications.

(i) $w = 0$: Just as a cross-check, we will re-examine the dust case. Then the differential equation reads

$$a^{1/2}\, da \,=\, \pm\, c\, a_0^{1/2}\, dt\,,$$

$$\frac{2}{3}\, a^{3/2} \,=\, \pm\, c\, a_0^{1/2}\, \big(t - t_i\big)$$

with an integration constant t_i. If we want to have a universe with a big bang at $t = 0$, we have to choose $t_i = 0$ and the plus sign,

$$a(t) \,=\, \Big(\frac{9\, a_0\, c^2}{4}\Big)^{1/3}\, t^{2/3} \,=\, b\, t^{2/3}\,.$$

This gives the Einstein-deSitter universe, as we had known before,

$$g \;=\; -\,c^2\,dt^2 \;+\; b^2\,t^{4/3}\Big(d\chi^2 \;+\; \chi^2\,d\Omega^2\Big)\,.$$

(ii) $w = -1$: This is a perfect fluid mimicking a cosmological constant. The differential equation reads

$$a^{-1}\,da \;=\; \pm\,c\,a_0^{-1}\,dt\,,$$
$$\ln(a) \;=\; \pm\,c\,a_0^{-1}\,t \;-\; C\,.$$

If we choose $C = \ln(a_0)$ and the plus sign we get

$$a(t) \;=\; a_0\exp\Big(\frac{c\,t}{a_0}\Big)$$

which is, indeed, the deSitter universe with a flat slicing,

$$g \;=\; -\,c^2\,dt^2 \;+\; a_0^2\exp\Big(\frac{2\,c\,t}{a_0}\Big)\,\Big(d\chi^2 \;+\; \chi^2\,d\Omega^2\Big)\,.$$

(iii) $w = 1/3$: For a universe filled with radiation, the differential equation reads

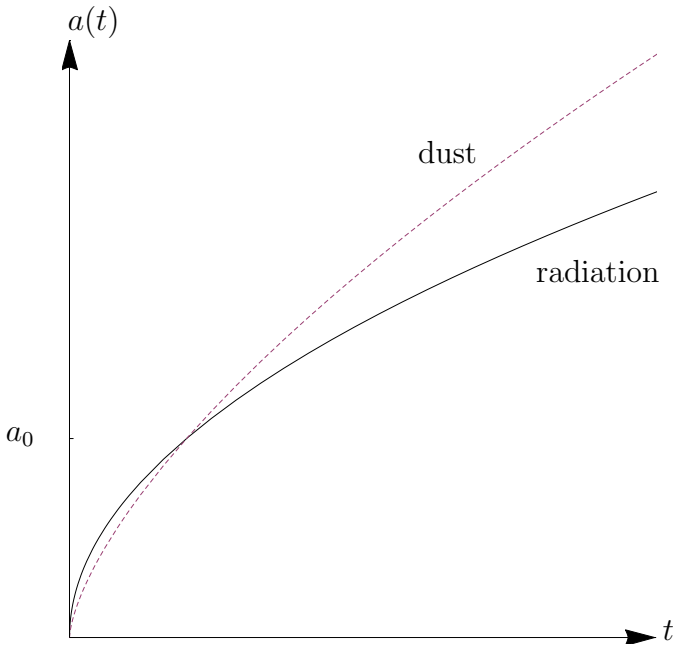
$$a\,da \;=\; \pm\,c\,a_0\,dt\,,$$

$$\frac{1}{2}\,a^2 \;=\; \pm\,c\,a_0\,(t - t_i)\,.$$

If we want to have a universe with a big bang at $t = 0$ we have to choose $t_i = 0$ and the plus sign,

$$a(t) \;=\; \sqrt{2\,a_0\,c\,t}\;.$$

So in a radiation-filled universe the scale factor grows with $t^{1/2}$, in comparison to a dust-filled universe where it grows with $t^{2/3}$, see the diagram.



Anticipating later discussions, we may assume as a rather realistic model of our universe a total energy-momentum tensor that is composed of two perfect fluids, one for a dust (modelling ordinary matter and also "cold" dark matter) $T^m_{\rho\sigma}$, and one for a photon gas (modelling the cosmic background radiation) $T^r_{\rho\sigma}$. If we also allow for the cosmological constant (as the simplest way of modelling dark energy), the field equation reads

$$R_{\rho\sigma} \,-\, \frac{R}{2}\, g_{\rho\sigma} \,+\, \Lambda\, g_{\rho\sigma} \,=\, T^m_{\rho\sigma} \,+\, T^r_{\rho\sigma}\,.$$

We assume that the dust and the radiation are both at rest with respect to the standard observers of our Robertson-Walker spacetime,

$$T^m_{\rho\sigma} \,=\, \mu_m\, U_\rho\, U_\sigma\,, \quad T^r_{\rho\sigma} \,=\, \frac{\mu_r}{3}\left(4\, U_\rho\, U_\sigma \,+\, c^2 g_{\rho\sigma}\right)$$

with $U^\rho = \delta^\rho_t$. Then the first Friedmann equation reads

$$\frac{3\, c^2\, k}{a(t)^2} + \frac{3}{a(t)^2}\, a'(t)^2 - \Lambda\, c^2 = \kappa\, c^4 \left(\mu_m(t) \,+\, \mu_r(t)\right),$$

hence

$$\mu_m(t) \,+\, \mu_r(t) \,+\, \frac{\Lambda}{\kappa\, c^2} \,-\, \frac{3\, a'(t)^2}{\kappa\, c^4\, a(t)^2} \,=\, \frac{3\, k}{\kappa\, c^2\, a(t)^2}\,.$$

Evaluating this equation at a time t_o ("now") yields

$$\mu_m(t_o) \,+\, \mu_r(t_o) \,+\, \frac{\Lambda}{\kappa\, c^2} \,-\, \frac{3\, H(t_o)^2}{\kappa\, c^4} \,=\, \frac{3\, k}{\kappa\, c^2\, a(t_o)^2}\,.$$

If we introduce, again, the critical density

$$\mu_c(t_o) \,=\, \frac{3\, H(t_o)^2}{\kappa\, c^4}\,,$$

we can rewrite this equation as

$$\frac{\mu_m(t_o)}{\mu_c(t_o)} \,+\, \frac{\mu_r(t_o)}{\mu_c(t_o)} \,+\, \frac{\Lambda}{\kappa\, c^2\, \mu_c(t_o)} \,-\, 1 \,=\, \frac{3\, k}{\kappa\, c^2\, a(t_o)^2\, \mu_c(t_o)}\,.$$

With the density parameters for dust (matter), radiation and cosmological constant,

$$\Omega_m \,=\, \frac{\mu_m(t_o)}{\mu_c(t_o)}\,, \quad \Omega_r \,=\, \frac{\mu_r(t_o)}{\mu_c(t_o)}\,, \quad \Omega_\Lambda \,=\, \frac{\Lambda}{\kappa\, c^2\, \mu_c(t_o)}\,,$$

we get the famous relation

$$\Omega_m \,+\, \Omega_r \,+\, \Omega_\Lambda \begin{cases} < 1 & \text{if } k = -1\,, \\ = 1 & \text{if } k = 0\,, \\ > 1 & \text{if } k = 1\,. \end{cases}$$

We have seen that in a universe filled with radiation alone the density falls off as a^{-4}, while in a universe filled with a dust alone it falls off as a^{-3}. This indicates that in an expanding universe the radiation is important for the early universe but that its contribution becomes negligible for later times. We believe that at the present stage of our universe the radiation can be neglected. We also have good indications that our universe is spatially flat, $k = 0$. Then $\Omega_r(t_o)$ is negligibly small if t_o means "now", and $\Omega_m(t_o) \,+\, \Omega_\Lambda(t_o) \,=\, 1$. We will come back to this relation and its observational foundations later.

(d) Solutions with a scalar field as source

We have seen that, whenever a Robertson-Walker metric is plugged into Einstein's field equation, the energy-momentum tensor on the right-hand side has the perfect-fluid form,

$$T_{\rho\sigma} = \left(\mu + \frac{p}{c^2}\right) U_\rho U_\sigma + p\, g_{\rho\sigma}\,.$$

However, we may interpret this energy-momentum tensor in a different way. In this section we discuss the question of whether it can be interpreted as the energy-momentum tensor of a scalar field. This has important applications in cosmology. Several hypothetical scalar fields (or hypothetical "particles" in the quantised version) are discussed which may have a strong influence on the dynamics of the universe. The three most important of them are:

The Higgs field: In the basic version of gauge theories, all fields are massless. The Higgs field was invented to allow for massive fields. The mass terms come about by the interaction with the Higgs field. In 2012 a particle was detected at the Large Hadron Collider of CERN that is believed to be the Higgs particle. This won Peter Higgs and François Englert the Nobel Prize in Physics 2013. The Higgs field is a complex-valued scalar field that might have played an important role in cosmology at an early stage.

The inflaton: The inflaton field is a scalar field, in most theories assumed to be real-valued, that drives inflation. The idea is that it acted, for a period at a very early stage of the universe (something like 10^{-36} to 10^{-33} seconds after the big bang) like an enormously big cosmological constant, producing an exponential growth of the scale factor. The mechanism must be tuned in a way that the action of the inflaton field was then switched off so that it played no role for the later development of the universe. This is often called the "graceful exit" of inflation. The motivation for introducing an inflationary phase was that this would explain several things:

- The horizon problem: How could the universe become homogeneous at a time when, because of the existence of particle horizons, its different parts had had no time to interact?
- The monopole problem: Why is it that we do not observe a large number of magnetic monopoles although they are thought to have come into existence during phase transitions in the early universe?
- The flatness problem: Is it not highly unlikely that we live in a universe with spatial curvature K very close to zero if we think that our universe started with a random initial condition?

If there was an inflationary phase, different parts of the universe would have come into causal contact much earlier, the magnetic monopoles would have been diluted, and the blowing-up of the universe would bring the curvature K to a very small value whatever the initial conditions have been. Inflation was invented in 1980 independently by A. Guth and A. Starobinsky. It was further developed by A. Linde, A. Albrecht, P. Steinhardt and many others. To date there exists a large variety of inflationary scenarios. The basic idea of inflation will be explained below in terms of the simplest of these scenarios ("slow-roll inflation").

The quintessence: Since the late 1990s we have strong evidence that the expansion of our universe is accelerated. The agent which produces this acceleration is called *dark energy*. The simplest way of modelling dark energy is by identifying it with a positive cosmological constant. As we know, this may be re-interpreted as a perfect fluid with the equation of state $p = -c^2\mu$. We will see below that another possibility is to re-interpret the cosmological term as being produced by a (real-valued) scalar field. If we adopt this interpretation of dark energy, we call it the quintessence field. In contrast to the inflaton, the quintessence field produces an exponential growth that is much smaller and becomes relevant at a much later time.

Several other hypothetical scalar fields ("phantom fields", "chameleon fields", "galileon fields" ...) have been suggested, but it seems fair to say that their existence is highly speculative. The three fields mentioned above provide the main motivation for us to study scalar fields and their coupling to gravity on a Robertson-Walker spacetime.

To that end we begin with the simplest type of a scalar field equation. We only consider real-valued fields ϕ. On Minkowski spacetime, the Klein-Gordon equation

$$\Box\phi \,-\, m^2\,\phi \,=\, 0\,, \quad \Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi \,=\, \Delta\phi \,-\, \frac{1}{c^2}\,\partial_t^2\phi$$

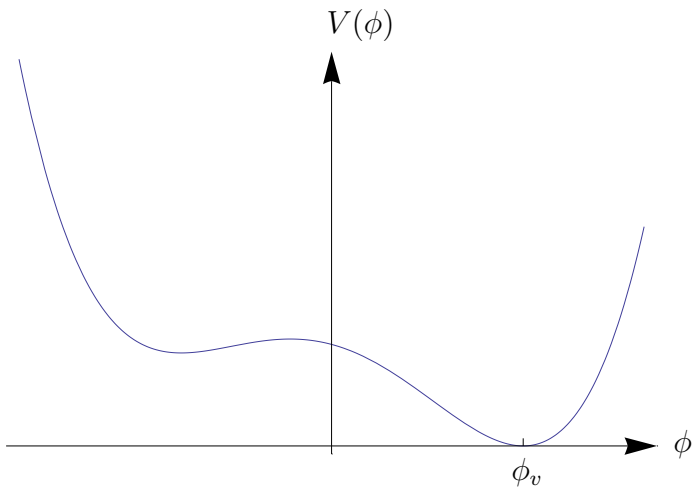
is the unique Lorentz-invariant linear field equation of second order for a scalar field. Here m is a constant that is related to the mass M associated with the field by $m = M/(c\hbar)$. (To put this another way, $1/m$ is the Compton wave-length of the field.) By the rule of minimal coupling, this generalises on a curved spacetime to the equation

$$\Box\phi \,-\, m^2\,\phi \,=\, 0\,, \quad \Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \,=\, g^{\mu\nu}\left(\partial_\mu\partial_\nu\phi \,-\, \Gamma^\rho{}_{\mu\nu}\,\partial_\rho\phi\right).$$

In the following we consider a generalised Klein-Gordon equation,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \,-\, V'(\phi) \,=\, 0\,, \qquad \text{(KG)}$$

with a potential $V(\phi)$. For the time being, V is largely arbitrary. We will only assume that it is bounded below, $V(\phi) \geq V_{\min}$, and that there is a ϕ_v with $V(\phi_v) = V_{\min}$. This "ground state" or "vacuum state" ϕ_v need not be unique. As the differential equation (KG) is unchanged if we add a constant to V, we may assume without loss of generality that $V_{\min} = 0$.



With the scalar field we associate the energy-momentum tensor

$$T_{\rho\sigma} \;=\; \nabla_\rho\phi\,\nabla_\sigma\phi \;-\; \left(\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,g_{\rho\sigma}\,.$$

We will derive this energy-momentum tensor from a variational principle below. Before doing this, we evaluate the energy conservation law $\nabla^\rho T_{\rho\sigma} = 0$ for this energy-momentum tensor:

$$\nabla^\rho T_{\rho\sigma} \;=\; \nabla^\rho\left(\nabla_\rho\phi\,\nabla_\sigma\phi \;-\; \left(\frac{1}{2}\,g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,g_{\rho\sigma}\right)$$

$$=\; \nabla^\rho\nabla_\rho\phi\,\nabla_\sigma\phi + \nabla_\rho\phi\,\nabla^\rho\nabla_\sigma\phi - \left(\frac{1}{2}\,g^{\mu\nu}\left\{\nabla^\rho\nabla_\mu\phi\,\nabla_\nu\phi \;+\; \nabla_\mu\phi\,\nabla^\rho\nabla_\nu\phi\right\} \;+\; V'(\phi)\nabla^\rho\phi\right)\,g_{\rho\sigma}$$

$$=\; \nabla^\rho\nabla_\rho\phi\,\nabla_\sigma\phi \;+\; \nabla_\rho\phi\,\nabla^\rho\nabla_\sigma\phi \;-\; \frac{1}{2}\,g^{\mu\nu}\left\{\nabla_\sigma\nabla_\mu\phi\,\nabla_\nu\phi \;+\; \nabla_\mu\phi\,\nabla_\sigma\nabla_\nu\phi\right\} \;-\; V'(\phi)\nabla_\sigma\phi$$

$$=\; \nabla^\rho\nabla_\rho\phi\,\nabla_\sigma\phi \;+\; \nabla^\mu\phi\,\nabla_\mu\nabla_\sigma\phi \;-\; \nabla^\mu\phi\,\nabla_\sigma\nabla_\mu\phi \;-\; V'(\phi)\nabla_\sigma\phi\,.$$

As covariant derivatives commute if they are applied to a scalar (!) field, the middle terms cancel, $\nabla_\mu\nabla_\sigma\phi = \nabla_\sigma\nabla_\mu\phi$, so

$$\nabla^\rho T_{\rho\sigma} \;=\; \nabla_\sigma\phi\left(\nabla^\rho\nabla_\rho\phi \;-\; V'(\phi)\right).$$

So we see that the generalised Klein-Gordon equation (KG) implies the conservation law, $\nabla^\rho T_{\rho\sigma} = 0$, and that the converse is also true if we assume that $\nabla_\sigma\phi \neq 0$.

We may derive the energy-momentum tensor of the scalar field and, thereby, the coupling of the scalar field to gravity via Einstein's field equation from a variational principle. Quite generally, the action for Einstein's field equation coupled to any matter source (the so-called *Einstein-Hilbert action*) reads

$$\mathcal{W} \;=\; \int_\Omega\left(\frac{R}{2} \;-\; \Lambda \;+\; \kappa\,\mathcal{L}_{\mathrm{mat}}\right)\,\sqrt{-\gamma}\,d^4x\,,$$

where

$$\gamma \;=\; \det\big((g_{\mu\nu})\big)\,,\quad d^4x \;=\; dx^0dx^1dx^2dx^3\,,$$

and Ω is a compact spacetime region with boundary. The matter Lagrangian for the scalar field is

$$\mathcal{L}_{\mathrm{mat}} \;=\; -\,\frac{1}{2}\,\nabla^\mu\phi\,\nabla_\mu\phi \;-\; V(\phi)\,. \qquad \text{(LS)}$$

Einstein's field equation follows by varying the Einstein-Hilbert action with respect to the metric. More precisely, Einstein's field equation results if one requires that $\delta\mathcal{W} = 0$ for all variations that vanish on the boundary of Ω. To work this out, we need the variation of the Ricci scalar and of the determinant of the metric. With some algebra one finds

$$\sqrt{-\gamma}\,\delta R \;=\; \sqrt{-\gamma}\,\delta\big(R_{\mu\nu}g^{\mu\nu}\big) \;=\; \sqrt{-\gamma}\,R_{\mu\nu}\delta g^{\mu\nu} \;+\; \underbrace{\partial_\sigma\Big(\sqrt{-\gamma}\,\big(g^{\mu\nu}\,\delta\Gamma^\sigma{}_{\mu\nu} \;-\; g^{\mu\sigma}\,\delta\Gamma^\rho{}_{\rho\mu}\big)\Big)}_{\text{boundary}\quad\text{term}}$$

$$\delta\gamma \;=\; \frac{\partial\gamma}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} \;=\; -\,\gamma\,g_{\mu\nu}\,\delta g^{\mu\nu}\,,$$

where "boundary term" means that the term can be converted with the Stokes theorem into an integral over the boundary of Ω which gives zero because the metric is kept fixed on the boundary. With the help of these results we find for the variation of the Einstein-Hilbert action:

$$\delta\mathcal{W} \;=\; \int_\Omega \left\{\delta\left(\frac{R}{2} \;-\; \Lambda\right)\sqrt{-\gamma} \;+\; \left(\frac{R}{2} \;-\; \Lambda\right)\delta\sqrt{-\gamma} \;+\; \kappa\,\delta\Big(\mathcal{L}_{\text{mat}}\sqrt{-\gamma}\Big)\right\} d^4x$$

$$=\; \int_\Omega \left\{\frac{\delta R}{2}\sqrt{-\gamma} \;+\; \left(\frac{R}{2} \;-\; \Lambda\right)\frac{\gamma\,g_{\mu\nu}}{2\sqrt{-\gamma}}\,\delta g^{\mu\nu} \;+\kappa\,\frac{\partial\Big(\mathcal{L}_{\text{mat}}\sqrt{-\gamma}\Big)}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu}\right\} d^4x$$

$$=\; \int_\Omega \left\{R_{\mu\nu} \;-\; \left(\frac{R}{2} \;-\; \Lambda\right)g_{\mu\nu} \;+\kappa\,\frac{2}{\sqrt{-\gamma}}\,\frac{\partial\Big(\mathcal{L}_{\text{mat}}\sqrt{-\gamma}\Big)}{\partial g^{\mu\nu}}\right\}\frac{\delta g^{\mu\nu}}{2}\sqrt{-\gamma}\,d^4x$$

The variational principle requires $\delta\mathcal{W}$ to be zero for all $\delta g^{\mu\nu}$ that vanish on the boundary of Ω. This is true if and only if the term inside the curly bracket vanishes,

$$R_{\mu\nu} \;-\; \frac{R}{2}\,g_{\mu\nu} \;+\; \Lambda\,g_{\mu\nu} \;+\kappa\,\frac{2}{\sqrt{-\gamma}}\,\frac{\partial\Big(\mathcal{L}_{\text{mat}}\sqrt{-\gamma}\Big)}{\partial g^{\mu\nu}} \;=\; 0\,.$$

This is Einstein's field equation,

$$R_{\mu\nu} \;-\; \frac{R}{2}\,g_{\mu\nu} \;+\; \Lambda\,g_{\mu\nu} \;=\; \kappa\,T_{\mu\nu}\,,$$

where the energy-momentum tensor is given as

$$T_{\mu\nu} \;=\; -\,\frac{2}{\sqrt{-\gamma}}\,\frac{\partial\Big(\mathcal{L}_{\text{mat}}\sqrt{-\gamma}\Big)}{\partial g^{\mu\nu}}\,.$$

The energy-momentum tensor constructed in this way from a variational principle is known as *Hilbert's energy-momentum tensor*. Note that it is automatically symmetric.

For the matter Lagrangian (LS) of the scalar field we find

$$T_{\rho\sigma} \;=\; \frac{2}{\sqrt{-\gamma}}\,\frac{\partial}{\partial g^{\rho\sigma}}\left\{\left(\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,\sqrt{-\gamma}\right\}$$

$$=\; \frac{\cancel{2}}{\sqrt{-\gamma}}\left\{\frac{1}{\cancel{2}}\,\nabla_\rho\phi\,\nabla_\sigma\phi\,\sqrt{-\gamma} \;+\; \left(\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,\frac{1}{\cancel{2}\,\sqrt{-\gamma}}\,\frac{\partial\gamma}{\partial g^{\rho\sigma}}\right\}$$

$$=\; \nabla_\rho\phi\,\nabla_\sigma\phi \;-\; \left(\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,\frac{\cancel{\gamma}\,g_{\rho\sigma}}{\cancel{\gamma}}$$

which is, indeed, the energy-momentum tensor given above.

We will now investigate if the energy-momentum tensor of a scalar field is of the same form as that for a perfect fluid. As we know that for a Robertson-Walker spacetime the energy-momentum tensor *always* has the form of a perfect fluid, this is a necessary step if we want to consider scalar fields on a Robertson-Walker spacetime.

