

GRAVITATING NON-ABELIAN SOLITONS AND HAIRY BLACK HOLES IN HIGHER DIMENSIONS

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This is a short review of classical solutions with gravitating Yang-Mills fields in $D > 4$ spacetime dimensions. The simplest SO(4) symmetric particlelike and SO(3) symmetric vortex type solutions in the Einstein-Yang-Mills theory in $D = 5$ are considered, and their various generalizations with or without an event horizon, for other symmetries, in more general theories, and also in $D > 5$ are described. In addition, supersymmetric solutions with gravitating Yang-Mills fields in string theory are discussed.

1. Introduction

In this mini-review the following two questions will be addressed:

- What is known about solutions with gravity-coupled non-Abelian gauge fields in $D > 4$ spacetime dimensions ?
- Why should one study such solutions ?

In answering these questions we shall briefly consider the simplest gravitating non-Abelian solutions in $D = 5$ and shall then discuss their known generalizations.

2. Einstein-Yang-Mills theory in D=4

Before going to higher dimensions, it is worth reminding the situation in $D = 4$. It is well known that neither pure gravity nor pure Yang-Mills theory in $D = 4$ admit solitons, by which we mean stationary and globally regular (but not necessarily stable) solutions with finite total energy.

In the case of pure Einstein gravity with the action

$$S_E = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x \quad (1)$$

the statement is the content of the theorem¹ of Lichenrowicz. The vacuum gravity admits however solutions describing localized stationary objects with finite mass, but these are not globally regular: black holes.

The action for the pure Yang-Mills (YM) theory with a compact and simple gauge group \mathcal{G} is given by

$$S_{\text{YM}} = -\frac{1}{4Ke^2} \int \text{tr} F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x, \quad (2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Here $A_\mu = A_\mu^a T_a$ and T_a are the gauge group generators normalized such that $\text{tr}(T_a T_b) = K \delta_{ab}$. If $\mathcal{G} = \text{SU}(2)$ then $a = 1, 2, 3$ and $T_a = \frac{1}{2} \tau_a$ where τ_a are the Pauli matrices. The gauge coupling constant is denoted

by e . Since this theory is scale invariant, the non-existence of stationary solitons, expressed by the statement “there are no classical glueballs”, essentially follows²⁻⁴ from the scaling arguments.

The physical reason for the non-existence of solitons in the above two cases is clear: one cannot have equilibrium objects in systems with only attractive (gravity) or only repulsive (Yang-Mills) interactions. However, in systems with both gravity and Yang-Mills fields described by the Einstein-Yang-Mills (EYM) action

$$S_{\text{EYM}} = S_{\text{E}} + S_{\text{YM}} \quad (3)$$

both attractive and repulsive forces are present, and so the existence of solitons is not excluded. At the same time, the existence of solitons in this case is not guaranteed either, and so it was a big surprise when such solutions were constructed⁵ by Bartnik and McKinnon. As this was actually the first known example of solitons in self-gravitating systems, a lot of interest towards the Einstein-Yang-Mills system (3) has been triggered. This interest was further increased by the discovery in the same system of the first known example⁶⁻⁸ of hairy black holes which are not uniquely characterized by their conserved charges and so violate manifestly the no-hair conjecture.⁹ Further surprising discoveries followed when these static EYM solitons and black holes were generalized¹⁰⁻¹² to the non-spherically symmetric and non-static cases. This has revealed that the EYM solutions provide manifest counter-examples also to a number of classical theorems originally proven for the Einstein-Maxwell system but believed to be generally true, as for example the theorems of staticity, circularity, the uniqueness theorem, Israel’s theorem and so on (see¹³ for a review). As a result, the EYM theory has become an interesting topic of studies.

3. Pure gravity and pure Yang-Mills in $D=5$

Going to higher dimensions, let us first consider separately some simplest solutions for pure gravity and pure Yang-Mills theory in $D = 5$. We shall denote the spacetime coordinates by x^M where $M = 0, 1, 2, 3, 4$.

