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The transition from inertial to viscous flow in capillary rise

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Abstract

We investigate the initial moments of capillary rise of liquids in a tube. In this period both inertia and viscous flow losses balance the pressure generated by the meniscus curvature (capillary pressure). It is known that the very first stage is purely dominated by inertial forces, where subsequently the influence of viscosity increases (visco-inertial flow). Finally the effect of inertia vanishes and the flow becomes purely viscous. In this study we derive the times and meniscus heights at which the transition between the time periods occur. This is done in an attempt to provide a method to determine a priori which terms of the momentum balance are relevant for a given problem. Analytic solutions known from previous literature are discussed and the time intervals of their validity compared. The predicted transition times and the calculated heights show good agreement with experimental results from literature. The results are also discussed in dimensionless form and the limitations of the calculations are pointed out.

Key words: Capillary tube, Analytic solution, Capillary rise, Lucas-Washburn equation, Washburn equation, Imbibition, Liquid penetration, Flow regimes

1. Introduction

When dealing with the problem of capillary rise it is of great interest to know which forces (e.g. inertia, viscous forces, gravity) are dominant. This is due to the fact that all equations that can be used to predict the meniscus height have underlying assumptions. These assumptions are mostly the neglect of certain forces. This, however, limits the validity of the derived equations to certain time intervals where these forces can actually be neglected [1–4]. Stange et al. [5,6] separate the individual time stages by means of dimensionless numbers. There are also approaches to solve the full momentum balance numerically as done in [7,8]. Ichikawa and Satoda [9] compare several previous works, present experimental results and conduct a dimensional analysis. Quere et al. [10,11] investigate the inertia dominated flow period. Some publications focus on the effect of the dynamic contact angle [12–14]. Subsequent time stages ($t \gg 0$) with influence of gravity are discussed in [15,16].

In this paper we now want to shed some light on the different stages of capillary rise and the transitions between them. The momentum balance of a liquid inside a capillary tube shows that the capillary pressure must be balanced by the inertial forces, the viscous forces and the hydrostatic pressure (e.g.[4,15])

$$\frac{2\sigma\cos\theta}{R} = \frac{d\,(\rho h\dot{h})}{dt} + \frac{8\mu h}{R^2}\dot{h} + \rho g h. \qquad (1)$$

σ refers to the surface tension, R to the inner tube radius, ρ to the fluid density, g to gravity and μ to the fluid viscosity. Hereby the assumptions are made that there are no inertia or entry effects in the liquid reservoir and that the viscous pressure loss inside the tube is given by the Hagen-Poiseuille law. Most importantly it is assumed that the capillary pressure is constant, and it is calculated using a static contact angle θ and the tube radius R. For a more detailed discussion of this topic please refer to the chapter "Limitations of the model".

2. Analytic solutions for defined time stages

In following several approaches to obtain analytical solutions to the momentum balance are discussed.

2.1. Purely inertial time stage

For the very first moments after the contact of the tube with the liquid Quere [10] takes following approach: Neglecting the viscous and the gravity term in Eq.(1) gives

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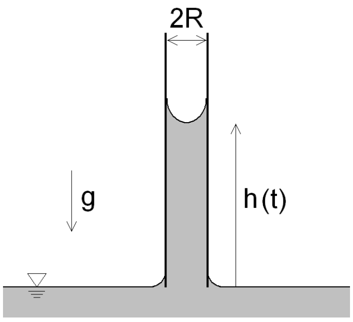

Fig. 1. Liquid rise in a capillary tube

$$\frac{2\sigma\cos\theta}{\rho R} = \frac{d\,(h\dot{h})}{dt} = \dot{h}^2 + h\ddot{h}. \quad (2)$$

