Gravitating Non-Abelian Solitons and Hairy Black Holes in Higher Dimensions

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FRANCE

Contents

- What is known about non-Abelian solutions in $D > 4$?
- Why should one study them ?

Einstein-Yang-Mills in D=4

Pure gravity /pure attraction/

$$\mathcal{L}_E = -\frac{R}{16\pi G}$$

has no solitons /Lichenrowitz/, there are black holes. Pure Yang-Mills /purely repulsion/

$$\mathcal{L}_{YM} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$$

is scale invariant $\Rightarrow$ no solitons /Deser, Coleman/.
Gravity + Yang-Mills = attraction+repulsion

$$\mathcal{L}_{EYM} = -\frac{R}{16\pi G} - \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$$

EYM solutions in D=4

- Solitons of Bartnik-McKinnon $\Rightarrow$ the first example of globally regular gravitational solitons.
- EYM black holes $\Rightarrow$ the first example of hairy black holes /Gal'tsov+M.S.V./
- Generalizations: non-spherically symmetric, non-static solitons/black holes, coupling to other fields . . . /Kleihaus-Kunz+. . ./

Manifest counter-examples to a number of electroweak theorems /uniqueness, staticity, circularity, Israel's theorem, no-hair theorems . . ./

/D.V.Gal'tsov, M.S.V. Phys.Rep. **319**, 1 (1999)/

What happens in D=5 ?

Pure gravity:

- black holes /SO(4) symmetry/

$$g_{MN}dx^M dx^N = N dt^2 - \frac{dr^2}{N} - r^2 d\Omega_3^2, \qquad N = 1 - \left(\frac{r_g}{r}\right)^2$$

- black strings: if $\partial/\partial x^4$ is a symmetry and $g_{\mu\nu}$ is 4D Ricci flat (black hole) $\Rightarrow$

$$g_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu - (dx^4)^2$$

if $\partial/\partial t$ is a symmetry and Euclidean $g^E_{\mu\nu}$ is 4D Ricci flat /gravitational instanton/ $\Rightarrow$

$$g_{MN}dx^M dx^N = dt^2 - g^E_{\mu\nu}dx^\mu dx^\nu$$

YM particles

Pure Yang-Mills is not scale invariant in $D \neq 4$ (the coupling g is dimensionful) $\Rightarrow \exists$ solitons. If $A^a_\mu(x^\nu)$ is a solution of 4D Euclidean YM equations, then

$$A^a_M = (0, A^a_\mu(x^\nu)) \qquad /\mu = 1, 2, 3, 4/$$

will be a soliton in 5D with the energy

$$E = \frac{1}{4g^2} \int (F^a_{\mu\nu})^2 d^4x \geq \frac{8\pi^2 |n|}{g^2}$$

Self-dual 4D YM instantons $\Rightarrow$ ‘YM particles’ in D=5.

YM vortices

If $\partial/\partial x^4$ is a symmetry $\Rightarrow$

$$A^a_M = (0, A^a_i(x^k), H^a(x^k)), \qquad /i, k = 1, 2, 3/$$

the energy per unit x^4,

$$E = \frac{1}{2g^2} \int ((\partial_i H^a + \varepsilon_{abc} A^b_i H^c)^2 + \frac{1}{2}(F^a_{ik})^2) d^3x \geq \frac{4\pi|n|}{g^2}$$

coincides with the energy of the D=3 YM-Higgs system $\Rightarrow$
$\Rightarrow$ monopoles, when lifted back to D=5 they become

'YM vortices'.

Question

What happens to the 5D ‘YM particles’ and ‘YM vortices’ if one couples them to the 5D gravity ?

/M.S.Volkov, Phys.Lett. B524 (2002) 369/

Gravitating YM particles

$$S = \int \left(-\frac{1}{16\pi G} R - \frac{1}{4g^2} F^a_{MN} F^{aMN} \right) \sqrt{g}\, d^5x \,.$$

$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \varepsilon_{abc} A^b_M A^c_N \; (a = 1, 2, 3),$
$[G^{1/3}] = [g^2] = [\text{length}] \Rightarrow$ dimensionless coupling

$$\kappa = \frac{8\pi G}{g^6} \qquad /\kappa = 2\alpha^2/$$

SO(4)-symmetry: if θ^a are the invariant forms on S^3,

$$ds^2 = \sigma(r)^2 N(r) dt^2 - \frac{dr^2}{N(r)} - r^2\, d\Omega_3^2 \,, \quad A^a = (1 + w(r))\, \theta^a \,,$$

the length scale is g^2.

Field equations

$$r^2 N w'' + r\, w' + \kappa\,(m - (w^2-1)^2)\frac{w'}{r} = 2\,(w^2-1)w\,,$$

$$rm' = r^2 N w'^2 + (w^2-1)^2\,, \qquad \boxed{N \equiv 1 - \kappa m(r)/r^2}$$

$$\sigma' = \kappa\,\frac{w'^2}{r}\,\sigma$$

If $\kappa = 0 \Rightarrow \sigma = N = 1$, the YM particle with $M_{\mathrm{ADM}} = \frac{8}{3}$

$$w = \frac{1 - b\,r^2}{1 + b\,r^2},$$

$b \in R$ is a scale parameter. What happens if $\kappa \neq 0$?