3.1. *Pure gravity*

Vacuum gravity in $D > 4$ has been much studied and many interesting solutions have been obtained (see¹⁴ for a recent review), but we shall just mention a couple of the simplest ones. If $g_{\mu\nu}^E(x^\sigma)$, where $\mu, \nu, \sigma = 1, 2, 3, 4$, is a Euclidean Ricci flat metric (gravitational instanton), then the 5D geometry

$$ds^2 = -dt^2 + g_{\mu\nu}^E dx^\mu dx^\nu$$

will also be Ricci flat. One obtains in this way static and globally regular vacuum solutions in $D = 5$. However, since there are no¹⁵ asymptotically Euclidean gravitational instantons, these 5D solutions will not be asymptotically flat (although they can be asymptotically locally flat). There exist also localized and asymptotically

flat solutions – black holes – but these are not globally regular. In the simplest case with the SO(4) spatial symmetry this is the Schwarzschild black hole,

$$ds^2 = -Ndt^2 + \frac{dr^2}{N} + r^2 d\Omega_3^2, \quad N = 1 - \left(\frac{r_g}{r}\right)^2.$$

If $\partial/\partial x^4$ is a symmetry and $g_{\mu\nu}$ is a Ricci flat Lorentzian metric $/\mu = 0, 1, 2, 3/$ then the 5D metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + (dx^4)^2$$

will also be Ricci flat. Choosing $g_{\mu\nu}$ to be a 4D black hole metric gives then a one-dimensional linear mass distribution in $D = 5$ – black string.

3.2. Pure Yang-Mills theory: particles and vortices

Since the Yang-Mills theory in $D = 5$ is not scale invariant, one can have localized and globally regular object made of pure gauge field. We shall call them ‘Yang-Mills particles’, these are essentially the 4D YM instantons uplifted to D=5. Specifically, if $A_\mu^a(x^\nu)$ is a solution of the 4D Euclidean YM equations then

$$A_M^a = (0, A_\mu^a(x^\nu)) \quad / \mu = 1, 2, 3, 4/$$

will be a solution of the 5D YM equations describing a static object whose 5D energy coincides with the 4D instanton action¹⁶

$$E = \frac{1}{4e^2} \int (F_{\mu\nu}^a)^2 d^4x \geq \frac{8\pi^2 |n|}{e^2}.$$

The equality here is attained for self-dual configurations, with $F_{\mu\nu} = *F_{\mu\nu}$, in which case $n \in \mathbb{Z}$ is the number of the ‘YM particles’. Their mutual interaction forces being exactly zero, the particles can be located anywhere in the 4-space.

If $\partial/\partial x^4$ is a symmetry, then choosing

$$A_M^a = (0, A_i^a(x^k), H^a(x^k)), \quad /i, k = 1, 2, 3/$$

the YM energy per unit x^4 ,

$$E = \frac{1}{2e^2} \int ((\partial_i H^a + \varepsilon_{abc} A_i^b H^c)^2 + \frac{1}{2} (F_{ik}^a)^2) d^3x \geq \frac{4\pi |n|}{e^2},$$

coincides with the energy¹⁷ of the D=3 YM-Higgs system. Its absolute minima for a given n are the BPS monopoles, and when lifted back to D=5 they become one-dimensional objects – ‘YM vortices’. The number of vortices, n , can be arbitrary, and they can be located anywhere in the 3-space of x^k .

We shall now describe the self-gravitating analogs¹⁸ for the YM particles and also for the YM vortices. Practically all other known solutions with gravitating gauge fields in higher dimensions are generalizations of these two simplest types.

4. Gravitating YM particles¹⁸

The EYM theory with gauge group SU(2) in D=5 is defined by the action

$$S = \int \mathcal{L}_5 \sqrt{{}^{(5)}g} d^5x = \int \left(\frac{1}{16\pi G} {}^{(5)}R - \frac{1}{4e^2} F_{MN}^a F^{aMN} \right) \sqrt{{}^{(5)}g} d^5x, \quad (4)$$

where $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + \varepsilon_{abc} A_M^b A_N^c$ ($a = 1, 2, 3$). The Newton constant G and gauge coupling e being both dimensionful, $[G^{1/3}] = [e^2] = [\text{length}]$, one can define the dimensionless coupling parameter as

$$\kappa = \frac{8\pi G}{e^6}.$$

Let us try to construct the 5D counterparts of the 4D solitons⁵ of Bartnik and McKinnon: particle-like objects localized in 4 spatial dimensions. In the simplest case they are static and purely magnetic and have the SO(4) spatial symmetry. The fields are then given by

$$ds^2 = e^2 \left\{ -\sigma(r)^2 N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega_3^2 \right\}, \quad A^a = (1 + w(r)) \theta^a, \quad (5)$$