Quere solves the differential equation giving a capillary rise with constant velocity

$$h = t\sqrt{\frac{2\sigma\cos\theta}{\rho R}}. \quad (3)$$

2.2. *Visco-inertial time stage*

Bosanquet [4] finds a solution featuring the inertial and viscous term resulting in following differential equation

$$\frac{d}{dt}\left(h\dot{h}\right) + ah\dot{h} = b, \quad (4)$$

with

$$a = \frac{8\mu}{R^2\rho}, \quad (5)$$

and

$$b = \frac{2\sigma\cos\theta}{R\rho}. \quad (6)$$

He obtains

$$h^2 = \frac{2b}{a}\left[t - \frac{1}{a}(1 - e^{-at})\right], \quad (7)$$

which is also used by Ichikawa and Satoda [9] in dimensionless form. Note: For $t \to \infty$ Eq. (7) converges into the Lucas-Washburn equation which will be presented next.

2.3. *Purely viscous time stage*

For the intermediate flow period Lucas [1] and Washburn [2] neglect the influence of inertia and the influence of gravity. They find

$$h^2 = \frac{\sigma R\cos\theta}{2\mu}t. \quad (8)$$

2.4. *Viscous and gravitational time stage*

During the later stages of capillary rise gravity can no longer be neglected. Fries and Dreyer [16] show that for $h > 0.1\,h_{eq}$ gravity has to be considered. h_{eq} is the equilibrium height where the hydrostatic pressure balances the capillary pressure (see Eq. (13)). Analytic solutions (neglecting inertia) are given by Washburn [2] in implicit form

$$t(h) = -\frac{h}{\beta} - \frac{\alpha}{\beta^2}\ln\left(1 - \frac{\beta h}{\alpha}\right), \quad (9)$$

and by Fries and Dreyer [16] in explicit form

$$h(t) = \frac{\alpha}{\beta}\left[1 + W(-e^{-1-\frac{\beta^2 t}{\alpha}})\right]. \quad (10)$$

Here $W(x)$ is the Lambert W function. The constants

$$\alpha = \frac{\sigma R\cos\theta}{4\mu}, \quad (11)$$

and

$$\beta = \frac{\rho g R^2}{8\mu} \quad (12)$$

are used. Finally one can calculate the equilibrium height (where capillary pressure equals hydrostatic pressure) to be [1]

$$h_{eq} = \frac{\alpha}{\beta} = \frac{2\sigma\cos\theta}{\rho g R}. \quad (13)$$

3. Separation of time stages

One can derive three transition times (see Fig. 3):

- t_1 The transition time between the purely inertial and the visco-inertial stage,
- $t_{2,S}$ The time when the solution by Quere and the Lucas-Washburn equation provide the same rise rate [5],
- $t_{2,Q}$ The time when the solution by Quere and the Lucas-Washburn equation provide the same height [10],
- t_3 The transition time between visco-inertial and the purely viscous stage.

As stated the purely inertial flow period shows a rise with constant velocity. Both solutions by Quere Eq. (3) and Bosanquet Eq. (7) show this linear behavior in the beginning. At some point - in contrast to the solution by Quere - Eq. (7) deviates to lower values as viscous effects become more important. We find that point where viscous effects have to be taken into account by following approach: To obtain the time when both solutions have reached a certain level of disagreement (e.g. 3% deviation) we write

$$0.03 = \frac{h_{Quere}(t_1) - h_{Bosanquet}(t_1)}{h_{Quere}(t_1)}. \quad (14)$$

By rearranging one finds

$$t_1 = \frac{0.1856}{a} = \frac{0.0232R^2\rho}{\mu}, \quad (15)$$

and using Eq. (7) provides

$$h_1 = \frac{0.1800\sqrt{b}}{a} = 0.0318\sqrt{\frac{R^3\rho\,\sigma\cos\theta}{\mu^2}}. \quad (16)$$

Stange et al. [5] find the time when the rise rates of the equation by Quere and Lucas-Washburn are equal ($\dot{h}_{Quere} = \dot{h}_{Lucas-Washburn}$) to be