We have to identify the two expressions

$$T_{\rho\sigma} \;=\; \nabla_\rho\phi\,\nabla_\sigma\phi \;-\; \left(\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi \;+\; V(\phi)\right)\,g_{\rho\sigma}$$

and

$$T_{\rho\sigma} \;=\; \left(\mu \;+\; \frac{p}{c^2}\right)\,U_\rho U_\sigma \;+\; p\,g_{\rho\sigma}\,.$$

Obviously, this is possible only if the first terms on the right-hand sides coincide. This requires

$$\nabla_\rho\phi \;=\; s\,U_\rho$$

with a scalar factor s. This factor is determined by the normalisation condition on the four-velocity, $U^\rho U_\rho = -c^2$. We find

$$\nabla^\rho\phi\,\nabla_\rho\phi \;=\; -c^2\,s^2\,.$$

So the identification requires that either $\nabla^\rho\phi$ is timelike (and $s \neq 0$) or $\nabla^\rho\phi = 0$ (and $s = 0$).

We can now determine p and μ in terms of the scalar field by equating the two expressions of the energy-momentum tensor. Comparing the coefficients in front of $g_{\rho\sigma}$ yields

$$p \;=\; -\frac{1}{2}\,\nabla^\mu\phi\nabla_\mu\phi \;-\; V(\phi)\,. \qquad \text{(PS)}$$

Equating the first terms requires

$$\left(\mu \;+\; \frac{p}{c^2}\right)\,U_\rho\,U_\sigma \;=\; s^2\,U_\rho\,U_\sigma\,, \qquad c^2\,\mu \;+\; p \;=\; -\,\nabla^\mu\phi\,\nabla_\mu\phi\,,$$

$$c^2\,\mu \;=\; -\,\frac{1}{2}\,\nabla^\mu\phi\,\nabla_\mu\phi \;+\; V(\phi)\,. \qquad \text{(DS)}$$

We summarise our results in the following way: The energy-momentum tensor of a scalar field is of the form of a perfect fluid whenever the gradient of the scalar field is timelike or zero. The corresponding pressure and the corresponding density are then given by (PS) and (DS).

Now we specify to the case that we are on a Robertson-Walker spacetime,

$$g_{\mu\nu}\,dx^\mu\,dx^\nu \;=\; -\,c^2\,dt^2 \;+\; a(t)^2\left(d\chi^2 \;+\; \eta(\chi)^2\left(d\vartheta^2+\sin^2\vartheta\,d\varphi^2\right)\right).$$

Because of the spatial homogeneity of the metric, the scalar field must be independent of the spatial coordinates for consistency,

$$\phi \;=\; \phi(t)\,.$$

This guarantees that $\nabla_\rho\phi = \delta^t_\rho\,d\phi/dt$ is indeed timelike or zero. Then the equations for pressure and density, (PS) and (DS), simplify to

$$p \;=\; -\,\frac{1}{2}\,g^{tt}\left(\frac{d\phi}{dt}\right)^2 \;-\; V(\phi)\,,\qquad c^2\,\mu \;=\; -\,\frac{1}{2}\,g^{tt}\left(\frac{d\phi}{dt}\right)^2 \;+\; V(\phi)\,.$$

With $g^{tt} = 1/g_{tt} = -1/c^2$ this results in

$$p \;=\; \frac{1}{2\,c^2}\left(\frac{d\phi}{dt}\right)^2 \;-\; V(\phi)\,,\qquad c^2\,\mu \;=\; \frac{1}{2\,c^2}\left(\frac{d\phi}{dt}\right)^2 \;+\; V(\phi)\,.$$

Clearly, p and μ are not in general related by an equation of state. An equation of state results only in two very special cases:

If $V(\phi) \;=\; 0$ we have $p \;=\; w\,c^2\,\mu$ with $w \;=\; 1$.

If $d\phi/dt \;=\; 0$ we have $p \;=\; w\,c^2\,\mu$ with $w \;=\; -\,1$.

Recall that $w = -1$ is the case of a perfect fluid mimicking a cosmological constant. As $V(\phi) \geq 0$, we have in any case

$$-\,1 \;\leq\; \frac{p}{c^2\,\mu} \;=\; \frac{\dfrac{1}{2\,c^2}\left(\dfrac{d\phi}{dt}\right)^2 \;-\; V(\phi)}{\dfrac{1}{2\,c^2}\left(\dfrac{d\phi}{dt}\right)^2 \;+\; V(\phi)} \;\leq\; 1\,.$$

With μ and p expressed in terms of the scalar field, the Friedmann equations read

$$\frac{3c^2k}{a^2} \;+\; \frac{3}{a^2}\left(\frac{da}{dt}\right)^2 \;-\; \Lambda\,c^2 \;=\; \kappa\,c^2\left(\frac{1}{2\,c^2}\left(\frac{d\phi}{dt}\right)^2 \;+\; V(\phi)\right),\qquad \text{(F1s)}$$

$$-\,k \;-\; \frac{1}{c^2}\left(\frac{da}{dt}\right)^2 \;-\; \frac{2\,a}{c^2}\frac{d^2a}{dt^2} \;+\; \Lambda\,a^2 \;=\; \kappa\,a^2\left(\frac{1}{2\,c^2}\left(\frac{d\phi}{dt}\right)^2 \;-\; V(\phi)\right).\qquad \text{(F2s)}$$

We know that Einstein's field equation implies $\nabla^\rho T_{\rho\sigma} = 0$ and that for a Robertson-Walker spacetime this equation takes the form of the energy balance equation (C1). As in the case of a scalar field the equation $\nabla^\rho T_{\rho\sigma} = 0$ is equivalent to the generalised Klein-Gordon equation (KG), provided that $\nabla_\rho \phi \neq 0$, we can derive the special form of the equation (KG) for a Robertson-Walker spacetime by evaluating the energy balance (C1). This spares us the trouble of calculating the Christoffel symbols for the Robertson-Walker spacetime.

Recall that the energy balance law, which is just a version of the First Law of Thermodynamics, reads

$$\frac{d}{dt}\Big(c^2\,\mu\,a^3\Big) \;=\; -\,p\frac{d}{dt}a^3\,,$$

hence

$$\frac{d(c^2\,\mu)}{dt}\,a^3 \;+\; c^2\,\mu\,3\,a^2\,\frac{da}{dt} \;=\; -\,p\,3\,a^2\,\frac{da}{dt}\,,$$

$$\frac{d(c^2\,\mu)}{dt} \;=\; -\,\frac{3}{a}\Big(c^2\,\mu \;+\; p\Big)\frac{da}{dt}\,.$$

Inserting the expressions for μ and p in terms of the scalar field yields

$$\frac{d}{dt}\Big(\frac{1}{2\,c^2}\,\Big(\frac{d\phi}{dt}\Big)^2 \;+\; V(\phi)\Big) \;=\; -\,\frac{3}{a\,c^2}\,\Big(\frac{d\phi}{dt}\Big)^2\,\frac{da}{dt}\,,$$

$$\frac{1}{c^2}\,\frac{d\phi}{dt}\,\frac{d^2\phi}{dt^2} \;+\; V'(\phi)\,\frac{d\phi}{dt} \;=\; -\,\frac{3}{a\,c^2}\,\Big(\frac{d\phi}{dt}\Big)^2\,\frac{da}{dt}\,.$$

If we divide by $d\phi/dt$, assuming that this expression is non-zero, we get the generalised Klein-Gordon equation on a Robertson-Walker spacetime,

$$\frac{1}{c^2}\,\frac{d^2\phi}{dt^2} \;+\; \frac{3}{c^2}\,\frac{1}{a}\,\frac{da}{dt}\,\frac{d\phi}{dt} \;+\; V'(\phi) \;=\; 0\,. \qquad \text{(KGs)}$$

This equation is of the same form as the equation of motion for a particle in the one-dimensional potential V, with a damping term. The damping is proportional to the Hubble function $H(t) = a(t)^{-1}da(t)/dt$. (Strictly speaking, it is a damping only in the case that $H(t)$ is positive.) So we may visualise the dynamics of the scalar field as the dynamics of a particle in the potential V with friction. This analogy is habitually used for scalar fields in cosmology, where one says that "the field is rolling down a slope of the potential" and so on.

From our consideration in the preceding section we know that (F2) is a consequence of (F1) and the conservation law except for static Robertson-Walker spacetimes where $da/dt = 0$. We will exclude the static case in the following. So in the case at hand we only need to consider (F1s) and (KGs); the other Friedmann equation is then automatically satisfied. These two equations give us a system of coupled ordinary differential equations for the two unkown functions $a(t)$ and $\phi(t)$.

We will first demonstrate how a scalar field can act as a cosmological constant; we will then briefly discuss the "slow-roll inflation" scenario. For imitating a cosmological constant it is, of course, reasonable to consider the equations with $\Lambda = 0$. In addition, we restrict to the spatially flat case, so we assume

$$k \,=\, 0\,, \quad \Lambda \,=\, 0\,.$$

It is our goal to mimic the cosmological constant with a constant scalar field,

$$\phi(t) \,=\, \phi_0 \,=\, \text{constant}\,.$$

Then (KGs) requires

$$V'\big(\phi_0\big) \,=\, 0\,,$$

i.e., the scalar field must sit in an extremum of the potential. This may be a minimum, a maximum or a saddle. If we want the solution to be stable with respect to small perturbations, we have to choose a local minimum.

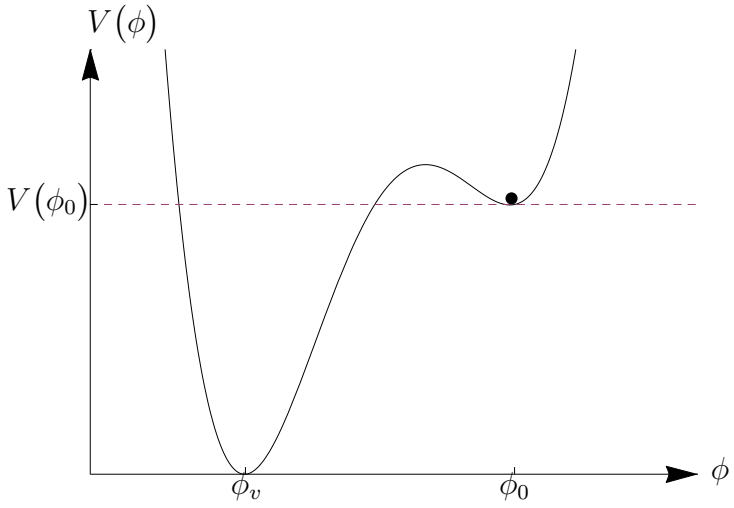
(F1s) reduces to

$$\frac{1}{a^2}\Big(\frac{da}{dt}\Big)^2 \,=\, \frac{\kappa}{3}\,c^2\,V\big(\phi_0\big)\,,$$

$$\frac{da}{a} \,=\, \pm\,c\,\sqrt{\frac{\kappa}{3}\,V\big(\phi_0\big)}\,dt\,,$$

$$\ln\big(a\big) \,-\, \ln\big(a_0\big) \,=\, \pm\,c\,\sqrt{\frac{\kappa}{3}\,V\big(\phi_0\big)}\,t\,,$$

$$a(t) \,=\, a_0\exp\left(\pm\,\sqrt{\frac{\kappa}{3}\,V\big(\phi_0\big)}\,c\,t\right)$$



If we have chosen the vacuum state, $\phi_0 = \phi_v$, we have $V\big(\phi_0\big) = 0$ and the scale factor is constant, $a(t) = a_0$. As we are considering the case $k = 0$, this gives us Minkowski spacetime. If, however,

$$V\big(\phi_0\big) \,>\, 0\,,$$

we may choose the solution with the plus sign and the integration constant as

$$a_0 \,=\, \sqrt{\frac{3}{\kappa\,V\big(\phi_0\big)}}\,.$$

Then we get deSitter spacetime with the flat slicing (i.e., half of the hyperboloid, see the picture on p. 33),

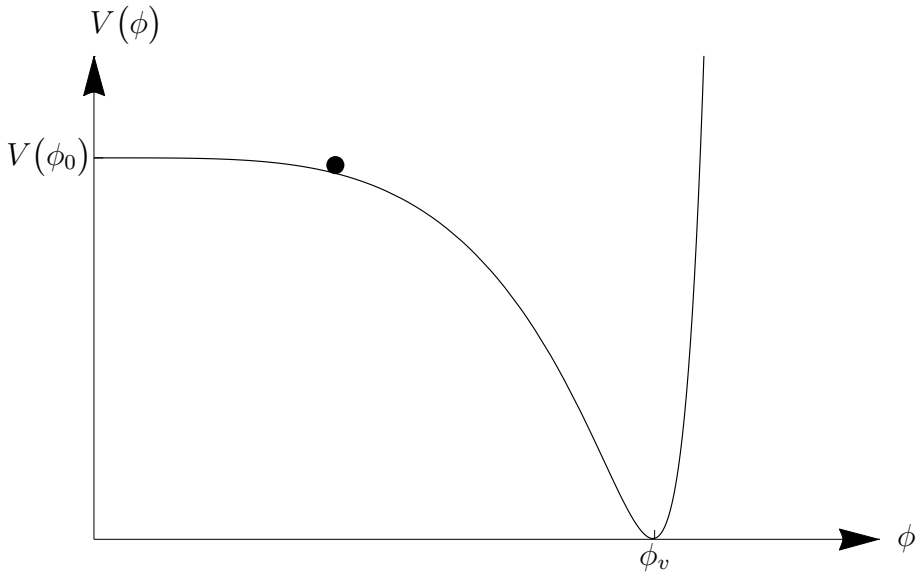
$$a(t) \,=\, \sqrt{\frac{3}{\kappa\,V\big(\phi_0\big)}}\,\exp\left(\sqrt{\frac{\kappa}{3}\,V\big(\phi_0\big)}\,c\,t\right)\,. \qquad (*)$$

The exponential growth of the scale factor is now driven by the scalar field. It has the same effect as a positive cosmological constant,

$$\Lambda \,\widehat{=}\, \kappa\,V\big(\phi_0\big)\,.$$

To explain the accelerated expansion of the universe *now* as indicated by observations of Supernovae Ia (see next chapter), we need a cosmological constant $\sqrt{\Lambda} \approx 10^{-26}\mathrm{m}^{-1}$. We have already seen that this can be re-interpreted as the effect of a perfect fluid ("dark energy") with the exotic equation of state $p = -c^2\mu$. We have now another possible interpretation: We may say that what seems to be a positive cosmological constant is actually a constant scalar field. This scalar field is called "quintessence". This makes the cosmological constant dynamical in the sense that it is not necessarily exactly a constant: The quintessence field may change in the course of time. If this is true, the future of our universe is not predictable; if there is a cosmological constant in the strict sense, it will dominate all other sources (radiation and matter) in the course of time so that the universe asymptotically approaches the deSitter spacetime.

The quintessence field is a possible re-interpretation of the cosmological constant but there is no compelling reason for introducing it. You may leave the cosmological term on the left-hand side of the field equation, or you may write it on the right-hand side and interpret it as a perfect fluid. As long as all observations can be explained with a cosmological constant that is really a constant, this is largely a matter of taste. The situation is different with the inflaton: Just as the quintessence field, the inflaton field is supposed to create exponential growth, but at a much, much earlier stage of the universe and by a much, much bigger factor. Most importantly, the inflaton field should switch itself off at the end of the inflationary period. So here we are not re-interpreting a cosmological constant, we are rather considering a scalar field that acts *approximately* as a cosmological term only over a certain period of time. So we need a potential that is almost constant (i.e., very flat) over a certain ϕ interval. While the scalar field is "slowly rolling" down this potential, our calculation for a constant scalar field ϕ_0 holds *approximately*, i.e., the scale factor grows *approximately* exponentially according to $(*)$. If the scalar field rolls into the vacuum state ϕ_v, the exponential growth stops, because after a few oscillations the "damping term" in the generalised Klein-Gordon equation forces the scalar field to settle at ϕ_v which, as demonstrated above, leads to a constant scale factor. In this way, the inflaton switches itself off and plays no role in the later development of the universe. This is what one calls the "graceful exit" of inflation. By choosing $V(\phi_0)$ sufficiently big, the exponential growth rate during the inflationary period can be as large as we wish, see $(*)$. We will discuss in the next chapter how big this growth rate should be in order to explain the observations.



This so-called "slow-roll" inflationary scenario is the simplest way in which an inflationary stage can be produced. It was suggested by A. Linde and independently by A. Albrecht and P. Steinhardt in 1982. Already earlier, other inflationary scenarios had been introduced by A. Guth and by A. Starobinsky. Guth wanted to identify the field that drives inflation with the Higgs field and he suggested a certain tunnelling process for the transition into the vacuum state. Starobinsky used a different aproach, based not on Einstein's field equation coupled to a scalar field but rather on a modified field equation which contained a curvature coupling that was motivated by ideas from quantum gravity. In the mean-time there is a big number of different inflationary scenarios. By now, none of them is unanimously accepted. It should also be mentioned that inflation is still a hypothetical concept, not directly verified by observations. It is believed by a majority of cosmologists that an inflationary period took place at an early stage of the universe, but there are also outspoken critics, e.g. Roger Penrose.

We have now discussed the dynamics of Robertson-Walker universes for various matter sources. According to the model that is favourised by the majority of cosmologists ("concordance model"), different matter sources were dominating at different stages. In the following we list these different stages; we will discuss in the next chapter on what observational evidence the concordance model is based and at what times the different periods took place.

- Big bang: We believe that the universe began with a hot bing bang, a state of extreme density. The time immediately after the big bang is not yet theoretically understood. A (not yet existing) quantum theory of gravity is probably needed.

- Inflationary period: We conjecture that there was a very short period during which the scale factor grew exponentially by an enormous factor (10^{30} at least). This conjecture solves the horizon problem, the monopole problem and the flatness problem. Inflation was driven by a hypothetical scalar field called the inflaton.

- Radiation-dominated period: After the graceful exit from inflation radiation gives the dominating contribution to the density. The density parameters of matter and of the cosmological constant are negligibly small in comparison to that of radiation. The universe expands $\sim t^{1/2}$, i.e., decelerating.

- ΛCDM period: At present, the universe is matter-dominated and the effect of a positive cosmological constant has to be taken into account. The matter content can be modelled as a dust, i.e., as "cold matter". It comprises the usual ("baryonic") matter and a mysterious "dark matter" whose nature is unknown as of now. (CDM stands for "cold dark matter".) The density parameter of radiation is now negligible; recall that the density of radiation falls off with a^{-4} while the density of matter (dust) falls off with a^{-3}. The cosmological constant may be re-interpreted as a perfect fluid with equation of state $p = -c^2\mu$ ("dark energy") or as a scalar field ("quintessence"). Because of the cosmological constant, the expansion of the universe is accelerating.

- Asymptotic deSitter period: If the cosmological constant is really a constant, it will dominate the dynamics at late times. Our universe will then asymptotically approach the deSitter spacetime. If the cosmological constant is actually a scalar field, it may change with time and the future of the universe cannot be predicted.

4. Observations

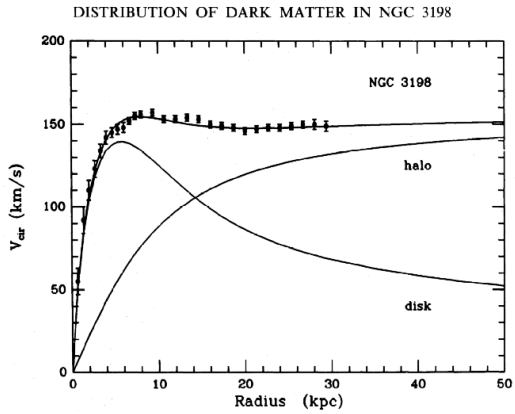
In this chapter we will summarize the observational facts on which the concordance model is based. The most important information comes from the cosmic background radiation and from the distance-redshift relation, but lensing and some other observational facts also provide important restrictions on the model.

4.1 Evidence for dark matter

There are several observational facts indicating that a large part of matter in the universe is dark and therefore detectable only by way of its gravitational field. As long as Einstein's general relativity theory (and, where applicable, the Newtonian approximation) is considered as the theoretical basis, these observational facts are compelling. We list them in chronological order.

- Velocity distribution in galaxy clusters: In 1933 and 1937 F. Zwicky wrote two papers in which he gave the first evidence for the existence of dark matter. He analysed the dynamics inside the Coma cluster, a galaxy cluster with more than 1000 galaxies. He found that the galaxies are too fast for the cluster to form a gravitationally bound system if one assumes that the total mass of the cluster is given by the luminous matter we are observing. He conjectured that more than 99.8 % of the matter is dark. (In the 1933 article, which is written in German, he speaks of "dunkle (kalte) Materie".) We believe today that this number is too big. The reason is that at this time the distance ladder was wrongly calibrated: Zwicky assumed that the Coma cluster is about 15 Mpc away; actually it is more than 100 Mpc. If one re-analyses his calculation with a corrected distance scale, one finds that about 95 % of the matter in galaxy clusters must be dark. Zwickys prediction of the existence of dark matter was largely ignored at the time.

- Rotation curves of galaxies: In the 1970s V. Rubin analysed rotation curves in a large sample of spiral galaxies, partly in collaboration with K. Ford. She looked at galaxies which are seen almost edge-on. Then, because of the rotation of the stars about the centre of the galaxy, the spectral lines are red-shifted on one side and blue-shifted on the other. Measuring these shifts in different parts of the galaxy gives the so-called rotation curve, i.e., the orbital velocity as a function of the radius. If all the mass of the galaxy were in the centre, the stars would move in a potential $\sim r^{-1}$ according to Newtonian gravity (which is a valid approximation here). This would give an orbital velocity $\sim 1/\sqrt{r}$. If one takes the actual distribution of the matter in the bulge and in the disc into account, according to the visible masses, one gets a curve that increases in the inner part, but then falls off like $1/\sqrt{r}$ in the outer part. This fall-off was not what V. Rubin observed: Actually, the rotation curves remained almost flat. This could be explained, on the basis of Einstein's general relativity and its Newtonian approximation, only by the assumption that the galaxy is embedded in a "dark matter halo". V. Rubin estimated, that it should make up about 50 % of the galaxy's mass. Later observations indicated that it should be at least 85 %. The picture on the next page shows the example of the galaxy NGC 3198. (The rotation curves of all galaxies show the same tendency.) The graph labelled "disk" is what one would see if there were only the visible matter. The graph labelled "halo" shows the difference between this graph and the observed rotation curve which is interpreted as the effect of the dark matter halo.



picture from T. S. van Albada, J. N. Bahcall, K. Begeman, R. Sancisi: Astrophys. J. 295, 305 (1985)

The dark matter halo is usually modelled as spherically symmetric and several density profiles have been suggested, e.g.

