LG (LANDAU-GINZBURG) in GL (GREGORY-LAFLAMME)

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We report a study of the Gregory-Laflamme instability of black strings, or more precisely of the order of the transition, being either first or second order, and the critical dimension which separates the two cases. First, we describe a novel method based on the Landau-Ginzburg perspective for the thermodynamics that somewhat improves the existing techniques. Second, we generalize the computation from a circle compactification to an arbitrary torus compactifications. We explain that the critical dimension cannot be lowered in this way, and moreover in all cases studied the transition order depends only on the number of extended dimensions. We discuss the richer phase structure that appears in the torus case.

In the presence of a compact dimension Gregory and Laflamme (GL) discovered that uniform black strings are perturbatively unstable below a certain critical dimensionless mass density.[1] The order of the transition can be computed by following perturbatively the branch of non-uniform solutions which emanates from the critical GL string, as first shown by Gubser in the case of a five-dimensional spacetime[2] where the transition is first order. That calculation was generalized by one of us (ES) to arbitrary spacetime dimensions with the surprising result that the transition is first order only for $D < D^* =$ "13.5" while it is second order for higher dimensions.[3] Here first order means a transition between two distinct configurations, while a second order transition is smooth – the uniform string changes smoothly into a slightly non-uniform string. Kudoh and Miyamoto[4] repeated the calculation in the economical Harmark-Obers coordinates,[5] confirmed previous results and observed that in the canonical ensemble the critical dimension actually changes from $D^* =$ "13.5" to $D^*_{can} =$ "12.5". All this data is crucial in the construction of the phase diagram for this system (see[6] and[7] for a review).

The present report includes two main results. First, we show how to somewhat improve the existing method of calculating the transition order by employing a Landau-Ginzburg perspective (the basic idea was described already in Appendix A of[8]). Secondly, we generalize from the usual $\mathbf{S}^1 \equiv \mathbf{T}^1$ compactification to an arbitrary torus compactification T^p.

Landau-Ginzburg improvement to the method. In the Landau-Ginzburg (LG) theory of phase transitions one expands the free energy of the system around the critical point in powers of order parameters. In particular, it is known that as long as the coefficient of a certain cubic term in the free energy is non-vanishing then the transition is first order. If the cubic term vanishes, for instance due to a parity symmetry such as in our case, then it is the sign of the coefficient of a certain quartic term, which we denote by $\mathcal{C}$, that determines whether the transition is first order or higher (of course if this term vanishes one has to go to higher terms).

Before we can compare the Landau-Ginzburg method with Gubser's method, we recall the features of the latter. There one computes order by order the metric of the static non-uniform string branch emanating from the critical GL string. The first order is nothing but the GL mode. At the second order one computes the back-reaction. Finally the third order is computed, or more precisely only the first harmonic along the compact dimension, from which one can finally compute the leading coefficients of the changes in mass and entropy, $\eta_1,\ \sigma_2$ of the new branch. The sign of these two quantities determines the order of thetransition.

At first sight the two methods look quite different. However, we show that in the LG method one also needs to precisely compute the second order back-reaction to the metric. *The third order however is not required in LG (thereby avoiding the solution of a set of linear differential equations with sources). Rather one needs only to expand the action to an appropriate quartic order, to substitute in the results from the first and second orders and perform certain integrals that add up to the constant whose sign determines the order.*

A way to understand the simplification is the following: in Gubser's method one computes the third order, but it turns out that all that is really needed is the projection of the third order onto the GL mode. That is precisely the reason why the first harmonic sufficed (as the GL mode is in the first harmonic). Our substitution into the quartic order of the free-energy achieves exactly that, without the need to compute other properties of the third order. We perform the "Landau-Ginzburg" calculation for an $\mathbf{S}^1$ compactification in various dimensions and verify that we get the same bottom-line coefficients $\eta_1,\ \sigma_2$ as in the previous method.

Torus compactification. It is interesting to generalize the compactifying manifold, and the simplest option beyond the circle $\mathbf{S}^1 \equiv \mathbf{T}^1$, is a product of circles, or more generally a p-torus $\mathbf{T}^p$. The number of extended spacetime dimensions is denoted by d and the total spacetime dimension is $D = d + p$. The critical GL density for such a torus compactification is easily found to be given in terms of the shortest vector in the reciprocal lattice.[9]

Before we proceed to analyze the transition order we note that it sufficies to restrict to square torii. Basically, we view the space of torii as having two boundaries – on the one hand highly asymmetrical torii, where one (or more) dimensions are much larger than the rest, and on the other hand highly symmetrical torii such as the square torus. Since the limit of a highly asymmetrical torus reduces to the case of a lower dimensional torus (mostly the well-understood case of $\mathbf{S}^1$ compactification), we argue that by studying the opposite limit of a highly symmetrical torus, we achieve an understanding of both limits and thereby also some understanding of the intermediate region of general torii.

For a square $\mathbf{T}^p$ torus compactification, p modes turn marginally tachyonic at the same (GL) point. We find that the constant $\mathcal{C}$ is replaced by a $p \times p$ quadratic form $\mathcal{C}^{ij}$, in order to allow for the various possible directions in the (marginally) "tachyon space", and that the transition is second order iff $\mathcal{C}^{ij}$ is positive for all directions. Namely, it is enough that there is a single direction in tachyon space which sees a first order transition for the transition to be one. Taking into account the $\mathbf{T}^1$ results we may immediately deduce that the critical dimension cannot be lower than in the $\mathbf{T}^1$ case with the same number of extended dimensions, d.

Due to the high degree of symmetry of the square torii $\mathcal{C}^{ij}$ consists only of two independent entries: all the diagonal entries are the same, denoted by $\mathcal{C}^{=}$, and all the off-diagonal entries are the same, denoted by $\mathcal{C}^{\neq}$. Since the diagonal term is precisely the one computed in the T^1 compactification, we set to compute the off-diagonal term. Due to the symmetry $\mathcal{C}^{\neq}$ is the same for all p and for that purpose it suffices to consider $p = 2$, namely we consider the square T^2 torus. The only parameter remaining is the number of extended dimensions. Once we have chosen the diagonal direction in tachyon space we are not bothered any longer by the presence of several (marginally) tachyonic modes. However, the number of metric components involved in the calculation (back reaction and quartic coefficient) is larger than in the $\mathbf{T}^1$ case. Certain discrete symmetries are found to be helpful in simplifying the calculation.

As a result we find that for all the studied values of d where the T^1 transition is second order, the T^p transition is also second order for all p. Combining this result with observation mentioned above we conclude that *the transition order for square torii shows some robustness in that it depends only on d, the number of extended dimensions, and not on p.*

In addition we note some subtler implications, including the finding that for almost all d the diagonal direction in tachyon space is disfavored relative to turning on a tachyon in a single compact dimension, and in this sense we have spontaneous symmetry breaking.

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