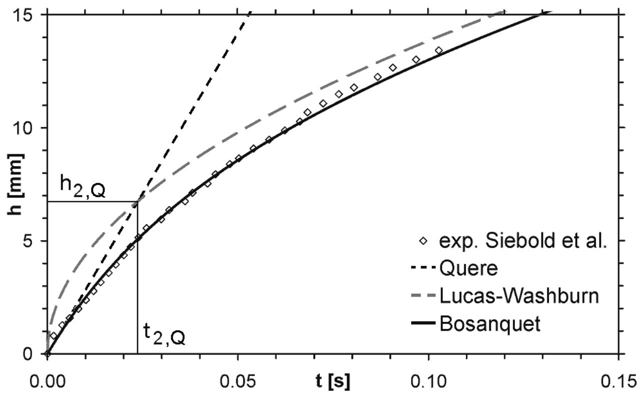

Fig. 2. Comparison of analytic solutions with experimental results by Siebold et al. (pentane in a glass tube with 191 μm radius). A constant contact angle of 73° (as found by Siebold et al.) has been used. Note: This angle differs from the static contact angle.

$$t_{2,S} = \frac{1}{2a} = \frac{R^2\rho}{16\mu}. \tag{17}$$

Quere [10] calculates the time when the heights of his solution and the Lucas-Washburn equation intersect ($h_{Quere} = h_{Lucas-Washburn}$, see Fig. 2). By equating the heights he obtains

$$t_{2,Q} = \frac{2}{a} = \frac{R^2\rho}{4\mu}, \tag{18}$$

and Eq. (3) or (8) give

$$h_{2,Q} = \frac{2\sqrt{b}}{a} = 0.3536\sqrt{\frac{R^3\rho\,\sigma\cos\theta}{\mu^2}}. \tag{19}$$

Quere denotes our $t_{2,Q}$ as t^*. We will however use $t_{2,Q}$ to prevent confusion with the dimensionless time introduced in the next chapter. $t_{2,S}$ and $t_{2,Q}$ are feasible, "general" indicators for the transition from the inertial to the viscous time period. However, they do not provide information on when the influence of inertia is negligible and the Lucas-Washburn equation is sufficient to describe the capillary rise. To obtain such a measure we take the equation given by Bosanquet [4] (visco-inertial stage) and the Lucas-Washburn solution (purely viscous). One can show that for $t \to \infty$ both solutions converge into each other. One may find the time of 3% deviation in the predicted heights by writing

$$0.03 = \frac{h_{LucasWashburn}(t_3) - h_{Bosanquet}(t_3)}{h_{LucasWashburn}(t_3)}. \tag{20}$$

By rearranging we find

$$t_3 = \frac{16.921}{a} = \frac{2.1151R^2\rho}{\mu}. \tag{21}$$

Using Eq. (7) gives

$$h_3 = \frac{5.6429\sqrt{b}}{a} = 0.9975\sqrt{\frac{R^3\rho\,\sigma\cos\theta}{\mu^2}}. \tag{22}$$

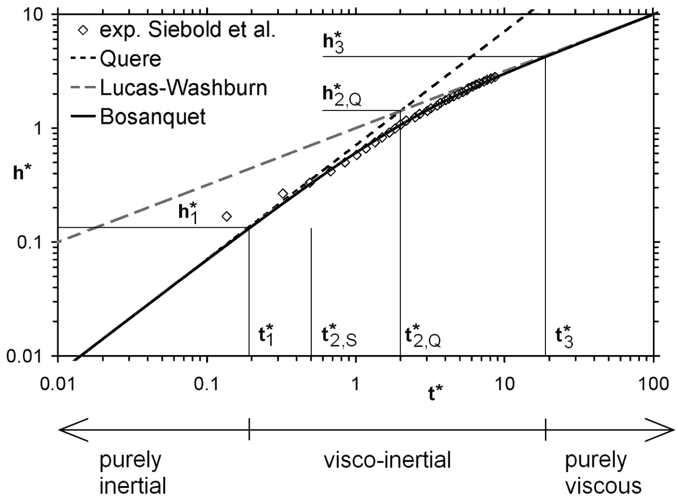

Fig. 3. Dimensionless diagram showing an overview of the initial time stages of capillary rise.