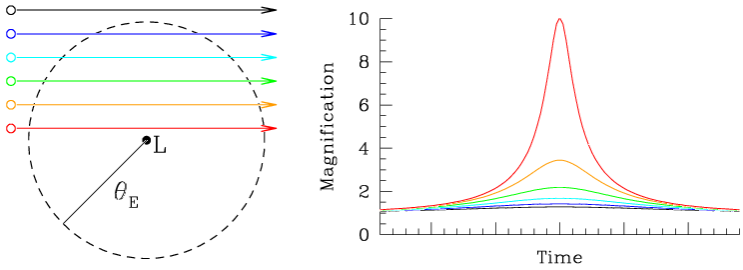
Non-singular isothermal sphere: $\mu(r) = \dfrac{\mu_0 r_0^2}{r_0^2 + r^2}$,

Navarro-Frenk-White profile: $\mu(r) = \dfrac{\mu_0 r_0^4}{r^2\big(r_0^2 + r^2\big)}$,

Einasto profile: $\mu(r) = \mu_0 \exp\big(-Ar^\alpha\big)$,

where μ_0, r_0, A and α are parameters that can be fitted to observations. The Navarro-Frenk-White profile is the most commonly used for the density of dark matter, not only in galaxies but also in galaxy clusters. It was found not just by guess-work but rather by numerical N-body simulations.

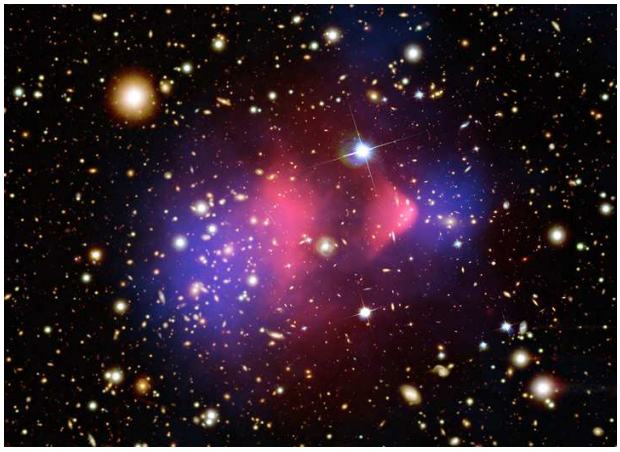
- Microlensing: The first candidates for dark matter in the halo of our galaxy that come to mind are "Massive Compact Halo Objects" (MaCHOS) such as black holes, brown dwarfs and planets. They cannot be seen directly, because they do not emit light, but they can be observed by the influence of their gravitational field on light:
If a star passes behind such a compact object, the light is focussed towards the observer, i.e., one sees a light curve that goes up and then down again. The closer the star comes to the line of sight, the bigger is the effect.



This is called microlensing. More precisely, the word microlensing is used for lensing situations where multiple images are created but cannot be resolved; so what one sees is just a change of brightness of the compound image. Microlensing events are routinely observed since the early 1990s. They are very common (several hundreds per year), and in the majority of cases the observations are made towards the halo of our galaxy. (Observations are also made towards the bulge of our galaxy, towards the Magellanic clouds and towards the Andromeda galaxy.) These observations give an upper bound on the total mass of the MaCHOs in our galaxy. The microlensing surveys came to the conclusion that not more than 20 % of the dark matter that is needed for explaining the rotation curves can be MaCHOs.

- Weak lensing: While microlensing is the most important tool for detecting dark matter in our galaxy or nearby, weak lensing is the most important tool for detencting dark matter in distant galaxy clusters. What one observes is the deformation of background galaxies by the lensing effect of the cluster. For understanding the basic idea, let us assume for a moment that the background galaxies were perfectly spherical. Then we would see each galaxy distorted into an ellipse on the sky, and the orientations and the eccentricities of the ellipses would tell us where the deflecting mass is located in the sky and how big its surface mass density (mass density projected onto the plane perpendicular to the view-line) is. Unfortunately, galaxies are not spherical. For most of them the shape can be approximated reasonably well by an ellipsoid. So when we see an ellipse in the sky we have to distinguish the intrinsic shape from the distortion effect produced by lensing. This can be done statistically, based on the assumption that the intrinsic shapes are distributed randomly. A sophisticated numerical method has been established to deduce from the observed shapes of background galaxies the surface mass density of a galaxy cluster. This has been worked out since the 1980s for a large number of galaxy clusters. If Einstein's theory describes the effect of gravitational fields on light correctly, all these observations confirm Zwicky's prediction that the majority of matter in galaxy clusters is dark.

  The most famous example is the *bullet cluster*. These are two colliding galaxy clusters. The picture consists of three different contributions which are overlayed: An ordinary photograph in the optical made with the Hubble space telescope (all the white or yellowish spots which are actually galaxies), an X-ray picture taken by the X-ray satellite Chandra (the two red clouds) and the surface mass density calculated from weak lensing observations (the two blue clouds).

  picture from apod.nasa.gov

  It is the shape of the red cloud on the right that gave the name to the bullet cluster, because it looks like a bullet rushing into a target. The interpretation is as follows: We see two colliding galaxy clusters. The stars in the galaxies move through each other largely without any effect, because collisions are rare. The hot gases, however, strongly decelerate each other when colliding, so they stay behind; this is what the red clouds are showing. As the majority of visible masses in a galaxy cluster is in the form of hot gases, one would expect the blue clouds to coincide with the red clouds. However, this is not what we see: The blue clouds have not been decelerated by the collision, so the majority of the gravitating mass must consist of a kind of dark matter that is more or less frictionless. The bullet cluster gives compelling evidence for the existence of dark matter in galaxy clusters, provided one accepts Einstein's general relativity theory, and it gives strong restrictions on the way this dark matter can interact with itself and with other matter. Since the discovery of the bullet cluster a few other pairs of colliding clusters with similar properties have been found.

Taking the evidence from the velocity distributions in galaxy clusters, from the rotation curves in galaxies and from weak lensing together, we are more or less forced to assume that about 90 % of the matter is dark. As the bullet cluster shows most clearly, the mysterious dark matter can interact only very weakly with other things and with itself. Several hypothetical particles have been suggested as dark matter candidates, e.g.

- weakly interacting massive particles (WIMPs)
- axions,
- new types of neutrinos,

  ...

In spite of intensive searches, none of them has been detected so far. So the present status is: We do not know what this dark matter is, but we have to assume that it is there in order to explain the observations.

The only alternative to accepting the existence of dark matter seems to be a modification of the gravitational theory. As the observations that led us to postulating dark matter are mainly done at a level where the Newtonian approximation is valid, it would be necessary to modify already the Newtonian theory (and then Einstein's theory in a way that it gives the modified Newtonian theory in the appropriate limit). Several modified theories of this kind have been brought forward:

- Modified Newtonian Dynamics (MoND)

  This theory was suggested by M. Milgrom in 1983. It modifies the Newtonian equation of motion (Newton's Lex Secunda) from $\vec{F} = m\,\vec{a}$ to

  $$\vec{F} \;=\; m\,\mu\Big(\frac{a}{a_0}\Big)\,\vec{a}\,.$$

  Here a_0 is a hypothetical constant of Nature with the dimension of an acceleration and μ is a function that is to be chosen in a way that the old version of the Lex Secunda is still valid if the acceleration a is much bigger than a_0, i.e.,

  $$\mu(x) \;\approx\; 1 \quad \text{for } x \gg 1\,.$$

  Milgrom has demonstrated that the rotation curves of galaxies can be explained in a quite satisfactory manner if

  $$a_0 \;\approx\; 10^{-10}\,\mathrm{m/s^2}$$

  and

  $$\mu(x) \;=\; \frac{1}{1 \,-\, \dfrac{1}{x}}$$

  or

  $$\mu(x) \;=\; \frac{1}{\sqrt{1 \,-\, \dfrac{1}{x^2}}}\,.$$

  Of course, MoND cannot be considered as anything else but a Newtonian-like limit of a "true theory" which generalises Einstein's theory.

- Tensor-Vector-Scalar (TeVeS) theory

  It took about 20 years to find a generalisation of general relativity that reduces to MoND in the appropriate limit. It was found by J. Bekenstein in 2004. In this theory the gravitational field is not just described by a tensor field, as in Einstein's theory, but in addition by vector and scalar fields. The field equations are extremely complicated and the geometrical appeal of Einstein's theory is largely destroyed. TeVeS (and, thus, MoND) has problems to explain the observations of binary pulsars and of the cosmic background radiation. However, the greatest challange for this theory is the bullet cluster. Milgrom admitted that he was not able to fully explain the observations of the bullet cluster within TeVeS/MoND.

- Conformal gravity

  In 1989 P. Mannheim suggested that the rotation curves of galaxies can be explained in a theory where the gravitational field is still given by a metric tensor, as in general relativity, but the field equation is modified. Instead of Einstein's field equation, which derives from an action given by the Ricci scalar, this field equation derives from an action given by the square of the conformal curvature tensor (also known as the Weyl tensor). As a result, the left-hand side of the field equation is conformally invariant, i.e., it does not change if the metric is multiplied with a positive function. The same field equation was suggested already in 1920 by R. Bach. The theory suffers from severe conceptual problems. In particular, the conformal symmetry has to be broken by some mechanism in order to alllow for non-zero masses because an energy-momentum tensor can be conformally invariant only in the case that it describes matter made up of massless particles (such as a photon gas).

4.2 The distance-redshift relation

Recall that we have found various versions of a "Hubble law" in Robertson-Walker spacetimes. Without using the field equation, we could demonstrate the following.

- There is an *exact* linear relation between proper distance D_p and proper velocity dD_p/dt, see p. 22. This, however, is of no relevance in view of observations because D_p cannot be measured.

- The expression for each of the distance measures D_T, D_p, D_A and D_L can be expanded as a power series in z and we calculated the two leading-order terms for each of them which are determined by $H(t_o)$ and $q(t_o)$. In particular, we did this for the luminosity distance D_L as a function of z, see p.26. This can be linked to observations if standard candles are available.

When using the field equation, we could establish stronger results:

- For dust solutions without a cosmological constant, we gave the relation between scale factor and time analytically in parametric form. We derived an *exact* relation between luminosity distance and redshift which is known as the Mattig relation, see p.48. Just as the approximate second-order formula for an arbitrary Robertson-Walker universe, the Mattig formula is determined by $H(t_o)$ and $q(t_o)$. Recall from Worksheet 6 that in a dust universe $\Omega_m(t_o) = 2q(t_o)$, so $q(t_o)$ can be replaced with the density parameter $\Omega_m(t_o)$.

- For dust solutions with a cosmological constant, we did not give the relation between scale factor and time in analytical form, although this is possible in terms of elliptic integrals. We just qualitatively discussed the influence of Λ on the scale factor. With the exact analytical solution one can derive generalised Mattig relations, see M. Dąbrowski and J. Stelmach, Astron. J. 92, 1272 (1986). However, we did not (and will not) work them out because they are very complicated. The distance-redshift relation in a dust universe with a cosmological constant is usually evaluated numerically. Keep in mind that in involves $\Omega_m(t_o)$ and $\Omega_\Lambda(t_o)$ and that the case $k = 0$ is characterised by the equation $\Omega_m(t_o) + \Omega_\Lambda(t_o) = 1$.

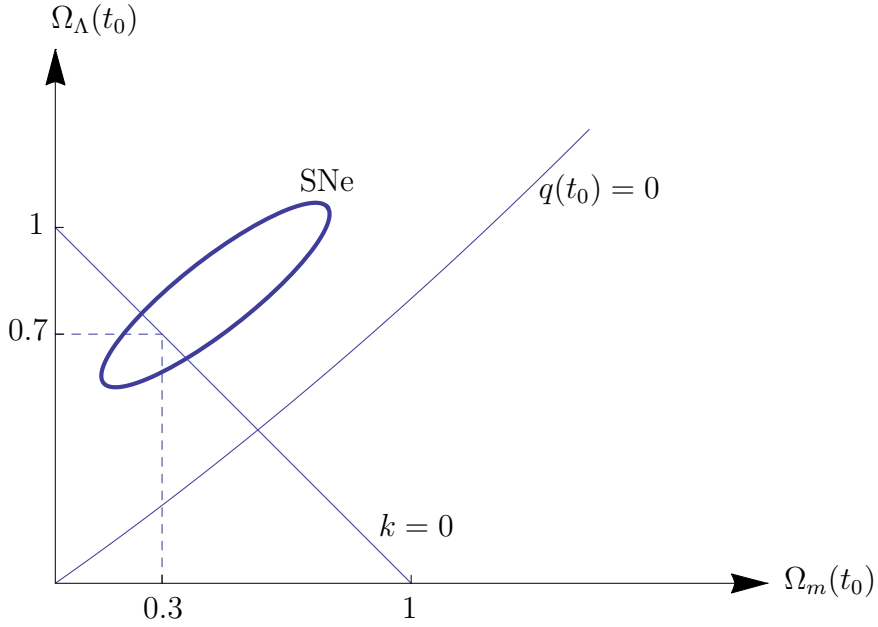
We will now link these mathematical results to observations, following the historic development.

In 1929 E. Hubble claimed that there is a linear relation between luminosity distance and redshift. This claim was based on a sample of about 25 galaxies whose redshifts had been measured before. He used Cepheids (variable stars whose period is related to their luminosity) as standard candles in combination with some rather rough estimates. He announced that the "K factor" (that's what we now call the Hubble constant) had a value of about 500 (km/s)/Mpc.

In the 1950s it was realised that the distance ladder, based on observations of Cepheids, had to be recalibrated. This reduced the Hubble constant $H(t_0)$, which up to this time had be generally assumed to be bigger than 200 (km/s)/Mpc, by a factor of 2. Until the 1990s, there was a controversy between two groups of cosmologists, one advocating a Hubble constant of about 50 (km/s)/Mpc and the other advocating a Hubble constant of about 100 (km/s)/Mpc. The deceleration parameter $q(t_0)$ was generally believed to be positive (corresponding to a decelerating expansion of the universe), but actually the observations were to inaccurate for determining $q(t_o)$. On theoretical grounds, many cosmologists were in favour of the Einstein-deSitter universe where the deceleration parameter is independent of time and equal to 1/2.

In 1998/1999 the results from two groups were published which both used supernovae of type Ia as standard candles. These supernovae are believed to be white dwarfs in a binary system. If so much mass from the companion has been accreted onto the white dwarf that its mass exceeds the Chandrasekhar limit of 1.44 $M_\odot$, the white dwarf becomes unstable and explodes as a supernova. As this instabiliy occurs at a fixed value of the mass, there is a universal relation between the shape of the light curve and the luminosity. This is why these supernovae are good standard candles. (Actually, the situation is a bit more complicated. Supernovae of type Ia are characterised by the fact that their spectra show no hydrogen lines but a silicon line. Not all supernovae of this type can be used as standard candles; one has to exclude some subclasses for which the relation between the light curve and the luminosity is different.) From observations of about 45 supernovae of type Ia in galaxies at redshifts up to $z \approx 0.9$ both groups independently found that the data *cannot* be matched to the distance-redshift relation of a universe with $q(t_0) > 0$, in particular not to a dust universe without a cosmological constant. For a dust universe with a positive cosmological constant, however, it worked, see the diagram on the next page. If one assumes a spatially flat universe ($k = 0$), as suggested by inflation and supported by the cosmic background radiation (see below), the best fit to the supernovae Ia data suggests that the density parameters should be $\Omega_m = 0.3$ for matter (90 % of which is assumed to be dark matter) and $\Omega_\Lambda = 0.7$ for the cosmological constant (which may be re-interpreted as dark energy or quintessence). The Hubble constant came out as $H(t_0) \approx 65$(km/s)/Mpc; the present data, also including observations of the cosmic background radiation, are in favour of a slightly

bigger value of about $H(t_0) \approx 70(\mathrm{km/s})/\mathrm{Mpc}$. The precise value of the deceleration parameter $q(t_0)$ is still unclear, but the supernovae Ia observations showed that it must be negative with a confidence of $7\,\sigma$. For the observation that our universe is accelerating S. Perlmutter, A. Riess and B. Schmidt won the Nobel Prize in Physics 2011. The determination of the distance-redshift relation with the help of supernovae of type Ia has been extended to redshifts bigger than 1 in the years after 2000. In addition to ground-based observations, a satellite project SNAP (SuperNova Acceleration Probe) had been proposed which later became a sub-project of WFIRST (Wide Field Infrared Survey Telescope). This NASA satellite could be launched around 2020.



4.3 The cosmic background radiation

We have already mentioned the chequered history of the cosmic background radiation. Recall that the officially recognised detection was made in the year 1964 by A. Penzias and R. Wilson who won the Nobel Prize in 1978. In the following years, the properties of the cosmic background radiation have been carefully investigated, in particular by several satellite missions. Note that the maximum of the cosmic background radiation is in the frequency range of microwaves where the radiation is largely blocked by the water vapour in our atmosphere. Therefore, observations of the cosmic background radiation are made with satellites, with balloons, or with ground-based telescopes at high altitude, in particular near the South Pole where because of the cold temperature the amount of water vapour in the atmosphere is low. The most important projects have been the following:

- COBE (Cosmic Background Explorer)

  This was a satellite that was launched in 1989. Data were released in 1992 and made a great impact. In particular, COBE found that the cosmic background radiation shows a perfect Planck spectrum and that it is isotropic to an extremely high degree, but it also found the first tiny deviations from isotropy. J. Mather and G. Smoot were awarded the Nobel Prize for these discoveries in 2006.

- Boomerang (Balloon Observations Of Millimetric Extragalactic Radiation and Geomagnetics)

  As the name suggests, this was a balloon experiment. It flew two times, in 1998 and in 2003 and its most important result was that it detected the first acoustic peak in the power spectrum (see below) at precisely the postition where it should be in a universe with $k = 0$.

- WMAP (Wilkinson Microwave Anisotropy Probe)

  This was a NASA satellite that took data over the unusually long period from 2001 to 2010. It mapped the anisotropies over the whole sky, confirmed the first acoustic peak in the power spectrum and found the next ones.

- Planck

  This European satellite was in operation from 2009 to 2013. The data analysis is still ongoing. Both the sensitivity and the resolution of the Planck satellite was even higher than that of WMAP, so Planck was able to determine the power spectrum to ever higher values of ℓ (see below).

Another project that made big headlines was BICEP2, a telescope at the South Pole. In March 2014 the BICEP2 team announced that their investigation of the polarisation of the cosmic background radiation showed distinctive signatures from primordial gravitational waves. In the following months it was found that the measurements were correct but the interpretation was wrong. The so-called B-modes that have been observed in the polarisation had not been produced by primordial gravitational waves (at a very early stage of the universe) but rather by the influence of dust on the propagation of the photons in the cosmic background radiation (at a much later time).

We now discuss the most important observed features of the cosmic background radiation.

(a) Planck spectrum

From elementary physics text-books we know that the Planck spectrumc is

$$dn \;=\; \frac{8\,\pi\,V\nu^2\,d\nu}{c^3\left(\exp\!\Big(\dfrac{h\nu}{kT}\Big)\;-\;1\right)}\,, \qquad\qquad \text{(P)}$$

where

$\nu =$ frequency, $\nu = \dfrac{\omega}{2\pi} = \dfrac{c|\vec{k}|}{2\pi} = \dfrac{c}{\lambda}$,

$n =$ photon number,

$V =$ volume,

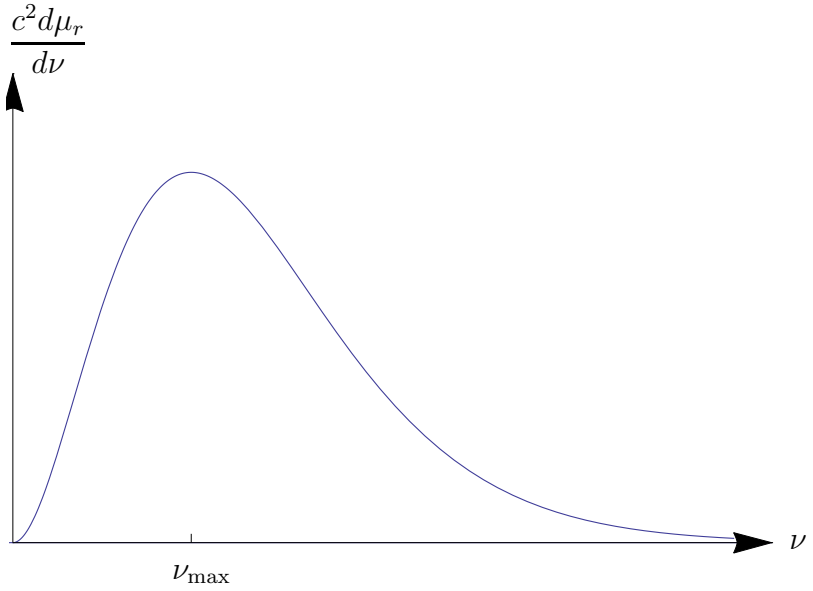
$T =$ temperature,

$h =$ Planck constant,

$k =$ Boltzmann constant.

The corresponding spectral energy density is

$$\frac{c^2\,d\mu_r}{d\nu} \;=\; \frac{c^2\,d\mu_r}{dn}\,\frac{dn}{d\nu} \;=\; \frac{h\,\nu}{V}\,\frac{8\,\pi\,V\nu^2}{c^3\left(\exp\!\Big(\dfrac{h\nu}{kT}\Big)\;-\;1\right)} \;=\; \frac{8\,\pi\,h\nu^3}{c^3\left(\exp\!\Big(\dfrac{h\nu}{kT}\Big)\;-\;1\right)}\,.$$



The maximum of the spectral energy density is at a frequency $\nu_{\max}$ which is determined by the temperature T.

Integrating over all frequencies gives the *Stefan-Boltzmann law*,

$$c^2\,\mu_r \;=\; \int_0^\infty c^2\,\frac{d\mu_r}{d\nu}\,d\nu \;=\; \int_0^\infty \frac{8\,\pi\,h\,\nu^3\,d\nu}{c^3\left(\exp\!\left(\dfrac{h\nu}{kT}\right)\,-\,1\right)}$$

$$=\; \frac{8\,\pi\,k^4\,T^4}{c^3\,h^3}\int_0^\infty \frac{x^3\,dx}{e^x-1} \;=\; \frac{8\,\pi\,k^4\,T^4}{c^3\,h^3}\,\frac{\pi^4}{15} \;=\; \frac{4\,\sigma}{c}\,T^4$$

where σ is the Stefan-Boltzmann constant. Inserting the numerical values for h, k and c results in the equation

$$c^2\,\mu_r \;\approx\; 7.6\times 10^{-16}\,\frac{\mathrm{J}}{\mathrm{m}^3}\,\frac{T^4}{\mathrm{K}^4}$$

which allows to calculate the energy density $c^2\mu_r$ from the temperature T.

