Counterterm Method in Lovelock Theory and Horizonless Solutions in Dimensionally Continued Gravity

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Abstract

In this paper we, first, generalize the quasilocal definition of the stress energy tensor of Einstein gravity to the case of Lovelock gravity, by introducing the tensorial form of surface terms that make the action well-defined. We also introduce the boundary counterterm that removes the divergences of the action and the conserved quantities of the solutions of Lovelock gravity with flat boundary at constant t and r. Second, we consider the metric of spacetimes generated by brane sources in dimensionally continued gravity which has no curvature singularity and no horizons, but have conic singularity and compute the conserved quantities of these solutions through the use of the counterterm method introduced in the first part of the paper.

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I. INTRODUCTION

The most natural extension of general relativity in higher dimensional spacetimes with the assumption of Einstein – that the left hand side of the field equations is the most general symmetric conserved tensor containing no more than second derivatives of the metric – is Lovelock theory. Lovelock [1] found the most general symmetric conserved tensor satisfying this property. The resultant tensor is nonlinear in the Riemann tensor and differs from the Einstein tensor only if the spacetime has more than 4 dimensions. Since the Lovelock tensor contains metric derivatives no higher than second order, the quantization of the linearized Lovelock theory is ghost-free [2].

Our first aim in this paper is to generalize the definition of the quasilocal stress energy tensor for computing the conserved quantities of a solution of Lovelock gravity. The concepts of action and energy-momentum play central roles in gravity. However there is no good local notion of energy for a gravitating system. A quasilocal definition of the energy and conserved quantities for Einstein gravity can be found in [3]. They define the quasilocal stress energy tensor through the use of the well-defined gravitational action of Einstein gravity with the surface term of Gibbons and Hawking [4]. Therefore the first step is to find the surface terms for the action of Lovelock gravity that make the action well-defined. These surface terms were introduced by Myers in terms of differential forms [5]. Here, we write down the tensorial form of the surface terms for Lovelock gravity, and then introduce the stress energy tensor via the quasilocal formalism. The explicit form of these surface terms for second and third order Lovelock gravity have been written in Refs. [6] and [7] respectively.

Of course, as in the case of Einstein gravity, the action and conserved quantities diverge when the boundary goes to infinity [3]. One way of eliminating these divergences is through the use of background subtraction [3, 8, 9], in which the boundary surface is embedded in another (background) spacetime, and all quasilocal quantities are computed with respect to this background, incorporated into the theory by adding to the action the extrinsic curvature of the embedded surface. Such a procedure causes the resulting physical quantities to depend on the choice of reference background; furthermore, it is not possible in general to embed the boundary surface into a background spacetime. For asymptotically AdS solutions, one can instead deal with these divergences via the counterterm method inspired by AdS/CFT correspondence [10]. This conjecture, which relates the low energy limit of string theory in asymptotically anti de-Sitter spacetime and the quantum field theory on its boundary, has attracted a great deal of attention in recent years. The equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computation on the other side. A dictionary translating between different quantities in the bulk gravity theory and their counterparts on the boundary has emerged, including the partition functions of both theories. In the present context this conjecture furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime [11, 12] by adding additional terms on the boundary that are curvature invariants of the induced metric. Although there may exist a very large number of possible invariants one could add in a given dimension, only a finite number of them are nonvanishing as the boundary is taken to infinity. Its many applications include computations of conserved quantities for black holes with rotation, NUT charge, various topologies, rotating black strings with zero curvature horizons and rotating higher genus black branes [13–15]. Although the counterterm method applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [16].

Our second aim in this paper is to apply counterterm method to asymptotically anti de Sitter (AdS) horizonless solution of dimensionally continued gravity.

In this paper we are dealing with the issue of the spacetimes generated by brane sources in D-dimensional continued gravity that are horizonless and have nontrivial external solutions. These kinds of solutions have been investigated by many authors in four dimensions. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions were considered in [18]. Similar static solutions in the context of cosmic string theory were found in [19]. All of these solutions [18, 19] are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. The extension to include the electromagnetic field has also been done [20, 21]. Here we present the D-dimensional solution in dimensionally continued gravity, and use the counterterm method to compute the conserved quantities of the system.

