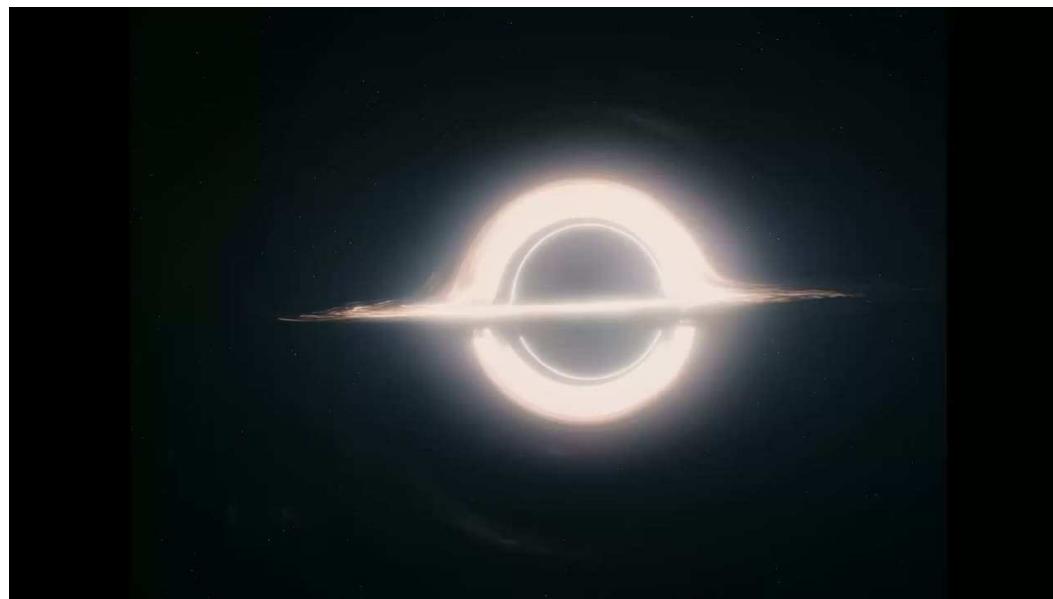


Gravitational lensing in the presence of a plasma

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from the movie “Interstellar”

Outline:

1. Light propagation on a general-relativistic spacetime

 1.1 Light propagation in a vacuum

 1.2 Light propagation in a plasma

2. Spherically symmetric and static spacetimes

 2.1 Schwarzschild lensing in a vacuum

 2.2 Schwarzschild lensing in a plasma

 2.1 Arbitrary spherically symmetric and static spacetimes

3. Kerr spacetime

 3.1 Kerr lensing in a vacuum

 3.2 Kerr lensing in a plasma

1. Light propagation on a general-relativistic spacetime

1.1 Light propagation in a vacuum

Consider a general-relativistic spacetime with metric $g_{ik}(x)$

Vacuum light rays are lightlike geodesics:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad g_{jk} \dot{x}^j \dot{x}^k = 0$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} \right)$$

Can be justified by the Ehlers-Pirani-Schild axiomatic approach to spacetime theory

J. Ehlers, F. A. E. Pirani and A. Schild, in L. O'Reiffeartaigh (ed.): "General Relativity. Papers in honour of J. L. Synge", Oxford, Clarendon Press (1972)
pp. 63–84

Can also be derived from Maxwell's equations

J. Ehlers: Z. Naturforschg. 22 a, 1328-1333 (1967)

Lagrange formalism for vacuum light rays:

$$\frac{d}{ds} \frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}^j} - \frac{\partial \mathcal{L}(x, \dot{x})}{\partial x^j} = 0, \quad \mathcal{L}(x, \dot{x}) = 0$$

with

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} g_{jk}(x) \dot{x}^j \dot{x}^k$$

Hamilton formalism for vacuum light rays

$$\dot{x}^i = \frac{\partial \mathcal{H}(x, p)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}(x, p)}{\partial x^i}, \quad \mathcal{H}(x, p) = 0$$

where

$$\mathcal{H}(x, p) = \frac{1}{2} g^{ik}(x) p_i p_k$$

Then

$$\dot{x}^i = g^{ij}(x) p_j, \quad g_{ij}(x) \dot{x}^i \dot{x}^j = 0$$

Choose an observer field U^i with $g_{ij}U^iU^j = -c^2$

Decompose $p_i = g_{ik}\dot{x}^k$ into frequency and wave vector

$$p_i = \frac{1}{c}\omega U_i + c k_i, \quad k_i U^i = 0, \quad \omega = -\frac{1}{c} p_i U^i$$

Dispersion relation $\mathcal{H} = 0$ becomes

$$-\omega^2 + c^2 g^{ij} k_i k_j = 0, \quad \omega = ck$$

Phase velocity

$$v_\varphi = \frac{\omega}{k} = c$$

Index of refraction

$$n = \frac{c}{v_\varphi} = 1$$

1.2 Light propagation in a plasma

In a non-magnetised pressure-free (“cold”) plasma consisting of ions and electrons, light rays are solutions of Hamilton’s equations

$$\dot{x}^i = \frac{\partial \mathcal{H}(x, p)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}(x, p)}{\partial x^i}, \quad \mathcal{H}(x, p) = 0$$

where

$$\mathcal{H}(x, p) = \frac{1}{2} \left(g^{ik}(x) p_i p_k + \omega_p(x)^2 \right),$$

plasma frequency: $\omega_p(x)^2 = \frac{e^2}{\epsilon_0 m_e} N(x)$

e : charge of the electron, m_e : mass of the electron
 $N(x)$: number density of the electrons

Can be derived from Maxwell’s equation with a two-fluid source

R. Breuer, J. Ehlers: Proc. Roy. Soc. London, A 370, 389 (1980), A 374, 65 (1981)

VP: “Ray Optics, Fermat’s Principle and Applications to General Relativity”
Springer (2000)

$$\dot{x}^i = \frac{\partial \mathcal{H}(x, p)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}(x, p)}{\partial x^i}, \quad \mathcal{H}(x, p) = 0$$

where

$$\mathcal{H}(x, p) = \frac{1}{2} \left(g^{ik}(x) p_i p_k + \omega_p(x)^2 \right)$$

Then

$$\dot{x}^i = g^{ij}(x) p_j, \quad g_{ij}(x) \dot{x}^i \dot{x}^j = -\omega_p(x)^2$$

Light rays are **timelike** curves

Actually, they are **timelike geodesics** of the conformally rescaled metric $\omega_p(x)^{-2} g_{ij}(x)$

Choose an observer field U^i with $g_{ij}U^iU^j = -c^2$

Decompose $p_i = g_{jk}\dot{x}^k$ into frequency and wave vector

$$p_i = \frac{1}{c}\omega U_i + c k_i, \quad k_i U^i = 0, \quad \omega = -\frac{1}{c} p_i U^i$$

Dispersion relation $\mathcal{H} = 0$ becomes

$$-\omega^2 + c^2 g^{ij} k_i k_j = -\omega_p^2, \quad \omega = \sqrt{c^2 k^2 + \omega_p^2}$$

Phase velocity

$$v_\varphi(x, \omega) = \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - \omega_p(x)^2}} > c$$

Index of refraction

$$n(x, \omega) = \frac{c}{v_\varphi(x, \omega)} = \sqrt{1 - \frac{\omega_p(x)^2}{\omega^2}} < 1$$

$$n(x, \omega) = \sqrt{1 - \frac{\omega_p(x)^2}{\omega^2}}$$

Propagation is possible only if $\omega > \omega_p(x)$

For $\omega \gg \omega_p(x)$ is the influence of the plasma negligible

- Solar corona near Solar rim: $\omega_p \approx 100 \text{ MHz}$
- Ionosphere: $\omega_p \approx 10 \text{ MHz}$
- Interstellar space: $\omega_p \approx 10 \text{ kHz}$
- Intergalactic space: $\omega_p \approx 10 \text{ Hz}$

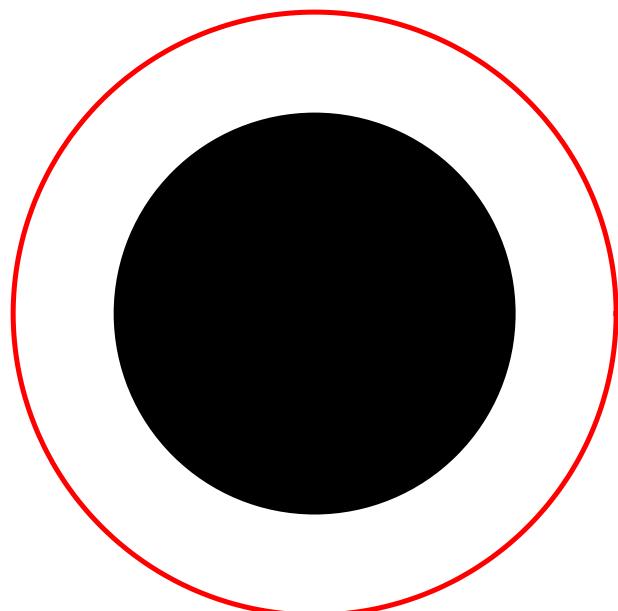
2. Spherically symmetric and static spacetimes

2.1 Schwarzschild lensing in a vacuum

Metric:

$$g_{ik}(x)dx^i dx^k = -\left(1 - \frac{2m}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

mass parameter $m = \frac{GM}{c^2}$



Horizon:

$$r_S = \frac{2GM}{c^2} = 2m$$

Light sphere (photon sphere)

$$\frac{3}{2}r_S = \frac{3GM}{c^2} = 3m$$

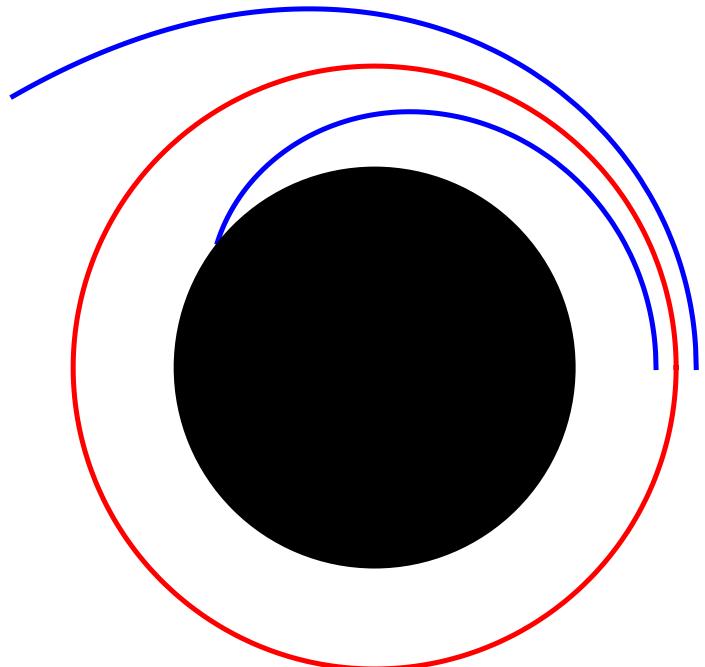
2. Spherically symmetric and static spacetimes

2.1 Schwarzschild lensing in a vacuum

Metric:

$$g_{ik}(x)dx^i dx^k = -\left(1 - \frac{2m}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

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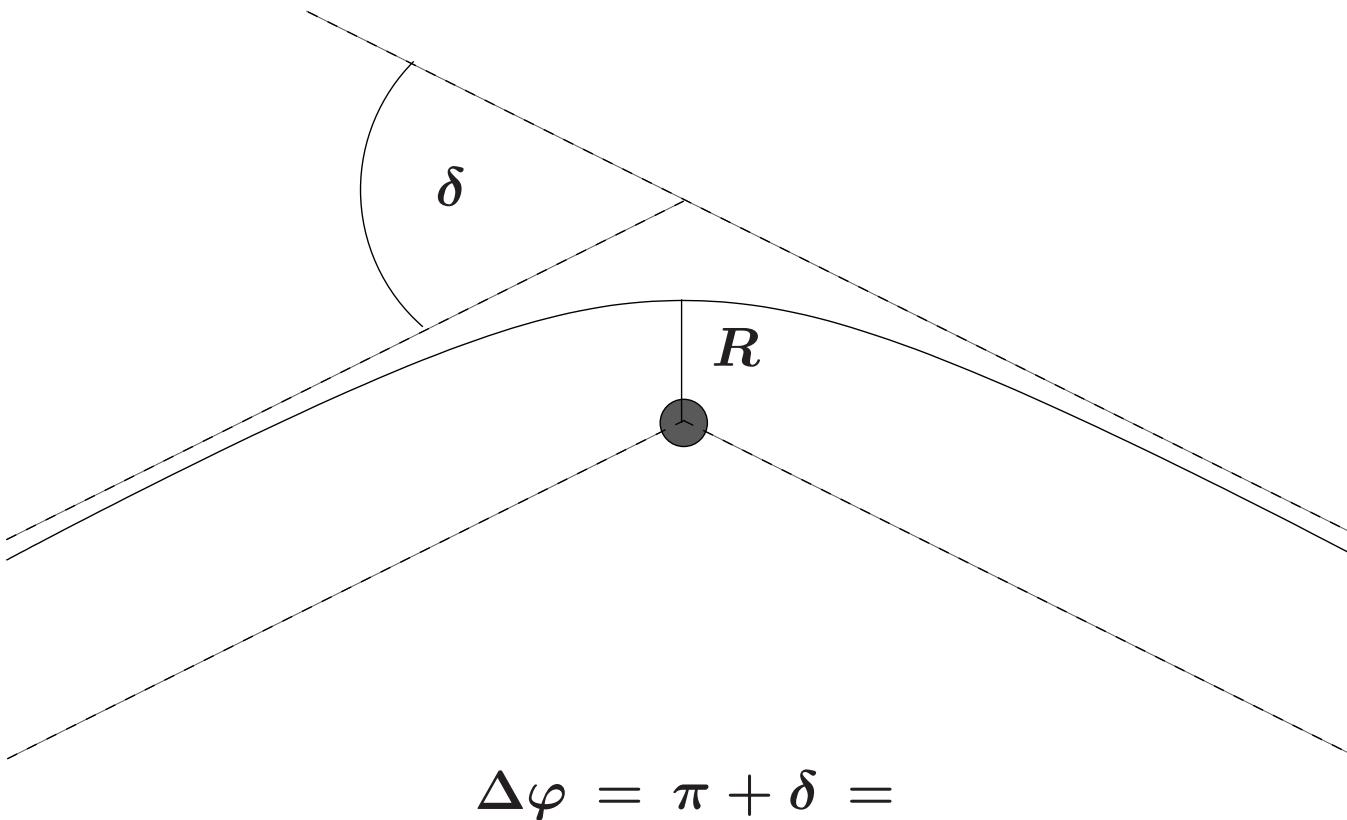
Horizon:

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Light sphere (photon sphere)

$$\frac{3}{2}r_S = \frac{3GM}{c^2} = 3m$$

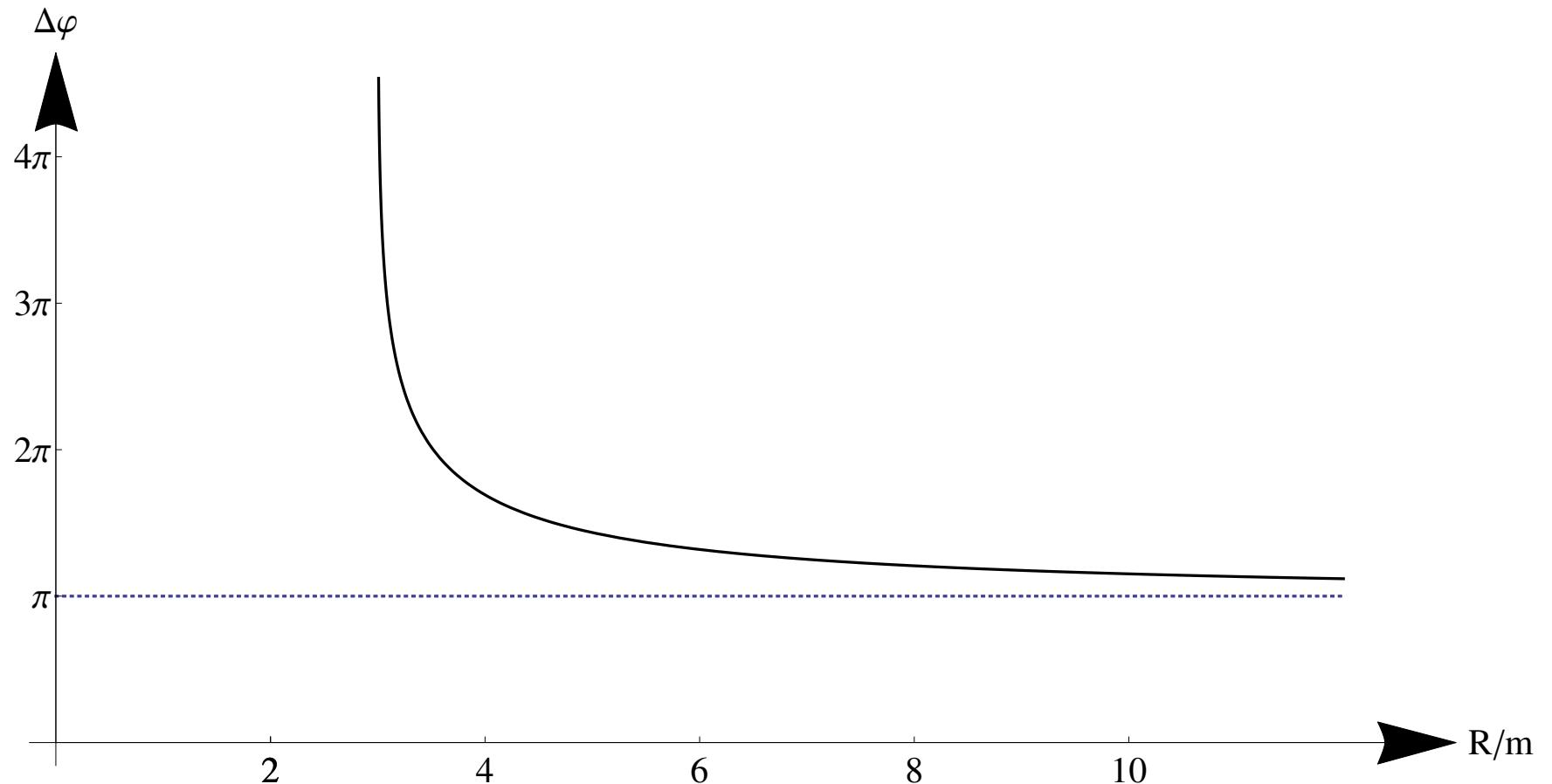
Deflection angle δ



$$2 \int_R^\infty \left(\frac{r^3(R - 2m)}{R^3(r - 2m)} - 1 \right)^{-1/2} \frac{dr}{\sqrt{r(r - 2m)}} =$$

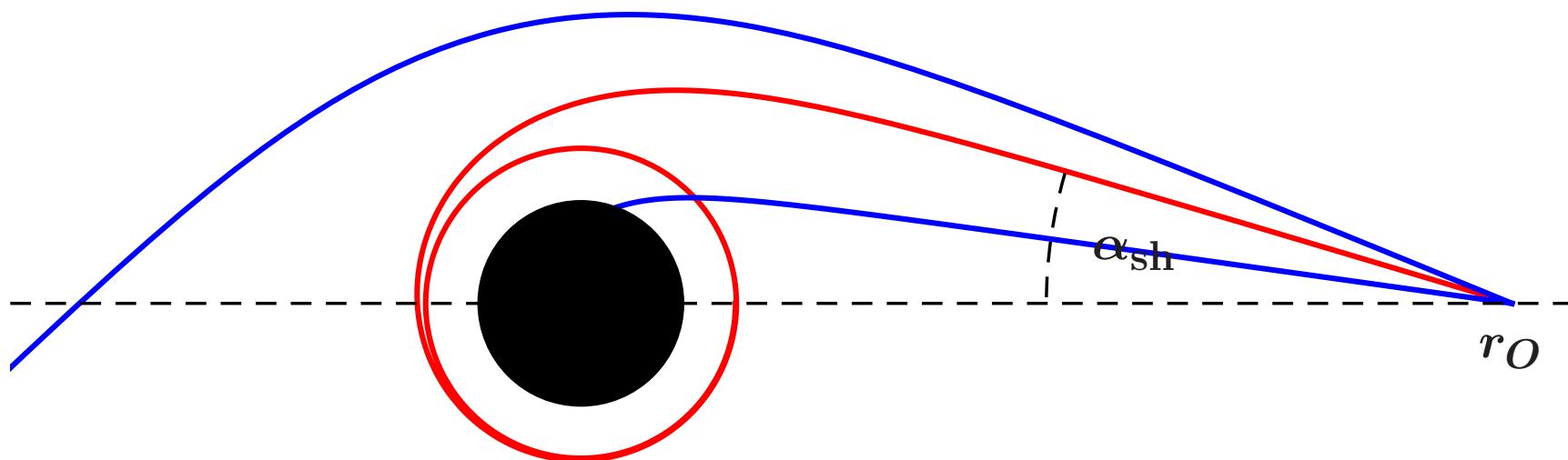
$$\pi + \frac{4m}{R} \left(1 + O\left(\frac{m}{R}\right) \right)$$

Deflection angle $\delta = \Delta\varphi - \pi$



Divergence of δ at $R = 3m$ is crucial for the construction of the shadow

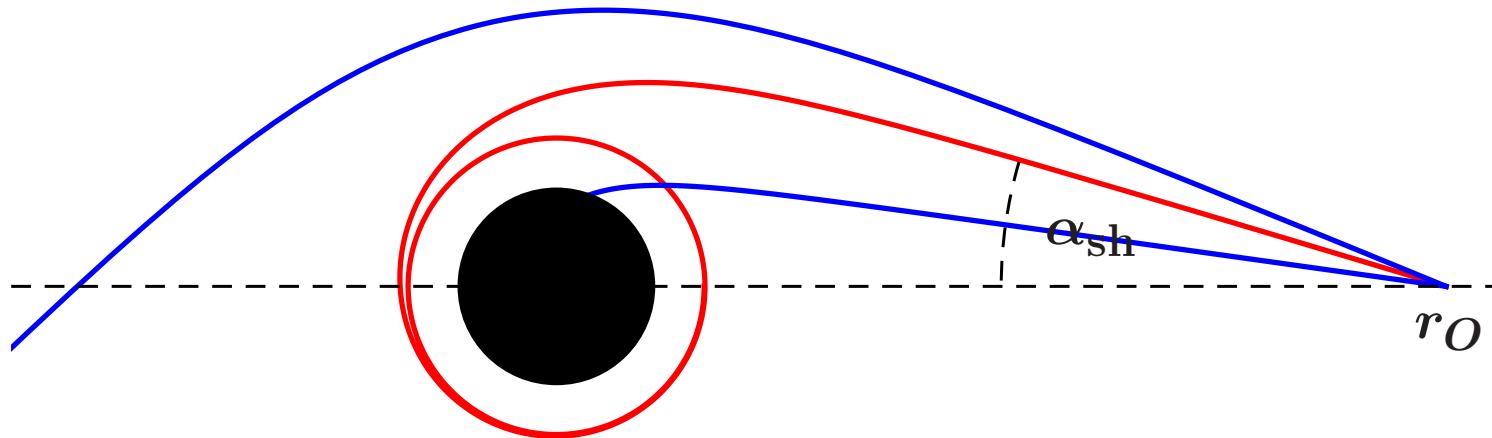
Assumption: Observer at r_O , light sources anywhere but not between the observer and the black hole



$\alpha < \alpha_{\text{sh}}$: darkness (interior of shadow)

$\alpha > \alpha_{\text{sh}}$: brightness (bright backdrop)

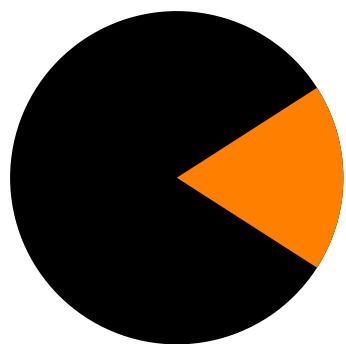
$\alpha = \alpha_{\text{sh}}$: boundary of the shadow



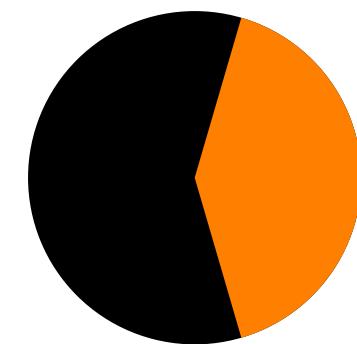
Angular radius α_{sh} of the “shadow” of a Schwarzschild black hole:

$$\sin^2 \alpha_{\text{sh}} = \frac{27 r_S^2 (r_O - r_S)}{4 r_O^3} = \frac{27 m^2}{r_O^2} \left(1 - \frac{2m}{r_O}\right)$$

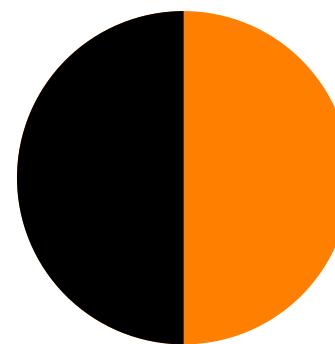
[J. L. Synge, Mon. Not. R. Astr. Soc. 131, 463 \(1966\)](#)



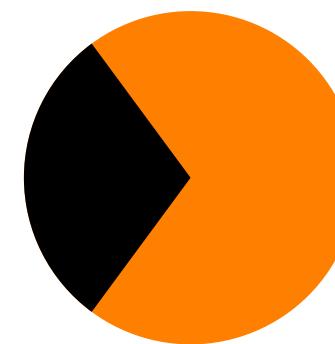
$r_O = 1.05 r_S$



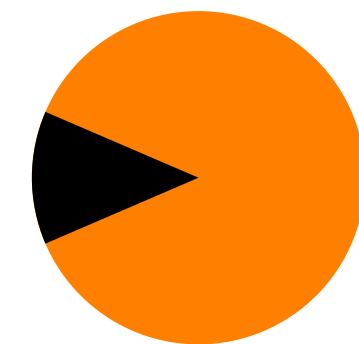
$r_O = 1.3 r_S$



$r_O = 3 r_S/2$

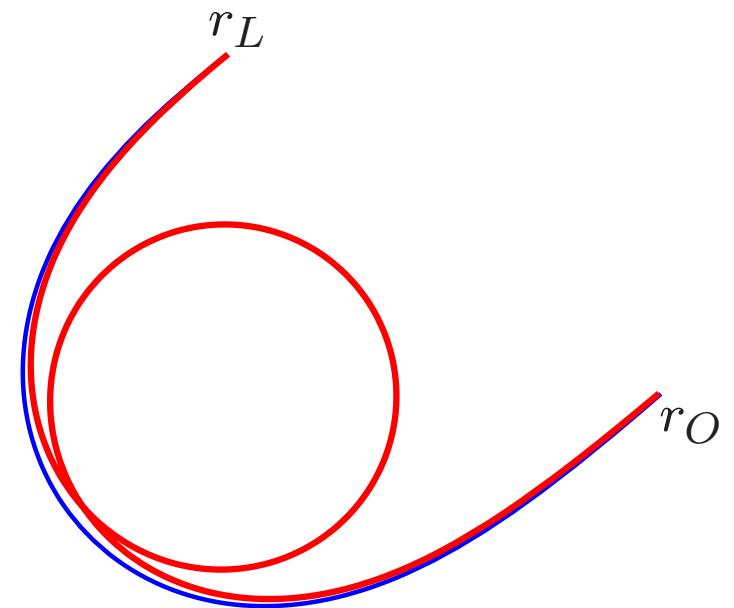
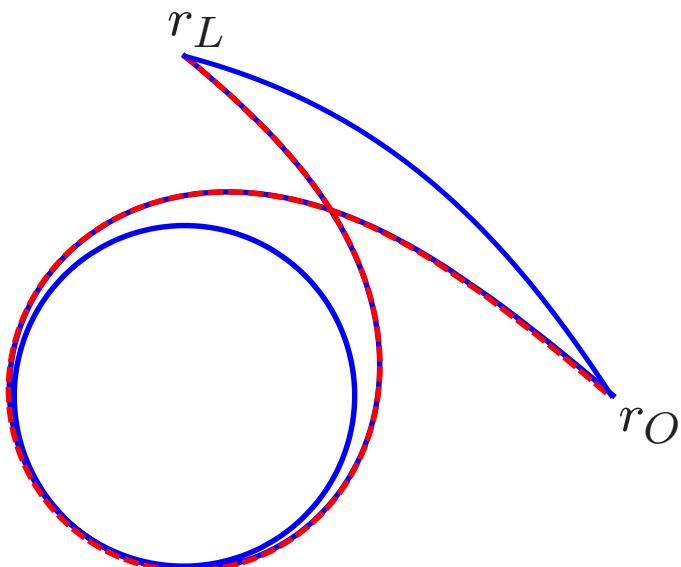


$r_O = 2.5 r_S$

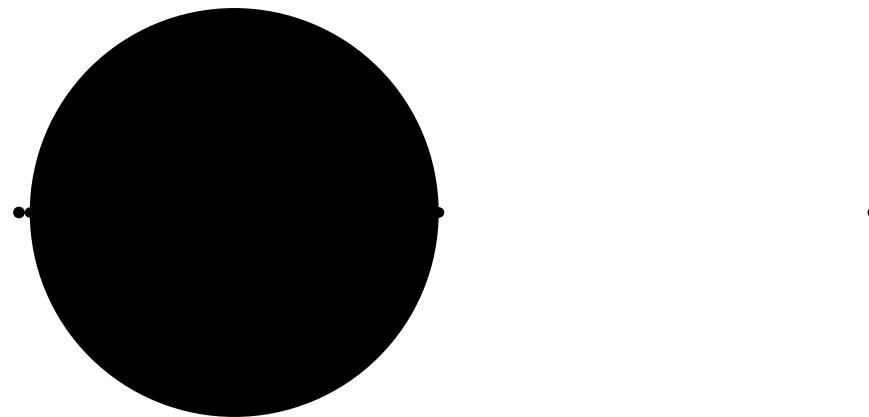


$r_O = 6 r_S$

**Schwarzschild black hole produces infinitely
many images:**



Imaging of a point source by a Schwarzschild black hole



Perspectives of observations

Object at the centre of our galaxy:

Mass = $4 \times 10^6 M_{\odot}$

Distance = 8 kpc

Angular diameter of the shadow by Synge's formula $\approx 54 \mu\text{as}$

(corresponds to a grapefruit on the moon)

Object at the centre of M87:

Mass = $3 \times 10^9 M_{\odot}$

Distance = 16 Mpc

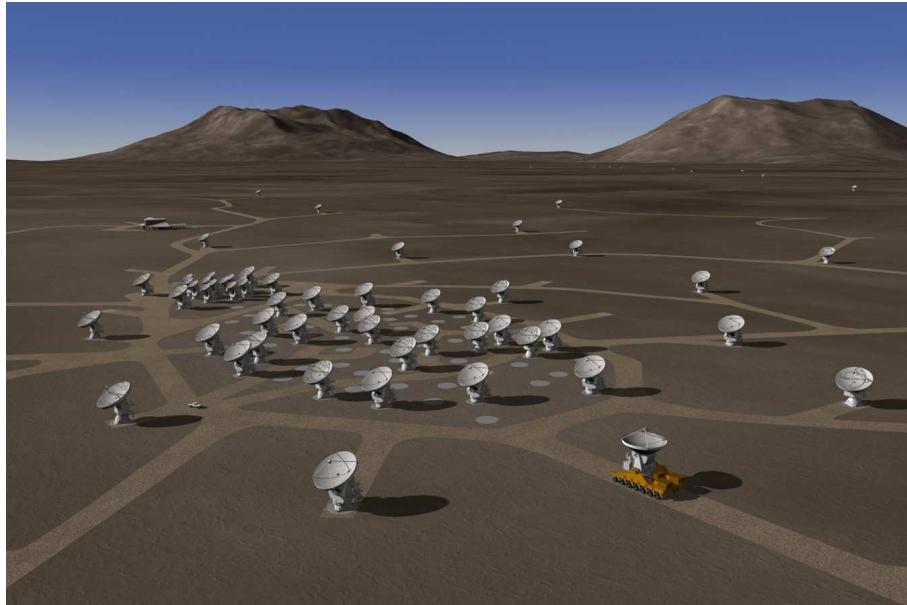
Angular diameter of the shadow by Synge's formula $\approx 20 \mu\text{as}$