where θ^a are the invariant forms on S^3 . With $N \equiv 1 - \kappa m(r)/r^2$, the EYM field equations for the action (4) reduce to

$$r^2 N w'' + r w' + \kappa (m - (w^2 - 1)^2) \frac{w'}{r} = 2(w^2 - 1)w, \quad (6)$$

$$r m' = r^2 N w'^2 + (w^2 - 1)^2, \quad (7)$$

$$\sigma' = \kappa \frac{w'^2}{r} \sigma \quad \Rightarrow \quad \sigma(r) = \exp \left(-\kappa \int_r^\infty \frac{w'^2}{r} dr \right). \quad (8)$$

If $\kappa = 0$ then $\sigma = N = 1$ and the geometry is flat. The solution of Eq.(6) is the 5D YM particle, which is the same as the 4D one-instanton solution:

$$w = \frac{1 - b r^2}{1 + b r^2}. \quad (9)$$

Here $b > 0$ is an integration constant that determines the size of the particle. This solution is everywhere regular and its energy is the ADM mass: $E = M_{\text{ADM}} = m(\infty) = \frac{8}{3}$.

The next question is what happens if $\kappa \neq 0$? Let us consider asymptotically flat solutions with finite ADM mass. If they are globally regular, then they will have a regular origin at $r = 0$, and for small r the local solution of Eqs.(6)–(8) will be given by

$$w = 1 - 2br^2 + O(r^4), \quad m = O(r^3). \quad (10)$$

One can also consider black hole solutions, in which case

$$\exists r_h > 0: \quad N(r_h) = 0, \quad N'(r_h) > 0, \quad w(r_h) < \infty. \quad (11)$$

In both cases, using the boundary condition (10) if $r_h = 0$ and (11) for $r_h > 0$ and integrating (7) gives the ADM mass functional

$$M_{\text{ADM}}[w(r)] = m(\infty) = \frac{r_h^2}{\kappa} + \int_{r_h}^{\infty} \frac{dr}{r} (r^2 w'^2 + (w^2 - 1)^2) \sigma(r). \quad (12)$$

This shows that one should have $w(\infty) = \pm 1$ for the mass to be finite. Let us suppose that there is a solutions with $M_{\text{ADM}} < \infty$ and let $w(r)$ be the corresponding gauge field amplitude. The first variation of the mass functional (12) should then vanish under all variations $w(r) \rightarrow w(r) + \delta w(r)$ respecting the boundary conditions. In particular, if we consider special variations $w(r) \rightarrow w(\lambda r)$ then one should have $\frac{d}{d\lambda} M[w(\lambda r)] = 0$ for $\lambda = 1$. However, (12) implies that $\frac{d}{d\lambda} M[w(\lambda r)] < 0$ for any λ . As a result, there are no solutions with finite ADM mass.

We therefore conclude that *the flat space YM particles do not admit self-gravitating generalizations with finite energy*. One can say that they get completely destroyed by gravity, as in some sense they resemble dust, since their energy momentum tensor is

$$T_{MN} = \epsilon(r) \delta_M^0 \delta_N^0$$

and in addition they can be scaled to an arbitrary size. As a result, when gravity is on, the repulsion and attraction are not balanced and so equilibrium states are not possible. If one nevertheless tries integrating Eqs.(6)–(8) starting from the regular origin, say, to see what happens, one finds a peculiar quasi-periodic behavior shown in Fig.1. This can be explained qualitatively: integrating Eq.(6) with the regular

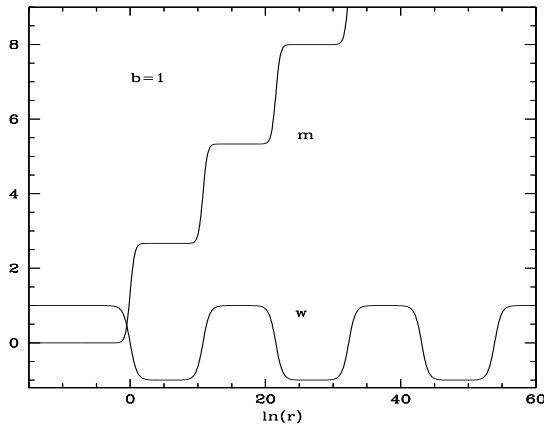


Fig. 1. Globally regular solution to Eqs.(6)–(8),(10) with $b = 1$, $\kappa = 10^{-8}$.

