Lovelock Gravity and The Counterterm Method

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In this paper we, first, generalize the quasilocal definition of the stress energy tensor of Einstein gravity to the case of Lovelock gravity, by introducing the tensorial form of surface terms that make the action well-defined. In order to compute the conserved quantities of the solutions of Lovelock gravity, We introduce the boundary counterterm that removes the divergences of the action with flat boundary at constant t and r.

1. Introduction

The most natural extension of general relativity in higher dimensional spacetimes with the assumption of Einstein – that the left hand side of the field equations is the most general symmetric conserved tensor containing no more than second derivatives of the metric – is Lovelock theory.[1] Since the Lovelock tensor contains metric derivatives no higher than second order, the quantization of the linearized Lovelock theory is ghost-free.[2]

Our aim in this paper is to generalize the definition of the quasilocal stress energy tensor for computing the conserved quantities of a solution of Lovelock gravity. The concepts of action and energy-momentum play central roles in gravity. However there is no good local notion of energy for a gravitating system. A quasilocal definition of the energy and conserved quantities for Einstein gravity can be found in.[3] They define the quasilocal stress energy tensor through the use of the well-defined gravitational action of Einstein gravity with the surface term of Gibbons and Hawking.[4] Therefore the first step is to find the surface terms for the action of Lovelock gravity that make the action well-defined. Of course, as in the case of Einstein gravity, the action and conserved quantities diverge when the boundary goes to infinity.[3] For asymptotically AdS solutions, one can deal with these divergences via the counterterm method inspired by AdS/CFT correspondence.[5] This conjecture relates the low energy limit of string theory in asymptotically anti de-Sitter spacetime and the quantum field theory on its boundary. In the present context this conjecture furnishes a means for calculating the action and conserved quantities intrinsically by adding additional terms on the boundary that are curvature invariants of the induced metric.

The outline of our paper is as follows. In Sec. 2, we give the tensorial form of the surface terms that make the action well-defined, generalize the Brown York energy-momentum tensor for Lovelock gravity, and introduce the counterterm method for calculating the finite action and conserved quantities of solutions of Lovelock gravity with flat boundary. We finish our paper with some concluding remarks.

2. Lovelock Gravity and the Counterterm Method

We consider a D-dimensional spacetime manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. In this spacetime, The gravitational action satisfying the assumption of Einstein is precisely of the form proposed by Lovelock:[1]

$$I_G = \kappa \int d^D x\sqrt{-g}\sum_{p=0}^{n}\alpha_p \mathcal{L}_p \tag{1}$$

where $n \equiv [(D-1)/2]$ and $[z]$ denotes the integer part of z, α_p is an arbitrary constant and $\mathcal{L}_p$ is the Euler density of a $2p$-dimensional manifold

$$\mathcal{L}_p = \frac{1}{2^p}\delta^{\mu_1\nu_1\cdots\mu_p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma_p} R_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots R_{\mu_p\nu_p}{}^{\rho_k\sigma_k} \tag{2}$$

In Eq. (2) $\delta^{\mu_1\nu_1\cdots\mu p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma p}$ is the generalized totally anti-symmetric Kronecker delta and $R_{\mu\nu}{}^{\rho\sigma}$ is the Riemann tensor of the Manifold $\mathcal{M}$.

The Einstein-Hilbert action (with $\alpha_p = 0$ for $p \geq 2$) does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of $\delta g_{\mu\nu}$ normal to the timelike boundary $\partial\mathcal{M}$. These normal derivative terms are canceled by the variation of the Gibbons-Hawking surface term[4]

$$I_b^{(1)} = 2\kappa\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-\gamma}\Theta \tag{3}$$

where γ_{ab} is induced metric on the boundary $r =$ const. and Θ is trace of extrinsic curvature of this boundary. The surface terms that make the variational principle of Lovelock gravity well-defined are known in terms of differential forms.[6] The tensorial form of these surface terms may be written as[7]

$$I_b = -2\kappa\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-\gamma}\sum_{p=0}^{n}\sum_{s=0}^{p-1}\frac{(-1)^{p-s}p\alpha_p}{2^s(2p-2s-1)}\mathcal{H}^{(p)} \tag{4}$$

where α_p is the Lovelock coefficients and $\mathcal{H}^{(p)}$ is

$$\mathcal{H}^{(p)} = \delta^{[a_1\ldots a_{2p-1}]}_{[b_1\ldots b_{2p-1}]} R^{b_1b_2}{}_{a_1a_2}\cdots R^{b_{2s-1}b_{2s}}{}_{a_{2s-1}a_{2s}}\Theta^{b1}_{a1}\cdots\Theta^{b_{2p-1}}_{a_{2p-1}} \tag{5}$$