In an expanding universe the volume at a time t_e is related to the volume at a time t_o according to

$$\frac{V(t_o)}{V(t_e)} \;=\; \frac{a(t_o)^3}{a(t_e)^3}\,.$$

With the redshift law for Robertson-Walker spacetimes,

$$\frac{\nu(t_e)}{\nu(t_o)} \;=\; 1\,+\,z \;=\; \frac{a(t_o)}{a(t_e)}\,,$$

this can be rewritten as

$$\frac{V(t_o)}{V(t_e)} \;=\; \frac{\nu(t_e)^3}{\nu(t_o)^3}\,,$$

hence

$$V(t_o)\,\nu(t_o)^3 \;=\; V(t_e)\,\nu(t_e)^3\,.$$

Differentiation with respect to the frequency, keeping t_e and t_o fixed, yields

$$V(t_o)\,\nu(t_o)^2\,d\nu(t_o) \;=\; V(t_e)\,\nu(t_e)^2\,d\nu(t_e)\,,$$

i.e., the numerator in the Planck law (P) is time-independent. If the photon number is conserved, this implies that also the denominator must be time-independent,

$$\frac{\nu(t_o)}{T(t_o)} \;=\; \frac{\nu(t_e)}{T(t_e)}\,.$$

Invoking again the redshift law in Robertson-Walker spacetimes, this implies

$$\frac{T(t_e)}{T(t_o)} \;=\; \frac{a(t_o)}{a(t_e)}\,,$$

i.e., if the universe expands by a certain factor, the temperature drops by the same factor, which is quite intuitive. Note that the Stefan-Boltzmann law is in agreement with the fact that in a universe filled with radiation the density is inverse proportional to the fourth power of the scale factor, as we have seen before.

We observe that the cosmic background radiation reaches us now (at time t_o) with a perfect Planck spectrum whose temperature is $T(t_o) = 2.73\,\mathrm{K}$. The maximum of the radiation is at a frequency $\nu_{\max} \approx 160\,\mathrm{GHz}$ which corresponds to a wavelength of $\lambda_{\max} \approx 1.1\,\mathrm{mm}$. By the Stefan-Boltzmann law, the temperature gives us the energy density $c^2\,\mu_r(t_o)$ of the radiation,

$$\mu_r(t_o) \;\approx\; 4.6\times 10^{-31}\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,.$$

From the spectral distribution we can calculate that this corresponds to approximately 500 photons per cm^3. On the other hand, the critical density is determined by the Hubble constant which we know rather well,

$$\mu_c(t_o) \;=\; \frac{3\,H(t_o)^2}{\kappa\,c^4} \;=\; 1.9\times 10^{-26}\,h^2\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

where

$$H(t_o) \;=\; 100\times h\,\frac{\mathrm{km/s}}{\mathrm{Mpc}}\,.$$

With $h \approx 0.7$ we find that the density parameter of the radiation is

$$\Omega_r(t_o) \;=\; \frac{\mu_r(t_o)}{\mu_c(t_o)} \;<\; 10^{-4}$$

which can be ignored in comparison to the density parameters of the cosmologicall constant and of matter, $\Omega_\Lambda \approx 0.7$ and $\Omega_m \approx 0.3$.

The above analysis shows that a Planck spectrum remains a Planck spectrum if photons freely propagate in an expanding universe, with the temperature being proportional to the inverse of the scale factor. The obvious idea is that the cosmic background radiation has come into existence at some time t_e, when the scale factor was smaller and the temperature was higher, and from that time onwards the photons of the cosmic background radiation have propagated

more or less freely until we observe them today at time t_o. What can we say about the time t_e? Certainly, photons cannot propagate freely if the universe is densely filled with free electrons so that the photons undergo frequent scattering. As a rough approximation, the time t_e coincides with the time when electrons and ions formed neutral atoms, so $T(t_e)$ can be estimated from the condition that this temperature should approximately correspond to the energy where atoms are ionised,

$$k\,T(t_e) \;\approx\; E_{\text{ionisation}}\,.$$

A typical ionisation energy is in the order of some eV. (For hydrogen, e.g., it is 13.6 eV.) So for our rough estimate we may assume that

$$k\,T(t_e) \;\approx\; 1\,\text{eV}\,.$$

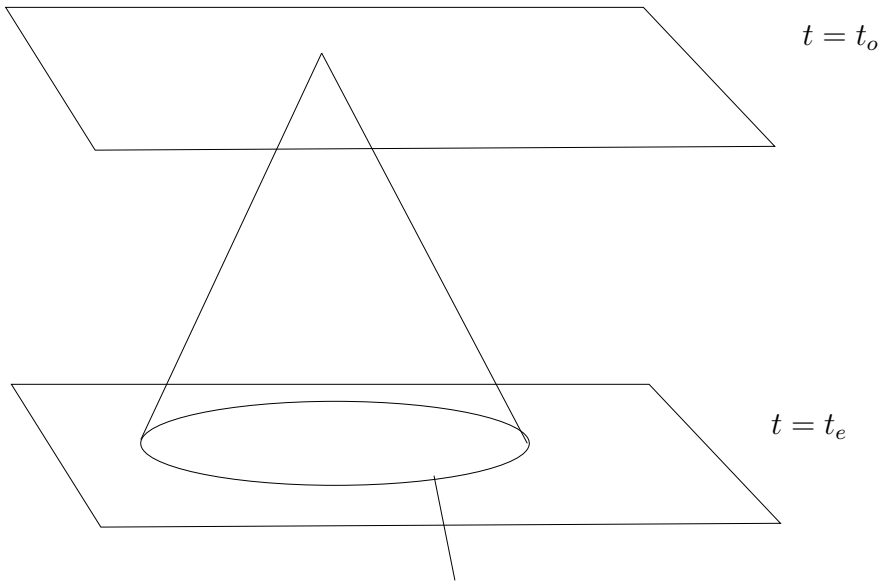
As the Boltzmann constant is

$$k \;=\; 0.86\times 10^{-4}\,\frac{\text{eV}}{\text{K}}\,,$$

this gives

$$T(t_e) \;\approx\; 10^4\,\text{K}\,.$$

A more detailed analysis shows that the universe became transparent at a temperature of 3000 – 4000 K. This corresponds to a redshift of $z \approx 1100$. We refer to the hypersurface $t = t_e$ with this temperature $T(t_e)$ as to the *hypersurface of last scattering*. The intersection of this hypersurface with the past light-cone of an observer here and now is called the *surface of last scattering*.



It is usual to refer to the time when electrons and ions formed neutral atoms as to the time of *recombination*. Actually, this is a misnomer because it is for the first time in the universe that electrons and ions combine. Of course, the time of recombination was not one precise moment but rather a time interval. Similarly, the hypersurface of last scattering $t = t_e$ is an idealised model for a spacetime region with a certain temporal extension.

We have said that in the time before recombination photons underwent frequent scattering processes with free electrons. In the rest system of the electron, the photon has a certain initial energy $E_{\gamma i}$ and a certain final energy $E_{\gamma f}$. One speaks of

– Compton scattering if $E_{\gamma i} > E_{\gamma f}$,

– Thomson scattering if $E_{\gamma i} = E_{\gamma f}$,

– inverse Compton scattering if $E_{\gamma i} < E_{\gamma f}$.

In the time after recombination, the photons of the cosmic background radiation are scattered only very rarely. When they pass through the hot gas (plasma) in a galaxy cluster, these rare scattering processes can lead to a tiny distortion of the Planck spectrum. This is known as the *Sunyaev-Zel'dovich effect*

(b) Anisotropy

The cosmic background radiation shows a Planck spectrum, so we can associate to it a temperature T. This temperature is isotropic, i.e., independent of the direction from which the radiation comes, to an extremely high degree (if we subtract the dipole term, see below). However, it is not perfectly isotropic, there are tiny anisotropies in the order of $\Delta T/T \lessapprox 10^{-5}$. These tiny anisotropies give important information on the universe. They are usually modelled with the help of an expansion into spherical harmonics

$$Y_\ell^m(\vartheta,\varphi) \;=\; \sqrt{\frac{(2\ell+1)}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\vartheta)\,e^{im\varphi}\,,$$

where the P_ℓ^m are the associated Legendre polynomials,

$$P_\ell^m(x) \;=\; \frac{(-1)^m}{2^\ell \ell!}\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

and the $P_\ell(x)$ are the Legendre polynomials,

$$P_\ell(x) \;=\; \frac{1}{2^\ell \ell!}\,\frac{d^\ell}{dx^\ell}\Big(\left(x^2-1\right)^\ell\Big)\,.$$

The temperature is a function on the sky, i.e. $T : S^2 \to \mathbb{R}^+$. The points on the sphere are in one-to-one correspondence with unit vectors $\vec{e}$ which can be represented with the help of standard spherical coordinates as

$$\vec{e} \;=\; \begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}\,.$$

The spherical harmonics form an orthonormal basis for square-integrable functions on the sphere, so we may write the temperature $T\big(\vec{e}\big)$ as

$$\frac{T\big(\vec{e}\big)}{T_0} \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m\,Y_\ell^m(\vartheta,\varphi)$$

where T_0 is the averaged temeperature,

$$T_0 \;=\; \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} T\big(\vec{e}\big)\sin\vartheta\,d\vartheta\,d\varphi\,.$$

For each ℓ,

$$a_\ell(\vartheta,\varphi) \;=\; \sum_{m=-\ell}^{\ell} a_\ell^m \, Y_\ell^m(\vartheta,\varphi)$$

is the multipole moment of degree ℓ of the temperature anisotropy. The monopole moment is

$$a_0 \;=\; 1$$

because we have divided by T_0. The dipole moment

$$a_1(\vartheta,\varphi) \;=\; \sum_{m=-1}^{1} a_1^m \, Y_1^m(\vartheta,\varphi)$$

was measured around 1970. It was found to be in the order of $\big|a_1(\vartheta,\varphi)\big| \lessapprox 10^{-3}$. It is understood as a result of the motion of the Earth with respect to the rest system of the cosmic background radiation: If the cosmic background radiation were perfectly isotropic with respect to the standard observers in a Robertson-Walker observer, any other observer would see a dipole anisotropy that can be explained as a Doppler effect resulting from the motion of this observer relative to the standard observers. In the forward direction the Doppler effect causes a blueshift of the photons which results in a Planck spectrum with a higher temeperature, in the backward direction the Doppler effect causes a redshift of the photons which results in a Planck spectrum with a lower temperature. When we talk about anisotropies in the cosmic background radiation we always subtract the dipole term, i.e., we consider the quantity

$$\delta^T\big(\vec{e}\big) \;=\; \frac{T\big(\vec{e}\big)}{T_0} \;-\; 1 \;-\; \sum_{m=-1}^{1} a_1^m Y_1^m(\vartheta,\varphi) \;=\; \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m \, Y_\ell^m(\vartheta,\varphi)\,.$$

It is this quantity for which observations give a bound of $\big|\delta^T\big(\vec{e}\big)\big| \lessapprox 10^{-5}$. It is natural to assume that δ^T is *Gaussian* (i.e., that values of δ^T measured over the sky show a Gauss distribution about the mean-value of zero) and *statistically isotropic* (i.e., that the values for all higher-order multipole moments vary randomly over the sky without distinguishing a particular direction). However, the recent satellite missions WMAP and Planck, which have measured the anisotropies in the cosmic background radiation with a high accuracy, have found some indications for non-Gaussianities and also for a distinguished axis in the sky (sometimes called the "axis of evil") with which the quadrupole moment and the octupole moment seem to be aligned. These observations have to be confirmed, so at the moment it is not yet clear if the assumptions of Gaussianity and of statistical isotropy really have to be dropped.

If we take a conservative view and assume that Gaussianity holds, the two-point autocorrelation function

$$C^T \;=\; \langle \delta^T\big(\vec{e}\big)\, \delta^T\big(\vec{e}\,'\big)\rangle$$

of the temperature anisotropy δ^T determines the correlation completely because for a Gaussian distribution all higher-order correlation functions are determined by the two-point correlation function. Moreover, if statistical isotropy holds, for any two points $\vec{e}$ and $\vec{e}\,'$ in the sky the correlation depends only on the angle between $\vec{e}$ and $\vec{e}\,'$,

$$C^T(\vartheta) \;=\; \langle \delta^T\big(\vec{e}\big)\, \delta^T\big(\vec{e}\,'\big)\rangle_{\vec{e}\cdot\vec{e}\,'=\cos\vartheta}\,.$$

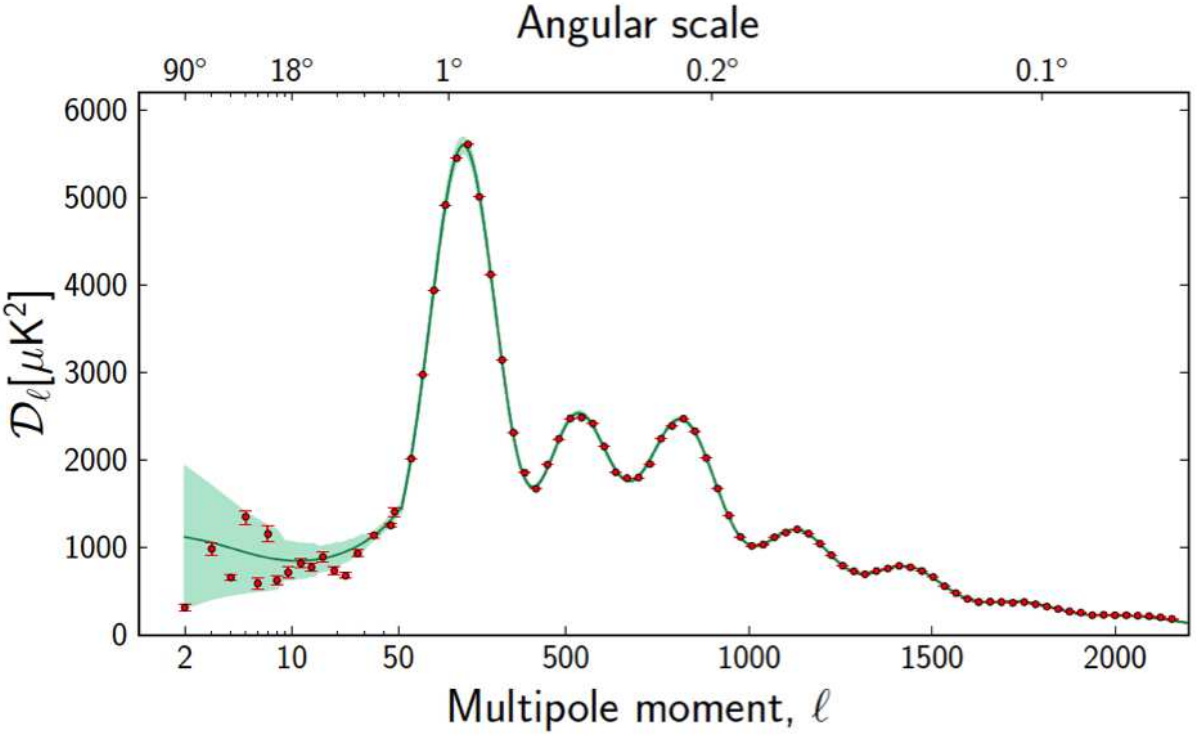
Because of statistical isotropy the ensemble average $\langle \cdot \rangle$ can be evaluated with $\vec{e}\,'$ kept fixed at the North Pole of the coordinate system; then the angle ϑ is just the polar coordinate ϑ of the point $\vec{e}$. As C^T depends only on ϑ, expansion of this function with respect to spherical harmonics involves only terms which are independent of φ, i.e., terms with $m = 0$. As Y_ℓ^0 is a multiple of P_ℓ, this results in an expansion in terms of the Legendre polynomials,

$$C^T(\vartheta) \;=\; \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, C_\ell^T\, P_\ell(\cos\vartheta)\,.$$

The coefficients C_ℓ^T give the *angular power spectrum* of the temperature anisotropy. (This is not the general definition of the angular power spectrum, but in the case at hand it is an equivalent definition.) High values of ℓ correspond to correlations on a small angular scale. Note that because the Legendre polynomials satisfy an orthogonality condition with respect to the L^2 scalar product, the expansion equation of $C^T(\vartheta)$ can be solved for the coefficients C_ℓ^T,

$$C_\ell^T \;=\; 2\,\pi \int_0^\pi C^T(\vartheta)\, P_\ell(\cos\vartheta)\, \sin\vartheta\, d\vartheta\,.$$

The measurement of the C_ℓ^T for many values of ℓ was one of the main goals of the satellite missions WMAP and Planck. Although ℓ is a discrete variable, taking only non-negative integer values, C_ℓ^T is usually plotted against ℓ as if ℓ could take all non-negative real values. The diagram shows the 2013 results of the Planck mission. The $\mathcal{D}_l$ plotted on the vertical axis has the dimension of temperature squared, while our C_ℓ^T are dimensionless. The difference lies in the fact that we didn't consider the temperature anisotropy but rather the temperature anisotropy divided by T_0.



Already the balloon mission Boomerang had observed a local maximum of C_ℓ^T near $\ell = 200$. WMAP and Planck found additional local maxima at higher values of ℓ, see the diagram. These local maxima are known as "acoustic peaks". Before they were observed, they had actually been predicted on the basis of the following theoretical consideration: In the era before recombination, ions, electrons and photons formed a kind of soup where with a certain statistical probability overdensities formed. Each overdensity grew over a certain time, because it gravitationally attracted neighbouring matter, until the pressure became so big that a (roughly spherical) wave expanded from the overdensity. This is quite similar to the formation of a sound wave in a gas, therefore one calls these waves "acoustic". The distance the photons, which are part of the soup, could travel before they decoupled from the matter at about the time of recombination, depends on the speed of sound. The latter can be theoretically calculated with the help of perturbation theory, see the next chapter. After the time of recombination, the photons decouple from the matter and just freely follow the expansion of the universe. This process results in the formation of roughly spherical shells of photons with a radius that can be theoretically predicted. Clearly, the existence of such shells results in a certain correlation of the anisotropy of the cosmic background radiation at a certain angular scale, with a maximum at a particular value of ℓ. On the basis of a universe with $k = 0$, as suggested by inflation, the first acoustic peak was predicted to occur near $\ell = 200$. This was precisely what the observations have shown. Also the discovery of the other peaks is in agreement with the assumption that $k = 0$. As these calculations are quite sensitive to the value of k, the location of the acoustic peaks give strong support to the idea that we live in a universe with $k = 0$.

Recall that the observations of the supernovae of type Ia could be explained by assuming a universe with a cosmological constant and a dust, i.e., with two density parameters $\Omega_\Lambda(t_o)$ and $\Omega_m(t_o)$. The data located the values for these density parameters within an elliptical area, see the picture on p. 74. If we combine this result with the evidence for $k = 0$, as it comes from the anisotropy of the cosmic background radiation, we have to intersect this elliptical area with the straight line where $\Omega_\Lambda(t_o) + \Omega_m(t_o) = 1$ which corresponds to $k = 0$. This gives the values of approximately $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$.

(c) Polarisation

The cosmic background radiation is unpolarised to a very high degree. Relative deviations are of the order $\lessapprox 10^{-6}$ which is even one order of magnitude smaller than that for the temperature anisotropy. Nonetheless, a slight degree of polarisation has been detected.

For a theoretical description, one uses again an expansion into spherical harmonics. However, this is now much more involved than for the temperature because the degree of polarisation cannot be described by a scalar variable: One usually uses the so-called Stokes parameters which can be combined to form a second-rank tensor field on the celestial sphere. Therefore, one cannot expand the polarisation measure in terms of the usual *scalar* spherical harmonics, one rather has to use *tensorial* spherical harmonics. There are two families of such tensorial spherical harmonics, one of them for the expansion of curl-free anisotropies and one for divergence-free anisotropies. Because of the analogy with electrodynamics, the curl-free anisotropies are called E-modes and the divergence-free anisotropies are called B-modes. (Note that this has nothing to do with the real electric or magnetic fields of which the cosmic background radiation consists!) Theoretically the slight degree of polarisation of the cosmic background radiation

can be explained by the influence of matter on the photons on their way from the surface of last scattering to us, i.e., as scattering or deflection (lensing) effects. E-modes that could be explained in this way have been observed for the first time in 2003 and B-modes that could be explained in this way have been observed in 2013. In 2014, it was announced that the telescope BICEP2 at the South Pole has observed a kind of B-modes that could be explained only as the result of primordial gravitational waves; this would have been a kind of anisotropy imprinted on the cosmic background radiation already when it left the surface of last scattering. If true, this would have given strong suppport for the idea of inflation because otherwise it would have been impossible to explain how the effect of primordial gravitational waves could have grown to a measurable size. Unfortunately, it was found out that the BICEP2 observations could very well be explained as the effect of dust ("foreground") on the cosmic background radiation. The BICEP2 team withdraw the announcement that they have detected primordial gravitational waves after a few months. Note that it is generally accepted that the *observations* of the BICEP2 team were correct; it is the *interpretation* of these observations that was wrong.

4.4 Other observations

Without going into details, we very briefly indicate that our cosmological models are also restricted by some other kind of observations.

- Number counts

  Let us assume we count all galaxies in the sky up to a certain magnitude, i.e., all galaxies whose flux is bigger than a certain chosen limit value F. How does the number N of these galaxies depend on the flux F? In a Euclidean static universe, N would grow with R^3 where R is the radius of the volume in which we count the galaxies. On the other hand, the flux falls off with R^{-2}, so we have $N \sim F^{3/2}$. As the log of F gives the magnitude, one usually writes this as

  $$\log N \;=\; \frac{3}{2}\log F \;+\; \text{constant}\,.$$

  For an expanding (and possibly non-Euclidean) universe, we get a different relation. In this way number counts give us a means for testing a chosen cosmological model.

  Unfortunately, this method is not very reliable. The reason is that galaxies develop over time. On average, a distant galaxy is seen at a younger stage of its life than a galaxy closer by. We do not know enough about the development of galaxies for accurately estimating the effect of age on the intrinsic luminosity.

- Baryonic Acoustic Oscillations (BAO)

  We have briefly mentioned the formation of acoustic peaks in the cosmic background radiation from spherical acoustic waves that formed at a time before recombination. Not only the photons take part in these acoustic waves, but also the baryons. So they should also form a spherical shell about each centre where an overdensity had formed. This should be visible in the two-point correlation function *for the matter density*. The Sloan Digital Sky Survey has revealed indications for these socalled Baryonic Acoustic Oscillations. They are in agreement with the ΛCDM model with $\Omega_\Lambda(t_o) = 0.7$ and $\Omega_m(t_o) = 0.3$.