boundary condition (10) gives

$$w'^2 - (w^2 - 1)^2 = - \int_{-\infty}^{\tau} (\ln(\sigma^2 N))' w'^2 d\tau, \quad (13)$$

with $' \equiv \frac{d}{d\tau} \equiv r\sqrt{N} \frac{d}{dr}$. This describes a particle moving with friction in the inverted double-well potential $U = -(w^2 - 1)^2$. At $\tau = -\infty$ the particle starts at the local maximum of the potential at $w = 1$, but then it loses energy due to the dissipation and cannot reach the second maximum at $w = -1$. Hence it bounces back. For large τ the dissipative term tends to zero, and the particle ends up oscillating in the potential well with a constant energy. After each oscillation the mass function m increases in a step-like fashion, and for large r one has $m(r) \sim \tau \sim \ln r$. As a result, there emerges an infinite sequence of static spherical shells of the YM energy in the D=5 spacetime. These arguments easily generalize to the black hole case.

4.1. Other particlelike solutions^{19–26}

A number of generalizations of the above results have been considered. It turns out that the divergence of the mass is the generic property of the static EYM solutions with the spherical symmetry in $D > 4$. In particular, it has been shown that for all static EYM solutions with gauge group $\mathcal{G} = \text{SO}(d)$ and with the spatial $\text{SO}(d)$ symmetry in $D > 4$ dimensions, where $d = D$ if D is even and $d = D - 1$ if D is odd, the ADM mass diverges. This concerns both globally regular and black hole solutions, and also solutions with a Λ -term.

It turns out, however, that one can nevertheless have finite mass particlelike solutions in $D > 4$ at the expense of adding the higher order curvature corrections to the EYM Lagrangian. The gravitational term of the action, R , is then ‘corrected’ by adding all possible Gauss-Bonnet invariants, while the Yang-Mills term, $\text{tr}(F \wedge *F)$, is augmented by adding terms of the type $\text{tr}(\mathcal{F} \wedge *\mathcal{F})$, where $\mathcal{F} = F \wedge \dots \wedge F$. It is important that after taking these corrections into account the field equations remain the second order. These equations admit localized solutions with a finite mass which have been studied^{19,21–26} quite extensively.

4.2. Gravitating Yang monopoles²⁷

Flat space 4D YM instantons with $\mathcal{G} = \text{SU}(2)$ can also be used to construct self-gravitating solutions in $D = 6$. The YM instanton lives in this case on the S^4 angular part of the $\text{SO}(5)$ -invariant 5-space, the ansatz for the metric and gauge field being

$$ds^2 = -\sigma(r)^2 N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 (d\xi^2 + \sin^2 \chi d\Omega_3^2), \quad A^a = (1 + w(\chi)) \theta^a.$$

YM equations decouple and admit a solution $w(\chi) = \cos \chi$, whose energy momentum tensor does not depend on r and determines the source for the Einstein equations. The latter give

$$\sigma = 1, \quad N = 1 - \frac{2Gm(r)}{r^3}, \quad m' = 8\pi, \quad m(r) = 8\pi r + m_0,$$

and so the mass is linearly divergent for large r . This solution can be generalized²⁷ also to $D = 2k + 2$ dimensions, choosing $\text{SO}(2k)$ as the gauge group. In this case

one finds $m(r) \sim r^{2k-3}$.

5. Gravitating YM vortices¹⁸

Let us return to the 5D EYM model defined by Eq.(4) and assume the existence of a hypersurface orthogonal Killing vector $\partial/\partial x^4$, such that the metric is

$$g_{MN}dx^M dx^N = e^{-\zeta}g_{\mu\nu}dx^\mu dx^\nu + e^{2\zeta}(dx^4)^2,$$

$$A_M^\alpha dx^M = A_\mu^\alpha dx^\mu + H^\alpha dx^4, \quad / \mu, \nu = 0, 1, 2, 3/. \quad (14)$$

Inserting this to (4) reduces the 5D EYM model to the 4D EYM-Higgs-dilaton theory,

$$\begin{aligned} \sqrt{{}^{(5)}g} \mathcal{L}_5 = & \left(\frac{{}^{(4)}R}{2\kappa e^6} - \frac{3}{\kappa e^6} (\partial_\mu \zeta)^2 \right. \\ & \left. - \frac{1}{2e^2} e^{-2\zeta} (D_\mu H^a)^2 - \frac{1}{4e^2} e^\zeta (F_{\mu\nu}^a)^2 \right) \sqrt{{}^{-(4)}g}, \end{aligned}$$