No finite mass solutions with $\kappa \neq 0$

$M_{\rm ADM} < \infty \Rightarrow w(\infty) = \pm 1$. Either

- $w = 1 - 2br^2 + O(r^4),\ m = O(r^3)$ as $r \to 0$ /solitons/ or
- $\exists r_h$: $N(r_h) = 0$, $N'(r_h) > 0$, $w(r_h) < \infty$ /black holes/

$$M_{\rm ADM}[w(r)] = m(\infty) =$$

$$= \frac{r_h^2}{\kappa} + \int_{r_h}^{\infty} \frac{dr}{r}\,(r^2 w'^2 + (w^2-1)^2)\exp\left(-\kappa \int_r^{\infty} \frac{w'^2}{r}\,dr\right)$$

One should have

$$\frac{d}{d\lambda} M[w(\lambda r)] = 0 \ \text{ for } \ \lambda = 1 \ \ \text{but} \ \ \frac{d}{d\lambda} M[w(\lambda r)] < 0 \quad \forall \lambda$$

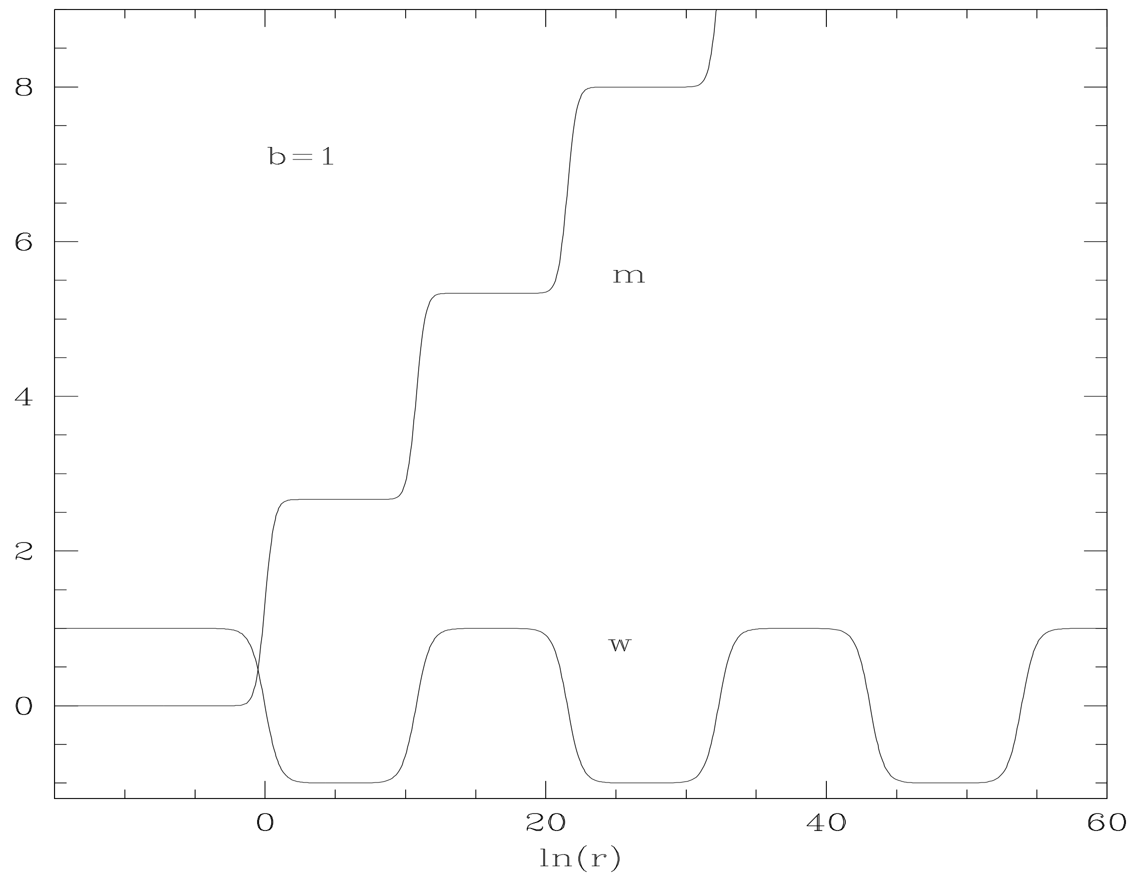
The SO(4) YM particles get completely destroyed by gravity. They resemble a dust, since

$$T_{\mu\nu} = \epsilon(r)\delta_{\mu}^{0}\delta_{\nu}^{0}$$

and they can be scaled to an arbitrary size $\Rightarrow$ repulsion and attraction are not balanced $\Rightarrow$ equilibrium states are not possible.

The globally regular solutions with infinite mass are quasi-periodic /infinite sequence of static spherical shells/ with the metric approaching the flat metric at large r (but not fast enough).

Quasi-periodic solutions



$m(r) \sim \ln r$

Gravitating Yang monopole in D=6

YM field is again = 4D instanton, but this time on S^4

$$ds^2 = \sigma(r)^2 N(r) dt^2 - \frac{dr^2}{N(r)} - r^2 (d\xi^2 + \sin^2\chi \, d\Omega_3^2) \,, \quad A^a = (1 + w(\chi))\, \theta$$

YM equations decouple $\Rightarrow w(\chi) = \cos\chi$, Einstein eq-s $\Rightarrow$ $\sigma = 1$,

$$N = 1 - \frac{2Gm(r)}{r^3}, \qquad m' = 8\pi, \qquad m(r) = 8\pi r + m_0,$$

$\Rightarrow$ mass is linearly divergent
/Gibbons, Townsend, hep-th/0604024/
$D = 2k + 2$, SO($2k$), $m \sim r^{2k-3}$.