- Gravitational lensing

  Microlensing plays an important role for estimating the dark matter that can exist in the form of Massive Compact Halo Objects and weak lensing is crucial for estimating the dark matter in galaxy clusters. This was outlined already in Section 4.1. In addition, lensing is also relevant for determining the matter in the universe at very large scales. The same kind of weak lensing observations that has been made in the direction of galaxy clusters has also been made in directions where no galaxy clusters are visible. Any deviation from a random distribution of the shapes of background galaxies would indicate a deforming influence of the matter distribution in the universe on the cross-sections of light bundles at very large scales. This so-called "cosmic shear" was detected around the year 2000. It restricts the possible ways in which we can model our universe as a Robertson-Walker universe with certain perturbations. As the weak-lensing observations can only determine the surface mass density (i.e., mass per area of a surface perpendicular to the line of sight) it cannot determine a 3D distribution of matter in the universe. However, if weak lensing is combined with other observations, it is possible to produce 3D maps of the distribution of matter. These maps show a strong tendency of the matter to form filaments at very large scales.

5. Perturbation theory

If we want to theoretically describe the anisotropies in the cosmic background radiation, and other anisotropies, we have to go beyond the homogeneous and isotropic cosmological models, i.e., beyond Robertson-Walker spacetimes. Cosmological perturbation theory is the usual mathematical setting for this kind of investigations.

General relativistic perturbation theory is based on an ansatz for the metric of the form

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$$

where $\overline{g}_{\mu\nu}$ is a given ("background") metric. One assumes that the perturbation is so small that it is justified to linearise all equations with respect to $h_{\mu\nu}$ and its derivatives. In this way, the nonlinear Einstein equation for the metric $g_{\mu\nu}$ is reduced to a linear differential equation for the perturbation $h_{\mu\nu}$. This linearised formalism is best known for the case that $\overline{g}_{\mu\nu}$ is the Minkowski metric. In an appropriate gauge (i.e., if the freedom of making coordinate transformations is used in an intelligent way), the resulting vacuum equation for $h_{\mu\nu}$ reduces to the ordinary wave equation and thus to the Laplace equation for the static case. In this setting Einstein derived the perihelion precession of Mercury, the light deflection at the Sun and the existence of gravitational waves. The linearised formalism is also well developped for the case that $\overline{g}_{\mu\nu}$ is the Schwarzschild metric. After decomposing the perturbation into two parts that transform differently under spatial inversion (parity), this leads to the Regge-Wheeler equation for one part and to the Zerilli equation for the other.

In cosmological perturbation theory, it is natural to choose $\overline{g}_{\mu\nu}$ to be a Robertson-Walker metric. This formalism dates back to a pioneering paper by Y. Lifshits (1946), but it developped into a powerful tool only after J. Bardeen (1980) wrote the perturbation functions in a way that is

invariant under coordinate transformations. As in perturbation theory a change of coordinates is somewhat similar to a gauge transformation in electrodynamics, it is usual to refer to Bardeen's formalism as to *gauge-invariant* perturbation theory. We will now explain the basic features of this formalism.

For the sake of simplicity, we restrict to the case that the background metric is a *spatially flat* Robertson-Walker universe, i.e., to the case $k = 0$. Then

$$\overline{g}_{\mu\nu} dx^\mu dx^\nu \;=\; -\,c^2\,dt^2 \;+\; a^2\,\Big(d\chi^2 \;+\; \chi^2\big(d\vartheta^2 \;+\; \sin^2\vartheta\,d\varphi^2\,\big)\Big)$$

where the scale factor a is a function of t. If we use the conformal time T, this metric can be rewritten as

$$\overline{g}_{\mu\nu} dx^\mu dx^\nu \;=\; a^2\,\Big(\,-\;c^2\,dT^2 \;+ d\chi^2 \;+\; \chi^2\big(d\vartheta^2 \;+\; \sin^2\vartheta\,d\varphi^2\,\Big)$$

where now a has to be viewed as a function of T. What we have in the bracket is just the Minkowski metric in spherical polars, so we may rewrite it in the form

$$\overline{g}_{\mu\nu} dx^\mu dx^\nu \;=\; a^2\,\Big(\,-\;c^2\,dT^2 \;+ \delta_{ij} dx^i dx^j\Big)\,.$$

As usual, latin indices $i, j, \ldots$ take values 1,2,3 and, for this section, we agree to lower and to raise latin indices with δ_{ij} and δ^{ij}, respectively. The metric is now, up to the conformal factor a^2, the Minkowski metric in usual inertial coordinates. Note, however, that cT and χ are dimensionless while usually, when writing the Minkowski metric in spherical polars, we use ct and r which have the dimension of a length. So we have to keep in mind that the "Minkowski-like" coordinates cT, x^1, x^2 and x^3 are dimensionless and that the correct dimension of $\overline{g}_{\mu\nu}$ is provided by the conformal factor a^2 which has the dimension of a length.

We now switch on the perturbation. If we label the components of $h_{\mu\nu}$ appropriately, this can be written as

$$g_{TT} \;=\; \overline{g}_{TT} \;+\; h_{TT} \;=\; -\,c^2\,a^2 dT^2 \;+ h_{TT} \;=\; -\,c^2\,a^2\,\big(\,1 \;+\; 2\,A\big)\,,$$

$$g_{Ti} \;=\; \overline{g}_{Ti} \;+\; h_{Ti} \;=\; 0 \;+\; h_{Ti} \;=\; a^2\,B_i\,,$$

$$g_{ij} \;=\; \overline{g}_{ij} \;+\; h_{ij} \;=\; a^2\,\delta_{ij} \;+\; h_{Ti} \;=\; a^2\,\Big(\delta_{ij} \;+\; \frac{h_{ij}}{a^2}\,\Big)\,.$$

Then the perturbed metric reads

$$g_{\mu\nu} dx^\mu dx^\nu \;=\; a^2\,\left(-c^2\big(1 + 2A\big) \;+\; 2\,B_i\,dx^i dT \;+\; \Big(\,\delta_{ij} \;+\; \frac{h_{ij}}{a^2}\,\Big)\,dx^i dx^j\right)\,.$$

Here it is convenient to further decompose the vectorial part, B_i, and the tensorial part, h_{ij}, of the perturbation. As we are on a spatially flat background, we can use ordinary vector calculus. It is well known that any vector field on Euclidean 3-space can be decomposed into a curl-free and a divergence-free vector field. This is known as the *Helmholtz decomposition theorem*, so we may write in the case at hand

$$B_i \;=\; \partial_i B \;+\; \hat{B}_i$$

where B is a scalar field and $\hat{B}_i$ is divergence-free, $\partial_i \hat{B}^i = 0$.

Sketch of proof of the Helmholtz decomposition theorem:

We want to write a given vector field $\vec{B}$ in the form

$$\vec{B} \,=\, \vec{\nabla} B \,+\, \vec{\hat{B}}$$

where $\vec{\nabla}\cdot\vec{\hat{B}} = 0$. If this equation holds, we must have

$$\vec{\nabla}\cdot\vec{B} \,=\, \Delta B\,.$$

For any smooth $\vec{B}$, this equation has a solution which is, of course, not unique. A particular solution may be written as

$$B\big(\vec{x}\,\big) \,=\, -\,\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\big(\vec{\nabla}\,'\cdot\vec{B}\big)\big(\vec{x}\,'\big)}{\big|\vec{x}-\vec{x}\,'\big|}\,d^3\vec{x}\,'\,,$$

as is well known from electrodynamics, provided that $\vec{B}$ falls off sufficiently strongly so that the integral exists. In any case, with a chosen solution B we may *define* $\vec{\hat{B}} := \vec{B} - \vec{\nabla} B$. Then $\vec{\nabla}\cdot\vec{\hat{B}} = 0$ and we are done.

Similarly to the Helmholtz decomposition of vector fields, one may also decompose tensor fields of second rank. We just give the result here, for details we refer to J. Ehlers' notes in General Relativity and Gravitation 39, 1929 (2007). One finds that the symmetric second-rank tensor field h_{ij} can be written as

$$\frac{h_{ij}}{a^2} \,=\, 2\,C\,\delta_{ij} \,+\, \Big(\partial_i\partial_j \,-\, \frac{1}{3}\,\delta_{ij}\,\Delta\Big)\,E \,+\, \partial_i\hat{E}_j \,+\, \partial_j\hat{E}_i \,+\, 2\,\hat{E}_{ij}$$

where C and E are scalar fields, $\hat{E}_i$ is a vector field with $\partial_i\hat{E}^i = 0$ and $\hat{E}_{ij}$ is a second-rank tensor field with $\partial_i\hat{E}^{ij} = 0$ and $\hat{E}_i{}^i = 0$. Following a general convention, we denote tensor fields that are divergence-free by a hat. This puts the perturbed metric into the following form:

$$g_{\mu\nu}dx^\mu dx^\nu \,=\, a^2\,\Big\{-\,c^2\,\Big(1\,+\,2\,A\Big)\,dT^2 \,+\, \Big(\partial_i B \,+\, \hat{B}_i\Big)dx^i dT$$

$$+\,\bigg((1+2C)\delta_{ij} \,+\, \Big(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\Big)E \,+\, 2\,\partial_i\hat{E}_j \,+\, 2\,\hat{E}_{ij}\bigg)\,dx^i dx^j\Big\}\,.$$

We have just relabelled the perturbation: In the beginning we had the $h_{\mu\nu}$ which form a symmetric 4×4 matrix, i.e., there are 10 independent scalar perturbation functions. After the relabelling we have

- 4 scalar fields A, B, C, E,
- two (co)vector fields $\hat{B}_i$ and $\hat{E}_i$,
- one symmetric second-rank tensor field $\hat{E}_{ij}$

which have $4 + 6 + 6 = 16$ scalar components.

They are restricted by the constraints

- $\partial_i \hat{E}^i = 0$, $\partial_i \hat{B}^i = 0$
- $\partial_i \hat{E}^{ij} = 0$,
- $\hat{E}_i{}^i = 0$

which are $2 + 3 + 1 = 6$ conditions. So altogether we have $16 - 6 = 10$ independent scalar perturbation variables which is indeed the same number as before.

The perturbations $\partial_i B$ and $\partial_i E$ are often called *longitudinal*, while the perturbations $\hat{B}_i$, $\hat{E}_i$ and $\hat{E}_{ij}$ are called *transverse*. This terminology refers to Fourier transformations: If we expand all terms with respect to the spatial variables as integrals $\int \ldots \exp(k_i x^i) d3\vec{k}$, there are terms proportional to $\vec{k}$ and terms perpendicular to $\vec{k}$. For obvious reasons, the former are called "longitudinal" while the latter are called "transverse". Note that Forier expansion requires square integrability of the perturbations which is usually assumed in cosmological perturbation theory.

You may ask what is the advantage of relabelling the perturbation in such a rather complicated form. The answer is that in terms of the new variables $A, B, C, E, \hat{E}_i, \hat{B}_i, \hat{E}_{ij}$ it is easier to find out which perturbations are gauge invariant and to decompose a general perturbation into scalar, vector and second-rank tensor parts.

To make this clear, we have to investigate how the perturbations transform under coordinate changes. As we are interested in perturbations only to within the linear approximation, it suffices to consider coordinate transformations $\left(T, x^1, x^2, x^3\right) \mapsto \left(\tilde{T}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3\right)$ where

$$\tilde{T} \,=\, T \,+\, \tau\,, \quad \tilde{x}^i \,=\, x^i \,+\, \xi^i \,=\, x^i \,+\, \partial^i \xi \,+\, \hat{\xi}^i$$

where, according to the Helmholtz theorem, ξ is a scalar field and $\partial_i \hat{\xi}^i = 0$. We have now to calculate how the metric coefficients transform under such a coordinate transformation. We work this out in detail for the time-time component:

$$g\Big(\frac{\partial}{\partial_T}, \frac{\partial}{\partial_T}\Big) \,=\, \Big(\frac{\partial \tilde{T}}{\partial T}\Big)^2 g\Big(\frac{\partial}{\partial_{\tilde{T}}}, \frac{\partial}{\partial_{\tilde{T}}}\Big) \,+\, 2\,\frac{\partial \tilde{T}}{\partial T}\,\frac{\partial \tilde{x}^i}{\partial T}\, g\Big(\frac{\partial}{\partial_{\tilde{T}}}, \frac{\partial}{\partial_{\tilde{x}^i}}\Big) \,+\, \frac{\partial \tilde{x}^i}{\partial T}\,\frac{\partial \tilde{x}^j}{\partial T}\, g\Big(\frac{\partial}{\partial_{\tilde{x}^i}}, \frac{\partial}{\partial_{\tilde{x}^j}}\Big)\,.$$

The second and the third term on the right-hand side are of second order and thus negligible. Hence

$$-\,c^2\,a(T)^2\,\big(\,1 \,+\, 2\,A\,\big) \,=\, -\,\Big(\,1 \,+\, \frac{\partial \tau}{\partial T}\Big)^2\, c^2\,a(T+\tau)^2\,\big(\,1 \,+\, 2\,\tilde{A}\,\big)\,,$$

$$\begin{aligned} a(T)^2\,\big(1 + 2\,A\big) \,&=\, \Big(\,1 \,+\, 2\,\frac{\partial \tau}{\partial T} \,+\, \ldots\,\Big)\Big(\,a(T) \,+\, \frac{da(T)}{dT}\,\tau \,+\, \ldots\,\Big)^2\,\big(\,1 \,+\, 2\,\tilde{A}\,\big) \\ &=\, \Big(\,1 \,+\, 2\,\frac{\partial \tau}{\partial T} \,+\, \ldots\,\Big)\,a(T)^2\,\Big(\,1 \,+\, \frac{2}{a(T)}\,\frac{da(T)}{dT}\,\tau \,+\, \ldots\,\Big)\,\big(\,1 \,+\, 2\,\tilde{A}\,\big) \\ &=\, a(T)^2\,\Big(\,1 \,+\, 2\,\frac{\partial \tau}{\partial T} \,+\, \frac{2}{a(T)}\,\frac{da(T)}{dT}\,\tau \,+\, 2\,\tilde{A} \,+\, \ldots\,\Big)\,, \end{aligned}$$

hence

$$1 + 2\,A = 1 + 2\,\frac{\partial\tau}{\partial T} + \frac{2}{a(T)}\,\frac{da(T)}{dT}\,\tau + 2\,\tilde{A} + \dots \Big)\,,$$

$$A = \frac{\partial\tau}{\partial T} + \mathcal{H}\,\tau + \tilde{A}\,,$$

where

$$\mathcal{H}(T) = \frac{1}{a(T)}\,\frac{da(T)}{dT}$$

is the Hubble "constant" with respect to conformal time.

Similarly, the transformation of all the other metric coefficients can be calculated. We find

$$\tilde{A} = A - \frac{\partial\tau}{\partial T} - \mathcal{H}\,\tau\,,$$

$$\tilde{B} = B + \tau - \frac{\partial\xi}{\partial T}\,,$$

$$\tilde{C} = C - \mathcal{H}\,\tau - \frac{1}{3}\,\Delta\xi\,,$$

$$\tilde{E} = E - \xi\,,$$

$$\tilde{\hat{B}}_i = \hat{B}_i - \frac{d\hat{\xi}_i}{dT}\,,$$

$$\tilde{\hat{E}}_i = \hat{E}_i - \xi_i\,,$$

$$\tilde{\hat{E}}_{ij} = \hat{E}_{ij}\,.$$

We see that by choosing the coordinate transformation (i.e., the scalar functions τ and ξ and the vector field $\hat{\xi}^i$) appropriately, we can achieve that

$$B = 0\,, \quad E = 0\,, \quad \hat{E}_i = 0\,,$$

or, alternatively,

$$B = 0\,, \quad E = 0\,, \quad \hat{B}_i = 0\,.$$

The latter choice has the advantage that the hypersurfaces $T =$ constant are then perpendicular to the T-lines, even in the perturbed spacetime. This is known as the *synchroneous gauge*. Mixed spatial-temporal components (i.e., components g_{Ti} in our setting) are usually called *gravitomagnetic terms*. This refers to an analogy to electromagnetism: Whereas rotating charges produce magnetic fields, rotating masses produce g_{Ti} terms. Our observation that B and $\hat{B}_i$ can be transformed to zero by a coordinate transformation implies that, within cosmological perturbation theory, gravitomagnetic terms are pure gauge terms.

Out of the perturbation variables $A, B, C, E, \hat{E}_i, \hat{B}_i$ and $\hat{E}_{ij}$ we can form the following *gauge-invariant* variables which were introduced by J. Bardeen in 1980:

$$\Psi \,=\, A \,-\, \mathcal{H}\Big(\, B \,-\, \frac{\partial E}{\partial T}\,\Big) \,+\, \frac{\partial}{\partial T}\Big(\, B \,-\, \frac{\partial E}{\partial T}\,\Big)\,,$$

$$\Phi \,=\, -\,C \,-\, \mathcal{H}\Big(\, B \,-\, \frac{\partial E}{\partial T}\,\Big) \,+\, \frac{1}{3}\,\Delta E\,,$$

$$\hat{\Phi}_i \,=\, \frac{\partial \hat{E}_i}{\partial T} \,-\, \hat{B}_i\,,$$

$$\hat{E}_{ij}\,.$$

It is easy to verify that these quantities are indeed unchanged, to within linear approximation, under a coordinate transformation. We may work in coordinates where $B = 0$, $E = 0$ and $\hat{E}_i = 0$ and express the metric perturbations in terms of the Bardeen variables $\Psi = A$, $\Phi = -C$, $\hat{\Phi}_i = -\hat{B}_i$ and $\hat{E}_{ij}$, i.e.

$$g_{\mu\nu}dx^\mu dx^\nu \,=\, a^2\Big(\,-\,\big(1+2\Psi\big)\,c^2dT^2 \,-\, 2\,\hat{\Phi}_i\,dx^i\,dT \,+\, \big((1-2\Phi)\delta_{ij} + 2\,\hat{E}_{ij}\big)dx^i\,dx^j\Big)\,.$$

In this way we work in a particular coordinate system, but the perturbation variables have a gauge-invariant meaning. Keep in mind that a depends only on T whereas the perturbation variables depend on all four coordinates T, x^1, x^2 and x^3.

Note that in the linearised formalism scalar, vector and tensor perturbations may be considered separately. A fairly large part of perturbation theory restricts to scalar perturbations, i.e., to the case that only the two scalar Bardeen potentials Ψ and Φ are non-zero. In this restricted formalism we cannot, of course, describe gravitational waves, because this requires tensor perturbations $\hat{E}_{ij} \neq 0$, but we may describe e.g. density perturbations.

We will now work out the linearised field equation for the case of scalar perturbations,

$$g_{\mu\nu}dx^\mu dx^\nu \,=\, a^2\Big(\,-\,\big(1+2\Psi\big)\,c^2dT^2 \,+\, (1-2\Phi)\delta_{ij}\,dx^i\,dx^j\Big)\,.$$

With the help of Mathematica (or some other computer programme) we calculate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ to within linear order with respect to Ψ, Φ and its derivatives. We find

$$G_{TT} \,=\, 3\,\mathcal{H}^2 \,+\, 2\,\Delta\Phi \,-\, 6\,\mathcal{H}\,\frac{\partial\Phi}{\partial T}\,,$$

$$G_{Ti} \,=\, 2\,\partial_i\Big(\,\frac{\partial\Phi}{\partial T} \,+\, \mathcal{H}\,\Psi\,\Big)\,,$$

$$G_{ij} \,=\, \Big(-\,2\,\frac{d\mathcal{H}}{dT} \,-\, \mathcal{H}^2 \,+\, \Delta\big(\Psi-\Phi\big) \,+\, 2\frac{\partial^2\Phi}{\partial T^2} \,+\, 2\,\Big(\,2\,\frac{d\mathcal{H}}{dT} \,+\, \mathcal{H}^2\,\Big)\,\big(\Phi+\Psi\big)$$

$$+\,2\,\mathcal{H}\,\frac{\partial\Psi}{\partial T} \,+\, 4\,\mathcal{H}\,\frac{\partial\Phi}{\partial T}\Big)\,\delta_{ij} \,+\, \partial_i\,\partial_j\big(\Phi-\Psi\big)\,.$$

We are interested in perfect-fluid solutions, so we have to find out on what conditions the field equation holds with an energy-momentum tensor of the form

$$T_{\rho\sigma} \,=\, \Big(\,\mu \,+\, \frac{p}{c^2}\,\Big)\,U_\rho\,U_\sigma \,+\, p\,g_{\rho\sigma}\,.$$

According to the rules of linear perturbation theory, we assume that

$$U_\rho \,=\, \overline{U}_\rho \,+\, \delta U_\rho \,=\, N\,\delta^T_\rho \,+\, \delta U_\rho\,,$$

$$\mu \,=\, \overline{\mu} \,+\, \delta\mu\,,$$

$$p \,=\, \overline{p} \,+\, \delta p\,,$$

where the overlined quantities refer to the perfect fluid associated with the unperturbed Robertson-Walker spacetime. The normalisation condition of the four-velocity requires

$$-\,c^2 \,=\, g^{\rho\sigma}U_\rho U_\sigma \,=\, g^{TT}\big(\,U_T\,\big)^2 \,+\, g^{ij}U_iU_j \,=\, \frac{1}{-\,c^2\,a^2\,(1\,+\,2\Psi)}\,\big(\,N^2 \,+\, 2\,N\,\delta U_T\big) \,+\, \ldots$$

$$=\, \frac{\big(1\,-\,2\Psi\,+\,\ldots\big)}{-\,c^2\,a^2}\,\big(\,N^2 \,+\, 2\,N\,\delta U_T\big) \,+\, \ldots \,=\, -\,\frac{N^2}{c^2\,a^2} \,+\, \frac{2\,\Psi\,N^2}{c^2\,a^2} \,-\, \frac{2\,N\,\delta U_T}{c^2\,a^2}\,.$$

Comparing zeroth order terms and first order terms yields

$$N \,=\, c^2\,a\,, \quad \delta U_T \,=\, c^2\,a\,\Psi\,,$$

i.e.,

$$U_\rho \,=\, c^2\,a\,\delta^T_\rho\big(1\,+\,\Psi\big) \,+\, \delta U_i\,\delta^\rho_i\,.$$

We linearise the energy-momentum tensor with respect to the perturbations,

$$T_{\rho\sigma} \,=\, \overline{T}_{\rho\sigma} \,+\, \delta T_{\rho\sigma} \,=\, \Big(\,\overline{\mu} \,+\, \frac{\overline{p}}{c^2}\,\Big)\,\overline{U}_\rho\,\overline{U}_\sigma \,+\, \overline{p}\,\overline{g}_{\rho\sigma}$$

$$+\,\Big(\,\delta\mu \,+\, \frac{\delta p}{c^2}\,\Big)\,\overline{U}_\rho\,\overline{U}_\sigma \,+\, \delta p\,\overline{g}_{\rho\sigma} \,+\, \Big(\,\overline{\mu} \,+\, \frac{\overline{p}}{c^2}\,\Big)\,\Big(\,\overline{U}_\rho\,\delta U_\sigma \,+\, \overline{U}_\sigma\,\delta U_\rho\,\Big) \,+\, \overline{p}\,h_{\rho\sigma}\,.$$

Decomposition into temporal and spatial components yields

$$T_{TT} \,=\, \big(\,\overline{\mu} \,+\, \delta\mu \,+\, 2\,\Psi\,\overline{\mu}\,\big)\,c^4\,a^2\,,$$

$$T_{Ti} \,=\, \Big(\,\overline{\mu} \,+\, \frac{\overline{p}}{c^2}\,\Big)\,c^2\,a\,\delta U_i\,,$$

$$T_{ij} \,=\, \big(\overline{p} \,+\, \delta p \,-\, 2\,\overline{p}\,\Phi\,\big)\,a^2\,\delta_{ij}\,.$$

We write Einstein's field equation,

$$G_{\mu\nu} \,+\, \Lambda\, g_{\mu\nu} \;=\; \kappa\, T_{\mu\nu}$$

and consider, in particular, the $ij-$components,

$$G_{ij} \,+\, \Lambda\, a^2\,\big(1-2\Phi\big)\,\delta_{ij} \;=\; \kappa\, T_{ij}\,.$$

With G_{ij} and T_{ij} inserted from above, we see that we get an equation of the form

$$\alpha\,\delta_{ij} \,+\, \partial_i\,\partial_j\big(\Phi-\Psi\big) \;=\; \beta\,\delta_{ij}$$

with some scalar functions α and β, hence

$$\partial_i\,\partial_j\big(\Phi-\Psi\big) \;=\; 0 \quad \text{for } i\neq j\,.$$

Solving this differential equation for all index combinations $i,j=1,2,3$ demonstrates that $\Phi-\Psi$ must be of the form

$$\big(\Phi \,-\, \Psi\big)(T,x^1,x^2,x^3) \;=\; f_1(T,x^1) \,+\, f_2(T,x^2) \,+\, f_3(T,x^3)\,.$$

If we require the perturbations to be square-integrable over $\mathbb{R}^3$, to allow for spatial Fourier expansion, this implies that

$$\Phi \,-\, \Psi \;=\; 0\,,$$

so we have only one Bardeen potential $\Phi=\Psi$ for perfect-fluid solutions.