Let us assume the SO(3) symmetry for the 4D fields,

$$e^{-\zeta(r)} ds^2 = -e^{2\nu(r)} dt^2 + dr^2 + R^2(r) d\Omega_2^2,$$

$$A_k^\alpha dx^k = (w(r) - 1) \epsilon_{aik} n^i dn^k, \quad H^a = n^a e^{\zeta(r)} h(r), \quad (15)$$

where n^k is the unit normal to the 2-sphere, $d\Omega_2^2 = dn^k dn^k$. The independent EYM equations can be rewritten¹⁸ in the form of a seven-dimensional dynamical system

$$\frac{d}{dr} y_k = F_k(y_s, \kappa) \quad (16)$$

with $y_k = \{w, w', h, h', Z = \zeta', R, R'\}$. Solutions of these equations show interesting features which can be qualitatively understood by studying the fixed points of the system. The system has the following fixed points.

I. The origin: $(w, h, Z, R) = (1, 0, 0, 0)$. Assuming that this fixed point is attained for $r = 0$, the local behavior of the solution for small r is

$$w = 1 - br^2 + O(r^2), \quad h = ar + O(r^3), \quad Z = O(r^2), \quad R = r + O(r^3). \quad (17)$$

II. Infinity: $(w, h, Z, 1/R) = (0, 1, 0, 0)$. This fixed point can be reached for $r \rightarrow \infty$, in which case

$$\begin{aligned} w &= A r^C e^{-r} + o(e^{-r}), \quad Z = \kappa Q r^{-2} + O(r^{-3} \ln r), \\ h &= 1 - C r^{-1} + O(r^{-2} \ln r), \\ R &= r - m \ln r + m^2 r^{-1} \ln r - r_0 + \gamma r^{-1} + O(r^{-2} \ln r). \end{aligned} \quad (18)$$

In these expressions $a, b, A, v, C, Q, r_\infty, \gamma$ are eight free parameters. The ADM mass is $M_{\text{ADM}} = 3(C + (2 + \kappa)Q)$.

III. “Warped” $AdS_3 \times S^2$. The fixed point values of the functions are expressed in this case in terms of roots of the cubic equation $4q^3 + 7q^2 + 11q = 1$ as

$$w^2 = q, \quad R^2 = \kappa \frac{(11q-1)(1-q)}{(4q^2-13q+1)}, \quad h^2 = \frac{1-q}{R^2}, \quad Z^2 = -\frac{4q^2-13q+1}{(4q+1)R^2}. \quad (19)$$

Evaluating,

$$w = 0.29, \quad h = \frac{1.27}{\sqrt{\kappa}}, \quad Z = \pm \frac{0.31}{\sqrt{\kappa}}, \quad R = 0.75\sqrt{\kappa}, \quad (20)$$

which gives an exact non-Abelian solution with the geometry

$$ds^2 = e^{2(1+\kappa h^2)Zr} dt^2 - dr^2 - e^{2Zr} (dx^4)^2 - R^2 d\Omega_2^2. \quad (21)$$

Linearizing Eqs.(16) around this fixed point gives the characteristic eigenvalues

$$\left(-\frac{2.77}{\sqrt{\kappa}}, -\frac{2.47}{\sqrt{\kappa}}, -\frac{2.12}{\sqrt{\kappa}}, -\frac{0.61}{\sqrt{\kappa}} \pm i \frac{1.24}{\sqrt{\kappa}}, +\frac{0.88}{\sqrt{\kappa}}, +\frac{1.54}{\sqrt{\kappa}} \right). \quad (22)$$

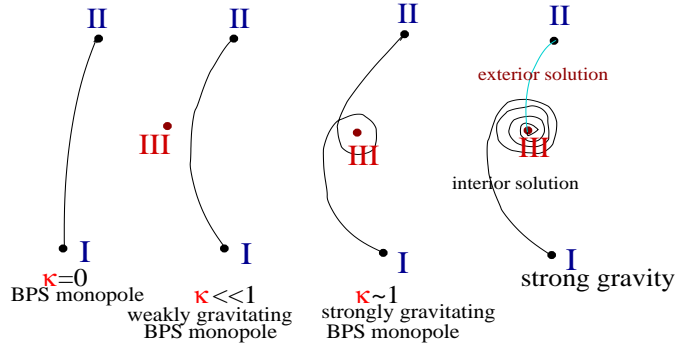


Fig. 2. Schematic behavior of solutions of Eqs.16 interpolating between fixed points I and II.