Gravitating YM vortices

If $\partial/\partial x^4$ is hypersurface orthogonal Killing vector, then

$$g_{MN}dx^M dx^N = e^{-\zeta} g_{\mu\nu} dx^\mu dx^\nu - e^{2\zeta}(dx^4)^2$$

$$A^a_M dx^M = A^a_\mu dx^\mu + H^a dx^4$$

reducing the 5D EYM to the 4D EYM-Higgs-dilaton theory

$$\begin{aligned} \sqrt{{}^{(5)}g}\mathcal{L}_{\rm EYM} &= \left(-\frac{{}^{(4)}R}{2\kappa g^6} + \frac{3}{\kappa g^6}(\partial_\mu \zeta)^2 \right. \\ &+ \left. \frac{1}{2g^2} e^{-2\zeta}(D_\mu H^a)^2 - \frac{1}{4g^2} e^{\zeta}(F^a_{\mu\nu})^2 \right) \sqrt{-{}^{(4)}g} \end{aligned}$$

SO(3) symmetry

$$e^{-\zeta(r)}ds^2 = \mathrm{e}^{2\nu(r)}dt^2 - dr^2 - R^2(r)\,d\Omega_2^2,$$

$$A_k^a dx^k = (w(r)-1)\epsilon_{aik}n^i dn^k, \qquad H^a = n^a e^{\zeta(r)}h(r)$$

the independent field equations can be represented as a seven-dimensional dynamical system

$$\frac{d}{dr}\,y_k = F_k(y_s, \kappa)$$

with $y_k = \{w, w', h, h', Z = \zeta', R, R'\}$.

Fixed points

I. The origin, $(w,h,Z,R)=(1,0,0,0)$; as $r\to 0$,

$$
\begin{aligned}
w &= 1-br^2+O(r^2), \quad h=ar+O(r^3),\\
Z &= O(r^2), \quad R=r+O(r^3).
\end{aligned}
\tag{1}
$$

II. Infinity, $(w,h,Z,1/R)=(0,1,0,0)$; as $r\to\infty$,

$$
\begin{aligned}
&w=A\,r^{C}\mathrm{e}^{-r}+o(\mathrm{e}^{-r}),\ Z=\kappa Qr^{-2}+O(r^{-3}\ln r),\\
&h=1-Cr^{-1}+O(r^{-2}\ln r),\\
&R=r-m\ln r+m^2r^{-1}\ln r-r_0+\gamma r^{-1}+O(r^{-2}\ln r)\,.
\end{aligned}
\tag{2}
$$

The ADM mass

$$M_{\mathrm{ADM}}=3(C+(2+\kappa)Q)$$

III. “Warped” $AdS_3 \times S^2$: If $4q^3 + 7q^2 + 11q = 1$, then $w^2 = q$,

$$R^2 = \kappa \frac{(11q-1)(1-q)}{(4q^2 - 13q + 1)}, \quad h^2 = \frac{1-q}{R^2}, \quad Z^2 = -\frac{4q^2 - 13q + 1}{(4q+1)R^2}$$

