Solitonic generation of solutions including five-dimensional black rings and black holes

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We present a solitonic solution-generating method to construct single-rotational, axisymmetric and stationary vacuum solutions in five-dimensional Einstein equations. Using this method we give further some new five-dimensional solutions corresponding to S^1-rotating black multi-ring.

Inspired by the remarkable discovery of a S^1-rotating black ring solution by Emparan and Reall,[1] numerous investigations have been concentrated on the five-dimensional black objects(strings, holes and rings). As further steps towards systematic construction of new solutions in five-dimensional case, we previously introduced the new solitonic solution-generating method in five dimensions and succeeded in obtaining the S^2-rotating black ring solution,[2,3] which is complementary to the S^1-rotating black ring solution. Successively we also rederived S^1-rotating black ring solution.[4] In this article, we give brief introduction of the solution-generating method developed by us and present how to superpose black rings and a black hole. Then we give the full metric of di-ring as an interesting example.

We concentrate on the spacetimes which satisfy the following conditions: (c1) five dimensions, (c2) asymptotically flat spacetimes, (c3) the solutions of vacuum Einstein equations, (c4) having three commuting Killing vectors including time translational invariance and (c5) having a single nonzero angular momentum component. Under the conditions (c1) – (c5), the following Weyl-Papapetrou metric form can be adopted as a starting point,[5]

$$ds^2 = -e^{2U_0}(dx^0 - \omega d\phi)^2 + e^{2U_1}\rho^2(d\phi)^2 + e^{2U_2}(d\psi)^2 + e^{2(\gamma+U_1)}\left(d\rho^2 + dz^2\right) \quad (1)$$

where U_0, U_1, U_2, ω and γ are functions of ρ and z and also U_1 is set to $-(U_0+U_2)$. After introducing new functions $S := 2U_0 + U_2$ and $T := U_2$, the metric form (1) is rewritten into

$$ds^2 = -e^{-T}\left[e^{S}(dx^0 - \omega d\phi)^2 + e^{-S}\rho^2(d\phi)^2 + e^{2(\gamma-S)}\left(d\rho^2 + dz^2\right)\right] + e^{2T}(d\psi)^2. \quad (2)$$

Then the five-dimensional Einstein equations are reduced to the following set of equations,

$$\text{(i)}\ \nabla^2 T = 0\,, \quad \text{(ii)}\ \begin{cases} \partial_\rho\gamma_T = \frac{3}{4}\rho\left[(\partial_\rho T)^2 - (\partial_z T)^2\right] \\ \partial_z\gamma_T = \frac{3}{2}\rho\left[\partial_\rho T\,\partial_z T\right] \end{cases} ,$$

$$\text{(iii)}\ \nabla^2\mathcal{E}_S = \frac{1}{\mathrm{Re}\,\mathcal{E}_S}\nabla\mathcal{E}_S\cdot\nabla\mathcal{E}_S\,, \quad \text{(iv)}\ \begin{cases} \partial_\rho\gamma_S = \frac{\rho}{2(\mathcal{E}_S+\bar{\mathcal{E}}_S)}\left(\partial_\rho\mathcal{E}_S\partial_\rho\bar{\mathcal{E}}_S - \partial_z\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S\right) \\ \partial_z\gamma_S = \frac{\rho}{2(\mathcal{E}_S+\bar{\mathcal{E}}_S)}\left(\partial_\rho\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S + \partial_\rho\bar{\mathcal{E}}_S\partial_z\mathcal{E}_S\right) \end{cases} ,$$

$$\text{(v)}\ (\partial_\rho\Phi,\,\partial_z\Phi) = \rho^{-1}e^{2S}\,(-\partial_z w,\,\partial_\rho w)\ ,\quad \text{(vi)}\ \gamma = \gamma_S + \gamma_T,$$

where Φ is defined through the equation (v) and the function $\mathcal{E}_S$ is defined by $\mathcal{E}_S := e^S + i\,\Phi$. The equation (iii) is exactly the same as the well-known Ernst equation, so that we call $\mathcal{E}_S$ Ernst potential. The most crucial point to obtain new metrics is to solve the equation (iii) because of it's non-linearity. For the actual analysis in the following, we take some generalized Weyl solutions[5] as seeds and use the formulas shown in the paper by Castejon-Amened and Manko.[6]