Perhaps observable soon with VLBI

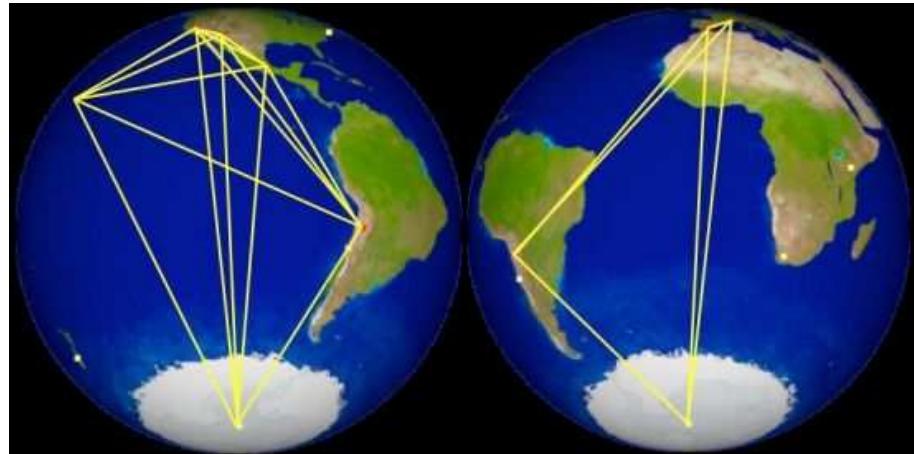
Projects to view the shadow with (sub-)millimeter VLBI:

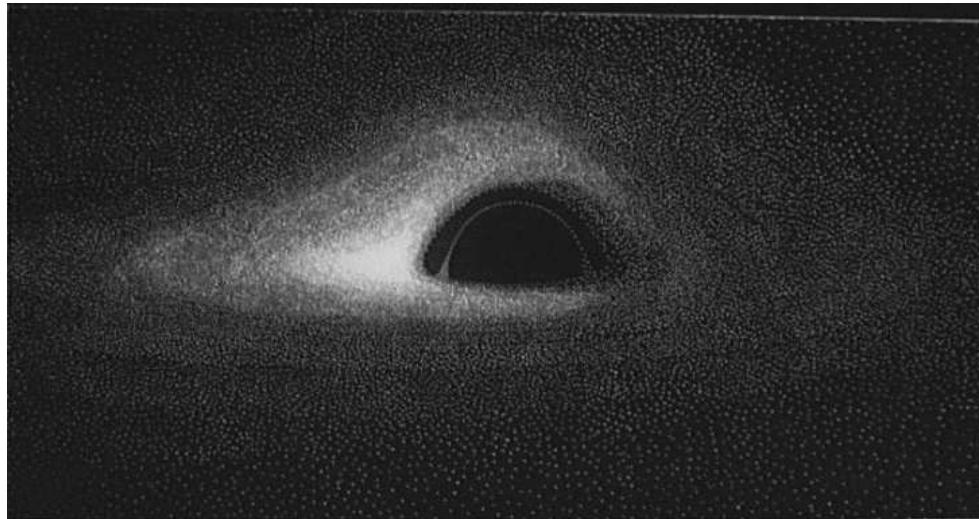
Event Horizon Telescope (EHT), BlackHoleCam

Using ALMA, NOEMA, LMT, South Pole Telescope ...

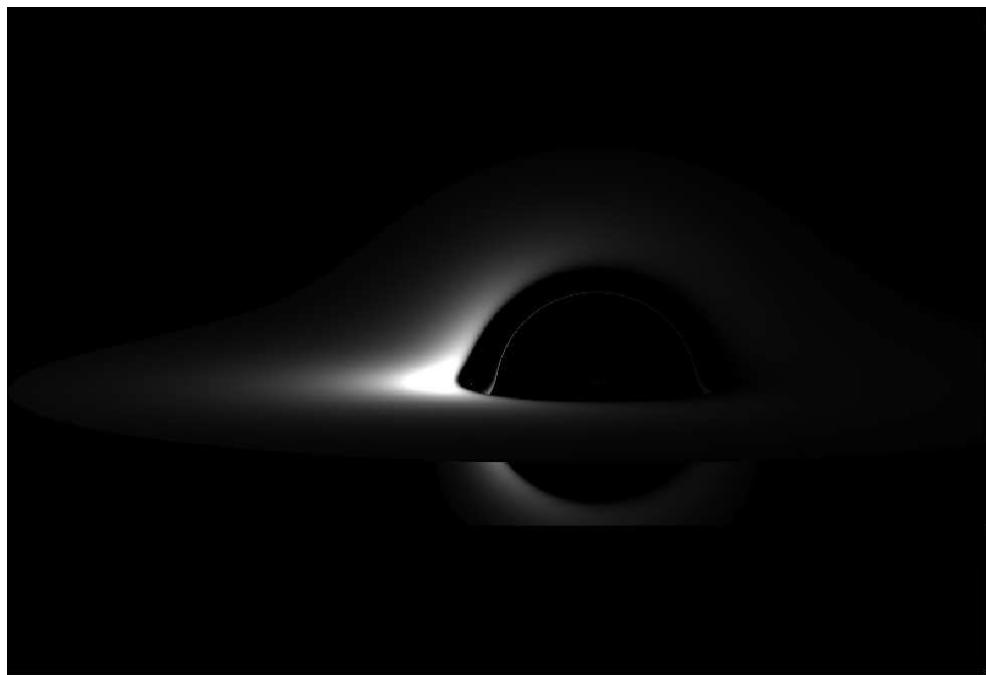


ALMA

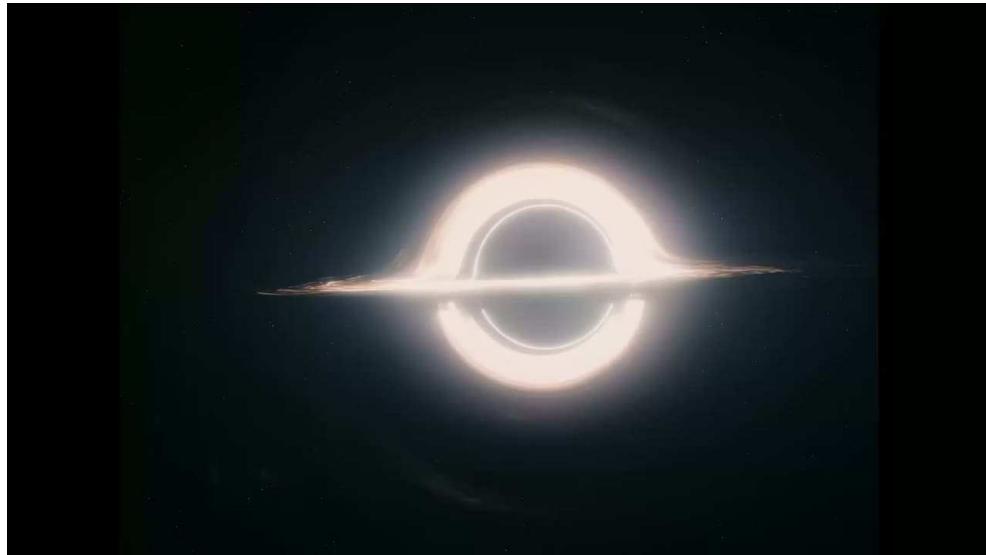




J.-P. Luminet (1979)



T. Müller (2012)



Interstellar (2014)

2.2 Schwarzschild lensing in a plasma

Metrik:

$$g_{ik}(x)dx^i dx^k = -\left(1 - \frac{2m}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{r}{2m}} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

Spherically symmetric and static plasma frequency: $\omega_p(r)$

Hamiltonian:

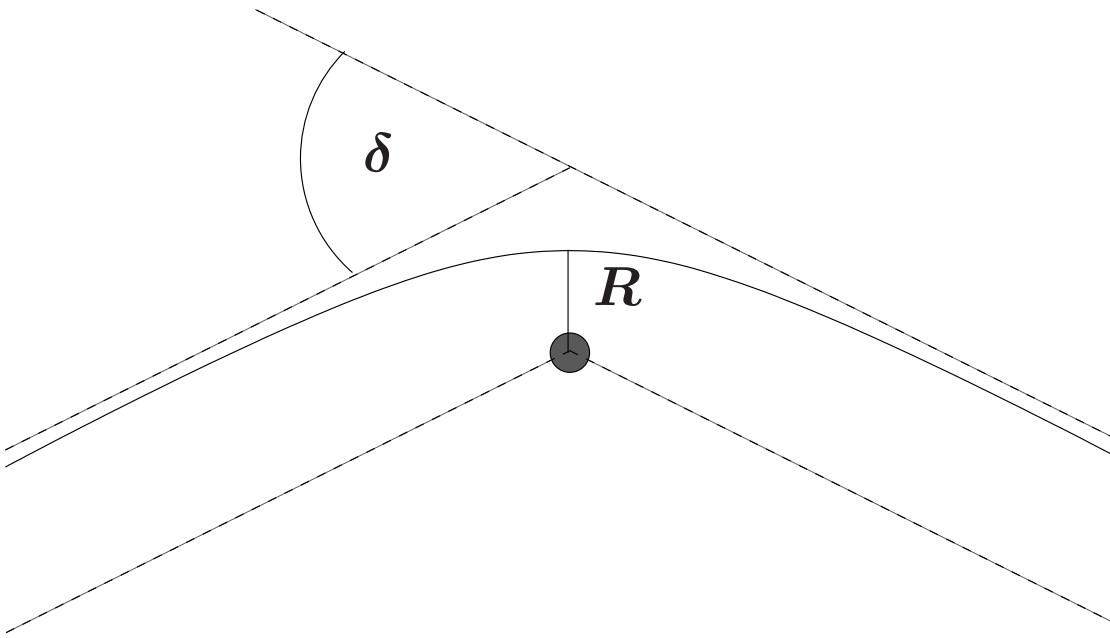
$$\mathcal{H}(x, p) = \frac{1}{2} \left(-\frac{p_t^2}{\left(1 - \frac{2m}{r}\right)c^2} + \left(1 - \frac{2m}{r}\right)p_r^2 + \frac{1}{r^2} \left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\right) + \omega_p(r)^2 \right)$$

Constant of motion: $\omega_0 = -\frac{p_t}{c}$

Frequency of photon with respect to $U^i = \delta_t^i \left(1 - \frac{2m}{r}\right)^{-1/2}$:

$$\omega(r) = -\frac{1}{c} p_i U^i = \omega_0 \left(1 - \frac{2m}{r}\right)^{-1/2}$$

Deflection angle δ



$$\Delta\varphi = \pi + \delta = 2 \int_R^\infty \left(\frac{r^2 \left(\frac{r}{r-2m} - \frac{\omega_p(r)^2}{\omega_0^2} \right)}{R^2 \left(\frac{R}{R-2m} - \frac{\omega_p(R)^2}{\omega_0^2} \right)} - 1 \right)^{-1/2} \frac{dr}{\sqrt{r}\sqrt{r-2m}}$$

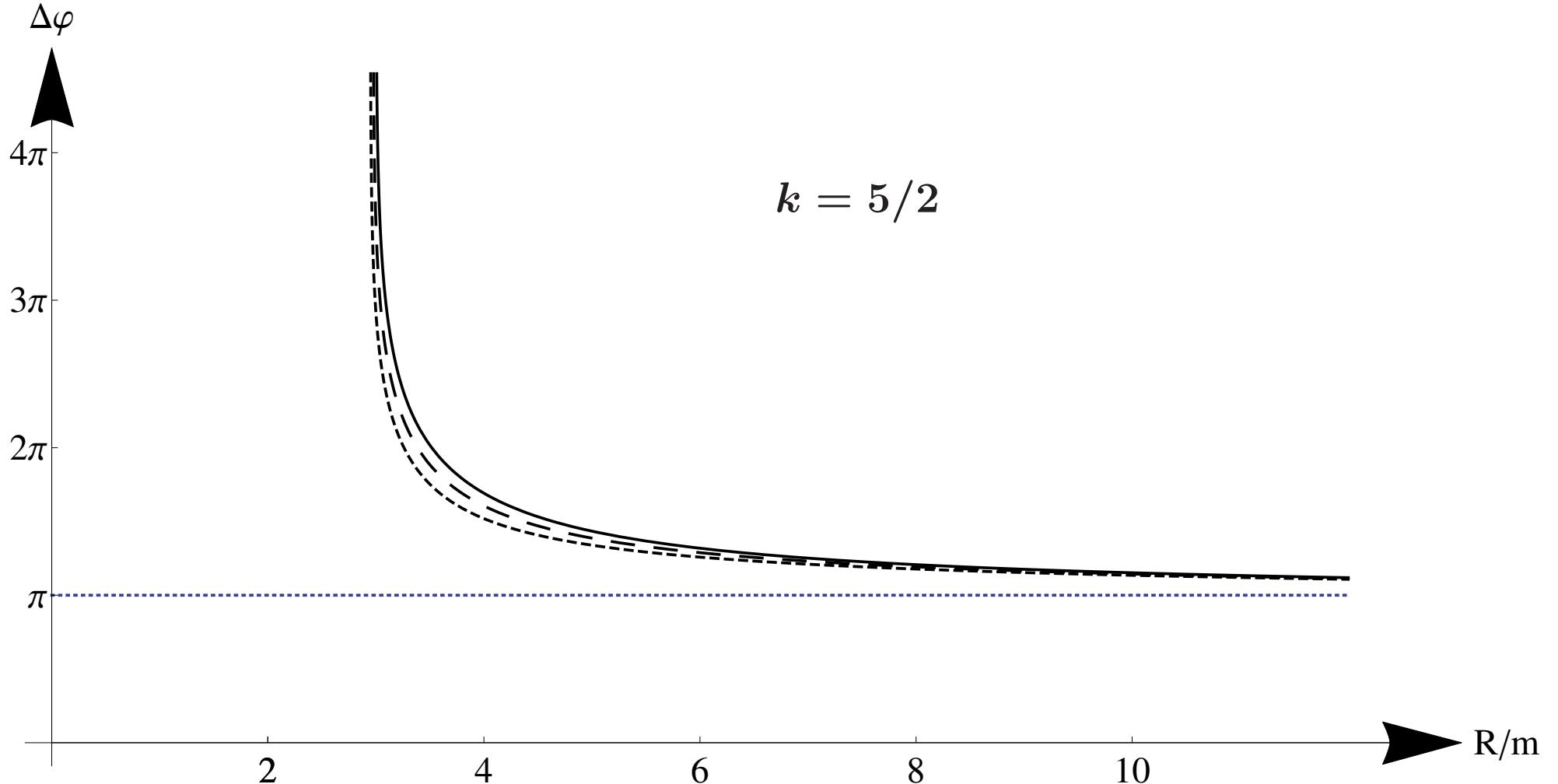
VP: “Ray optics, Fermat’s principle and applications to general relativity”
Springer (2000)

In the weak-field approximation:

D. O. Muhleman and I. D. Johnston: Phys. Rev. Lett. 17, 455 (1966)

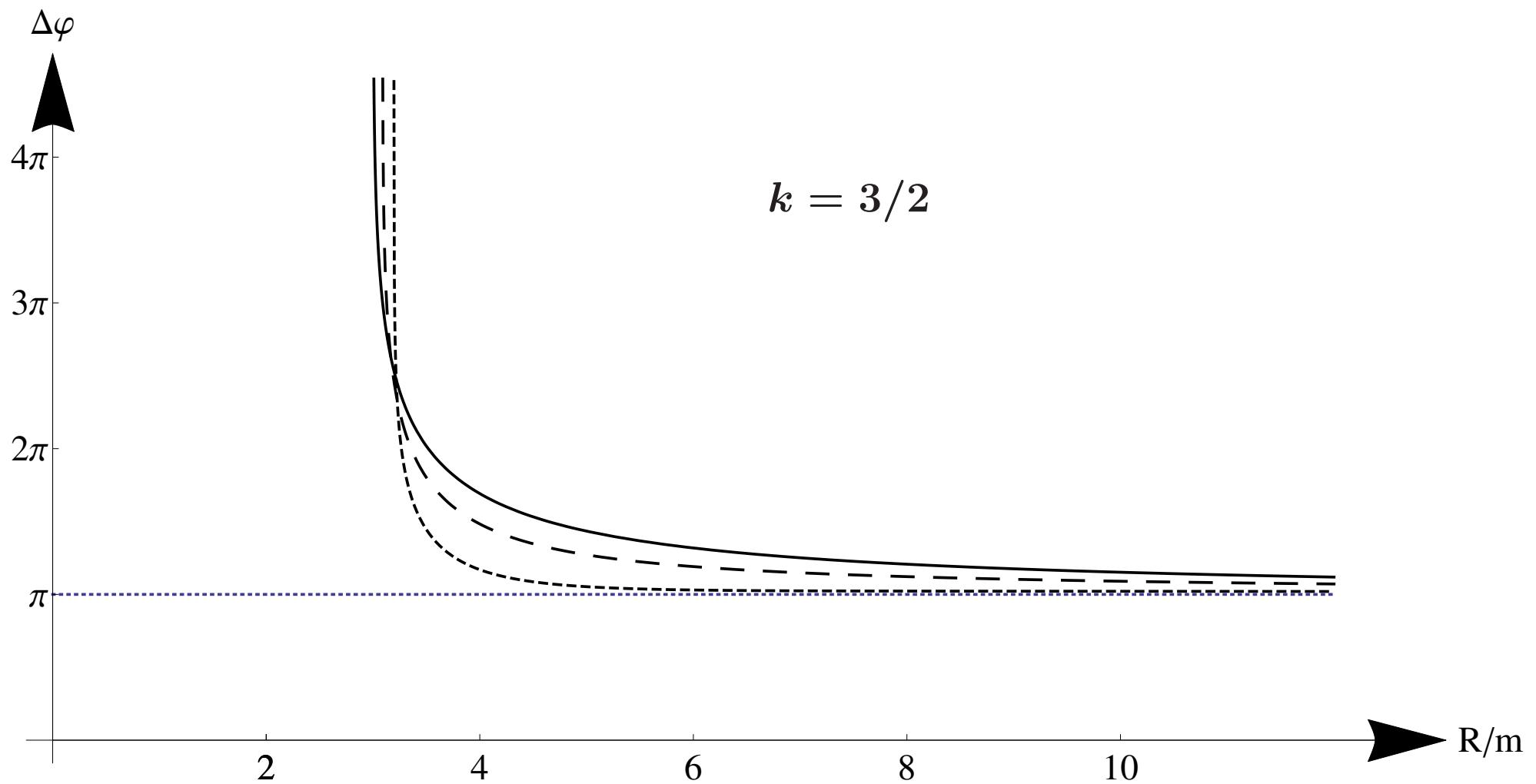
$$\frac{\omega_p(r)^2}{\omega_0^2} = \beta_0 \frac{m^k}{r^k}$$