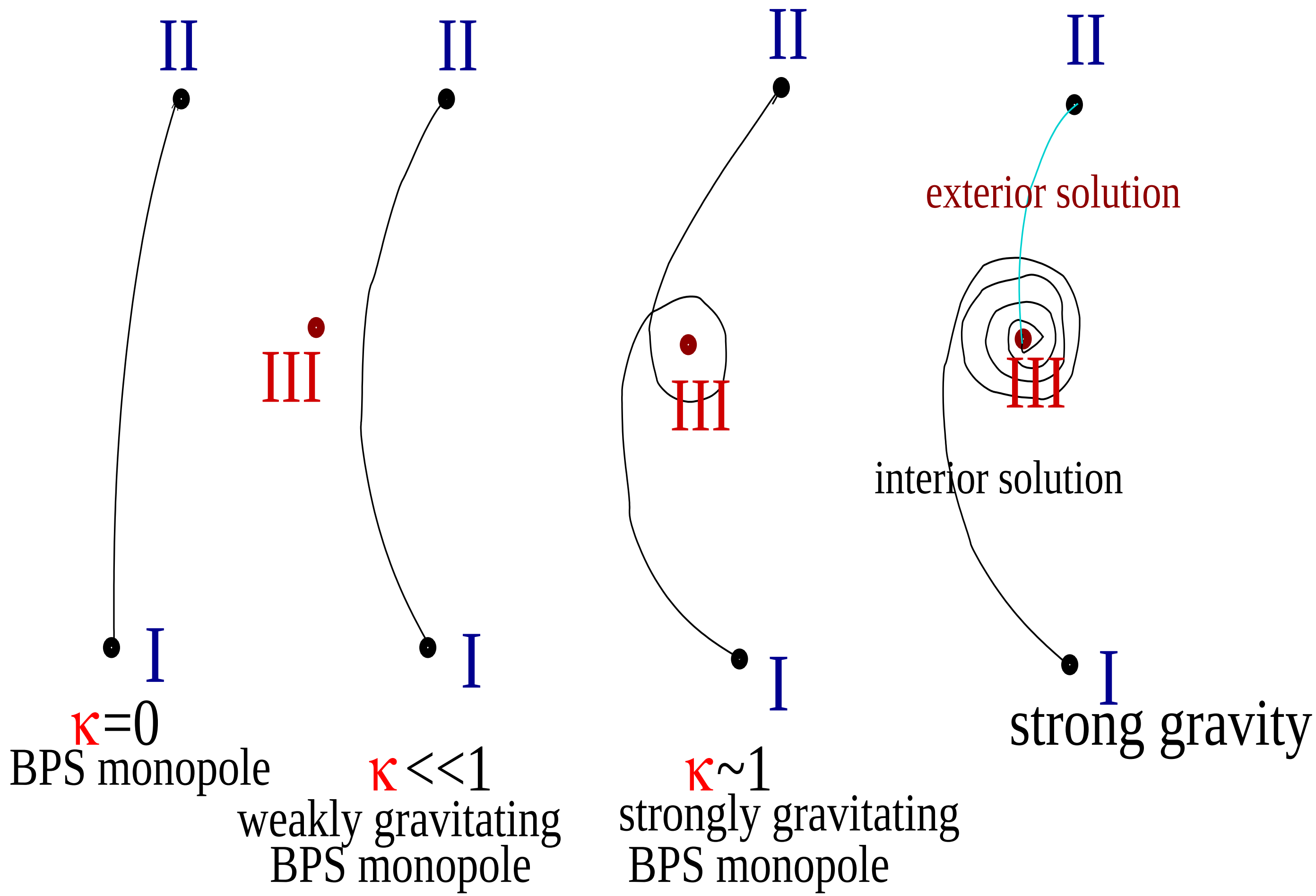
Evaluating, $w = 0.29, \; h = \frac{1.27}{\sqrt{\kappa}}, \; Z = \pm \frac{0.31}{\sqrt{\kappa}}, \; R = 0.75\sqrt{\kappa} \; \Rightarrow$ new exact non-Abelian solution with the geometry

$$ds^2 = \mathrm{e}^{2(1+\kappa h^2)Zr} dt^2 - dr^2 - \mathrm{e}^{2Zr} \, (dx^4)^2 - R^2 \, d\Omega_2^2$$

