HAMILTONIAN TREATMENT OF STATIC AND COLLAPSING SPHERICALLY SYMMETRIC CHARGED THIN SHELLS IN LOVELOCK GRAVITY

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Through a Hamiltonian treatment, charged thin shells in spherically symmetric spacetimes, containing black holes, or other specific type of solutions, in d dimensional Lovelock-Maxwell theory, are studied.

The total action of the theory I is the sum of the gravitational Lovelock action [1], plus the electrodynamic and the matter actions

$$I = \kappa \int_{\mathcal{M}} d^d x \sum_{p=0}^{[d/2]} \alpha_p \sqrt{-\mathfrak{g}}\, 2^{-p}\, \delta^{\mu_1 \cdots \mu_{2p}}_{\nu_1 \cdots \nu_{2p}}\, R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \cdots R^{\nu_{2p-1} \nu_{2p}}_{\mu_{2p-1} \mu_{2p}} \tag{1}$$
$$- \frac{1}{4\epsilon \Omega_{d-2}} \int_{\mathcal{M}} d^d\, x\, \sqrt{-\mathfrak{g}}\, F_{\mu\nu} F^{\mu\nu} - \frac{1}{\epsilon} \int_{\mathcal{M}} d^d\, x\, \sqrt{-\mathfrak{g}}\, J^{\mu} A_{\nu} + \int_{\mathcal{M}} d^d\, x \sqrt{-\mathfrak{g}}\, \mathcal{L}_m \,.$$

Here κ is proportional to Newton's constant G, $\mathcal{M}$ is the d-dimensional spacetime manifold, $\mathfrak{g}$ is the determinant of the spacetime metric $\mathfrak{g}_{\mu\nu}$, $\delta^{\mu_1 \cdots \mu_{2p}}_{\nu_1 \cdots \nu_{2p}}$ is totally anti-symmetric in the upper and lower indices, $R^{\rho\sigma}_{\mu\nu}$ is the Riemann tensor, ϵ is proportional to the vacuum electric permitivity ϵ_0, Ω_{d-2} is the surface area of the $d-2$ unit sphere, $F_{\mu\nu}$ is the Maxwell tensor, J^{μ} is the electromagnetic current, A^{μ} is the vector potential, and $\mathcal{L}_m$ is the matter Lagrangian. The energy-momentum tensor derived from $\mathcal{L}_m$ will be that of a general perfect fluid. The free coefficients α_p in the Lovelock theory are chosen to obtain a sensible theory, with a negative cosmological constant appearing naturally. After writing the action and the Lagrangian for a total spacetime comprised of an interior and an exterior regions, with a thin shell as a boundary in between, one finds the Hamiltonian using the ADM decomposition of the spacetime metric $ds^2 = -(N^{\perp})^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$, where the g_{ij} are the canonical coordinates intrinsic to the $(d-1)$-dimensional time foliation, and π^{ij} are the respective canonical momenta. The consequent ADM description of the action is $I = \int dt \int d^{d-1}x \left(\pi^{ij} \dot{g}_{ij} - \mathcal{H}\right)$, with $\mathcal{H} = N^{\perp}\mathcal{H}_{\perp} + N^i \mathcal{H}_i$ and $\mathcal{H}_{\perp} = \mathcal{H}^{(g)}_{\perp} + \mathcal{H}^{(e)}_{\perp} + \mathcal{H}^{(m)}_{\perp}$, $\mathcal{H}_i = \mathcal{H}^{(g)}_i + \mathcal{H}^{(e)}_i + \mathcal{H}^{(m)}_i$, where $N^{\perp}$, N^i are the Lagrange multipliers, and $\mathcal{H}_{\perp}$, $\mathcal{H}_i$ are the respective constraints [2,3]. To these one adds $E_{\varphi} \equiv p^r_{,r} - J^0 = 0$, the electrodynamic constraint [4], where p^r is proportional to the radial component of the electric field, its only non-zero component due to the symmetry of the geometric set-up, and is the radial component of the conjugate momentum to the electrodynamic canonical coordinate A_i. The respective Lagrange multiplier is $\varphi \equiv A_0$. Upon particularizing to spherically symmetric spacetimes, one reduces the relevant constraints to $\mathcal{H}_\perp = 0$, $\mathcal{H}_\perp$ being the time translation generator, and $E_\varphi = 0$, the electrodynamic constraint. Variation of the Hamiltonian with respect to the canonical coordinates and conjugate momenta, and the relevant Lagrange multipliers, yields the dynamic and constraint equations. The vacuum solutions [4,5],

$$f^2(r) = 1 + \frac{r^2}{l^2} - \chi \left(\frac{2G_k M + \delta_{d-2k,1}}{r^{d-2k-1}} - \frac{\epsilon\, G_k}{(d-3)} \frac{Q^2}{r^{2(d-k-2)}} \right)^{1/k}, \qquad (2)$$