We want to consider Einstein's field equation with $\Lambda=0$ for the special case that the unperturbed and the perturbed spacetime is a dust solution, i.e., $\overline{p}=0$ and $\delta p=0$. Then the background spacetime, being a solution to the Friedmann equation for a dust with $k=0$ and $\Lambda=0$, must be the Einstein-deSitter universe (recall p. 42),

$$a(T) \;=\; \frac{c^2\,a_0^2}{4}\,T^2\,,$$

$$\mathcal{H}(T) \;=\; \frac{1}{a(T)}\,\frac{da(T)}{dT} \;=\; \frac{2}{T}\,,$$

$$\frac{d\mathcal{H}}{dT} \;=\; -\,\frac{2}{T^2}\,,$$

By comparing first-order terms on both sides, we find for the TT, Ti and ij components of Einstein's field equation

$$2\,\Delta\Phi \,-\, \frac{12}{T}\,\frac{\partial\Phi}{\partial T} \;=\; \kappa\, c^4\, a^2\,\big(\,\delta\mu \,+\, 2\,\overline{\mu}\,\Phi\,\big)\,,$$

$$2\,\partial_i\Big(\,\frac{\partial\Phi}{\partial T} \,+\, \frac{2\Phi}{T}\,\Big) \;=\; \kappa\, c^2\, a\,\overline{\mu}\,\delta U_i\,,$$

$$2\,\frac{\partial^2\Phi}{\partial T^2} \,+\, \frac{12}{T}\,\frac{\partial\Phi}{\partial T} \;=\; 0\,,$$

respectively.

The last equation can be integrated: With

$$u \;=\; \frac{\partial \Phi}{\partial T}$$

we have to solve

$$\frac{\partial u}{\partial T} \;=\; -\,\frac{6}{T}\,u\,.$$

As this equation involves only differentiation with respect to T, we can solve it by separation of variables, keeping parametric dependence on x^1, x^2 and x^3 in mind.

$$\frac{du}{u} \;=\; -\,\frac{6\,dT}{T}\,,$$

$$\ln u \;=\; -\,6\ln T \;+\; \text{constant}$$

where the integration "constant" depends on $\vec{x} = (x^1, x^2, x^3)$, hence

$$u\big(T\,,\,\vec{x}\,\big) \;=\; \frac{-\,5\,C_1\big(\,\vec{x}\,\big)}{T^6}\,,$$

$$\Phi\big(T\,,\,\vec{x}\,\big) \;=\; \frac{C_1\big(\,\vec{x}\,\big)}{T^5} \;+\; C_2\big(\,\vec{x}\,\big)\,.$$

The first term falls off very strongly in the course of time. If we wait sufficiently long, the perturbation is given, to within a good approximation, by the second term which is time-independent, i.e., the perturbation is "frozen". We summarise these observations in the following way: In a dust universe, scalar perturbations become time-independent for late times. This was considered to be crucial at a time when people believed that we live in a dust universe without a cosmological constant. Now we believe that there is a cosmological constant that will become dominating for late times. Then the statement that scalar perturbations become "frozen" is no longer true.

As an application, we calculate the influence of scalar perturbations on the redshift formula and, thereby, on the cosmic background radiation. As a prerequisite, we need the redshift formula in an arbitrary general-relativistic spacetime.

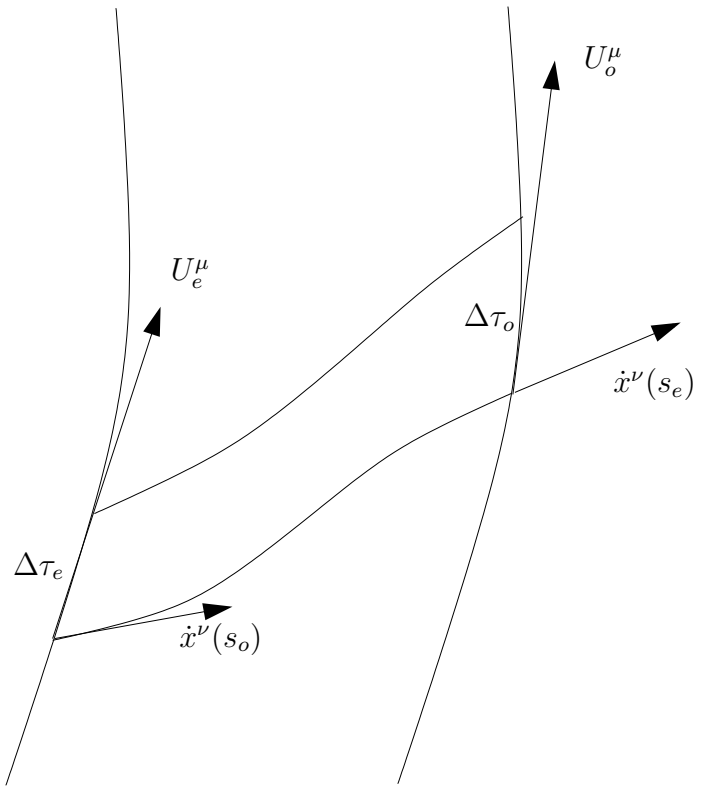
Consider an emitter whose worldline is parametrised by proper time τ_e and an observer whose worldline is parametrised by proper time τ_o, see the diagram on the next page. Denote the four-velocities (i.e., the tangent vector fields to the worldlines) by U_e and U_o, respectively. If two light rays are emitted at times τ_e and $\tau_e + \Delta\tau_e$, they are received at time τ_o and $\tau_o + \Delta\tau_o$, where the frequency ratio

$$1 \;+\; z \;=\; \lim_{\Delta\tau_e \to 0} \frac{\Delta\tau_o}{\Delta\tau_e} \;=\; \frac{d\tau_o}{d\tau_e}\,.$$

The general redshift formula says that

$$1 \;+\; z \;=\; \frac{g_{\mu\nu}\,\dot{x}^\nu(s_e)\,U_e^\mu}{g_{\rho\sigma}\,\dot{x}^\sigma(s_o)\,U_o^\rho}\,.$$

Here $x^\mu(s)$ is the light ray that starts at parameter value $s = s_e$ at the emitter and arrives at the parameter value $s = s_o$ at the observer.



Proof of the general redshift formula:

The following proof is borrowed from D. Brill. It can be found, e.g., in N. Straumann's book [N. Straumann: "General Relativity and Relativistic Astrophysics" Springer (1984)]. We consider the two-surface (possibly with self-intersections) spanned by the light rays from the emitter to the receiver. This two-surface can be labelled by two parameters, s and τ. We choose s as the affine parameter along each light ray, and τ in such a way that it coincides with proper time τ_e on the emitter worldline. Because of the redshift, τ will then *not* coincide with proper time τ_o on the observer wordline. We calculate

$$\partial_s\Big(g\big(\partial_s,\partial_\tau\big)\Big) \;=\; g\Big(\nabla_{\partial_s}\partial_s,\partial_\tau\Big) \;+\; g\Big(\partial_s,\nabla_{\partial_s}\partial_\tau\Big)\,.$$

The first term vanishes, because the light rays are geodesics

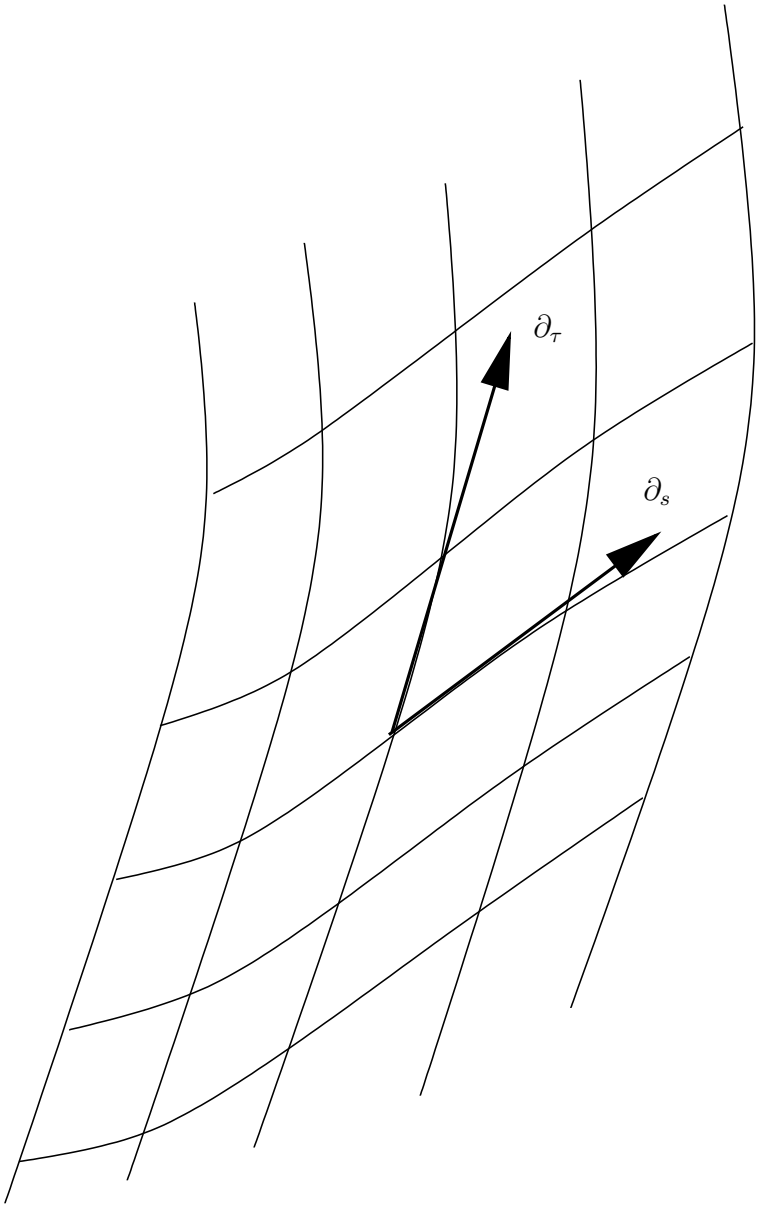
$$\nabla_{\partial_s}\partial_s = 0\,.$$

The second term can be rewritten with the help of the fact that the Levi-Civita connection ∇ is torsion-free, $\nabla_{\partial_s}\partial_\tau = \nabla_{\partial_\tau}\partial_s$. This results in

$$\partial_s\Big(g\big(\partial_s,\partial_\tau\big)\Big) \;=\; g\Big(\partial_s,\nabla_{\partial_\tau}\partial_s\Big) \;=\; \frac{1}{2}\,\partial_\tau\Big(g\big(\partial_s,\partial_s\big)\Big) \;=\; 0$$

because the light rays are lightlike, $g(\partial_s,\partial_s)=0$. We have thus found that

$$g\big(\partial_s,\partial_\tau\big)\big|_{s_e} \;=\; g\big(\partial_s,\partial_\tau\big)\big|_{s_o}\,.$$



Switching to coordinate notation, this equation reads

$$g_{\mu\nu}\,\dot{x}^\mu(s_e)\,U_e^\nu \;=\; g_{\rho\sigma}\,\dot{x}^\rho(s_o)\,U_o^\sigma\,\frac{d\tau_o}{d\tau_e}$$

which gives the redshift formula.

We want to evaluate this formula for a Robertson-Walker spacetime with a scalar perturbation of the form

$$g_{\mu\nu}dx^\mu dx^\nu \;=\; a^2\,\Big(\;-\;(1+2\Phi)\,c^2\,dT^2\;+\;(1-2\Phi)\,\delta_{ij}\,dx^i\,dx^j\Big)\,.$$

The wordlines of observer and emitter are supposed to be integral curves of the four-velocity vector field

$$U^\rho \;=\; g^{\rho\mu}\,U_\mu$$

where

$$U_\mu \,=\, c^2\, a\, \delta^T_\mu \big(1+\Phi\big) \,+\, \delta U_i\, \delta^\mu_i\,,$$

see above. Hence

$$U^\rho \,=\, g^{\rho\mu}\, \Big(c^2\, a\, \delta^T_\mu \big(1+\Phi\big) \,+\, \delta U_i\, \delta^\mu_i \Big)$$

$$=\, \delta^\rho_T\, g^{TT}\, c^2\, a\, \big(1+\Phi\big) \,+\, \delta^\rho_j\, g^{ij}\, \delta U_i$$

$$=\, -\,\delta^\rho_T\, \frac{c^2\, a}{c^2\, a^2\, \big(1+2\Phi\big)}\, \big(1+\Phi\big) \,+\, \delta^\rho_j\, \frac{\delta^{ij}}{a^2}\, \delta U_i$$

$$=\, -\,\delta^\rho_T\, \frac{1}{a}\, \big(1-\Phi\big) \,+\, \delta^\rho_j\, \frac{\delta^{ij}}{a^2}\, \delta U_i$$

The lightlike geodesic we have to consider may be written in the form

$$T(s) \,=\, \overline{T}(s) \,+\, \delta T(s)\,, \quad x^i(s) \,=\, \overline{x}^i(s) \,+\, \delta x^i(s)$$

where the overlined quantities are the coordinates of a lightlike geodesic in the unperturbed background spacetime and s is an affine parameter. The Lagrangian for the geodesics is

$$\mathcal{L}\big(\, x\,,\, \dot{x}\,\big) \,=\, \frac{1}{2}\, g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu$$

$$=\, \frac{a^2}{2}\, \Big(\, -\, \big(1+2\Phi\big)\, c^2\, \dot{T}^2 \,+\, \big(1-2\Phi\big)\, \delta_{ij}\, \dot{x}^i\, \dot{x}^j \,\Big)\,.$$

For lightlike geodesics we must have $\mathcal{L}=0$, hence

$$\delta_{ij}\, \dot{x}^i\, \dot{x}^j \,=\, \frac{1+2\Phi}{1-2\Phi}\, c^2\, \dot{T}^2 \,=\, \big(1+4\Phi\big)\, c^2\, \dot{T}^2 \,+\, \ldots$$

The Euler-Lagrange equation

$$\frac{d}{ds}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}^\rho}\Big) \,=\, \frac{\partial \mathcal{L}}{\partial x^\rho}$$

reads

$$\frac{d}{ds}\Big(g_{\rho\sigma}\dot{x}^\sigma\Big) \,=\, 0 \,-\, \frac{\partial \Phi}{dx^\rho}\, a^2\, \Big(c^2\, \dot{T}^2 \,+\, \delta_{ij}\, \dot{x}^i\, \dot{x}^i\Big) \,=\, -\,2\, \frac{\partial \Phi}{\partial x^\rho}\, a^2\, c^2\, \dot{\overline{T}}^2$$

where we have used that $\mathcal{L}=0$. To zeroth order, this gives the equation for lightlike geodesics in the background spacetime: For $x^\rho = T$ and $x^\rho = k$ we find

$$\frac{d}{ds}\Big(\, a^2\, c^2\, \dot{\overline{T}}\,\Big) \,=\, 0$$

and

$$\frac{d}{ds}\Big(\, a^2\, \delta_{kj}\, \dot{\overline{x}}^j\,\Big) \,=\, 0\,,$$

respectively.

For determining the redshift we calculate how $g_{\mu\nu}\, \dot{x}^\mu\, U^\nu$ changes along the light ray:

$$\frac{d}{ds}\Big(\, g_{\rho\sigma}\, \dot{x}^\sigma\, U^\rho\,\Big) \,=\, \frac{d}{ds}\Big(\, g_{\rho\sigma}\, \dot{x}^\sigma\,\Big)\, U^\rho \,+\, g_{\rho\sigma}\, \dot{x}^\sigma\, \frac{dU^\rho}{ds}\,.$$

We find:

$$\frac{d}{ds}\Big(a g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big) = \left(\frac{d}{ds}g_{\rho\sigma}\dot{x}^\rho\right) a\,U^\sigma + g_{\rho\sigma}\dot{x}^\rho\frac{d}{ds}\Big(a\,U^\sigma\Big)$$

$$= 2\,\frac{\partial\Phi}{dx^\sigma}\,a^2\,c^2\,\dot{\overline{T}}^2\,\delta^\sigma_T + g_{\rho\sigma}\dot{x}^\rho\,\frac{d}{ds}\left(-\big(1-\Phi\big)\,\delta^\sigma_T\,+\,\frac{1}{a}\,\delta U^i\delta^\sigma_i\right)$$

$$= 2\,\frac{\partial\Phi}{\partial T}\,a^2\,c^2\,\dot{\overline{T}}^2\,+\,\overline{g}_{TT}\,\dot{\overline{T}}\,\frac{d\Phi}{ds}\,+\,\overline{g}_{ij}\dot{\overline{x}}^j\,\frac{d}{ds}\Big(\frac{1}{a}\delta U^i\Big)$$

$$= \cancel{2}\,\frac{\partial\Phi}{\partial T}\,a^2\,c^2\,\dot{\overline{T}}^2\,-\,c^2\,a^2\,\dot{\overline{T}}\left(\frac{\partial\Phi}{\cancel{\partial T}}\cancel{\dot{\overline{T}}}\,+\,\frac{\partial\Phi}{\partial x^i}\,\dot{\overline{x}}^i\right)\,+\,a^2\delta_{ij}\dot{\overline{x}}^j\,\frac{d}{ds}\Big(\frac{1}{a}\delta U^i\Big)$$

$$= a^2\,c^2\,\dot{\overline{T}}\underbrace{\left\{\frac{\partial\Phi}{\partial T}\,\dot{\overline{T}}\,-\,\frac{\partial\Phi}{\partial x^i}\,\dot{\overline{x}}^i\,+\,k_j\,\frac{d}{ds}\Big(\frac{1}{a}\delta U^i\Big)\right\}}_{=:Q(s)}.$$

Integration from an observation event (index o) to an emission event (index e) yields

$$a\,g_{\rho\sigma}\,\dot{x}^\rho\,U^\sigma\Big|_e\;=\;a\,g_{\rho\sigma}\,\dot{x}^\rho\,U^\sigma\Big|_o\;+\;a^2\,c^2\,\dot{\overline{T}}\Big|_o\int_{s_o}^{s_e}Q(s)\,ds$$

where we have used our result that $a^2\,c^2\,\dot{\overline{T}}$ is constant. Hence

$$\frac{a(T_e)g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big|_e}{a(T_o)g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big|_o}\;=\;1\;+\;\frac{a^2\,c^2\,\dot{\overline{T}}\Big|_o}{a(T_o)g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big|_o}\int_{s_o}^{s_e}Q(s)\,ds\,.$$

As the integral on the right-hand side is of first order, we may truncate the factor in front of it after the zeroth order,

$$\frac{g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big|_e}{g_{\rho\sigma}\dot{x}^\rho U^\sigma\Big|_o}\;=\;\frac{a(T_o)}{a(T_e)}\left(1\;+\;\frac{a^2\,c^2\,\dot{\overline{T}}\Big|_o}{a(T_o)\overline{g}_{\rho\sigma}\dot{\overline{x}}^\rho\overline{U}^\sigma\Big|_o}\int_{s_o}^{s_e}Q(s)\,ds\right).$$

Now we insert on the left-hand side the general redshift formula from p. 92 and on the right-hand side we use that $\overline{U}^\sigma=\delta^\sigma_T$, hence $\overline{g}_{\rho\sigma}\dot{\overline{x}}^\rho\overline{U}^\sigma=-a^2c^2\dot{\overline{T}}$,

$$1\;+\;z\;=\;\frac{a(T_o)}{a(T_e)}\left(1\;-\;\int_{s_o}^{s_e}Q(s)\,ds\right).$$

To zeroth order we recover, of course, the familiar redshift law in an unperturbed Robertson-Walker universe. We see that the first-order correction is given by an integral over the unperturbed light ray $\big(\overline{T}(s),\overline{x}^i(s)\big)$. This first-order correction is known as the *integrated Sachs-Wolfe effect*. We have restricted our calculation here to the case of scalar perturbations with Bardeen potential $\Phi=\Psi$. Sachs and Wolfe calculated this effect in 1967 for arbitrary (i.e. scalar, vector and tensor) perturbations; they didn't use the Bardeen variables (which didn't exist at this time) and rather worked in a gauge where $\delta U^i=0$.

The integrated Sachs-Wolfe effect is the major influence of a perturbation on anisotropies in the cosmic background radiation for large ℓ, i.e., on large angular scales. There is also an effect (sometimes called the "non-integrated" Sachs-Wolfe effect) resulting from the fact that, because of the perturbation, the temperature of the cosmic background radiation is non-uniform already when it comes into existence at the hypersurface of last scattering.