4. Discussion in dimensionless form

We use the dimensionless scaling provided by Ichikawa and Satoda [9]. They obtain (here shown in rearranged form and written with the parameters a and b, see Eqs. (5) and (6))

$$t^* = at = \frac{8\mu t}{\rho R^2}, \tag{23}$$

and

$$h^* = \frac{ah}{\sqrt{2b}} = \sqrt{\frac{16\mu^2h^2}{\rho R^3\sigma\cos\theta}}. \tag{24}$$

In Fig. 3 the different equations introduced in the previous chapters are plotted in logarithmic scale. The points of transition between the time periods are shown.

Using the presented scalings we can give the points of transition (Eqs. (15-22)) in dimensionless form, see Table 1.

Table 1
Dimensionless values of the transition points.

t_1^*	h_1^*	$t_{2,S}^*$	$t_{2,Q}^*$	$h_{2,Q}^*$	t_3^*	h_3^*
0.1856	0.1273	0.5000	2.0000	1.4142	16.921	3.9901

5. Limitations of the model

For all discussed calculations the influence of the dynamic contact angle is neglected. This assumption may be especially critical for the initial moments of capillary rise as the flow velocities reach their maximum value there [12–14]. Empirical equations are available for the dynamic contact angle θ_d; Jiang et al. [17] (based on data by Hoffman [18]) give

$$\frac{\cos\theta_d - \cos\theta_s}{\cos\theta_s + 1} = -\tanh(4.94\text{Ca}^{0.702}), \tag{25}$$

Bracke et al. [19] find

$$\frac{\cos\theta_d - \cos\theta_s}{\cos\theta_s + 1} = -2\mathrm{Ca}^{0.5}. \quad (26)$$

Hereby the capillary number can be rearranged to be

$$\mathrm{Ca} = \frac{\mu\dot{\mathrm{h}}}{\sigma} = \mu\sqrt{\frac{2\cos\theta_\mathrm{s}}{\sigma\rho\,\mathrm{R}}}, \quad (27)$$

if one uses the maximum theoretical velocity (differentiation of Eq. (3), θ_s as conservative assumption). The obtained capillary number now allows to calculate the dynamic contact angle using Eq. (25) or Eq. (26). However, one should keep in mind that even for cases with high initial velocities these slow down fairly fast and assuming a constant contact angle becomes feasible for later time stages again.

Another restriction is that the model only applies for cases where gravity can be neglected. Using large, vertical capillaries one may find that the visco-inertial time stage can be directly followed by a stage where viscous, inertial and gravitational forces are dominant. For this special case there is no purely viscous stage and decaying oscillations around the equilibrium height can be observed. Thus, to have a clear separation of inertia and gravity we can state the following: h_3, the height below which inertia has to be taken into account, has to be smaller that the height from which on gravity has to be considered. Using the criterion given in [16] we can write

$$h_3 < 0.1\, h_{eq}. \quad (28)$$

Rearranging gives

$$\frac{R^5\rho^3 g^2}{\mu^2\sigma\cos\theta} < 0.0402. \quad (29)$$

It is interesting to note that the left hand side of Eq. (29) is dimensionless and equal to the Bond number Bo multiplied with the Galileo number Ga defined by

$$\mathrm{Bo} = \frac{\rho g R^2}{\sigma\cos\theta} \sim \frac{\text{gravitational force}}{\text{surface tension force}}, \quad (30)$$

and

$$\mathrm{Ga} = \frac{g R^3\rho^2}{\mu^2} \sim \frac{\text{gravitational force}}{\text{viscous force}}. \quad (31)$$

6. Conclusion

In this note we discuss the different time stages during the early stages of capillary rise. It is concluded that the purely inertial and the purely viscous flow period are separated by a visco-inertial stage where both effects have to be considered. By means of mathematical rearrangement we derive the times and heights where the transition between the time periods occur. This provides a tool which allows to calculate which terms of the momentum balance have to be taken into account to obtain a solution of sufficient precision. Up to now the time where the solution for the inertial and the viscous rise provided the same height has been used as a measure. However, we can now state that it takes about 8 times that time for the flow to become independent of inertial effects.