The outline of our paper is as follows. In Sec. II, we give the tensorial form of the surface terms that make the action well-defined, generalize the Brown York energy-momentum tensor for Lovelock gravity, and introduce the counterterm method for calculating the finite action and conserved quantities of solutions of Lovelock gravity with flat boundary. In

Sec. III we introduce the D-dimensional asymptotically AdS horizonless solutions of dimensionally continued gravity in odd and even dimensions and compute the finite conserved quantities of them. We finish our paper with some concluding remarks.

II. LOVELOCK GRAVITY AND THE COUNTERTERM METHOD

We consider a D-dimensional spacetime manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. We denote the timelike and spacelike boundaries of $\mathcal{M}$ by $\partial\mathcal{M}$ and Σ respectively. The metric and the extrinsic curvature of the timelike boundary $\partial\mathcal{M}$ are denoted by γ_{ab} and Θ_{ab}, while those of the spacelike hypersurface Σ are denoted by h_{ij} and K_{ij}. In this D-dimensional spacetime, the most general action which keeps the field equations of motion for the metric of second order, as the pure Einstein-Hilbert action, is Lovelock action. This action is constructed from the dimensionally extended Euler densities and can be written as

$$I_G = \kappa \int d^D x \sqrt{-g} \sum_{p=0}^{n} \alpha_p \mathcal{L}_p \tag{1}$$

where $n \equiv [(D-1)/2]$ and $[z]$ denotes the integer part of z, α_p is an arbitrary constant and $\mathcal{L}_p$ is the Euler density of a $2p$-dimensional manifold

$$\mathcal{L}_p = \frac{1}{2^p} \delta^{\mu_1\nu_1\cdots\mu_p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma_p} R_{\mu_1\nu_1}{}^{\rho_1\sigma_1} \cdots R_{\mu_p\nu_p}{}^{\rho_k\sigma_k} \tag{2}$$

In Eq. (2) $\delta^{\mu_1\nu_1\cdots\mu p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma p}$ is the generalized totally anti-symmetric Kronecker delta and $R_{\mu\nu}{}^{\rho\sigma}$ is the Riemann tensor of the Manifold $\mathcal{M}$. We note that in D dimensions, all terms for which $p > [D/2]$ are total derivatives, and the term $p = D/2$ is the Euler density. Consequently only terms for which $p < D/2$ contribute to the field equations.

The Einstein-Hilbert action (with $\alpha_p = 0$ for $p \geq 2$) does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of $\delta g_{\mu\nu}$ normal to the boundary $\partial\mathcal{M}$. These normal derivative terms do not vanish by themselves, but are canceled by the variation of the Gibbons-Hawking surface term [4]

$$I_b^{(1)} = 2\kappa \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{-\gamma}\Theta \tag{3}$$

where γ_{ab} is induced metric on the boundary $r =$ const. and Θ is trace of extrinsic curvature of this boundary. The main difference between higher derivative gravity and Einstein gravity is that the surface term that renders the variational principle well-behaved is much more complicated. However, the surface terms that make the variational principle of Lovelock gravity well-defined are known in terms of differential forms [5]. The tensorial form of these surface terms may be written as

$$I_b = -2\kappa \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-\gamma}\sum_{p=0}^{n}\sum_{s=0}^{p-1}\frac{(-1)^{p-s}p\alpha_p}{2^s(2p-2s-1)}\mathcal{H}^{(p)} \tag{4}$$

where α_p is the Lovelock coefficients and $\mathcal{H}^{(p)}$ is

$$\mathcal{H}^{(p)} = \delta^{[a_1 \ldots a_{2p-1}]}_{[b_1 \ldots b_{2p-1}]} R^{b_1 b_2}{}_{a_1 a_2} \cdots R^{b_{2s-1} b_{2s}}{}_{a_{2s-1} a_{2s}} \Theta^{b1}_{a1} \cdots \Theta^{b_{2p-1}}_{a_{2p-1}} \tag{5}$$