6. Bianchi models

If we want to go beyond the assumptions of homogeneity and isotropy which are inherent to the Robertson-Walker models, we have two possibilities. The first is to use perturbation theory, which in general yields models without any symmetries, but it has the disadvantage that linearising with respect to the perturbations destroys essential features of Einstein's field equation which is non-linear. The other is to work with exact solutions that have less symmetries than the Robertson-Walker models. In particular, models that are homogeneous but not isotropic have been extensively studied.

For studying homogeneous cosmological models we need the notion of Killing vector fields. Recall that a vector field $K^\mu\partial_\mu$ is called a *Killing vector field* if it satisfies the *Killing equation*

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\,.$$

(In coordinate-free notation the Killing equation can be rewritten as $L_K g = 0$ where $L_K g$ is the *Lie derivative* of the metric with respect to the Killing vector field K.) Killing vector fields describe symmetries of the spacetime: In Worksheet 2 we have shown that, near every point where a Killing field $K^\mu\partial_\mu$ is non-zero, we may find a coordinate system such that $K^\mu = \delta^\mu_1$ and the $g_{\rho\sigma}$ are independent of x^1.

It is obvious that a linear combination $c_1K_1 + c_2K_2$ of two Killing vector fields with *constant* coefficients is again a Killing vector field, and it is not difficult to verify that the Lie bracket $[K_1, K_2]$ of two Killing vector fields is again a Killing vector field. (The Lie bracket of two vector fields is their commutator, where we have to view the vector fields as derivative operators acting on a scalar function, $[K_1, K_2]f = K_1K_2f - K_2K_1f$.) The set of all Killing vector fields on a pseudo-Riemannian manifold is, thus, a Lie algebra. On an n-dimensional manifold, the maximal dimension of this Lie algebra is $n(n+1)/2$.

By definition, a spacetime (i.e., a 4-dimensional Lorentzian manifold) is spatially homogeneous if it admits an algebra of Killing vector fields with 3-dimensional spacelike orbits. (The orbit of a point is the union of all integral curves of Killing vector fields through this point.) The dimension of this Lie algebra cannot be smaller than 3 and it cannot be bigger than $3(3+1)/2 = 6$. We will consider the case that the dimension is equal to 3. The resulting spacetime models are known as *Bianchi models*. The name refers to the fact that L. Bianchi had classified in the 1890s all 3-dimensional Lie algebras. If the dimension is 4, the spacetime is called *Locally Rotationally Symmetric* (LRS). The case that the dimension is 5 is impossible, and if it is 6 we have a Robertson-Walker model; the latter are special cases of Bianchi models because their 6-dimensional Lie algebra always admits a 3-dimensional subalgebra of Killing vector fields that generate the 3-dimensional orbits.

We briefly review Bianchi's classification of 3-dimensional Lie algebras. Given a 3-dimensional Lie algebra, we may choose a basis (K_1, K_2, K_3). The Lie bracket of two basis vectors must then be a linear combination of the basis vectors,

$$[K_i, K_j] = C_{ij}^{\ell} K_\ell \,,$$

where the socalled *structure constants* C_{ij}^{ℓ} are real numbers. (As always, we use the summation convention for latin indices $i, j, \ldots = 1, 2, 3$). As the commutator of two operators is antisymmetric,

$$[K_i, K_j] = -[K_j, K_i] \,,$$

and satisfies the *Jacobi identity*

$$[\,[K_i, K_j], K_\ell\,] + [\,[K_j, K_\ell], K_i\,] + [\,[K_\ell, K_i], K_j\,] = 0 \,,$$

the structure constants must satisfy

$$C_{ij} = -C_{ji}$$

and

$$\varepsilon^{ijk} C_{ij}^{m} C_{km}^{\ell} = 0$$

where ε^{ijk} is a totally antisymmetric non-zero tensor. We may fix ε^{ijk} by requiring that in the chosen basis $\varepsilon^{123} = 1$. Then any other totally antisymmetric non-zero tensor is given by multiplying ε^{ijk} with a non-zero factor. The antisymmetry of the structure constants with repect to the lower indices implies that the same information as in the C_{ij}^{ℓ} is in the second-rank tensor

$$t^{ij} = \varepsilon^{imn} C_{mn}^{j} \,.$$

We decompose t^{ij} into symmetric and antisymmetric parts,

$$t^{ij} = n^{ij} + \varepsilon^{ijk} a_k \,, \quad n^{ij} = n^{ji} \,.$$

With a bit of algebra one verifies that then the Jacobi identity is satisfied if and only if

$$n^{ij} a_j = 0 \,.$$

One says that a 3-dimensional Lie algebra is of

- Bianchi Class A if $(a_1, a_2, a_3) = (0, 0, 0)$,
- Bianchi Class B if $(a_1, a_2, a_3) \neq (0, 0, 0)$.

A change of the basis,

$$\tilde{K}_i = L_i{}^j K_j \,,$$

preserves the condition $\varepsilon^{123} = 1$ if $\det\big(L_i{}^j\big) = 1$. Under such a transformation,

$$\tilde{n}^{ij} = \Big(L^{-1}\Big)^{i}{}_{k} \Big(L^{-1}\Big)^{j}{}_{\ell}\, n^{k\ell} \,, \quad \tilde{a}_i = L_i{}^j a_j \,. \quad \text{(T1)}$$

We may also change to another totally antisymmetyric tensor, $\hat{\varepsilon}^{ijk} = \lambda\, \varepsilon^{ijk}$. Then

$$\hat{n}^{ij} = \lambda\, n^{ij} \,, \quad \hat{a}_j = a_j \,. \quad \text{(T2)}$$

In the case of Bianchi Class A, a transformation (T1) with an orthogonal matrix $\left(L_i{}^j\right)$ may be used to diagonalise the matrix $\left(n^{ij}\right)$,

$$(n^{ij}) = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix} .$$

This may be followed by a transformation (T1) with a diagonal matrix $\left(L_i{}^j\right)$ to set all non-zero diagonal elements of $\left(n^{ij}\right)$ equal in magnitude. Finally, a transformation (T2) may be used to set all the non-zero diagonal elements equal to 1 or -1. We may choose the sign of λ such that the number of negative diagonal elements is smaller than or equal to the number of positive ones. We are then left with the following Bianchi types within Class A:

n_1	n_2	n_3	Bianchi type
0	0	0	I
1	0	0	II
0	1	-1	VI_0
0	1	1	VII_0
1	1	-1	VIII
1	1	1	IX

For Lie algebras of Bianchi Class B we may choose a transformation (T1) with $\left(L_i{}^j\right)$ orthogonal such that we achieve the form

$$(n^{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix} , \qquad (a_i) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} .$$

Here we have used that n^{ij} is symmetric and that $n^{ij}a_j = 0$. This form is preserved under transformations (T1) with

$$\left(L_i{}^j\right) = \begin{pmatrix} \dfrac{1}{b_2 b_3} & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} .$$

If $n_2 = n_3 = 0$, we can use such a transformation for setting a equal to 1. If $n_2 = 0$ and $n_3 \neq 0$, we do the same thing and simultaneously transform n_3 to 1 or -1. We may then use a transformation (T2) for setting n_3 equal to 1. The situation is more difficult if $n_2 n_3 \neq 0$. Then we see that the remaining transformations (T1) leave

$$h := \frac{a^2}{n_2 n_3}$$

invariant. We may choose such a tranformation (T1) for setting n_2 and n_3 equal to 1 or -1, and we may use a transformation (T2) for transforming the case that both are negative to the case that both are positive. Then, however, there is no further freedom for normalising a; the resulting Bianchi class will depend on the parameter h that may take all real values (non-zero for Bianchi Class B). This gives us the following Bianchi types for Class B.

a	n_1	n_2	n_3	Bianchi type	
1	0	0	0	V	
1	0	0	1	IV	
$\sqrt{-h}$	0	1	-1	VI_h	$(h<0)$
$\sqrt{h}$	0	1	1	VII_h	$(h>0)$

Bianchi type III is missing in the table because it is the same as VI_{-1}.

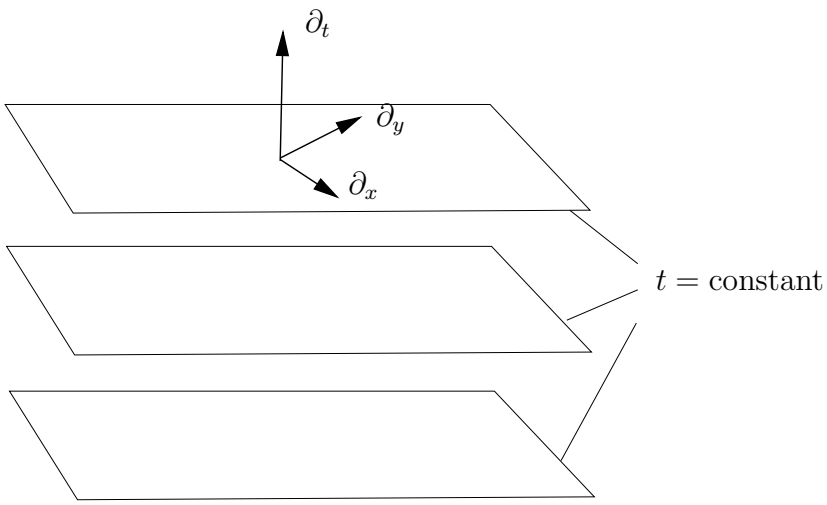
It is our goal to study Bianchi models of type I in some detail, first for vacuum and then for dust. Bianchi I is the simplest type; all the structure constants are zero, i.e., if we choose a basis (K_1, K_2, K_3) of Killing vector fields that generate the Bianchi symmetry, we have

$$[K_i, K_j] = 0\,.$$

Note that K_1, K_2 and K_3 must be linearly independent at each point because they are assumed to generate 3-dimensional spacelike hypersurfaces. Then the condition that the Lie bracket vanishes implies that we can choose, on each of these 3-dimensional hypersurfaces, coordinates (x, y, z) such that

$$K_1 = \partial_x\,,\ K_2 = \partial_y\,,\ K_3 = \partial_z\,.$$

As the fourth coordinate, we choose proper time t along the timelike curves perpendicular to the homogeneous slices. As ∂_x, ∂_y and ∂_z are Killing vector fields, the metric coefficients are functions of t only. For one time t, we may choose the spatial coordinate axes such that $g_{ij} = 0$ for $i \neq j$. We try to find solutions to the field equations, first for vacuum and then for dust, such that the metric remains diagonal for all times, i.e., we assume that the metric is of the form



$$g = -c^2\,dt^2 \,+\, X(t)^2\,dx^2 \,+\, Y(t)^2\,dy^2 \,+\, Z(t)^2\,dz^2\,.$$

In a Robertson-Walker universe we had one scale factor $a(t)$, now we have three scale factors $X(t)$, $Y(t)$, $Z(t)$. Correspondingly, there are three Hubble parameters which we denote

$$A(t) \,=\, \frac{X'(t)}{X(t)}\,,\quad B(t) \,=\, \frac{Y'(t)}{Y(t)}\,,\quad Z(t) \,=\, \frac{Z'(t)}{Z(t)}\,.$$

We also use the abbreviation

$$\theta(t) \,=\, A(t) \,+\, B(t) \,+\, C(t)\,.$$

For the Ricci tensor of our Bianchi I metric we find

$$R_{tt} \,=\, -\,\theta' \,-\, A^2 \,-\, B^2 \,-\, C^2\,,$$

$$R_{xx} \,=\, \frac{X^2}{c^2}\,\big(A' + \theta\,A\,\big)\,,$$

$$R_{yy} \,=\, \frac{Y^2}{c^2}\,\big(B' + \theta\,B\,\big)\,,$$

$$R_{zz} \,=\, \frac{Z^2}{c^2}\,\big(C' + \theta\,C\,\big)\,.$$

The off-diagonal elements vanish. The Ricci scalar reads

$$R \,=\, \frac{2}{c^2}\,\Big(\theta' \,+\, A^2 \,+\, B^2 \,+\, C^2 \,+\, A\,B \,+\, B\,C \,+\, C\,A\Big)\,.$$

We first determine the general solution to the vacuum field equation without a cosmological constant,

$$R_{\mu\nu} \,=\, 0\,.$$

Then we must have

$$0 \,=\, R_{tt} \,+\, \frac{c^2}{2}\,R \,=\, A\,B \,+\, B\,C \,+\, C\,A\,.$$

This implies that

$$\theta^2 \,=\, (A + B + C)^2 \,=\, A^2 \,+\, B^2 \,+\, C^2\,. \qquad \text{(C1)}$$

Inserting this result into the equation $R_{tt} = 0$ yields a differential equation for θ,

$$\theta' \,+\, \theta^2 \,=\, 0\,.$$

For solving this differential equation we have to distinguish two cases:

If $\theta = 0$, equation (C1) implies that $A = B = C = 0$, i.e., X, Y and Z are constants. This gives Minkowski spacetime.

If $\theta \neq 0$, the differential equation can be solved by separation of variables,

$$\frac{d\theta}{\theta^2} \,=\, -\,dt\,, \quad -\,\frac{1}{\theta} \,=\, -\,\big(\,t \,-\, t_i\,\big)\,.$$

As we are free to choose the origin of the t coordinate where we like, we choose the integration constant t_i equal to 0, i.e.,

$$\theta(t) \,=\, \frac{1}{t}\,. \qquad \text{(C2)}$$

With this result at hand, we can evaluate the equation $R_{xx} = 0$ which yields

$$A' \,+\, \frac{A}{t} \,=\, 0\,.$$

Again, this can be solved by separation of variables,

$$\frac{dA}{A} \,=\, -\,\frac{dt}{t}\,, \quad \ln A \,=\, -\ln t \,+\, \ln p$$

with an integration constant p, hence

$$A(t) \,=\, \frac{p}{t}\,.$$

As $A = X'/X$, this may be rewritten as

$$\frac{dX}{X} \,=\, p\,\frac{dt}{t}\,, \quad \ln X \,=\, p\ln t \,-\, p\ln t_0$$

with an integration constant t_0, hence

$$X(t) \,=\, \Big(\frac{t}{t_0}\Big)^p\,.$$

Similarly, evaluation of the equations $R_{yy} = 0$ and $R_{zz} = 0$ yields

$$Y(t) \,=\, \Big(\frac{t}{t_0}\Big)^q\,, \quad Z(t) \,=\, \Big(\frac{t}{t_0}\Big)^r\,,$$

with integration constants q and r. Note that we could choose the same integration constant t_0 for all three components because we are free to shift the origin of the spatial coordinate system.

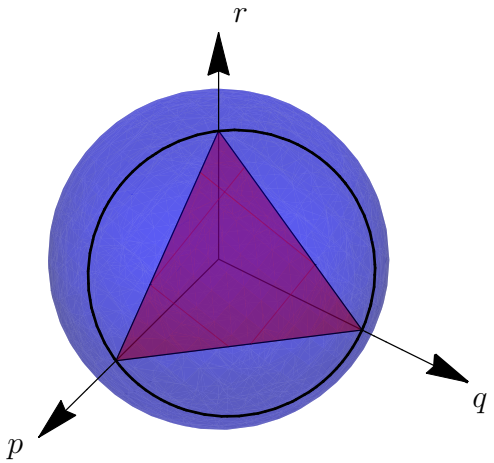
The constants p, q and r are not independent of each other: Condition (C2) requires

$$p \,+\, q \,+\, r \,=\, 1\,, \quad \text{(K1)}$$

and condition (C1) requires

$$p^2 \,+\, q^2 \,+\, r^2 \,=\, 1\,. \quad \text{(K2)}$$

In (p, q, r) space, (K1) determines a plane and (K2) determines a sphere, so the values of (p, q, r) are restricted to a circle, see the picture. The vacuum solution



$$g \,=\, -\,c^2\,dt^2 \,+\, \Big(\frac{t}{t_0}\Big)^{2p} dx^2 \,+\, \Big(\frac{t}{t_0}\Big)^{2q} dy^2 \,+\, \Big(\frac{t}{t_0}\Big)^{2r} dz^2$$

is known as the *Kasner solution*. It was found by the US American mathematician E. Kasner in 1921, without referring to the Bianchi classification. Depending on the signs of the coefficients p, q and r, the Kasner universe may be expanding in some spatial directions and contracting in others. The Kasner relations (K1) and (K2) restrict the possible cases. As the case $p = q = r = 0$ is in obvious contradiction with the Kasner relations, we have to distinguish the cases that one, two or all three Kasner coefficients are non-zero.

- $p \neq 0$, $q = r = 0$: Then the Kasner relations require $p = 1$. The metric reads

$$g \;=\; -\,c^2\,dt^2 \;+\; \Big(\frac{t}{t_0}\Big)^2\,dx^2 \;+\; dy^2 \;+\; dz^2\,.$$

  A coordinate transformation

$$\tilde{t} \;=\; t\cosh\frac{x}{c\,t_0}\,, \quad \tilde{x} \;=\; c\,t\sinh\frac{x}{c\,t_0}\,, \quad \tilde{y} \;=\; y\,, \quad \tilde{z} \;=\; z\,,$$

  reveals that this is Minkowski spacetime,

$$g \;=\; -\,c^2\,d\tilde{t}^2 \;+\; d\tilde{x}^2 \;+\; d\tilde{y}^2 \;+\; d\tilde{z}^2\,.$$

  The original coordinates are known as *Rindler coordinates*. They cover the "wedge" $\tilde{x} > c\,\big|\,\tilde{t}\,\big|$ and the t-lines are the worldlines of observers with constant acceleration ("Rindler observers"). An analogous result holds, of course, for $(p, q, r) = (0, 1, 0)$ and $(p, q, r) = (0, 0, 1)$.

- $p\,q \neq 0$, $r = 0$: Then the Kasner relations require $p + q = 1$ and $p^2 + q^2 = 1$. Squaring the first condition and subtracting the second yields $p\,q = 0$ which is in contradiction to the assumption. So this case is impossible.

- $p\,q\,r \neq 0$: Squaring the Kasner relation (K1) and subtracting (K2) yields

$$p\,q \;+\; q\,r \;+\; r\,p \;=\; 0\,.$$

  This equation demonstrates that the three coefficients cannot be all positive or all negative. Two of them must have the same sign, say $p\,q > 0$, and the third one must have the opposite sign, $p\,r < 0$ and $q\,r < 0$. Then

$$\begin{aligned} r^2 \;&=\; 1 \;-\; p^2 \;-\; q^2 \;=\; \big(1 \;-\; p \;-\; q\big)^2\,, \\ 1 \;-\; p^2 \;-\; q^2 \;&=\; 1 \;+\; p^2 \;+\; q^2 \;-\; 2\,p \;-\; 2\,q \;+\; 2\,p\,q\,, \\ 0 \;&=\; 2\,p^2 \;+\; 2\,q^2 \;-\; 2\,p \;-\; 2\,q \;+\; 2\,p\,q\,, \\ p \;+\; q \;&=\; p^2 \;+\; q^2 \;+\, p\,q\,. \end{aligned}$$

  As the right-hand side is positive, p and q must be positive and r must be negative,

$$p \;\geq\; q \;>\; 0 \;>\; r\,.$$

We summarise the features of the Kasner solutions in the following way: Each point of the Kasner circle corresponds to a particular Kasner solution. The intersections of the Kasner circle with the coordinate axes are special. (These are the corners of the triangle in the picture on the previous page.) There the Kasner metric is the Minkowski metric in Rindler coordinates. At all other points of the Kasner circle, two of the Kasner coefficients are positive and one is negative. This gives a universe with a singularity. If we consider the time interval $0 < t < \infty$, it is an initial singularity. This may be called a "big bang", but in contrast to the singularity in a Robertson-Walker universe the Kasner singularity is anisotropic. If it is approached backwards in time only two of the dimensions shrink to zero while the third one blows up. This is called a *cigar singularity*.

We will now discuss the Bianchi I universes with a dust source. It is our main goal to study the effect of the anisotropy on the initial singularity. As the cosmological constant is relevant only for the late universe, we may restrict ourselves to the case $\Lambda = 0$, i.e., to the field equation

$$R_{\rho\sigma} \,-\, \frac{R}{2}\, g_{\rho\sigma} \,=\, \kappa\, T_{\rho\sigma}$$

where

$$T_{\rho\sigma} \,=\, \mu\, U_\rho\, U_\sigma\,.$$

We will assume that the four-velocity of the dust is perpendicular to the homogeneous slices, i.e.,

$$U^\rho \,=\, \delta^\rho_t\,, \quad U_\sigma \,=\, g_{\sigma\rho}\, U^\rho \,=\, g_{\sigma t} \,=\, -\,c^2\,\delta^t_\sigma\,.$$

With the components of the Ricci tensor and the Ricci scalar given on p. 101, the (tt), (xx), (yy) and (zz) components of the field equation read

$$A\,B \,+\, B\,C \,+\, C\,A \,=\, \kappa\, c^4\, \mu\,, \qquad \text{(E1)}$$

$$A' \,+\, \theta\,A \,-\, \theta' \,-\, \theta^2 \,+\, A\,B \,+\, B\,C \,+\, C\,A \,=\, 0\,, \qquad \text{(E2)}$$

$$B' \,+\, \theta\,B \,-\, \theta' \,-\, \theta^2 \,+\, A\,B \,+\, B\,C \,+\, C\,A \,=\, 0\,, \qquad \text{(E3)}$$

$$C' \,+\, \theta\,C \,-\, \theta' \,-\, \theta^2 \,+\, A\,B \,+\, B\,C \,+\, C\,A \,=\, 0\,. \qquad \text{(E3)}$$

The off-diagonal components of the field equation reduce to the triviality $0 = 0$. Following the same strategy as in the vacuum case, we first derive a differential equation for θ alone, then we solve for the other unknown quantities.