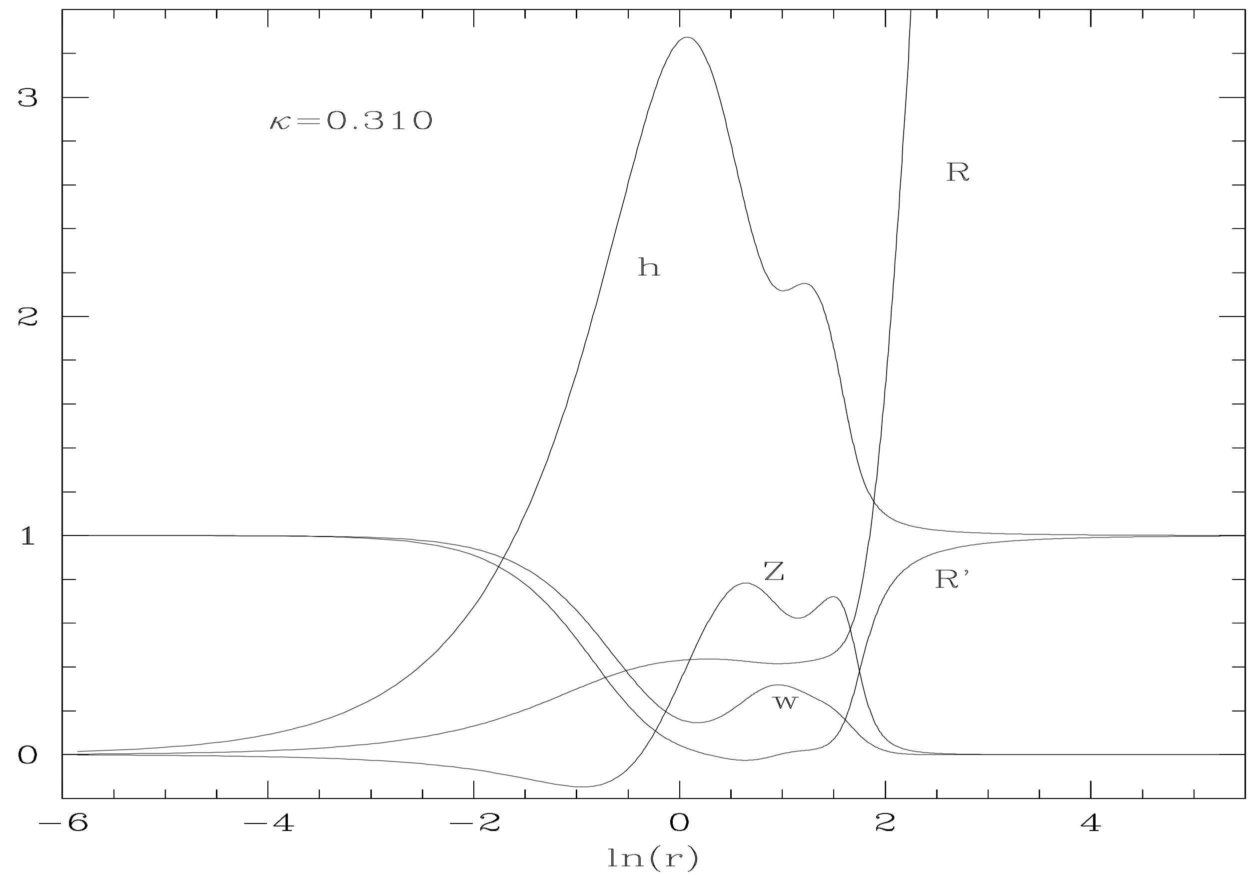
The characteristic eigenvalues

$$(-\frac{2.77}{\sqrt{\kappa}}, -\frac{2.47}{\sqrt{\kappa}}, -\frac{2.12}{\sqrt{\kappa}}, -\frac{0.61}{\sqrt{\kappa}} \pm i\frac{1.24}{\sqrt{\kappa}}, +\frac{0.88}{\sqrt{\kappa}}, +\frac{1.54}{\sqrt{\kappa}})$$

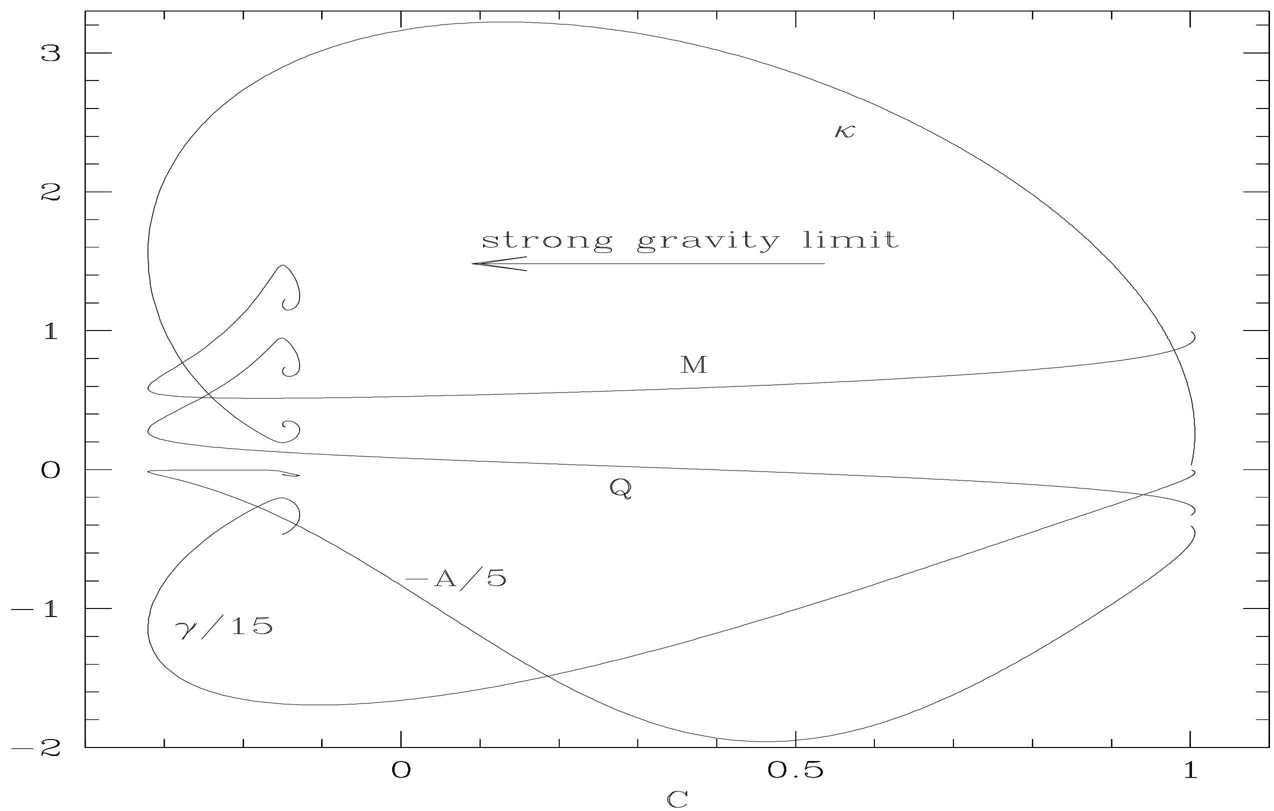
Global solutions



Strongly gravitating solutions



Strong gravity limit



For $\kappa_{max} = 3.22 > \kappa > \kappa_{min} = 0.11$ one has more than one solution (‘branches’).