In Eq. (5) $R^{ab}{}_{cd}(g)$'s are the boundary components of the Riemann tensor of the Manifold $\mathcal{M}$, which depend on the velocities through the Gauss–Codazzi equations

$$R_{abcd} = \widehat{R}_{abcd} + \Theta_{ac}\Theta_{bd} - \Theta_{ad}\Theta_{bc} \tag{6}$$

where $\widehat{R}_{abcd}(\gamma)$ are the components of the intrinsic curvature tensor of the boundary. The explicit form of the second and third surface terms of Eq. (4) may be written as [6, 7]

$$\begin{aligned} I_b^{(2)} \;=\; & 2\kappa \int_{\delta\mathcal{M}} d^{D-1}x\sqrt{-\gamma}\left\{2\alpha_2\left(J - 2\widehat{G}^{(1)}_{ab}\Theta^{ab}\right)\right. \\ & \left. +3\alpha_3\left(P - 2\widehat{G}^{(2)}_{ab}\Theta^{ab} - 12\widehat{R}_{ab}J^{ab} + 2\widehat{R}J - 4\Theta\widehat{R}_{abcd}\Theta^{ac}\Theta^{bd} - 8\widehat{R}_{abcd}\Theta^{ac}\Theta^{b}_{e}\Theta^{ed}\right)\right\} \end{aligned} \tag{7}$$

where $\widehat{G}^{(1)}_{ab}$ is the n-dimensional Einstein tensor of the metric γ_{ab}, J is the trace of

$$J_{ab} = \frac{1}{3}(2\Theta\Theta_{ac}\Theta^{c}_{b} + \Theta_{cd}\Theta^{cd}\Theta_{ab} - 2\Theta_{ac}\Theta^{cd}\Theta_{db} - \Theta^2\Theta_{ab}), \tag{8}$$

$\widehat{G}^{(2)}_{ab}$ is the second order Lovelock tensor for the boundary metric γ_{ab}:

$$G^{(2)}_{\mu\nu} = 2(\widehat{R}_{acde}\widehat{R}_{b}^{\;cde} - 2\widehat{R}_{acbd}\widehat{R}^{cd} - 2\widehat{R}_{ac}R^{c}{}_{b} + RR_{ab}) - \frac{1}{2}(\widehat{R}_{cdef}\widehat{R}^{cdef} - 4\widehat{R}_{cd}\widehat{R}^{cd} + \widehat{R}^2)\gamma_{ab} \tag{9}$$

and P is the trace of

$$\begin{aligned} P_{ab} \;=\; & \frac{1}{5}\left\{\left[\Theta^4 - 6\Theta^2\Theta^{cd}\Theta_{cd} + 8\Theta\Theta_{cd}\Theta^{d}_{e}\Theta^{ec} - 6\Theta_{cd}\Theta^{de}\Theta_{ef}\Theta^{fc} + 3(\Theta_{cd}\Theta^{cd})^2\right]\Theta_{ab}\right. \\ & -(4\Theta^3 - 12\Theta\Theta_{ed}\Theta^{ed} + 8\Theta_{de}\Theta^{e}_{f}\Theta^{fd})\Theta_{ac}\Theta^{c}_{b} - 24\Theta\Theta_{ac}\Theta^{cd}\Theta_{de}\Theta^{e}_{b} \\ & \left.+(12\Theta^2 - 12\Theta_{ef}\Theta^{ef})\Theta_{ac}\Theta^{cd}\Theta_{db} + 24\Theta_{ac}\Theta^{cd}\Theta_{de}\Theta^{ef}\Theta_{bf}\right\} \end{aligned} \tag{10}$$