Adding (E2), (E3) and (E4) together yields

$$\theta' \,+\, \theta^2 \,-\, 3\,\theta' \,-\, 3\,\theta^2 \,+\, 3\,\big(A\,B \,+\, B\,C \,+\, C\,A\big) \,=\, 0\,,$$

$$2\,\theta' \,+\, 2\,\theta^2 \,=\, 3\,\big(A\,B \,+\, B\,C \,+\, C\,A\big)\,. \qquad \text{(F1)}$$

Differentiating with respect to t results in

$$\big(\theta' \,+\, \theta^2\big)' \,=\, \frac{3}{2}\,\Big(A'\,B \,+\, A\,B' \,+\, B'\,C \,+\, B\,C' \,+\, C'\,A \,+\, C\,A'\Big)$$

$$=\, \frac{3}{2}\,\Big((B+C)\,A' \,+\, (C+A)\,B' \,+\, (A+B)\,C'\Big)\,.$$

With the help of (E2), (E3) and (E4) this can be rewritten as

$$\big(\theta' \,+\, \theta^2\big)' \,=\, \frac{3}{2}\,\Big((B+C)\,\big(\,-\,\theta\,A \,+\, \theta' \,+\, \theta^2 \,-\, A\,B \,-\, B\,C \,-\, C\,A\,\big)$$

$$+\,(C+A)\,\big(\,-\,\theta\,B \,+\, \theta' \,+\, \theta^2 \,-\, A\,B \,-\, B\,C \,-\, C\,A\,\big)$$

$$+\,(A+B)\,\big(\,-\,\theta\,C \,+\, \theta' \,+\, \theta^2 \,-\, A\,B \,-\, B\,C \,-\, C\,A\,\big)$$

$$= \frac{3}{2}\Big(-2\,\theta\Big(AB + BC + CA\Big) + 2\Big(\theta' + \theta^2 - AB - BC - CA\Big)\Big(A + B + C\Big)\Big)$$

$$= -3\,\theta\big(AB + BC + CA\big) + 3\big(\theta' + \theta^2 - AB - B - CA\big)\,\theta$$

$$= -6\,\theta\big(AB + BC + CA\big) + 3\big(\theta' + \theta^2\big)\,\theta$$

Inserting (E1) yields

$$\big(\theta' + \theta^2\big)' = -6\,\theta\,\kappa\,c^4\,\mu + 3\big(\theta' + \theta^2\big)\,\theta$$

and, with (F1),

$$\big(\theta' + \theta^2\big)' = -4\,\theta\big(\theta' + \theta^2\big) + 3\big(\theta' + \theta^2\big)\,\theta\,,$$

$$\theta'' + 2\,\theta\,\theta' = -\,\theta\,\theta' - \theta^3\,,$$

$$\theta'' + 3\,\theta\,\theta' + \theta^3 = 0\,.$$

We have thus achieved our goal of deriving a differential equation for θ alone. With the ansatz

$$\theta = \frac{v'}{v}\,, \qquad \theta' = \frac{v''}{v} - \frac{v'^{\,2}}{v^2}\,, \qquad \theta'' = \frac{v'''}{v} - \frac{3v''v'}{v^2} + \frac{2v'^{\,3}}{v^3}$$

the differential equation reads

$$\frac{v'''}{v} - \cancel{\frac{3v''v'}{v^2}} + \cancel{\frac{2v'^{\,3}}{v^3}} + \frac{3v'}{v}\left(\cancel{\frac{v''}{v}} - \cancel{\frac{v'^{\,2}}{v^2}}\right) + \cancel{\frac{v'^{\,3}}{v^3}} = 0\,,$$

i.e.,

$$v''' = 0\,.$$

The solution is

$$v(t) = \alpha\,t^2 + \beta\,t + \gamma$$

with constants α, β and γ, hence

$$\theta(t) = \frac{2\,\alpha\,t + \beta}{\alpha\,t^2 + \beta\,t + \gamma}$$

and

$$\frac{3}{2}\,\kappa\,c^4\,\mu = \theta' + \theta^2 = \frac{v''}{v} - \frac{v'^{\,2}}{v^2} + \frac{v'^{\,2}}{v^2} = \frac{2\,\alpha}{v}\,.$$

We see that $\alpha = 0$ gives the vacuum case $\mu = 0$ which was already covered, so we are only interested in the case that $\alpha \neq 0$. From the expression for $\theta(t)$ we see that then we may assume, without loss of generality, that $\alpha = 1$. Moreover, as we are free to choose the origin of the time coordinate as we like, we may set β equal to zero, hence

$$\theta(t) \;=\; \frac{2\,t}{t^2 \,+\, \gamma}$$

and

$$\frac{3}{2}\,\kappa\,c^4\,\mu(t) \;=\; \frac{2}{t^2 \,+\, \gamma}\,.$$

What remains to be done is to determine A, B and C from (E2), E(3) and (E4), respectively. With (F1), equation (E2) can be rewritten as

$$0 \;=\; A'(t) \,+\, \theta(t)\,A(t) \,-\, \frac{1}{2}\,\kappa\,c^4\,\mu(t)$$

$$=\; A'(t) \,+\, \frac{2\,t\,A(t)}{t^2 \,+\, \gamma} \,-\, \frac{2}{3\,\big(t^2 \,+\, \gamma\big)}\,,$$

$$\big(t^2 \,+\, \gamma\big)\,A'(t) \,+\, 2\,t\,A(t) \,-\, \frac{2}{3} \;=\; 0\,,$$

$$\frac{d}{dt}\Big(\big(t^2 \,+\, \gamma\big)\,A(t)\Big) \;=\; \frac{2}{3}\,,$$

$$\big(t^2 \,+\, \gamma\big)A(t) \;=\; \frac{2}{3}\big(t \,+\, \tilde{p}\big)\,,$$

$$A(t) \;=\; \frac{2\,\big(t \,+\, \tilde{p}\big)}{3\big(t^2 \,+\, \gamma\big)}\,.$$

Analogously we find from (E3) and (E4) that

$$B(t) \;=\; \frac{2\,\big(t \,+\, \tilde{q}\big)}{3\big(t^2 \,+\, \gamma\big)}\,,$$

$$C(t) \;=\; \frac{2\,\big(t \,+\, \tilde{r}\big)}{3\big(t^2 \,+\, \gamma\big)}\,.$$

The integration constants $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are not independent. As $A(t) \,+\, B(t) \,+\, C(t) \,=\, \theta(t)$, we must have

$$\frac{\cancel{2}\,\big(\,3\,t \,+\, \tilde{p} \,+\, \tilde{q} \,+\, \tilde{r}\,\big)}{3\,\cancel{\big(t^2 \,+\, \gamma\big)}} \;=\; \frac{\cancel{2}\,t}{\cancel{t^2 \,+\, \gamma}}\,,$$

hence

$$\tilde{p} \,+\, \tilde{q} \,+\, \tilde{r} \;=\; 0\,. \qquad\qquad (T1)$$

Moreover, by (E1) we must have

$$\frac{\cancel{A}\Big(3\,t^2\,+\,2\,\big(\tilde p\,+\,\tilde q\,+\,\tilde r\big)\,+\,\tilde p\,\tilde q\,+\,\tilde q\,\tilde r\,+\,\tilde r\,\tilde p\Big)}{3^{\cancel{2}}\big(t^2\,+\,\gamma\big)^{\cancel{2}}}\,=\,\frac{\cancel{A}}{\cancel{3\,\big(t^2\,+\,\gamma\big)}}\,,$$

$$3\,t^2\,+\,0\,+\,\tilde p\,\tilde q\,+\,\tilde q\,\tilde r\,+\,\tilde r\,\tilde p\,=\,3\big(t^2\,+\,\gamma\big)\,,$$

$$3\,\gamma\,=\,\tilde p\,\tilde q\,+\,\tilde q\,\tilde r\,+\,\tilde r\,\tilde p\,,$$

hence

$$0\,=\,\big(\tilde p\,+\,\tilde q\,+\,\tilde r\,\big)^2\,=\,\tilde p^2\,+\,\tilde q^2\,+\,\tilde r^2\,+\,2\big(\tilde p\,\tilde q\,+\,\tilde q\,\tilde r\,+\,\tilde r\,\tilde p\big)\,,$$

$$\tilde p^2\,+\,\tilde q^2\,+\,\tilde r^2\,=\,-\,6\,\gamma\,. \qquad\qquad (T2)$$

This equation demonstrates that γ cannot be negative. If $\gamma = 0$, we have $A(t) = B(t) = C(t) = 2/(3t)$ which gives the spatially flat Robertson-Walker dust universe without a cosmological constant, i.e., the Einstein-deSitter universe. As we know this case already sufficiently well, we assume in the following that $\gamma > 0$. We may then set

$$\gamma\,=\,-\,t_0^2\,,\quad t_0>0\,.$$

The scale factor $X(t)$ is then given by integrating the equation

$$\frac{X'(t)}{X(t)}\,=\,A(t)\,=\,\frac{2\,\big(t\,+\,\tilde p\,\big)}{3\,\big(t^2\,-\,t_0^2\big)}\,,$$

$$\frac{dX}{X}\,=\,\frac{2\,\big(t\,+\,\tilde p\,\big)\,dt}{3\,\big(t^2\,-\,t_0^2\big)}\,,$$

which gives an elementary integral that can be looked up in an integral table,

$$X(t)\,=\,\big(t\,-\,t_0\big)^p\,\big(t\,+\,t_0\big)^{\frac{2}{3}-p}\,,\quad p\,=\,\frac{1}{3}\Big(1\,+\,\frac{\tilde p}{t_0}\Big)\,.$$

Analogously,

$$Y(t)\,=\,\big(t\,-\,t_0\big)^q\,\big(t\,+\,t_0\big)^{\frac{2}{3}-q}\,,\quad q\,=\,\frac{1}{3}\Big(1\,+\,\frac{\tilde q}{t_0}\Big)\,,$$

$$Z(t)\,=\,\big(t\,-\,t_0\big)^r\,\big(t\,+\,t_0\big)^{\frac{2}{3}-r}\,,\quad r\,=\,\frac{1}{3}\Big(1\,+\,\frac{\tilde r}{t_0}\Big)\,.$$

The relations (T1) and (T2) of the coefficients $\tilde p$, $\tilde q$ and $\tilde r$ imply that p, q and r satisfy the Kasner relations (K1) and (K2) from p. 102,

$$p\,+\,q\,+\,r\,=\,\frac{1}{3}\Big(3\,+\,\frac{\tilde p+\tilde q+\tilde r}{t_0}\Big)\,=\,1\,+\,0\,,$$

$$p^2\,+\,q^2\,+\,r^2\,=\,\frac{1}{9}\Big(3\,+\,\frac{2\,\big(\tilde p+\tilde q+\tilde r\big)}{t_0}\,+\,\frac{\tilde p^2+\tilde q^2+\tilde r^2\big)}{t_0^2}\Big)\,=\,\frac{1}{9}\Big(3\,+\,0\,+\,\frac{6\,t_0^2}{t_0^2}\Big)\,=\,1\,.$$

The metric reads

$$g = -c^2\,dt^2 + \big(t-t_0\big)^{2p}\big(t+t_0\big)^{\frac{4}{3}-2p}\,dx^2 + \big(t-t_0\big)^{2q}\big(t+t_0\big)^{\frac{4}{3}-2q}\,dy^2 + \big(t-t_0\big)^{2r}\big(t+t_0\big)^{\frac{4}{3}-2r}\,dz^2\,.$$

The metric is regular on the interval $-\infty < t < -t_0$, on the interval $-t_0 < t < t_0$ and on the interval $t_0 < t < \infty$. We consider the latter case which is a universe with an initial singularity but no final singularity. The signs of p, q and r determine the behaviour of the scale factors if the singularity is approached. We have already discussed that the Kasner relations can be satisfied only in the following two cases:

(i) One Kasner coefficient is equal to 1 and the other two are zero. This is true at the corners of the triangle in the figure on p. 102. In the vacuum case the metric was then given by Minkowski spacetime in Rindler coordinates. In the dust case, we read from the expression of the metric that it is a universe in which, if the initial singularity is approached from the future, the scale factor shrinks to zero in one dimension and it stays finite in the other two dimensions. This is called a *pancake singularity*.

(ii) Two Kasner coefficients are positive and the third one is negative. This is the generic case, i.e., it is true at all points on the Kasner circle except at the corners of the triangle. As in the vacuum case, we read from the metric that then, if the singularity is approached from the future, the scale factor shrinks to zero in two spatial dimensions and it blows up in the third one. We have already mentioned that this is called a *cigar singularity*.

One might have thought that the initial singularity of Robertson-Walker universes is an artifact of the assumed isotropy. Now we see that this is not true. At least for a Bianchi I dust universe, we have demonstrated that dropping the assumption of isotropy does *not* avoid the formation of a singularity. The only difference in comparison to the Robertson-Walker case is in the fact that the singularity is approached in an anisotropic fashion. Generically, Bianchi I dust universes feature a cigar singularity, just as Bianchi I vacuum universes. We may thus say that, although in a Bianchi I dust universe the density $\mu(t)$ goes to infinity if the singularity is approached, generically the dust has no influence on the character of the singularity.

7. Singularity theorems

When studying Robertson-Walker unverses it seems likely that the occurrence of a singularity is an artifact of the high symmetry. This is analogous to the investigation of gravitational collapse where Oppenheimer and Snyder had shown in 1939 that a spherically symmetric ball of dust ends up in a singularity; also in this case, it seemed likely that there is no longer a singularity if spherical symmetry is broken. Our discussion of Bianchi I dust models has given a first indication that singularities might not be an artifact of high symmetries; Bianchi I models are still homogeneous but not isotropic, so one might have expected that they would avoid a singularity. However, we have seen that in the Bianchi I case only the character of the singularity is changed (from a point singularity generically to a cigar singularity), but not the fact that there is a singularity.

During the 1960s it became evident that the occurrence of singularities is a general feature of Einstein's field equation which has nothing to do with symmetries. There were two lines of research to this effect.

- In the Soviet Union, members of the Landau school, in particular V. Belinsky, I. Khalatnikov and E. Lifshitz (BKL), investigated the set of all initial conditions for Einstein's field equation that lead to a singularity. It was a characteristic feature of their work to concentrate on features that are independent of the matter model. (BKL considered cosmological solutions where the singularity is in the past of the initial hypersurface; for gravitational collapse it is in the future.) In an early paper, Lifshitz and Khalatnikov had claimed that almost all initial conditions lead to singularity-free solutions. Later the Russian scientists realised that this was an error and they found heuristic evidence that, on the contrary, singularities are the rule rather than the exception. However, they did not succeed in rigourously proving a theorem to this effect. Nonetheless, their work is very important because it gave some insight on how a singularity is approached.

- In the United Kingdom R. Penrose and S. Hawking proved a series of theorems demonstrating that singularities occur under rather generic conditions. There are four such theorems: The first one by Penrose (1965) is relevant for gravitational collapse, the second and third by Hawking (1967) are relevant for cosmology and the fourth one by Penrose and Hawking together (1970) is relevant for both situations.

In what follows we briefly summarise the content of Hawking's singularity theorems. The proofs are so involved that we will not even touch upon them. Details can be found in the book by S. Hawking and G. Ellis ["The large-scale structure of space-time", Cambridge University Press (1973)].

It is important to realise that the Penrose-Hawking singularity theorems do not prove the existence of a singularity in the sense that the energy density or a curvature invariant becomes infinite. What is proven is that there are timelike or lightlike geodesics that are incomplete (in the past for cosmological solutions and in the future for gravitational collapse). For timelike geodesics, this means that for a freely falling observer the world ends at a finite proper time which clearly indicates a pathological situation. For lightlike geodesics the affine parameter cannot be interpreted as the reading of a clock, but also incompleteness of a lightlike geodesic seems to be something pathological because a lightlike geodesic is the history of a photon. An example why it is not sufficient to study timelike incompleteness is given by the Reissner-Nordström metric which features a curvature singularity at $r = 0$ where no timelike but lightlike geodesics terminate. Of course, geodesics become incomplete in a trivial way if we remove points from a perfectly regular spacetime. Therefore, we say that a spacetime is singular if it is *inextendible* and contains an incomplete timelike or lightlike geodesic.

Having agreed on the definition of singularities, the task is to formulate hypotheses that are physically reasonable and predict the existence of a singularity. The Penrose-Hawking theorems use three types of hypotheses:

- First one needs a condition on the Ricci tensor which makes sure that gravity is attractive in the sense that it makes the worldlines of freely falling objects converge. In conjunction with Einstein's field equation, such a condition can be re-interpreted as an *energy condition*. For the Hawking singularity theorems the condition

$$R_{\mu\nu}K^\mu K^\nu \geq 0 \quad \text{if} \quad g_{\mu\nu}K^\mu K^\nu \leq 0$$

  is used. In view of the Jacobi equation (also known as the equation of geodesic deviation), this condition means that on averaging over directions gravity is attractive for freely falling

particles and photons. If Einstein's field equation for a perfect fluid without a cosmological constant is assumed, it can be rewritten as

$$\mu + \frac{p}{c^2} \geq 0\,, \quad \mu + \frac{3\,p}{c^2} \geq 0$$

and is known as the *strong energy condition*. It is obviously satisfied for a perfect fluid with positive energy density and positive pressure. It is violated, however, for "dark energy", i.e., for a perfect fluid mimicking a positive cosmological constant where $p = -c^2\mu$ is negative, recall Problem 2 of Worksheet 5.

- Then one needs a condition either on the topology or on the causal structure of spacetime. In one of the two Hawking theorems one considers a "closed universe", i.e., one assumes a compact spatial topology. In the other Hawking theorem one assumes that there are no closed timelike curves. It is widely accepted that closed timelike curves should be forbidden because they lead to the paradox that one could travel into one's own past and kill one's parents before one is borne. Actually, in the Hawking theorem a slightly stronger assumption is needed which is known as the "strong causality condition": Every neighbourhood of a point p contains a neighbourhood of p that no timelike or lightlike curve through p intersects more than once. This is a way of saying that it is not only forbidden for a timelike or lightlike curve to come back exactly to p but also to come back arbitrarily close to p.

- Finally, a third assumption is needed that makes sure that, at one time, the spacetime has the tendency to "contract" (towards the past for cosmology and towards the future for gravitational collapse). Such an initial condition is formulated with the help of a vector field $V = V^\mu\partial_\mu$ that satisfies $g(V,V) = -c^2$ and $\nabla_V V = 0$, i.e., its worldlines are timelike geodesics parametrised by proper time. For such a vector field, the scalar field $\theta = \nabla_\mu V^\mu$ is called the *expansion*. It measures if neighbouring worldlines approach each other ($\theta < 0$) or move away from each other ($\theta > 0$). The idea is to prove that, if a contracting initial condition is prescribed, and if the other assumptions of the theorem are satisfied, then the collapse cannot be stopped and will lead to a singularity.

We now give the precise formulation of the two Hawking theorems.

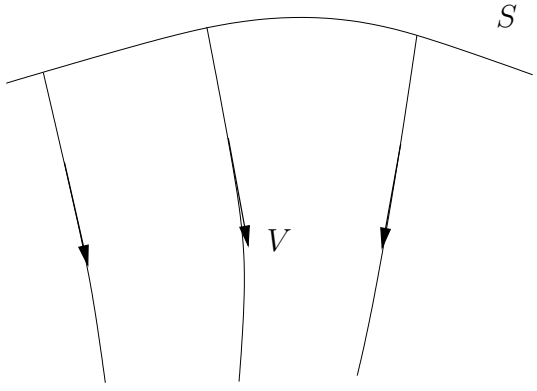
Theorem 1 (Hawking, 1967): Spacetime cannot be timelike and lightlike geodesically complete in the past if the following three assumptions hold:

(a) The strong energy condition is satisfied,

$$R_{\mu\nu}K^\mu K^\nu \geq 0 \quad \text{if} \quad g_{\mu\nu}K^\mu K^\nu \leq 0\,.$$

(b) There is a spacelike compact 3-dimensional submanifold S without boundary.

(c) Let $V = V^\mu\partial_\mu$ be the past oriented vector field with $g(V,V) = -c^2$ whose integral curves are the timelike geodesics orthogonal to S. Then the expansion $\theta = \nabla_\mu V^\mu$ satisfies $\theta\big|_S < 0$.



Note that in the theorem it is not assumed that the spacetime be inextendible. However, if we start with a spacetime where the assumptions (a), (b) and (c) hold, the theorem says that this spacetime cannot be extended to a timelike and lightlike geodesically complete spacetime without violating one of these three assumptions.

We have good observational evidence that we live in a universe that is expanding. If our universe is spatially compact and if the strong energy condition holds, then the theorem says that there must be a singularity in the sense that the world began for some freely falling particle at a finite time or for some photon at a finite affine parameter. It does *not* say that this is necessarily a curvature singularity or a state of infinite energy density. The strong energy condition might be considered now as more questionable than in 1967. Firstly, we now believe that there is "dark energy" which violates the strong energy condition. Secondly, and much more importantly for the early universe, most inflationary scenarios violate the strong energy condition. However, the (hypothetical) idea of inflation applies to the very early universe where it is questionable if our classical (i.e., non-quantum) spacetime model is still valid. So even if one accepts the idea of inflation, one might argue that Theorem 1 predicts a singularity in the regime where the model of a classical spacetime is applicable.

As we don't know if our universe is spatially compact, we would like to have another theorem which includes non-compact spatial topologies. This requires a more sophisticated formulation of the third condition, because then we do not have a spacelike compact submanifold from which the integral curves of our vector field V could start. Hawking's second theorem reads as follows.

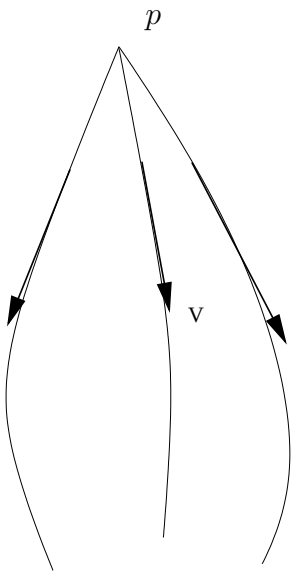
Theorem 2 (Hawking, 1967): Spacetime cannot be timelike and lightlike geodesically complete in the past if the following three assumtpions hold:

(a) The strong energy condition is satisfied,

$$R_{\mu\nu}K^\mu K^\nu \geq 0 \quad \text{if} \quad g_{\mu\nu}K^\mu K^\nu \leq 0\,.$$

(b) The strong causality condition is satisfied, i.e., every neighbourhood of a point p contains a neighbourhood that no timelike or lightlike curve through p intersects more than once.

(c) Let V be the past-oriented vector field whose integral curves are the timelike geodesics issuing from a point p and let $\theta = \nabla_\mu V^\mu$. Then there is a past-oriented timelike vector $w^\mu \partial_\mu\big|_p$ at p and a positive constant b such that on each past-oriented timelike geodesic from p the inequality $\theta < -3k/b$ holds within a proper time distance b from p, where the positive number k is defined by $k = -V^\mu\big|_p w_\mu$.



In contrast to Theorem 1, in Theorem 2 condition (c) is now not quite so easily connected with observations. In essence, however, it just says that the expansion in the past of some event must be negative and bounded away from zero by a certain amount.

The fact that solutions to Einstein's field equation with a reasonable matter content have a strong tendency to form singularities is viewed by many as the most serious problem of general relativity as a classical theory. It is widely believed that we will really understand what is going on near a singularity only if we have some quantum version of general relativity.