Limiting solution

Strongly gravitating solutions have a regular core connected to the asymptotic region by a long throat. For the limiting solution the throat becomes infinite and the solution splits up into the union of two different solutions

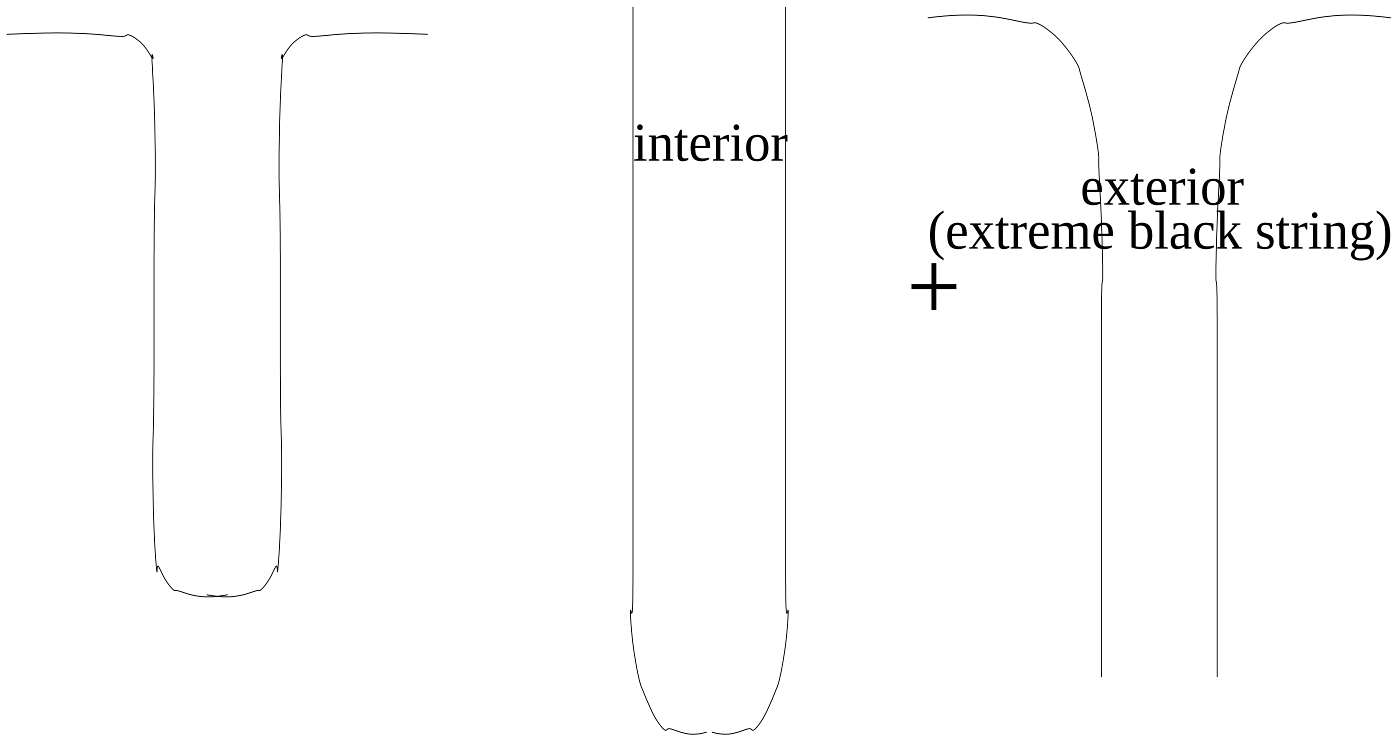
- Interior solution interpolates between the regular origin and the ‘warped ADS’.
- Exterior solution interpolates between the ‘warped ADS’ and infinity. Looks like an extreme non-Abelian black string: in the Schwarzschild gauge ($r = R$) one has

$$g_{rr} \sim (r - r_h)^{-2}, \quad g_{tt} \sim (r - r_h)^{2.02}, \quad g_{44} = e^{2\zeta} \sim (r - r_h)^{2.02}$$