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References

[1] R. Lucas. Ueber das Zeitgesetz des kapillaren Aufstiegs von Flüssigkeiten. *Kolloid-Zeitschrift*, 23:15–22, 1918.

[2] E.W. Washburn. The dynamics of capillary flow. *Physical Review*, 17(3):273–283, 1921.

[3] E.K. Rideal. On the flow of liquids under capillary pressure. *Philos. Mag. Ser. 6*, 44:1152–1159, 1922.

[4] C.H. Bosanquet. On the flow of liquids into capillary tubes. *Philos. Mag. Ser. 6*, 45:525–531, 1923.

[5] M. Stange, M.E. Dreyer, and H.J. Rath. Capillary driven flow in circular cylindrical tubes. *Phys. Fluids*, 15(9):2587–2601, September 2003.

[6] M. Stange. *Dynamik von Kapillarströmungen in Zylindrischen Rohren.* Cuvillier, Göttingen, 2004.

[7] J. Szekely, A.W. Neumann, and Y.K. Chuang. The rate of capillary penetration and the applicability of the washburn equation. *J. Colloid Interf. Sci.*, 35(2):273–278, 1971.

[8] G.L. Batten, Jr. Liquid imbibition in capillaries and packed beds. *J. Colloid Interf. Sci.*, 102(2):513–518, 1984.

[9] N. Ichikawa, Y. Satoda. Interface dynamics of capillary flow in a tube under negligible gravity condition. *J. Colloid Interf. Sci.*, 162:350–355, 1994.

[10] D. Quere. Inertial capillarity. *Europhys. Lett.*, 39(5):533–538, 1997.

[11] D. Quere, E. Raphael, and J.-Y. Ollitrault. Rebounds in a capillary tube. *Langmuir*, 15(10):3679–3682, 1999.

[12] A. Siebold, M. Nardin, J. Schultz, A. Walliser, and M. Oppliger. Effect of dynamic contact angle on capillary rise phenomena. *Colloids and Surfaces A*, 161(1):81–87, 2000.

[13] A. Hamraoui, T. Nylander. Analytical approach for the Lucas-Washburn equation. *J. Colloid Interf. Sci.*, 250:415–421, 2002.

[14] R. Chebbi. Dynamics of liquid penetration into capillary tubes. *J. Colloid Interface Sci.*, 315:255–260, 2007.

[15] B.V. Zhmud, F. Tiberg, K. Hallstensson. Dynamics of capillary rise. *J. Colloid Interface Sci.*, 228(2):263–269, 2000.

[16] N. Fries and M. Dreyer. An analytic solution of capillary rise restrained by gravity. *J. Colloid Interf. Sci.*, 320:259–263, 2008.

[17] T.-S. Jiang, S.-G. Oh, J.C. Slattery. Correlation for dynamic contact angle. *J. Colloid Interf. Sci.*, 69:74–77, 1979.

[18] R.L. Hoffman. A study of the advancing interface. I. Interface shape in liquid-gas systems. *J. Colloid Interf. Sci.*, 50:228–241, 1975.

[19] M. Bracke, F. De Voeght, P. Joos. The kinetics of wetting: the dynamic contact angle. *Progress in Colloid & Polymer Science*, 79:142–149, 1989.

Appendix

The scaling by Ichikawa and Satoda [9], discussed in section 4 of this paper, can be used to transform some of the analytic solutions into a dimensionless form. The solution by Quere [10] then reads

$$h^* = \frac{t^*}{\sqrt{2}}. \quad (32)$$

The equation by Bosanquet [4] changes to

$$h^* = \sqrt{t^* - (1 - e^{-t^*})}, \quad (33)$$

and the Lucas-Washburn equation [1,2] reads

$$h^* = \sqrt{t^*}. \tag{34}$$