If $k > 2$, bending angle is always smaller in the plasma



$\beta_0 = 0$ (solid), $\beta_0 = 5$ (wide-dashed), $\beta_0 = 10$ (narrow-dashed),

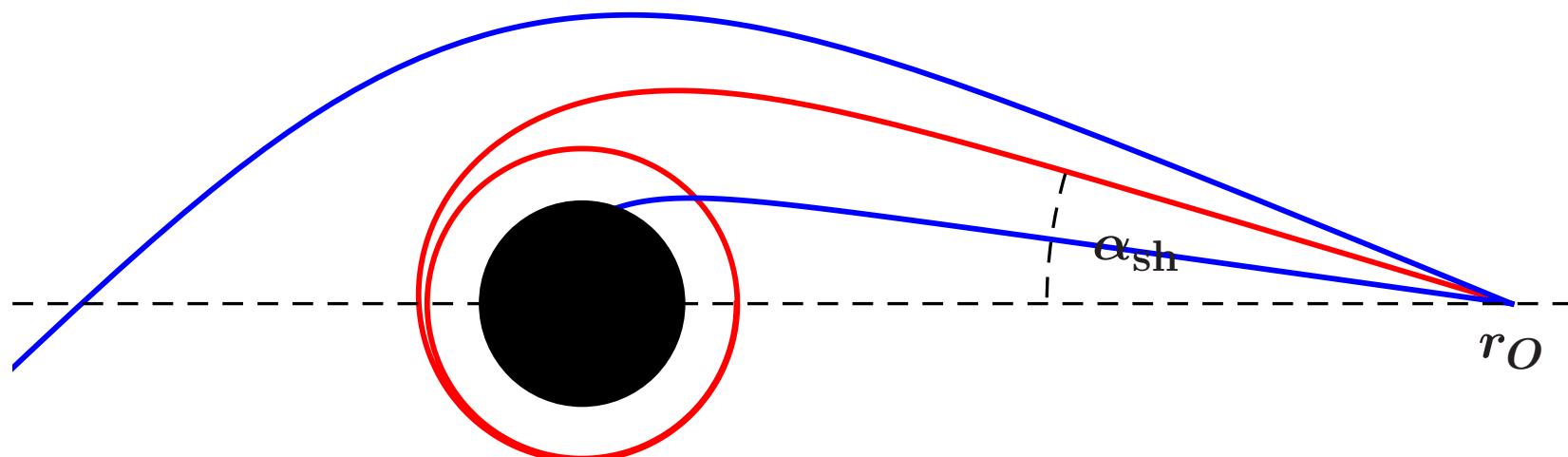
$$\frac{\omega_p(r)^2}{\omega_0^2} = \beta_0 \frac{m^k}{r^k}$$



$\beta_0 = 0$ (solid), $\beta_0 = 5$ (wide-dashed), $\beta_0 = 10$ (narrow-dashed),

Condition for photon sphere:

$$\left. \frac{dh(r)^2}{dr} \right|_{r=r_{\text{ph}}} = 0, \quad h(r)^2 = \frac{r^3}{r - 2m} - \frac{r^2 \omega_p(r)^2}{\omega_0^2}$$



Angular radius α_{sh} of shadow:

$$\sin^2 \alpha_{\text{sh}} = \frac{h(r_{\text{ph}})^2}{h(r_O)^2}$$

Example:

$$\frac{\omega_p(r)^2}{\omega_0^2} = \beta_0 \left(\frac{m}{r}\right)^{3/2}$$

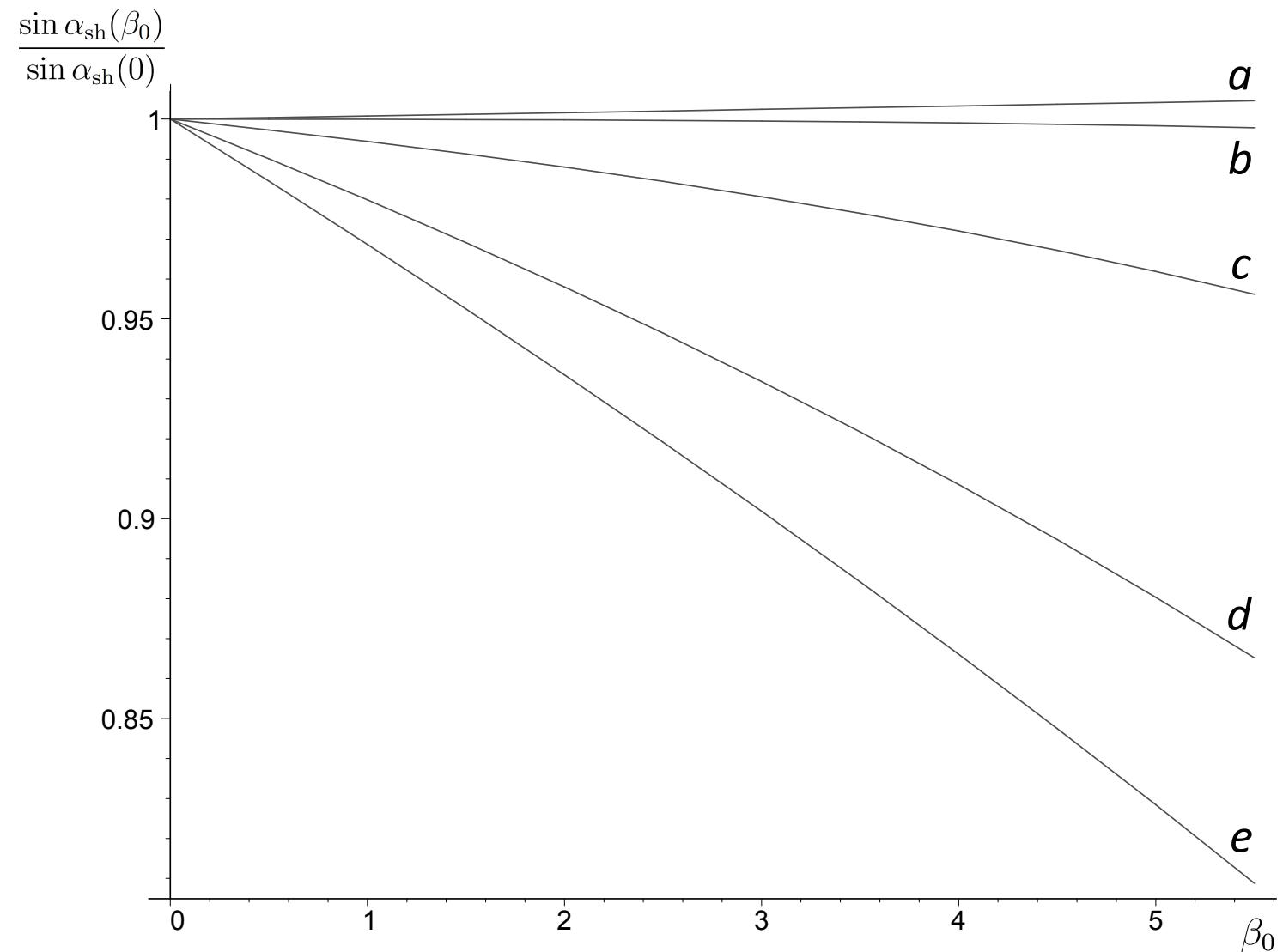
$$a : r_0 = 3.3 \text{ m}$$

$$b : r_0 = 3.8 \text{ m}$$

$$c : r_0 = 5 \text{ m}$$

$$d : r_0 = 10 \text{ m}$$

$$e : r_0 = 50 \text{ m}$$



2.3 Arbitrary spherically symmetric and static spacetime

Metrik:

$$g_{ik}(x)dx^i dx^k = -A(r)c^2 dt^2 + B(r)dr^2 + D(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

Spherically symmetric and static plasma frequency: $\omega_p(r)$

Hamiltonian:

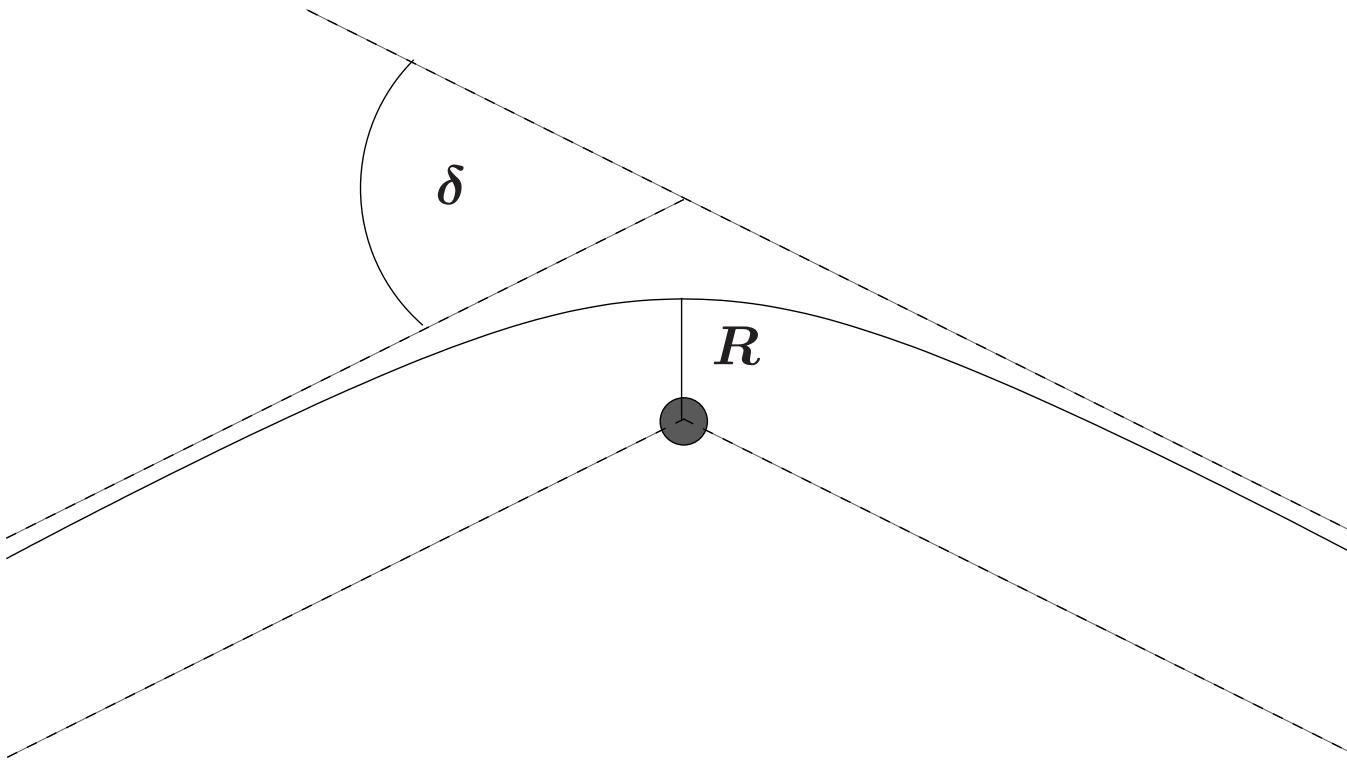
$$\mathcal{H}(x, p) = \frac{1}{2} \left(-\frac{p_t^2}{A(r)c^2} + \frac{p_r^2}{B(r)} + \frac{1}{D(r)} \left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} \right) + \omega_p(r)^2 \right)$$

Constant of motion: $\omega_0 = -\frac{p_t}{c}$

Frequency of photon with respect to $U^i = \delta_t^i A(r)^{-1/2}$:

$$\omega(r) = -\frac{1}{c} p_i U^i = \omega_0 A(r)^{-1/2}$$

Deflection angle δ

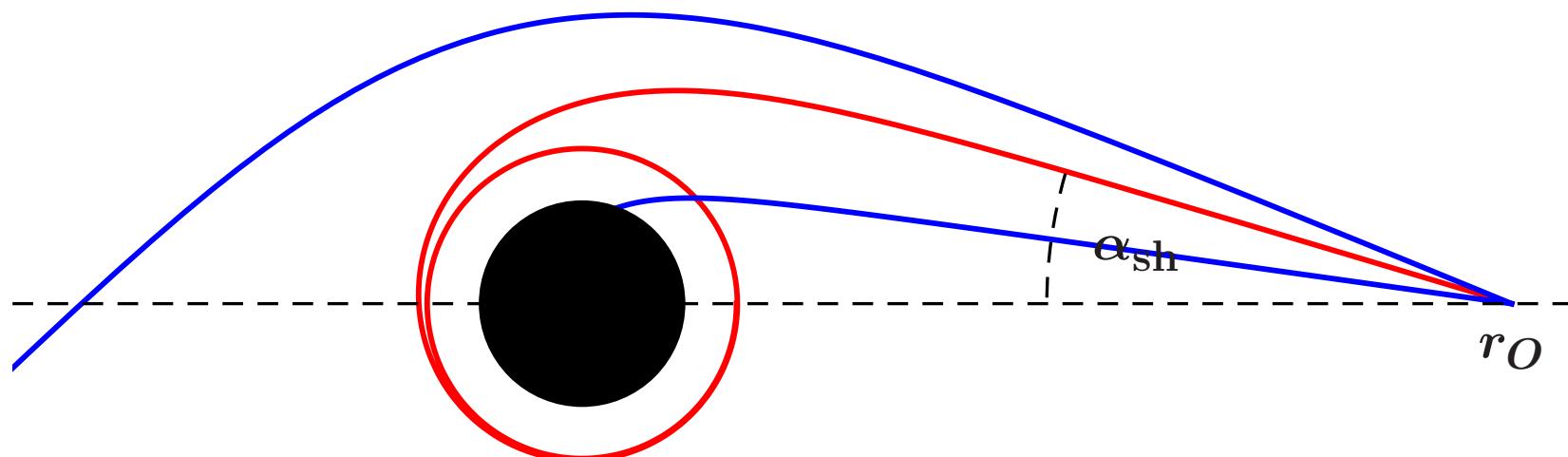


$$\Delta\varphi = \pi + \delta = 2 \int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{h(r)^2}{h(R)^2} - 1 \right)^{-1/2} dr$$

$$h(r)^2 = \frac{D(r)}{A(r)} \left(1 - A(r) \frac{\omega_p(r)^2}{\omega_0^2} \right)$$

Condition for photon sphere:

$$\left. \frac{dh(r)^2}{dr} \right|_{r=r_{\text{ph}}} = 0, \quad h(r)^2 = \frac{D(r)}{A(r)} \left(1 - A(r) \frac{\omega_p(r)^2}{\omega_0^2} \right)$$