$$N = N_\infty = 1\,, \qquad p(r) = \epsilon \frac{Q}{r^{(d-2)}}\,, \qquad \varphi(r) = \frac{\epsilon}{(d-3)} \frac{Q}{r^{(d-3)}}\,, \qquad (3)$$

yield a division of the theory into two branches, namely $d - 2k - 1 > 0$ (which includes general relativity, Born-Infeld type theories, and other generic gravities) and $d - 2k - 1 = 0$ (which includes Chern-Simons type theories), where k is the parameter giving the highest power of the curvature in the Lagrangian (cf. (2)). There appears an additional parameter $\chi = (-1)^{k+1}$, which gives the character of the vacuum solutions. For $\chi = 1$ the solutions, being of the type found in general relativity, have a black hole character. For $\chi = -1$ the solutions, being of a new type not found in general relativity, have a totally naked singularity character. Since there is a negative cosmological constant, the spacetimes are asymptotically anti-de Sitter (AdS), and AdS when empty. The integration of the constraint equation $\mathcal{H}_\perp = 0$ from the interior to the exterior vacuum regions, through the thin shell, takes care of the smooth junction, yielding the shell equation directly

$$\frac{1}{2} m\,(\gamma_+ + \gamma_-) = (M_+ - M_-) - \frac{\epsilon(Q_+^2 - Q_-^2)}{2(d-3)} \frac{1}{R^{d-3}}\,, \qquad (4)$$

where $m \equiv \sigma\,\Omega_{d-2} R^{d-2}$, and $M_\pm$, $Q_\pm$ are the masses and charges of the external/internal vacuum solutions, resp., and $\gamma_\pm \equiv \sqrt{f_\pm^2 + \dot{R}^2}$ are generalized Lorentz factors, with $f_\pm^2$ being the metric functions of the inner and outer spacetimes, respectively. The integration of $E_\varphi = 0$ yields charge conservation, $Q_+ - Q_- = q$, with $q \equiv \sigma_e \Omega_{d-2} R^{d-2}$, σ_e being surface charge density. The uncharged case is treated in [6]. It is interesting to note that Eq. (4) is formally the same as that of d-dimensional general relativity [7], however, the functions $f_\pm$ are different in general relativity and Lovelock gravity. Differentiating (4) with respect to the proper time of the thin shell, τ, one obtains the equation of the acceleration of the thin shell

$$m\ddot{R} = \frac{(Q_+^2 - Q_-^2)\gamma_+\gamma_- m}{2\,E\;R^{d-2}} + (d-2)R^{d-3}\Omega_{d-2} P \gamma_+\gamma_- - \frac{m^2}{4\,E} \times \left(\gamma_- \frac{df_+^2}{dR} + \gamma_+ \frac{df_-^2}{dR} \right), \qquad (5)$$

where P is the pressure of the perfect fluid thin shell, and E is the right hand side of (4). One can now study the static configurations of the thin-shell, determining the pressure at which the shell is held in stable static equilibrium, that is $\dot{R} = \ddot{R} = 0$, $P = -\frac{1}{(d-2) R_0^{d-3}\, \Omega_{d-2}\, \gamma_+ \gamma_-} \left[\frac{(Q_+^2 - Q_-^2)\, \gamma_+ \gamma_- m}{2\, E\, R_0^{d-2}} - \frac{m^2}{4\, E} \left(\gamma_- \left. \frac{df_+^2}{dR} \right|_{R_0} + \gamma_+ \left. \frac{df_-^2}{dR} \right|_{R_0} \right) \right]$, which can be reduced to the known result for general relativity in four dimensions $P = (G\, m^2)/(16\, \pi\, R_0^2 (R_0 - m))$[8].

Apart from static configurations, one can also study the collapse of the thin shell, in particular when it is made of dust, $P = 0$ (note that expanding shells are the time reversal of the collapsing shells). Following the division of the solutions into the two branches and the two possible characters, one concludes from (4) and (5) that the cosmic censorship holds, when the collapse is into an empty interior. More generally, for a collapsing charged shell into initially non singular spacetimes with generic character or empty interiors, it is proved that the cosmic censorship is definitely upheld. Also, when the spacetimes in question have the same character of those spacetimes provided by general relativity, ($\chi = 1$), the collapse of the thin shells in the backgrounds, black hole or otherwise, of each different type of Lovelock theory is in many ways similar to the collapse in general relativity itself, and when the spacetimes in question have the opposite character ($\chi = -1$), some other new behaviour shows up. This implies that if there are extra dimensions with a relative large size, as proposed in large extra dimension scenarios, then differences in the collapse of a thin shell in spacetimes with different characters can provide the signature to uncovering not only of actual spacetime dimension d, but also the value of the parameter k. For a detailed analysis of the above see [9].

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