In Eq. (5) $R^{ab}{}_{cd}(g)$'s are the boundary components of the Riemann tensor of the Manifold $\mathcal{M}$, which depend on the velocities through the Gauss–Codazzi equations.[7] The explicit form of the second and third surface terms of Eq. (4) have been written in.[8]

In general $I = I_G + I_b$ is divergent when evaluated on solutions, as is the Hamiltonian and other associated conserved quantities. In Einstein gravity, one can remove the non logarithmic divergent terms in the action by adding a counterterm action I_{ct} which is a functional of the boundary curvature invariants.[9] The issue of determination of boundary counterterms with their coefficients for higher-order Lovelock theories is at this point an open question. However for the case of a boundary with zero curvature $[\widehat{R}_{abcd}(\gamma)=0]$, it is quite straightforward. This is because all curvature invariants are zero except for a constant, and so the only possible boundary counterterm is one proportional to the volume of the boundary regardless of the number of dimensions:

$$I_{ct}=2\kappa\lambda\alpha_0\int_{\partial\mathcal{M}_\infty}d^{D-1}x\sqrt{-\gamma} \tag{6}$$

where λ is a constant which should be chosen such that the divergences of the action is removed.

Having the total finite action, one can use the quasilocal definition of Brown and York[3] to construct a divergence free stress-energy tensor as

$$T_b^a=-2\kappa\left\{\lambda\alpha_0\gamma_b^a+\sum_{p=0}^{n}\sum_{s=0}^{p-1}\frac{(-1)^{p-s}p\alpha_p}{2^s(2p-2s-1)}\mathcal{H}_b^{(p,s)a}\right\} \tag{7}$$

where $\mathcal{H}_b^{(p,s)a}$ is

$$\mathcal{H}_b^{(p,s)a}=\delta^{[a_1\ldots a_{2p-1}a]}_{[b_1\ldots b_{2p-1}b]}\widehat{R}^{b_1b_2}{}_{a_1a_2}\cdots\widehat{R}^{b_{2s-1}b_{2s}}{}_{a_2s-1a_2s}\Theta^{b_{2s+1}}_{a_{2s+1}}\cdots\Theta^{b_{2p-1}}_{a_{2p-1}}, \tag{8}$$

To compute the conserved mass of the spacetime, one should choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in Arnowitt-Deser-Misner (ADM) form:

$$\gamma_{ab}dx^a dx^a=-N^2dt^2+\sigma_{ij}\left(d\varphi^i+N^idt\right)\left(d\varphi^j+N^jdt\right)$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and N^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (7) can be written as

$$\mathcal{Q}(\xi)=\int_{\mathcal{B}}d^{D-2}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b \tag{9}$$

where σ is the determinant of the metric σ_{ab}, ξ and n^a are the Killing vector field and the unit normal vector on the spacelike boundary $\mathcal{B}$.

3. CLOSING REMARKS

The Lovelock action does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of $\delta g_{\mu\nu}$ normal to the boundary $\partial\mathcal{M}$. These normal derivative terms in Lovelock gravity are canceled by the variation of the surface terms that depend on the extrinsic and intrinsic curvature of the boundary $\partial\mathcal{M}$. we wrote down the tensorial form of these surface terms, and generalized the stress energy momentum tensor of Brown and York[3] to the case of Lovelock gravity. As in the case of Einstein gravity, the action is divergent when evaluated on the solutions. We, therefore, introduced a counterterm dependent only on the boundary volume which removed the divergences of the action and conserved quantities of the solutions of Lovelock gravity with zero curvature boundary.

We found that the counterterm (6) has only one term, since the boundaries of our spacetimes are curvature-free.

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