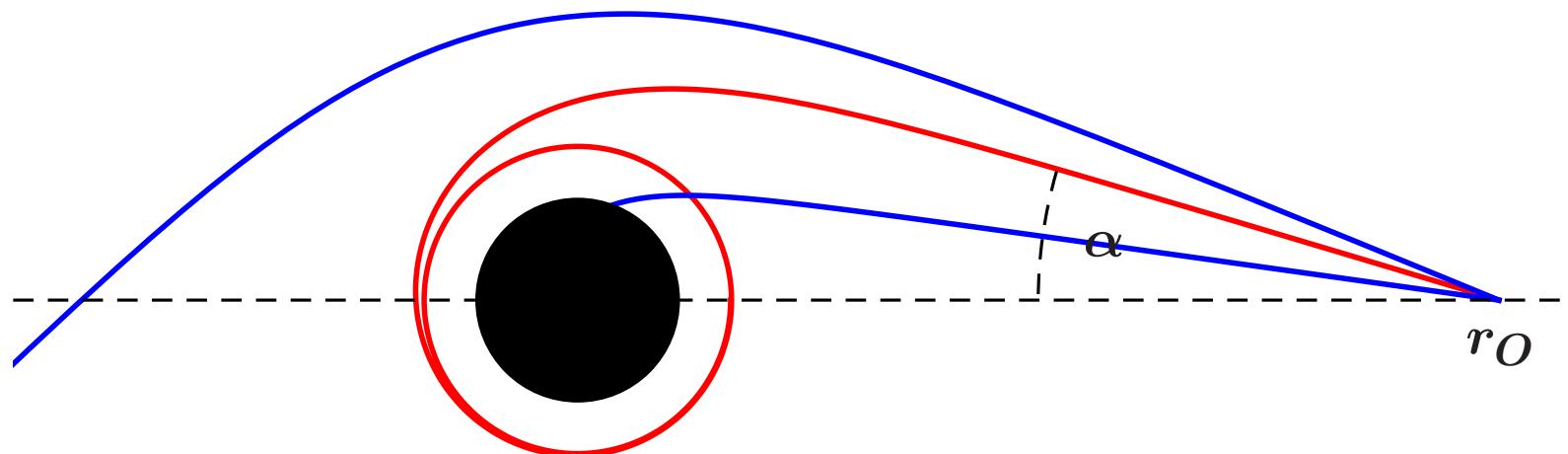


Angular radius α_{sh} of shadow:

$$\sin^2 \alpha_{\text{sh}} = \frac{h(r_{\text{ph}})^2}{h(r_O)^2}$$

Example 1: Ultracompact star

Spherically symmetric dark star with radius between $2m$ and $r_{\text{ph}} (= 3m)$



Ultracompact stars are very good “black-hole impostors”

Lensing features indistinguishable from Schwarzschild black hole

It seems that ultracompact objects are necessarily unstable, see

V. Cardoso, L. Crispino, C. Macedo, H. Okawa, P. Pani: Phys. Rev. D 90, 044069 (2014)

Example 2: Ellis wormhole

$$g_{ik}dx^i dx^k = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

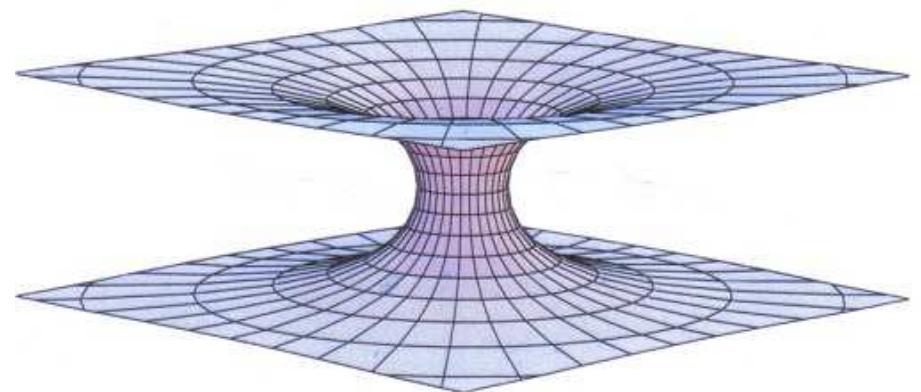
H. G. Ellis, J. Math. Phys. 14, 104 (1973)

Condition for photon sphere:

$$r_{\text{ph}} \left(1 - \frac{\omega_p(r_{\text{ph}})^2}{\omega_0^2} \right) = (r_{\text{ph}}^2 + a^2) \frac{\omega_p(r_{\text{ph}})\omega'_p(r_{\text{ph}})}{\omega_0^2}$$

Angular radius of shadow:

$$\sin^2\alpha_{\text{sh}} = \frac{(r_{\text{ph}}^2 + a^2)}{(r_O^2 + a^2)} \frac{(\omega_0^2 - \omega_p(r_{\text{ph}})^2)}{(\omega_0^2 - \omega_p(r_O)^2)}$$



Homogeneous plasma ($\omega_p(r) = \text{const.}$):

$$r_{\text{ph}} = 0, \quad \sin^2\alpha_{\text{sh}} = \frac{a^2}{r_O^2 + a^2}$$

3. Kerr spacetime

3.1 Kerr lensing in a vacuum

Metric in Boyer-Lindquist coordinates:

$$g_{ik}dx^i dx^k = \varrho(r, \vartheta)^2 \left(\frac{dr^2}{\Delta(r)} + d\vartheta^2 \right) + \frac{\sin^2 \vartheta}{\varrho(r, \vartheta)^2} \left(\textcolor{red}{a} dt - (r^2 + \textcolor{red}{a}^2)d\varphi \right)^2$$

$$- \frac{\Delta(r)}{\varrho(r, \vartheta)^2} \left(dt - \textcolor{red}{a} \sin^2 \vartheta d\varphi \right)^2$$

$$\varrho(r, \vartheta)^2 = r^2 + \textcolor{red}{a}^2 \cos^2 \vartheta, \quad \Delta(r) = r^2 - 2\textcolor{red}{m}r + \textcolor{red}{a}^2.$$

$$\textcolor{red}{m} = \frac{GM}{c^2} \quad , \quad \textcolor{red}{a} = \frac{J}{Mc}$$

Horizons at $r_{\pm} = m \pm \sqrt{m^2 - a^2}$ if $0 \leq a^2 \leq m^2$

Lightlike geodesics (with Mino parametrisation):

$$\dot{t} = a(L - Ea \sin^2 \vartheta) + \frac{(r^2 + a^2)((r^2 + a^2)E - aL)}{\Delta(r)},$$

$$\dot{\varphi} = \frac{L - Ea \sin^2 \vartheta}{\sin^2 \vartheta} + \frac{(r^2 + a^2)aE - a^2L}{\Delta(r)},$$

$$\dot{\vartheta}^2 = K - \frac{(L - Ea \sin^2 \vartheta)^2}{\sin^2 \vartheta} =: \Theta(\vartheta),$$

$$\dot{r}^2 = -K\Delta(r) + ((r^2 + a^2)E - aL)^2 =: R(r).$$

Constants of motion $E = -p_t, L = p_\varphi, K = \text{Carter constant}$

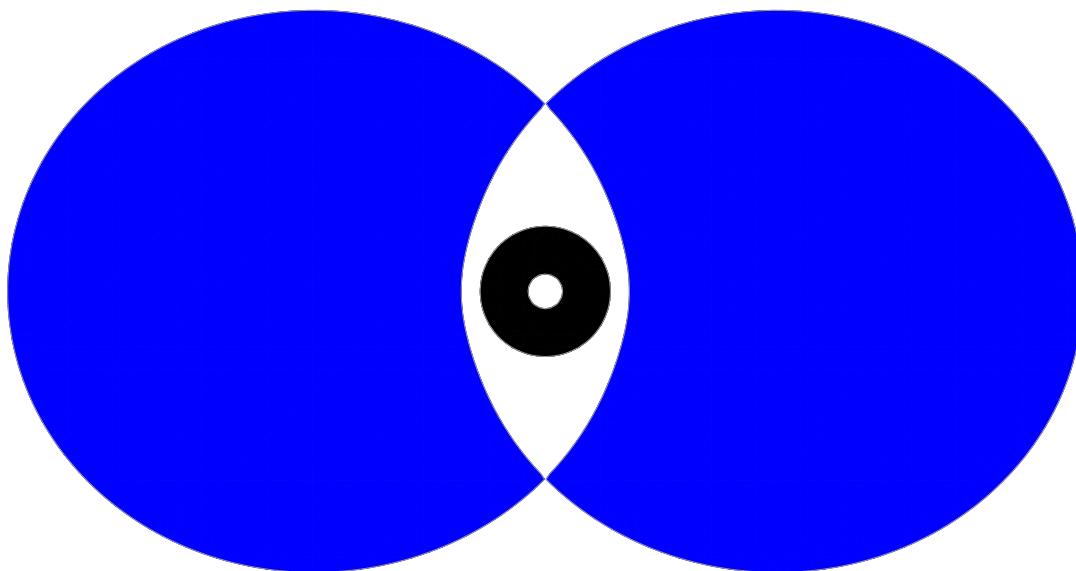
Spherical lightlike geodesics exist in the region where

$$R(r) = 0, \quad R'(r) = 0, \quad \Theta(\vartheta) \geq 0.$$

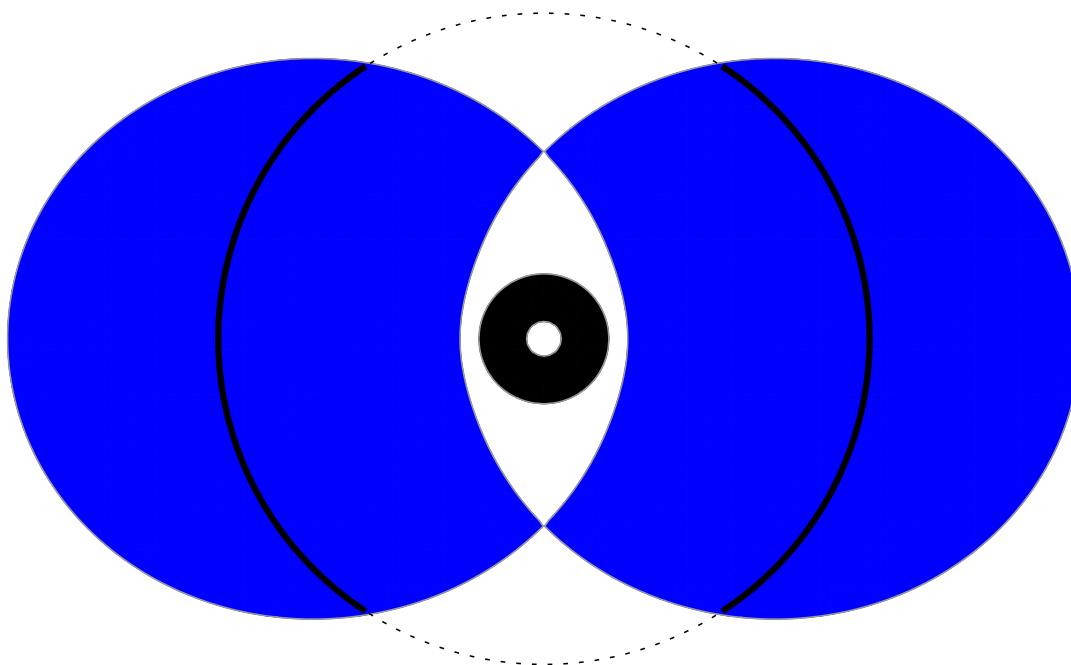
$$(2r\Delta(r) - (r - m)\varrho(r, \vartheta)^2)^2 \leq 4a^2r^2\Delta(r) \sin^2 \vartheta$$

(unstable if $R''(r) \geq 0$)

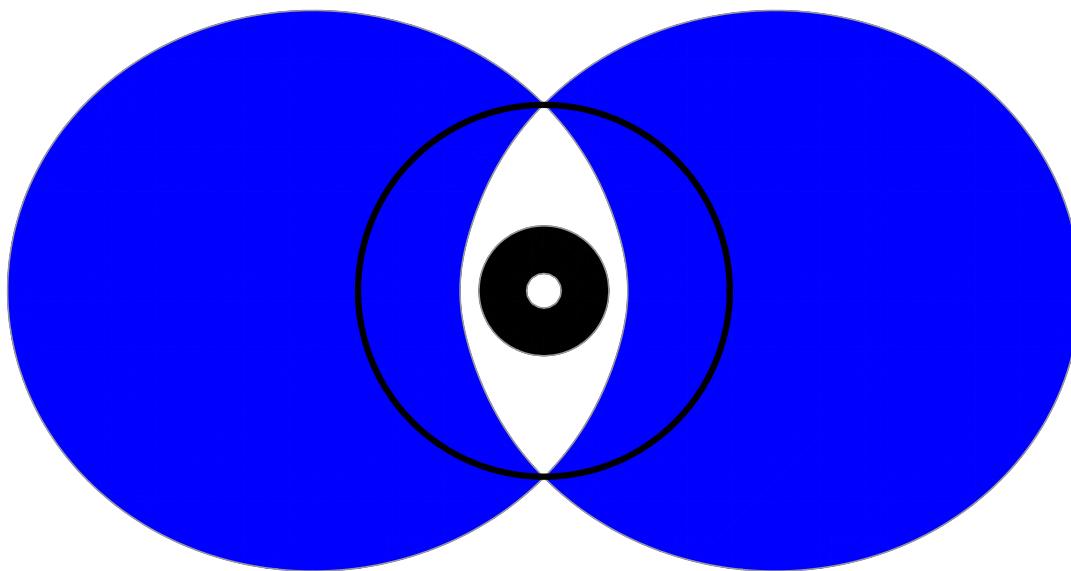
Photon region for Kerr black hole with $a = 0.75 m$



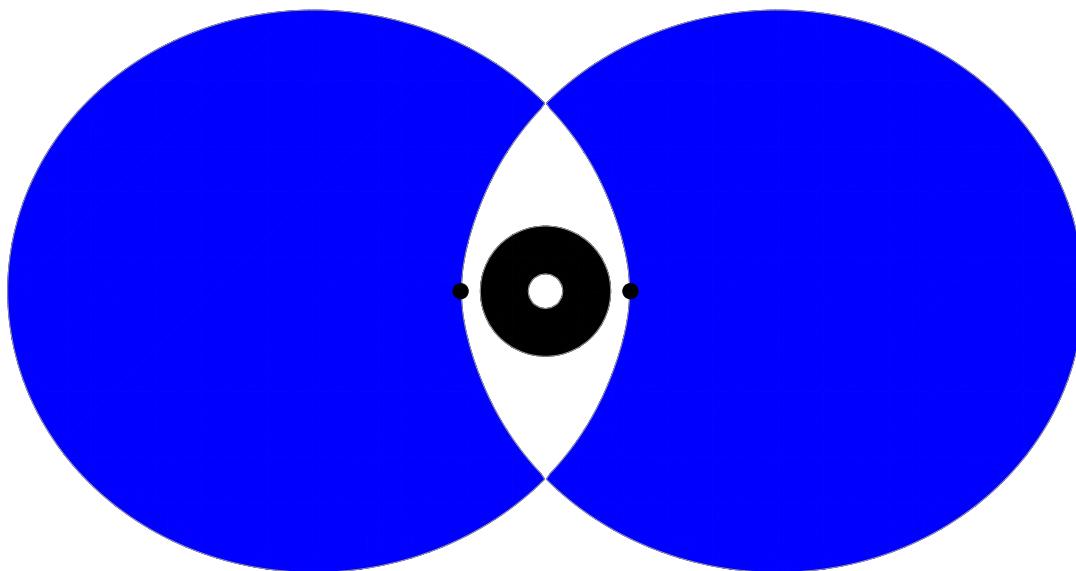
Photon region for Kerr black hole with $a = 0.75 m$