In general $I = I_G + I_b$ is divergent when evaluated on solutions, as is the Hamiltonian and other associated conserved quantities. In Einstein gravity, one can remove the non logarithmic divergent terms in the action by adding a counterterm action I_{ct} which is a functional of the boundary curvature invariants [22]. The issue of determination of boundary counterterms with their coefficients for higher-order Lovelock theories is at this point an open question. However for the case of a boundary with zero curvature $[\widehat{R}_{abcd}(\gamma)=0]$, it is quite straightforward. This is because all curvature invariants are zero except for a constant, and so the only possible boundary counterterm is one proportional to the volume of the boundary regardless of the number of dimensions:

$$I_{ct}=2\kappa\lambda\alpha_0\int_{\partial\mathcal{M}_\infty}d^{D-1}x\sqrt{-\gamma} \tag{11}$$

where λ is a constant which should be chosen such that the divergences of the action is removed.

Having the total finite action, one can use the quasilocal definition of Brown and York [3] to construct a divergence free stress-energy tensor as

$$T_b^a=-2\kappa\left\{\lambda\alpha_0\gamma_b^a+\sum_{p=0}^{n}\sum_{s=0}^{p-1}\frac{(-1)^{p-s}p\alpha_p}{2^s(2p-2s-1)}\mathcal{H}_b^{(p,s)a}\right\} \tag{12}$$

where $\mathcal{H}_b^{(p,s)a}$ is

$$\mathcal{H}_b^{(p,s)a}=\delta_{[b_1\ldots b_{2p-1}b]}^{[a_1\ldots a_{2p-1}a]}\widehat{R}^{b_1b_2}{}_{a_1a_2}\cdots\widehat{R}^{b_{2s-1}b_{2s}}{}_{a_2s-1a_2s}\Theta_{a_{2s+1}}^{b_{2s+1}}\cdots\Theta_{a_{2p-1}}^{b_{2p-1}}, \tag{13}$$

To compute the conserved mass of the spacetime, one should choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in Arnowitt-Deser-Misner (ADM) form:

$$\gamma_{ab}dx^adx^a=-N^2dt^2+\sigma_{ij}\left(d\varphi^i+N^idt\right)\left(d\varphi^j+N^jdt\right)$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and N^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (12) can be written as

$$\mathcal{Q}(\xi)=\int_{\mathcal{B}}d^{D-2}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b \tag{14}$$

where σ is the determinant of the metric σ_{ab}, ξ and n^a are the Killing vector field and the unit normal vector on the boundary $\mathcal{B}$. In the context of counterterm method, the limit in which the boundary $\mathcal{B}$ becomes infinite ($\mathcal{B}_\infty$) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface $\mathcal{B}$ in to a reference of spacetime is required and the quantities which are computed are intrinsic to the spacetimes.

III. HORIZONLESS SOLUTIONS IN DIMENSIONALLY CONTINUED GRAVITY

The dimensionally continued gravity is a special class of the Lovelock gravity, which may be regarded as the most natural generalization to higher dimensions of the Einstein gravity. In this theory, the arbitrary constants are reduced to two by embedding the Lorentz group $SO(D-1,1)$ into a larger AdS group $SO(D-1,2)$ [17]. The remaining two fundamental constants are the gravitational and cosmological constants. In odd dimensions it is possible to construct a Lagrangian invariant under the anti-de Sitter group by making a certain choice of the Lovelock coefficients, while it is not possible to construct a non-trivial action principle invariant under $SO(D-1,2)$ and it is necessary to break the symmetry down to the Lorentz group. Accordingly, Lovelock gravity is separated into two distinct type of branches for odd and even dimensions. In what follows, we will consider a particular choice of the Lovelock coefficients given by

$$\alpha_p = \begin{cases} (D-2p-1)! \begin{pmatrix} (D-1)/2 \\ p \end{pmatrix} l^{2p-D} & \text{for odd } D \\ (D-2p)! \begin{pmatrix} D/2 \\ p \end{pmatrix} l^{2p-D} & \text{for even } D \end{cases} \tag{15}$$

where l is a length. For later convenience, the units are chosen such that

$$\kappa = \begin{cases} -\frac{l^{D-2}}{2(D-3)!} & \text{for odd } D \\ -\frac{l^{D-2}}{2D(D-3)!} & \text{for even } D \end{cases} \tag{16}$$