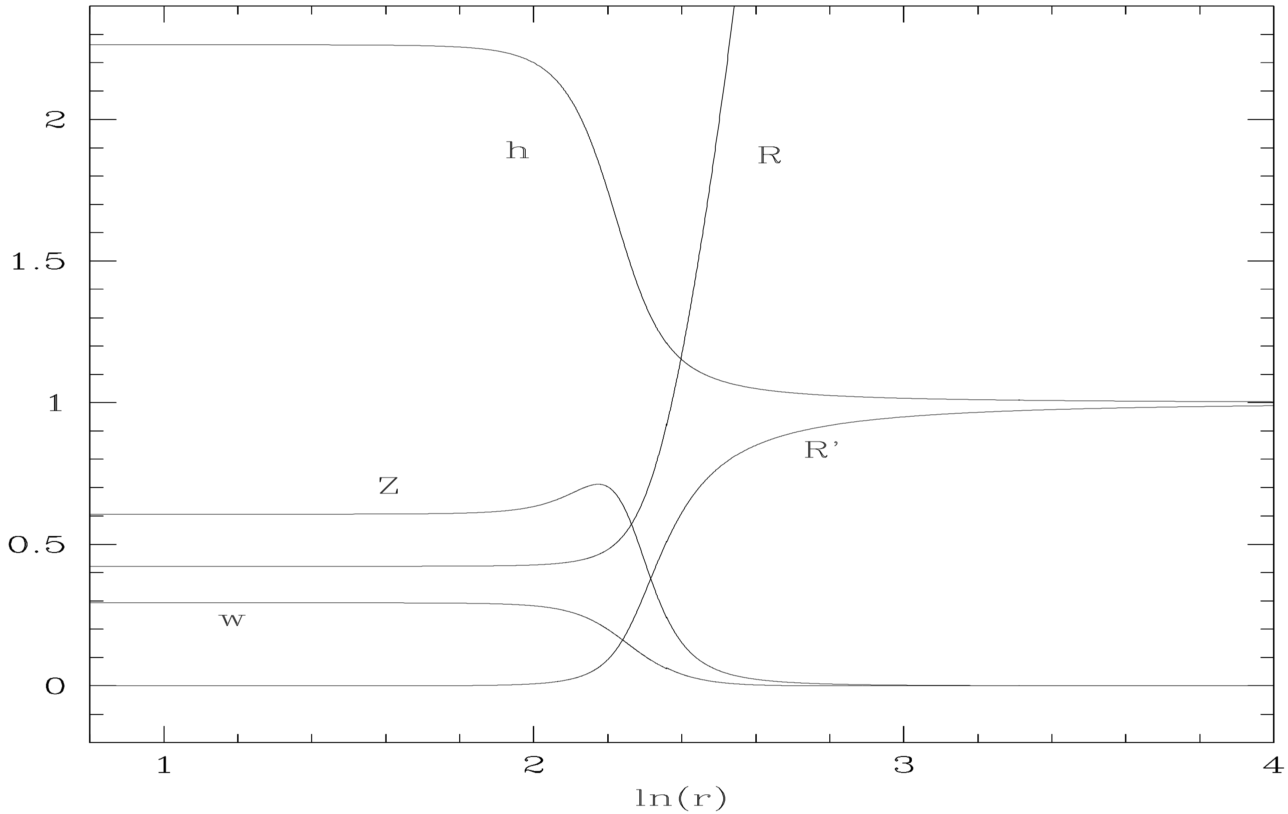
with $r_h = 0.42$, $\kappa = 0.316$.

Strong gravity limit

strong gravity | limiting solution



Limiting solution



Conclusions

- Gravitating YM vortices form a one-parameter family (fundamental branch) of globally regular solutions that interpolates between the flat space BPS monopole and the extreme non-Abelian black string.
- $\exists$ also excited solutions for which the YM field amplitude w oscillates around zero value – excited branches. These solutions do not have the flat space limit.

Generalizations of the fundamental YM vortices have been considered by Brihaye, Hartmann, Radu + other people. A good recent reference: /Y.Brihaye, B.Hartmann, E.Radu, Phys.Rev. D 72 (2005) 104008./ Excited solutions have not been considered so far.

YM black strings

From the 4D viewpoint the YM vortices are regular gravitating monopoles. Gravitating solitons can usually be generalized to include a small black hole with a non-degenerate horizon in the core /Kastor, Traschen '92/: replace the boundary condition at the origin, $r = 0$, by those at the regular horizon, $r = r_h$. This generalizes YM vortices to black strings /B.Hartmann '04/.

For a given κ there can be several YM vortices $\Rightarrow$ one finds several black string solutions with small r_h. As r_h increases, these solutions approach each other and finally merge for some maximal value $r_h^{\max}(\kappa)$. There are no black strings with $r_h > r_h^{\max}(\kappa) \Rightarrow$ black strings exists only for a finite domain of the $\kappa - r_h$ parameter plane. /Brihaye, Hartmann, Radu '05/.

‘Twisted’ solutions

/Brihaye, Radu ’05/ consider the case of a non-hypersurface orthogonal Killing vector $\partial/\partial x^4$:

$$g_{MN}dx^M dx^N = e^{-\zeta} g_{\mu\nu} dx^\mu dx^\nu - e^{2\zeta}(dx^4 + W_\mu dx^\mu)^2$$

$$A^a_M dx^M = A^a_\mu dx^\mu + H^a(dx^4 + W_\mu dx^\mu)$$

Upon the reduction to D=4 the twist W_μ becomes a U(1) vector fields $\Rightarrow$ 4D EYM-Higgs-dilaton+U(1) model. When the charge associated to the U(1) field vanishes, the solutions reduce to the YM vortices/black strings.

Deformed solutions

/Brihaye, Hartmann, Radu '05/ after the reduction to 4D choose the fields to be static, axially symmetric

$$ds^2 = f(r,\theta)dt^2 - m(r,\theta)(dr^2 + r^2\sin^2\theta) - l(r,\theta)d\varphi^2$$

the ansatz for A_μ^a, H^a contains two winding numbers, n, m. If $n = 1, m = 0$ the solutions are spherically symmetric. For $n > 1$, $m = 0$ one obtains multimonopoles, they are axially symmetric. Solutions with $m = 1$ describe monopole - antimonopole pairs. Both regular solutions and black strings are considered, the existence of several 'solution branches' is detected.
Lorentz boosting the solutions along the x^4 axis gives stationary spinning configurations.

Why studying gravitating YM in D>4 ?

String theory motivates this $\Rightarrow$ one can do as follows

- Take a 4D model, set D>4 and study solutions.

Why studying gravitating YM in D>4 ?