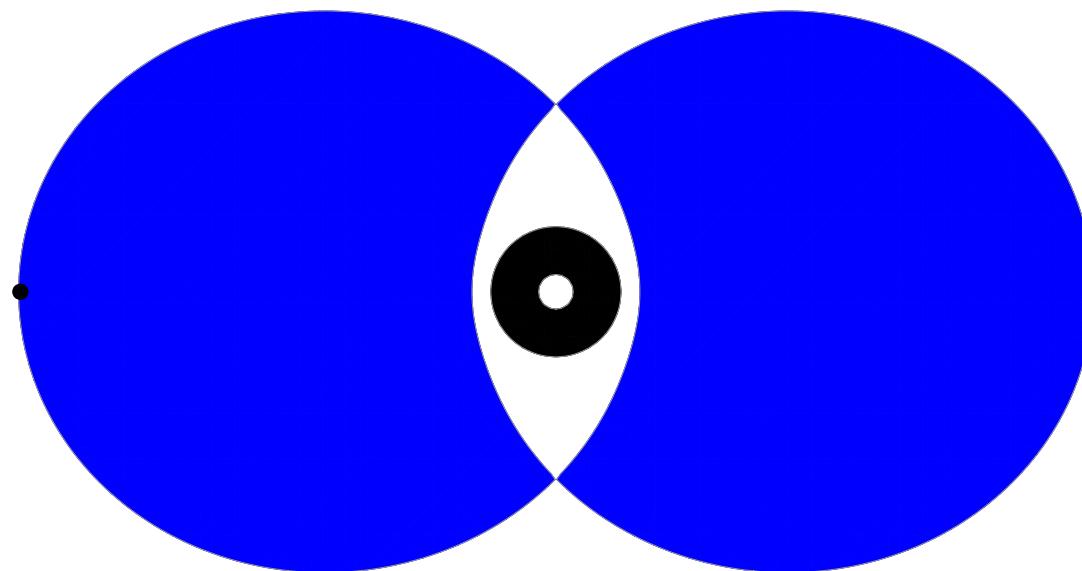
Photon region for Kerr black hole with $a = 0.75 m$



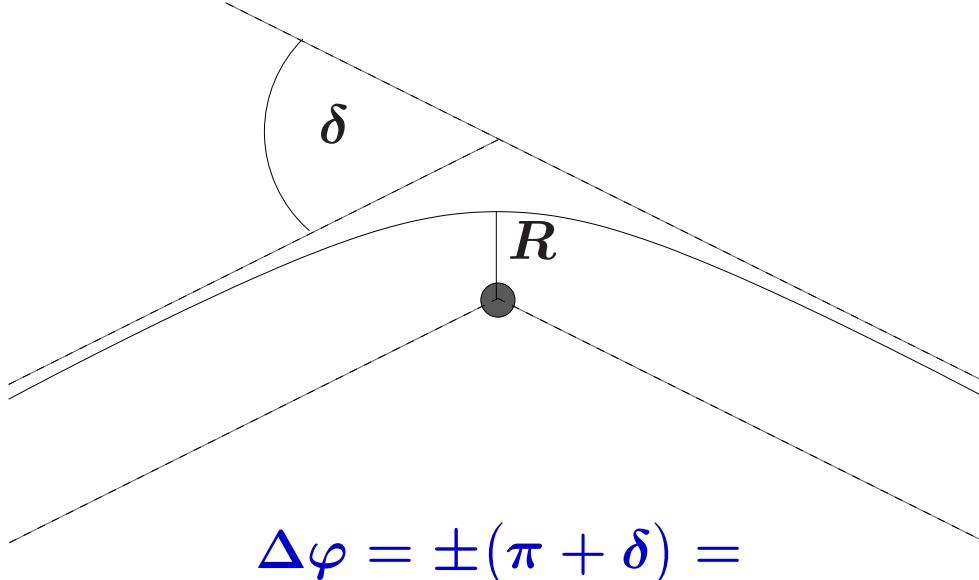
Photon region for Kerr black hole with $a = 0.75 m$



Photon region for Kerr black hole with $a = 0.75 m$



Bending angle for light rays in the equatorial plane



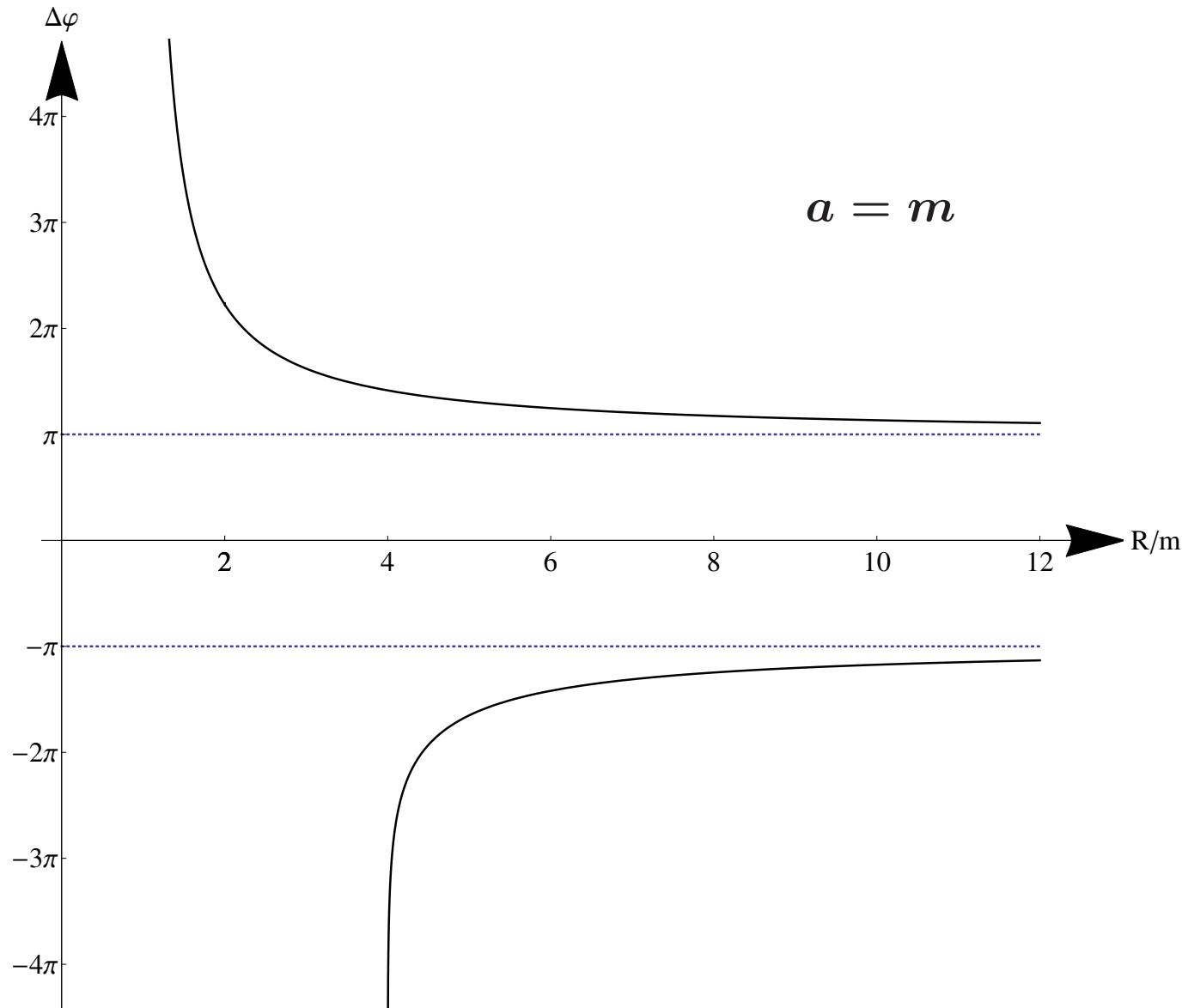
$$\pm 2 \int_R^\infty \left(\frac{(R - 2m)^2 r^2 \Delta(r)}{(2ma(R - r) \pm (r - 2m)R \sqrt{\Delta(r)})^2} - 1 \right)^{-1/2} \frac{\sqrt{r(r - 2m)} dr}{\Delta(r)}$$

δ diverges at corotating (r_{ph}^+) resp. counterrotating (r_{ph}^-) photon circle in the equatorial plane

$$r_{\text{ph}}^3 - 6mr_{\text{ph}}^2 + 9m^2r_{\text{ph}} - 4ma^2 = 0$$

$$m < r_{\text{ph}}^+ < 3m < r_{\text{ph}}^- < 4m$$

Bending angle in the equatorial plane of extreme Kerr spacetime



$\delta = |\Delta\varphi - \pi|$ diverges at $r_{\text{ph}}^+ = m$ and $r_{\text{ph}}^- = 4m$

Shadow no longer circular

Shape of the shadow of a Kerr black hole for observer at infinity:

J. Bardeen in C. DeWitt and B. DeWitt (eds.): “Black holes” Gordon & Breach (1973)

cf. S. Chandrasekhar: “The mathematical theory of black holes” Oxford UP (1983)

Shape and size of the shadow for black holes of the Plebański-Demiański class (including Kerr) for observer at coordinates (r_O, ϑ_O) (analytical formulas):

A. Grenzebach, VP, C. Lämmerzahl: Phys. Rev. D 89, 124004 (2014), Int. J. Mod. Phys. D 24, 1542024 (2015)

For observer at (r_O, ϑ_O) :

Relation between constants of motion $K_E = \frac{K}{E^2}$, $L_E = \frac{L}{E} - a$ and celestial coordinates θ, ψ :

$$\sin \theta = \frac{\sqrt{\Delta(r_O)} \ K_E}{r_O^2 - aL_E}, \quad \sin \psi = \frac{L_E + a \cos^2 \vartheta_O}{\sqrt{K_E} \ \sin \vartheta_O}$$

Expressing constants of motion $K_E = \frac{K}{E^2}$, $L_E = \frac{L}{E} - a$ in terms of the radius coordinate r of limiting spherical lightlike geodesic

$$K_E = \frac{16r^2 \Delta(r)}{(\Delta'(r))^2}, \quad aL_E = \left(r^2 - \frac{4r\Delta(r)}{\Delta'(r)} \right)$$

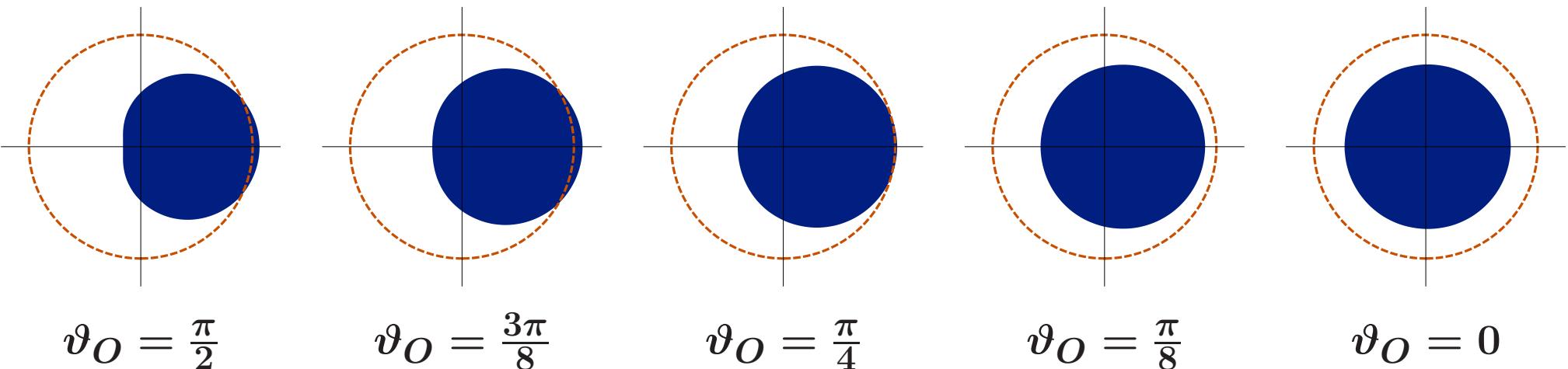
gives boundary curve of the shadow on the observer's sky $\theta(r), \psi(r)$

Vertical angular radius α_v of the shadow ($\vartheta_O = \pi/2$):

$$\sin^2 \alpha_v = \frac{27m^2 r_O^2 (a^2 + r_O(r_O - 2m))}{r_O^6 + 6a^2 r_O^4 + 3a^2(4a^2 - 9m^2)r_O^2 + 8a^6} = \frac{27m^2}{r_O^2} \left(1 + O(m/r_O)\right)$$

Up to terms of order $O(m/r_O)$, Synge's formula is still correct for the vertical diameter of the shadow

Shadow of black hole with $a = m$ for observer at $r_O = 5m$



3.2 Kerr lensing in a plasma

Metric in Boyer-Lindquist coordinates:

$$g_{ik}dx^i dx^k = \varrho(r, \vartheta)^2 \left(\frac{dr^2}{\Delta(r)} + d\vartheta^2 \right) + \frac{\sin^2 \vartheta}{\varrho(r, \vartheta)^2} \left(a dt - (r^2 + a^2)d\varphi \right)^2$$

$$- \frac{\Delta(r)}{\varrho(r, \vartheta)^2} \left(dt - a \sin^2 \vartheta d\varphi \right)^2$$

$$\varrho(r, \vartheta)^2 = r^2 + a^2 \cos^2 \vartheta, \quad \Delta(r) = r^2 - 2mr + a^2 .$$

Plasma frequency: $\omega_p(r, \vartheta)$

Hamiltonian:

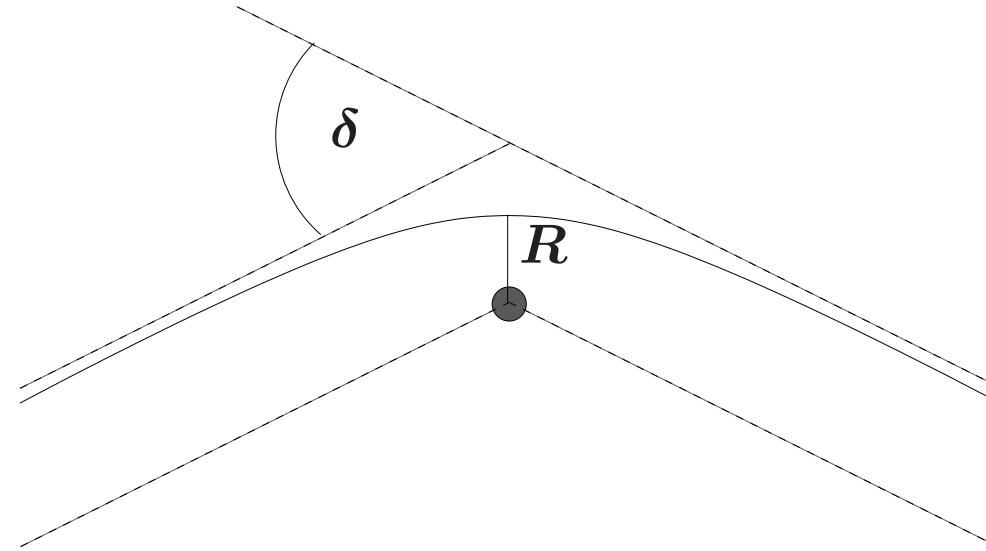
$$\mathcal{H}(x, p) = \frac{1}{2} \left(- \left(r^2 + a^2 + \frac{2mra^2 \sin^2 \vartheta}{\varrho(r, \vartheta)^2} \right) \frac{p_t^2}{c^2 \Delta(r)} - \frac{4mr a p_t p_\varphi}{c \varrho(r, \vartheta)^2 \Delta(r)} \right.$$

$$\left. + \left(1 - \frac{2mr}{\varrho(r, \vartheta)^2} \right) \frac{p_\varphi^2}{\Delta(r) \sin^2 \vartheta} + \frac{\Delta(r) p_r^2}{\varrho(r, \vartheta)^2} + \frac{p_\vartheta^2}{\varrho(r, \vartheta)^2} + \omega_p(r, \vartheta) \right)$$

Constant of motion: $\omega_0 = -p_t/c$

Bending angle for light rays in the equatorial plane for a plasma frequency $\omega_p(r)$:

$$\Delta\varphi = \pm(\pi + \delta)$$



$$= \pm 2 \int_R^\infty \sqrt{\frac{r(r-2m)}{(R-2m)^2 u(r)^2}} \frac{dr}{(\Delta(r))}$$

$$\sqrt{\frac{r(r-2m)}{(2ma(R-r) \pm (r-2m)u(R))^2} - 1}$$

$$u(r) = r \sqrt{\Delta(r)} \sqrt{1 - \frac{\omega_p(r)^2}{\omega_0^2} \left(1 - \frac{2m}{r}\right)}$$

VP: “Ray optics, Fermat’s principle and applications to general relativity”
Springer (2000)

Bending angle in the equatorial plane of extreme Kerr spacetime

Solid:

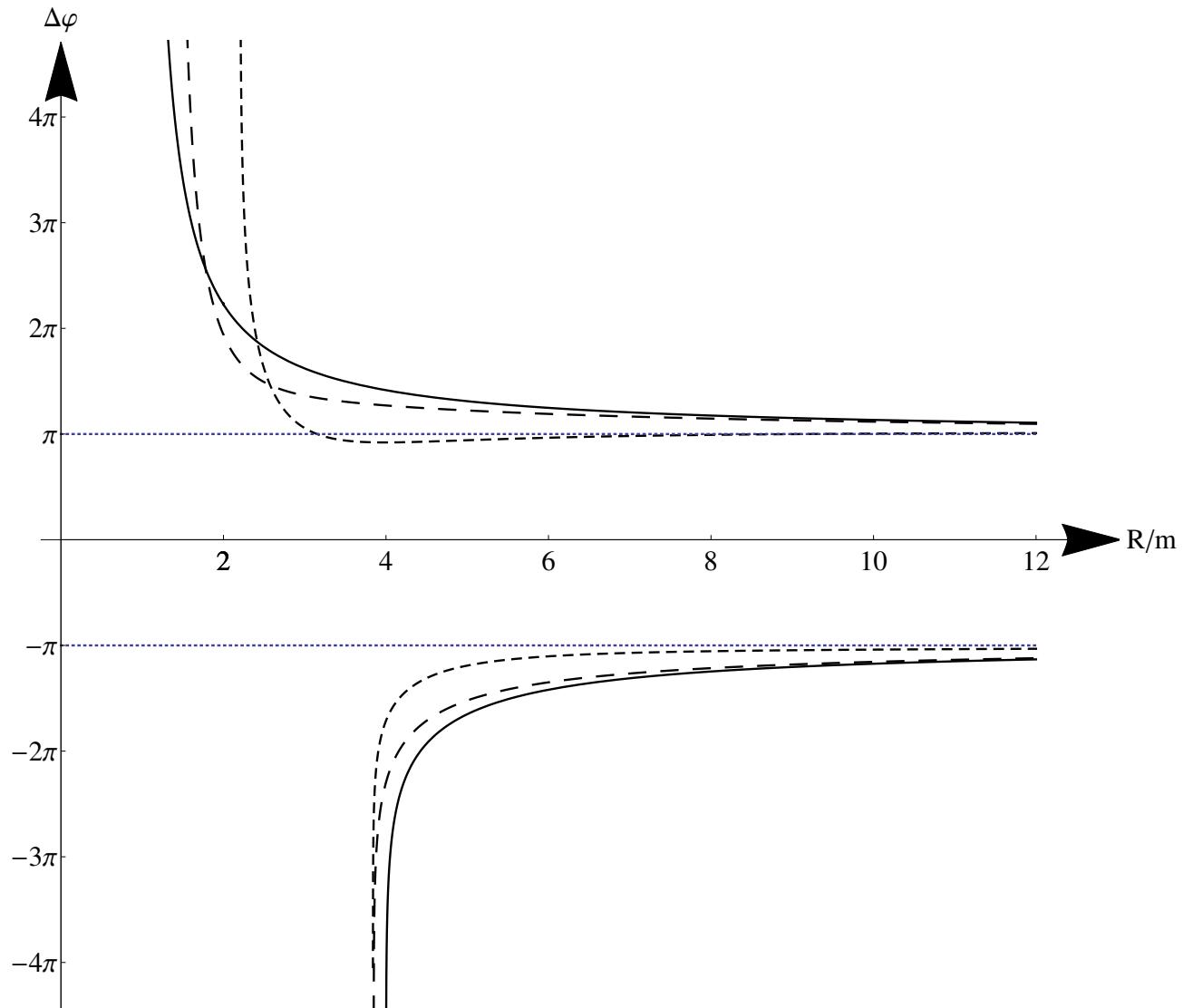
$$\omega_p(r)^2 = 0$$

Narrow-dashed:

$$\omega_p(r)^2 = 10 \omega_0^2 \left(\frac{m}{r}\right)^{3/2}$$

Wide-dashed:

$$\omega_p(r)^2 = 10 \omega_0^2 \left(\frac{m}{r}\right)^{5/2}$$



Construction of shadow in a plasma:

Condition for separability (existence of a Carter constant)

$$\omega_p(r, \vartheta)^2 = \frac{f_r(r) + f_\vartheta(\vartheta)}{r^2 + a^2 \cos^2 \vartheta}$$

Photon region:

$$\left(\frac{r^2 \Delta}{(r - m)^2} \left(1 \pm \sqrt{1 - f'_r(r) \frac{(r - m)}{2r^2 \omega_0^2}} \right)^2 - \frac{f_r(r) + f_\vartheta(\vartheta)}{\omega_0^2} \right) a^2 \sin^2 \vartheta \geq$$
$$\left(\frac{1}{(r - m)} \left(m(a^2 - r^2) \pm r \Delta \sqrt{1 - f'_r(r) \frac{(r - m)}{2r^2 \omega_0^2}} \right) + a^2 \sin^2 \vartheta \right)^2$$

For observer at (r_O, ϑ_O) , celestial coordinates as functions of constants of motion:

$$\sin \theta = \sqrt{\frac{(K - f_\vartheta(\vartheta_O))\Delta(r_O)}{\left((r_O^2 + a^2)\omega_0 + ap_\varphi\right)^2 - (f_r(r_O) + f_\vartheta(\vartheta_O))\Delta(r_O)}}$$

$$\sin \psi = \frac{-p_\varphi - a \sin^2 \vartheta_O \omega_0}{\sin \vartheta_O \sqrt{K - f_\vartheta(\vartheta_O)}}$$

Constants of motion as functions of radius coordinate of limit curve:

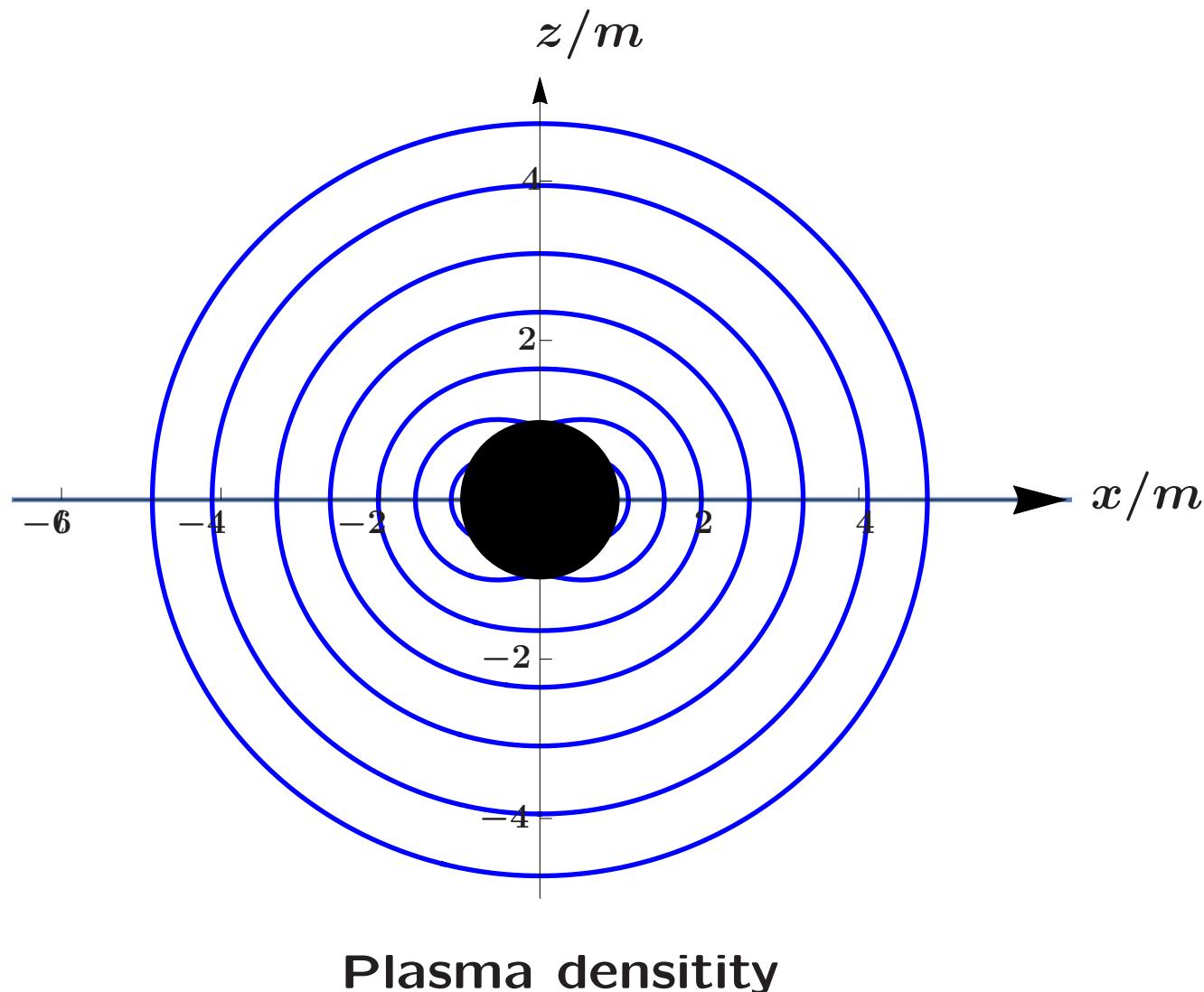
$$ap_\varphi = \frac{\omega_0}{(r - m)} \left(m(a^2 - r^2) \pm r\Delta \sqrt{1 - f'_r(r) \frac{(r - m)}{2r^2\omega_0^2}} \right)$$

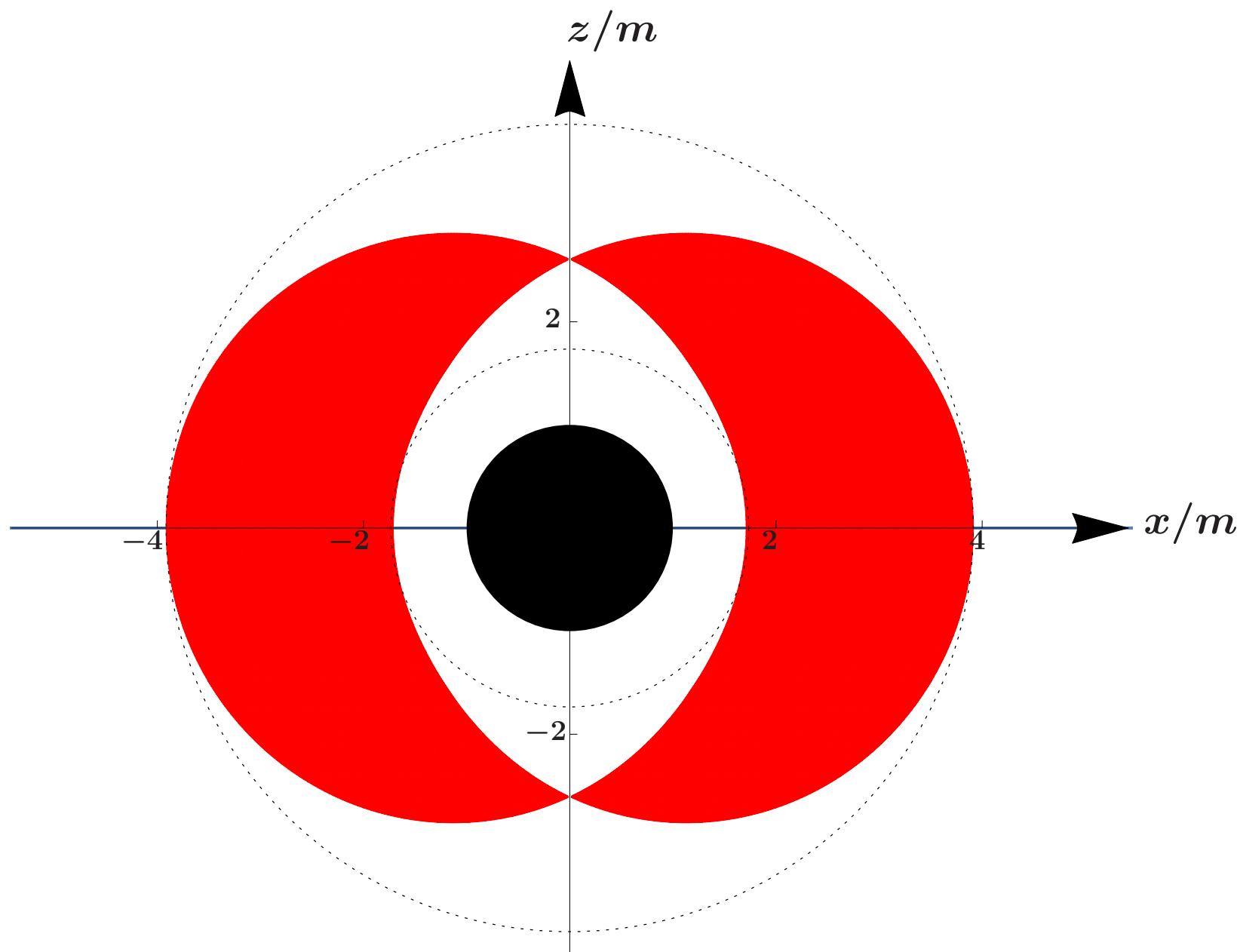
$$K = \frac{r^2 \Delta \omega_0^2}{(r - m)^2} \left(1 \pm \sqrt{1 - f'_r(r) \frac{(r - m)}{2r^2\omega_0^2}} \right)^2 - f_r(r)$$

Gives boundary curve of the shadow $(\theta(r), \psi(r))$

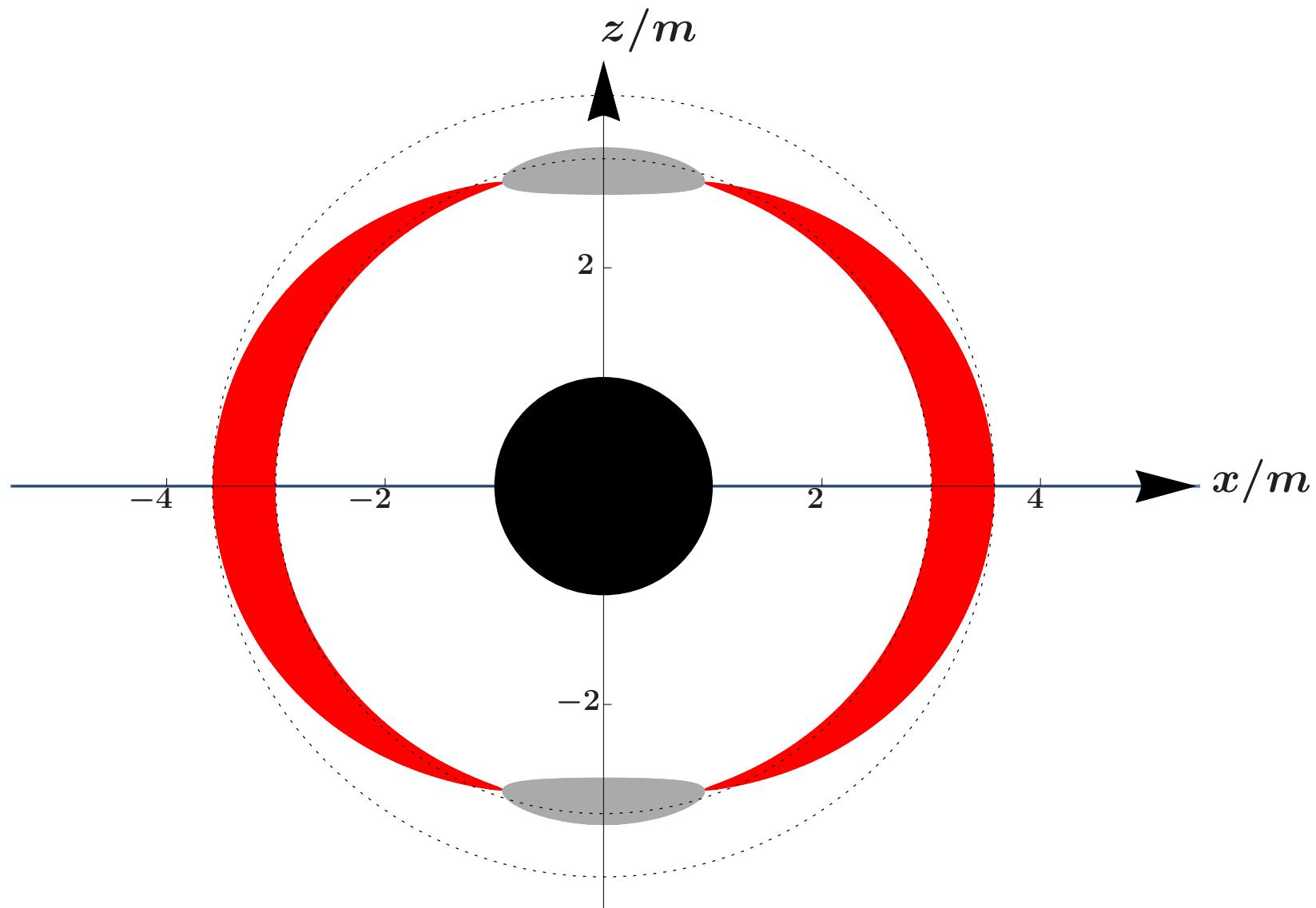
First example:

$$a = 0.999 \text{ m}, \quad \omega_p(r, \vartheta)^2 = \frac{\omega_c^2 \sqrt{m^3 r}}{r^2 + a^2 \cos^2 \vartheta}, \quad \omega_c = \text{constant}$$