The procedure to generate a new metric is the following: **(a)** choose an appropriate generalized Weyl solution as a seed; **(b)** extract seed functions $S^{(0)} = 2U_0^{(0)} + U_2^{(0)}$ and $T^{(0)} = U_2^{(0)}$ from the seed metric; **(c)** determine the auxiliary functions a and b by solving simple first-order linear differential equations[2] ; **(d)** introduce further the following new functions

$$\begin{cases} A := (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2 \\ B := [(x+1)+(x-1)ab]^2 + [(1+y)a+(1-y)b]^2 \\ C := (x^2-1)(1+ab)[(1-y)b-(1+y)a] + (1-y^2)(b-a)[x+1-(x-1)ab]\,, \end{cases}$$

where x and y are defined by $(R_\sigma + R_{-\sigma})/(2\sigma)$ and $(R_{-\sigma} - R_\sigma)/(2\sigma)$ with $R_{\pm\sigma} = \sqrt{\rho^2 + (z\mp\sigma)^2}$, respectively; **(e)** give the corresponding Ernst potential

$$\mathcal{E}_S = e^{S^{(0)}}\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)}\,,$$

(f) integrate (ii) and (iv) to determine the function γ (see the previous paper[3] for the explicit expression). Finally we can obtain the full metric with the following formulas

$$e^S = e^{S^{(0)}}\frac{A}{B}, \tag{3}$$

$$\omega = 2\sigma e^{-S^{(0)}}\frac{C}{A} + C_1, \tag{4}$$

$$e^{2\gamma} = C_2(x^2-1)^{-1}Ae^{2\gamma'}, \tag{5}$$

where γ' have been already given in the step **(f)** and C_1 and C_2 are fixed from some physical conditions.

It is very helpful to use the viewpoint of the rod-structure analysis[5,7] to describe what the method mentioned above does. By applying our method to a seed, the finite spacelike rods in the interval $-\sigma < z < \sigma$ is lifted from the ϕ-axis to the 'axis' of time-translation (i.e. horizon) and also ϕ-rotation is added to the timelike rods which correspond to the static black rings and holes around the interval.

Hence we can generate the spacetime of a black multi-ring and a black hole with the ϕ-rotation if the generalized Weyl solution corresponding to the spacetime with several static black rings (or a black hole) and also a local Kaluza-Klein bubble are appropriately prepared as a seed. It is noticed that the K-K bubble should be put on the interval $-\sigma < z < \sigma$, as a necessary condition to obtain a regular solution.

In the rest we mention the black di-ring system as the simplest case of black multi-ring and a hole system. The rod structure of the seed for the di-ring is given in the left figure of Fig.1. The resultant rod structure is given as in the right figure after applying the solitonic method.

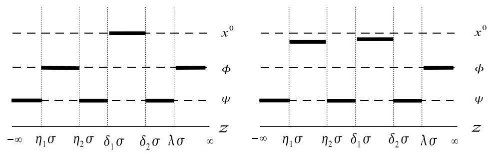

Fig. 1. Schematic pictures of rod structure of black di-ring and its seed.

The seed functions of the black di-ring can be easily extracted from Fig.1,

$$T^{(0)} = (\tilde{U}_{\lambda\sigma} - \tilde{U}_{\eta_2\sigma} + \tilde{U}_{\eta_1\sigma}) - (\tilde{U}_{\delta_2\sigma} - \tilde{U}_{\delta_1\sigma}),$$
$$S^{(0)} = (\tilde{U}_{\lambda\sigma} - \tilde{U}_{\eta_2\sigma} + \tilde{U}_{\eta_1\sigma}) + (\tilde{U}_{\delta_2\sigma} - \tilde{U}_{\delta_1\sigma}).$$

where
$\tilde{U}_d := \frac{1}{2}\ln\left[\sqrt{\rho^2+(z-d)^2}+(z-d)\right]$ and $U_d := \frac{1}{2}\ln\left[\sqrt{\rho^2+(z-d)^2}-(z-d)\right]$.
The auxiliary functions a and b appeared in the procedure **(c)** are given systematically by the formulas in the previous work.[3] Then we obtain the corresponding full metric following the procedure described above,

To extract the regular di-ring solutions, the further conditions must be introduced. We comment the conditions briefly. The constants C_1 and C_2 of Eq.(4) and (5) are fixed to eliminate the global rotation and adjust the periods of ϕ and ψ to 2π at the infinity. To assure the regularity of the metric component $g_{\phi\phi}$ and to eliminate the closed timelike curves, some parameters α and β of the auxiliary functions a and b must be adjusted. Another condition comes from curing conical singularities. Here we just say that some continuous set of parameters satisfies these conditions simultaneously and can be confirmed to assure the global regularity of the corresponding spacetime at least by numerical methods. The detailed analysis and discussion are given in the recent work.[8]

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