In order to obtain simplified equations of motion, it is more convenient to work in the Hamiltonian formalism. The Hamiltonian form of the action (1) is discussed in [23]. Here we consider the spacetimes generated by brane sources in D-dimensional spacetime that are horizonless and have nontrivial external solutions. We will work with the following ansatz for the metric:

$$ds^2 = -N^2(\rho)dt^2 + \frac{d\rho^2}{F(\rho)} + l^2F(\rho)d\phi^2 + \frac{\rho^2}{l^2}dX^2_{D-3} \tag{17}$$

where $dX^2_{D-3} = \sum_{i=0}^{D-3}(dx^i)^2$ is the Euclidean metric of $(D-3)$-dimensional submanifold. The parameter l^2 is appropriate constant proportional to cosmological constant Λ. The functions $N(\rho)$ and $F(\rho)$ need to be determined. The motivation for this metric gauge $[(g_{\rho\rho})^{-1} \propto g_{\varphi\varphi}]$ instead of the usual Schwarzschild gauge $[(g_{\rho\rho})^{-1} \propto g_{tt}]$ comes from the fact that we are looking for a string solution with conic singularity. by using Hamiltonian formalism and varying the Hamiltonian form of action with respect to $N(\rho)$ and $F(\rho)$, the metric functions may be computed as [24].

$$N(\rho) = C_1\rho + C_2, \tag{18}$$

$$F(\rho) = \begin{cases} \frac{\rho^2}{l^2} - (2c\rho + 2m)^{\frac{1}{n}} & \text{for odd } D \\ \frac{\rho^2}{l^2} - \left(2lc + \frac{2lm}{\rho}\right)^{\frac{1}{n}} & \text{for even } D \end{cases} \tag{19}$$

where m, c, C_1 and C_2 are integration constant. Since $N(\rho)$ is dimensionless and any constant may be absorbed in t, therefore we choose $C_1 = l^{-1}$ and $C_2 = 0$ without loss of generality.The properties and general structure of this solution was considered in [24].

A. Conserved Quantities

Now we apply the counterterm method to compute the conserved quantities of the solution (17). For the horizonless spacetime (17), the Killing vector is $\xi = \partial/\partial t$ and therefore its associated conserved charge is the total mass of the system enclosed by the boundary given as

$$M = \int_{\mathcal{B}} d^{D-2}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b \tag{20}$$

where T_{ab} is the stress energy tensor (12). It is a matter of calculation to show that the mass per unit volume ω_{D-2} is

$$M = m$$

IV. CLOSING REMARKS

The Lovelock action does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of $\delta g_{\mu\nu}$ normal to the boundary $\partial\mathcal{M}$. These normal derivative terms in Einstein gravity do not vanish by themselves, but are canceled by the variation of the Gibbons-Hawking surface term. Similarly in Lovelock gravity these normal derivatives can be canceled by surface terms that depend on the extrinsic and intrinsic curvature of the boundary $\partial\mathcal{M}$. First, we wrote down the tensorial form of these surface terms, and generalized the stress energy momentum tensor of Brown and York [3] to the case of Lovelock gravity. As in the case of Einstein gravity, I_G, and I_b of Eqs. (1) and (4) are divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. We, therefore, introduced a counterterm dependent only on the boundary volume which removed the divergences of the action and conserved quantities of the solutions of Lovelock gravity with zero curvature boundary.

Second, we considered the asymptotically AdS horizonless solutions in dimensionally continued gravity, which has no curvature singularity and no horizons, but have conic singularity at $r = 0$. These horizonless solutions have two fundamental constants which are the Newton's and cosmological constants. We applied the counterterm method to the case of our solutions in dimensionally continued gravity and calculated the finite mass of the spacetime. We found that the counterterm (11) has only one term, since the boundaries of our spacetimes are curvature-free. Other related problems such as the application of the counterterm method to the case of solutions of Lovelock gravity with nonzero curvature boundary remain to be carried out.

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