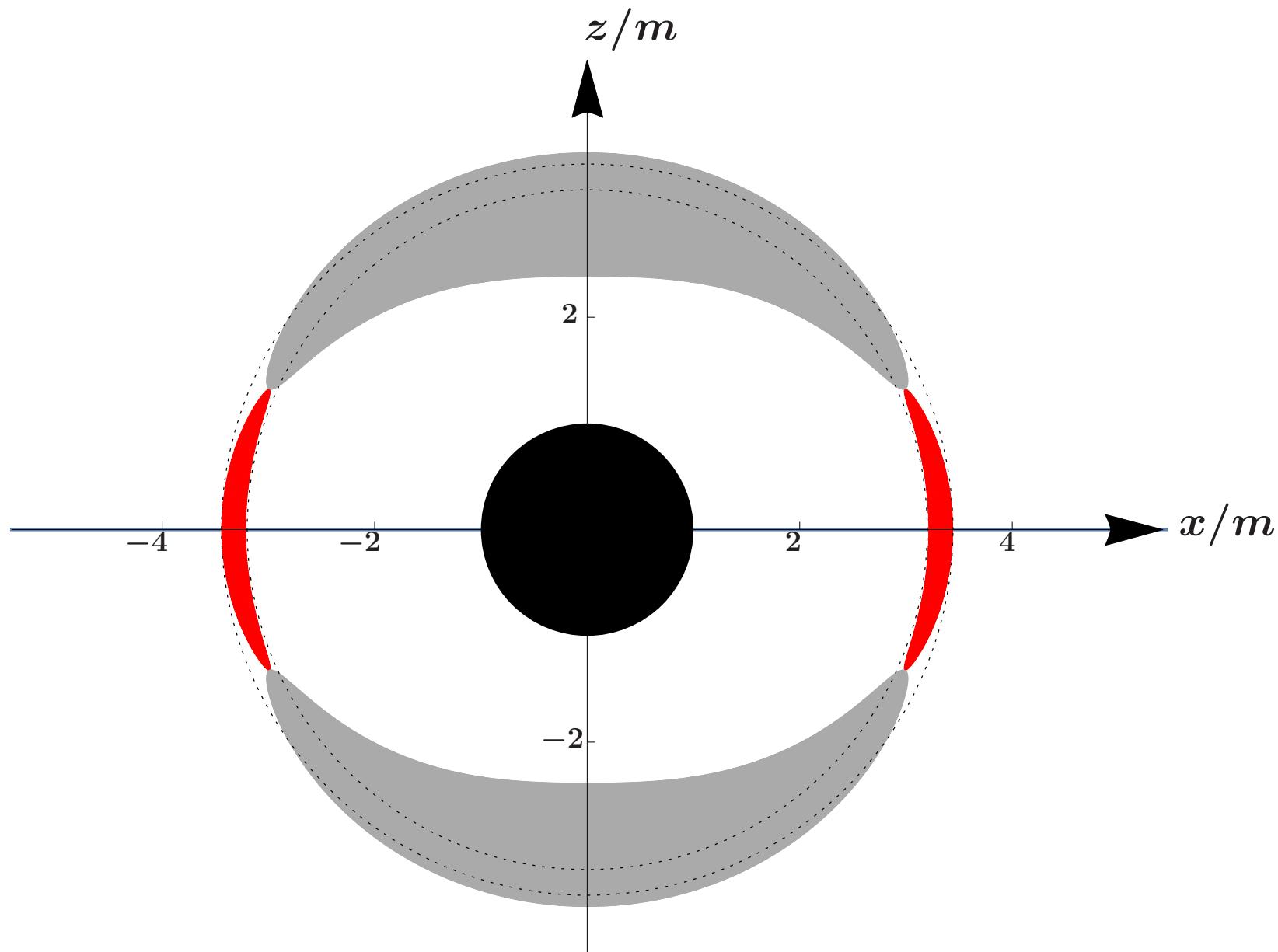




Photon region, $\omega_c^2/\omega_0^2 = 7$

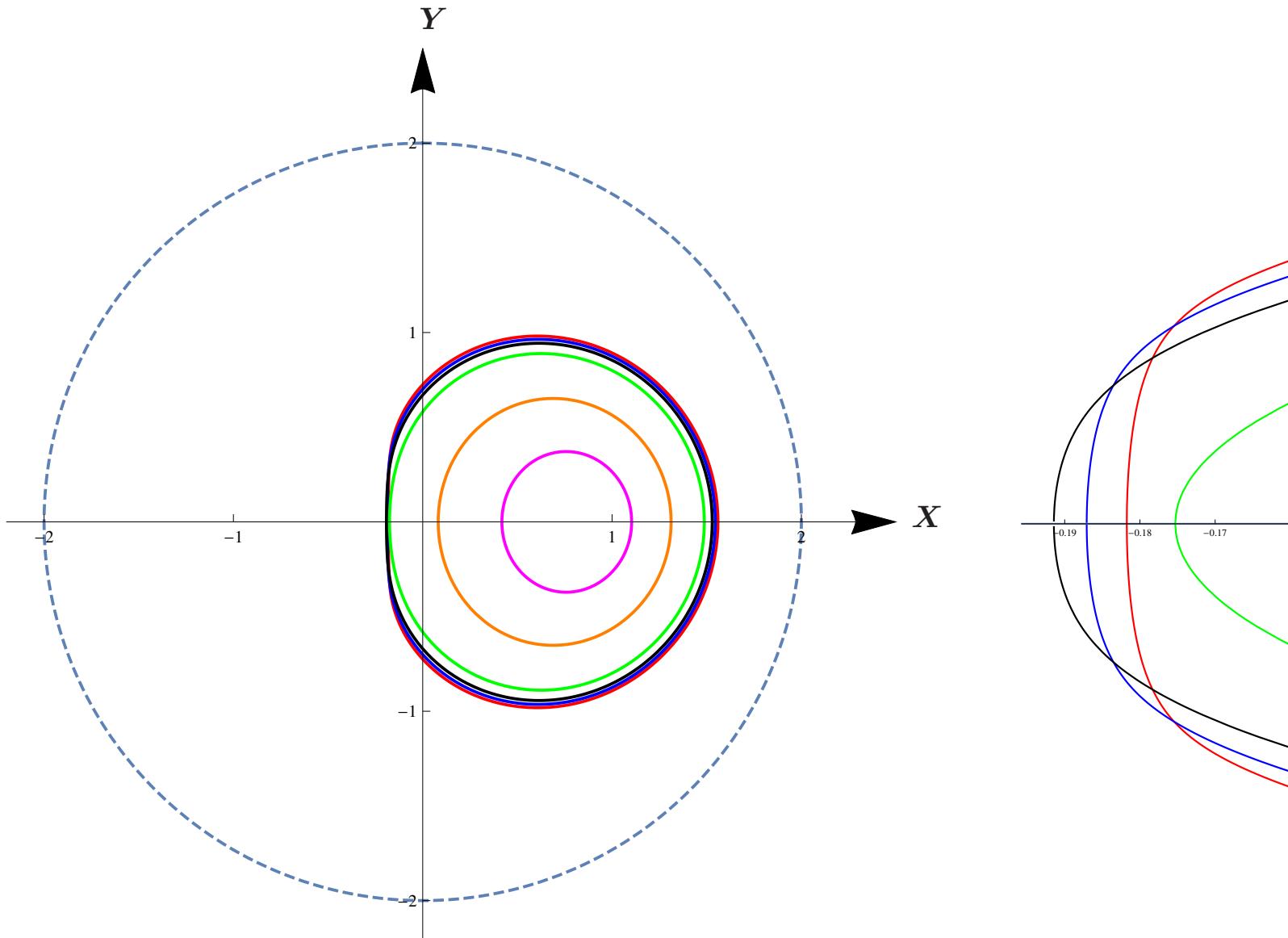


Photon region, $\omega_c^2/\omega_0^2 = 14.5$



Photon region, $\omega_c^2/\omega_0^2 = 15.1$

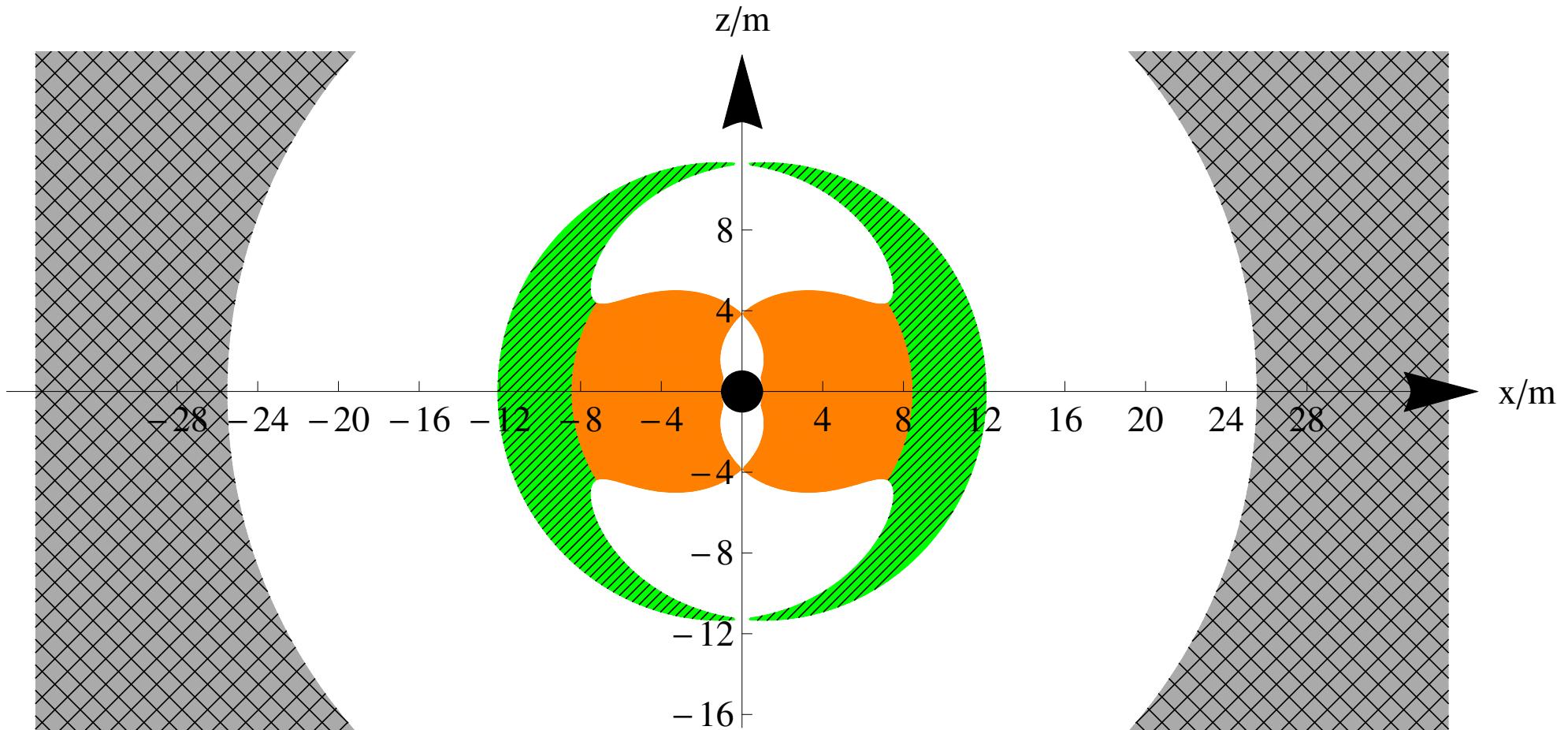
Shadow for $r_O = 5 \text{ m}$, $\vartheta_O = \pi/2$



Shadow shrinks with increasing ω_c^2/ω_0^2

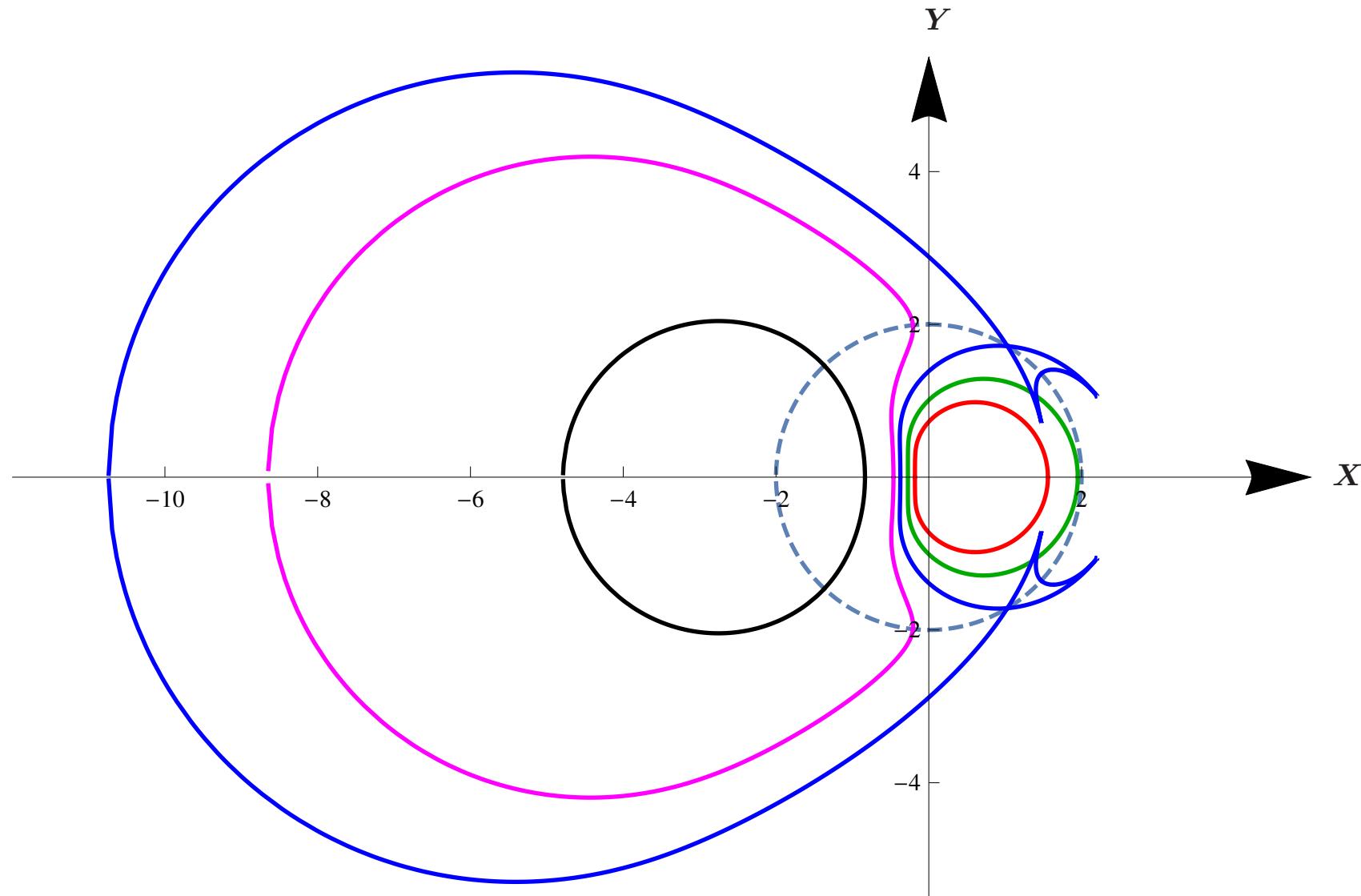
Second example:

$$a = 0.999 \text{ m}, \omega_p(r, \vartheta) = \omega_c = \text{constant}$$



Photon region, $\omega_c^2/\omega_0^2 = 1.085$

$$r_O = 5 \text{ m}, \vartheta_O = \pi/2$$



Shadow grows with increasing ω_c^2/ω_0^2 and forms “fishtails”