String theory motivates this $\Rightarrow$ one can do as follows

- Take a 4D model, set D>4 and study solutions.
- Aim at applications, say, of the Randall-Sundrum type. Example: a 3-brane in the D=7 EYM-Higgs-Λ theory with the monopole field in the orthogonal 3-space

$$ds^2 = A(r)\eta_{\mu\nu}dy^\mu dy^\nu - B(r)\delta_{ik}dx^i dx^k$$

$$A_k^a = \epsilon_{akj}x^j W(r), \quad \Phi^a = x^a H(r) \quad /r^2 = x^k x^k/,$$

solutions with $A(0) = 1$, $A(\infty) = 0 \Rightarrow$ gravity localization by the monopole /Shaposhnikov et al. '03/.

String vacua = solutions of SUGRA's

- Heterotic solitons: a 5-brane in the D=10 heterotic string theory with the YM instanton field in the orthogonal 4-space /Strominger '90/

$$ds^2 = A(x^\mu)\eta_{MN}dy^M dy^N - B(x^\nu)\delta_{\mu\nu}dx^\mu dx^\nu$$

$$A^a_\mu(x^\nu), \quad F_{\mu\nu} = \tilde{F}_{\mu\nu}, \quad \phi(x^\mu)$$

  compactifications $\Rightarrow$ many solitons /Duff, Khuri, Liu '95/

- Compactifications of type I,II $\Rightarrow$ gauged SUGRA's in D<10 = EYM-dilaton+moduli fields.

  Non-abelian solutions:

SUGRA monopole

SO(4), N=4 gauged SUGRA

$$\mathcal{L}_4 = \frac{1}{4}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\,e^{2\phi}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{8}\,(e^{-2\phi} + \xi^2 e^{2\phi} + 4\xi).$$

fermionic SUSY variations $/\mathcal{F} = \frac{1}{2}\alpha^a F^a_{\alpha\beta}\gamma^\alpha\gamma^\beta/$

$$\begin{aligned}\delta\chi &= \frac{1}{\sqrt{2}}\,\gamma^\mu\partial_\mu\phi\,\epsilon + \frac{1}{2}\,e^\phi\mathcal{F}\epsilon + \frac{1}{4}\,(e^{-\phi} - \xi e^\phi)\epsilon,\\ \delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \frac{1}{2\sqrt{2}}\,e^\phi\mathcal{F}\gamma_\mu\epsilon + \frac{1}{4\sqrt{2}}\,(e^{-\phi} + \xi e^\phi)\gamma_\mu\epsilon.\end{aligned}$$

$\boxed{\delta\chi = \delta\psi_\mu = 0}$ $\Rightarrow$ consistency conditions $\Rightarrow$ Bogomol'nyi equations $\Rightarrow$ string vacua /A.Chamseddine, M.S.V. '97-'04/

Bogomol'nyi equations

$$ds^2_{(4)} = -e^{2V(\rho)}dt^2 + e^{2\lambda(\rho)}d\rho^2 + r^2(\rho)d\Omega^2,$$

$$\tau^a A^a_\mu dx^\mu = \frac{i}{2}(1 - w(\rho))[T, dT], \quad \phi = \phi(\rho); \quad T = \tau^a n^a$$

$$V' - \phi' = \xi \frac{P}{\sqrt{2}N} e^{\phi+\lambda}, \qquad Q = e^{V+\phi}\frac{w}{N},$$

$$\phi' = \sqrt{2}\,\frac{BP}{N}\,e^{\lambda}, \qquad w' = -\frac{rwB}{N}e^{-\phi+\lambda},$$

$$N = \sqrt{w^2 + P^2}, \qquad r' = Ne^{\lambda}.$$

$$P = e^{\phi}\tfrac{1-w^2}{\sqrt{2}r} + \tfrac{r}{2\sqrt{2}}(e^{-\phi} + \xi e^{\phi}),\ B = -\tfrac{P}{\sqrt{2}r} + \tfrac{1}{2}e^{-\phi}.$$

Solutions

comprise a family labeled by ξ, all have N=1 SUSY.

- $\xi > 0$ asymptotically AdS
- $\xi < 0$ compact
- $\xi = 0 \Rightarrow$ SUGRA monopole

$$ds^2 = 2\,e^{2\phi}\left\{ dt^2 - d\rho^2 - R^2(\rho)\big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\big)\right\},$$

$$w = \pm\frac{\rho}{\sinh\rho}, \quad e^{2(\phi-\phi_0)} = \frac{\sinh\rho}{2\,R(\rho)}, \quad R(\rho) = \sqrt{2\rho\coth\rho - w^2 -$$

when uplifted to D=10 becomes

Solution in D=10

$$d\tilde{s}^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2 - d\rho^2 - R^2(\rho)\, d\Omega_2^2 - \Theta^a\Theta^a$$

$$H = \frac{1}{2\sqrt{2}}\, e^{-\frac{3}{4}\phi}\big(F^a \wedge \Theta^a + \epsilon_{abc}\Theta^a \wedge \Theta^b \wedge \Theta^c\big)$$

$\Theta^a = A^a - \theta^a$ /A.Chamseddine, M.S.V. '98/

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According to /Maldacena, Nunez '01/ this solution describes the NS-NS 5-brane wrapped on S^2 $\Rightarrow$ the dual description of N=1 SYM $\Rightarrow$ the dual description of confinement $\Rightarrow$ plenty of applications in string theory.

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Solution is generally known as ' Maldacena-Nunez solution'.

Conclusion

Solutions for gravity-coupled Yang-Mills fields can be very useful for those working in string theory.