Global solutions of Eqs.(16) can then be viewed as trajectories in the phase space interpolating between fixed point I (origin) and fixed point II (infinity). Their behavior is schematically sketched in Fig.2. For $\kappa = 0$ the 4D solution is the flat space BPS monopole and its 5D analog is the flat space YM vortex. For $\kappa \ll 1$ the solution is a weakly gravitating 5D vortex whose field configuration is only slightly deformed as compared to the $\kappa = 0$ case. The YM vortices therefore do generalize to curved space, unlike the YM particles.

5.1. Strong gravity limit

Further increasing the gravitational coupling κ deforms the YM vortex more and more and the phase space trajectory approaches closer and closer the third fixed point. Since the latter has complex eigenvalues with negative real part, the phase trajectory gets attracted by this fixed point and oscillates around it for a while before

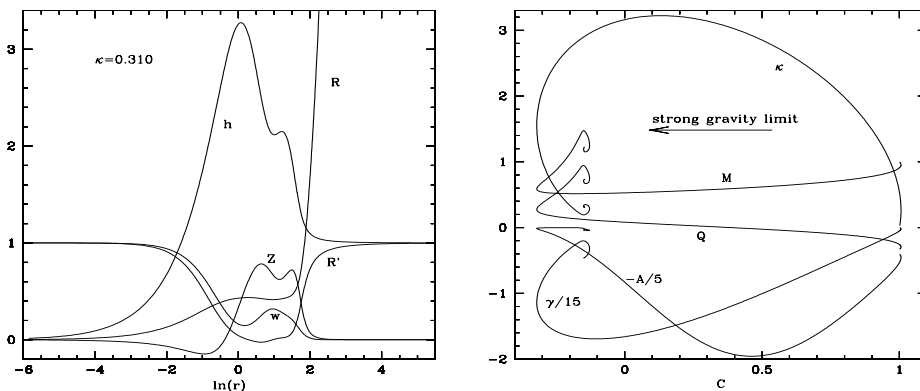


Fig. 3. Left: For strongly gravitating solutions the field amplitudes oscillate around the values (20) corresponding to the third fixed point. Right: The spiraling behavior of the solution parameters in the strong gravity limit. One sees that for a given value of $\kappa \in [0.11, 3.22]$ there exist several (at least two) different solutions.

going to infinity. These oscillations manifest themselves in the solution profiles; see Fig.3. A further increase of the gravitational force is not accompanied by an increase of the value of κ but can rather be achieved by increasing the value of $\xi = \kappa h'(0)$. The parameters $\kappa(\xi)$, $M_{\text{ADM}}(\xi)$, etc. then start developing spiraling oscillations around some limiting values; see Fig.3.

Such strongly gravitating solutions have a regular core connected to the asymptotic region by a long throat – the region where the radius of the two-sphere, R , is approximately constant and all other field amplitudes oscillate around the constant values (20). Increasing the value of $\xi = \kappa h'(0)$ this throat becomes longer and longer, and finally the solution splits up into the two independent solutions: interior and exterior, schematically shown in Fig.4.

The interior solution has a regular central core and approaches asymptotically the ‘warped ADS’ geometry (21). In Fig.2 this solution corresponds to the phase trajectory that starts at the fixed point I and after infinitely many oscillations around the fixed point III ends up there. The exterior solution, shown in Fig.4, interpolates between the ‘warped ADS’ and infinity. In Fig.2 this solution corresponds to the phase trajectory that interpolates between the fixed points III and II. It does not oscillate, since all ‘repelling’ eigenvalues (22) with positive real part are real. This solution is somewhat similar to an extreme non-Abelian black string, since passing to the Schwarzschild gauge where $r = R$ one has

$$g_{rr} \sim (r - r_h)^{-2}, \quad g_{tt} \sim g_{44} = e^{2\zeta} \sim (r - r_h)^{2.02}$$

where the event horizon radius is $r_h = 0.42$.

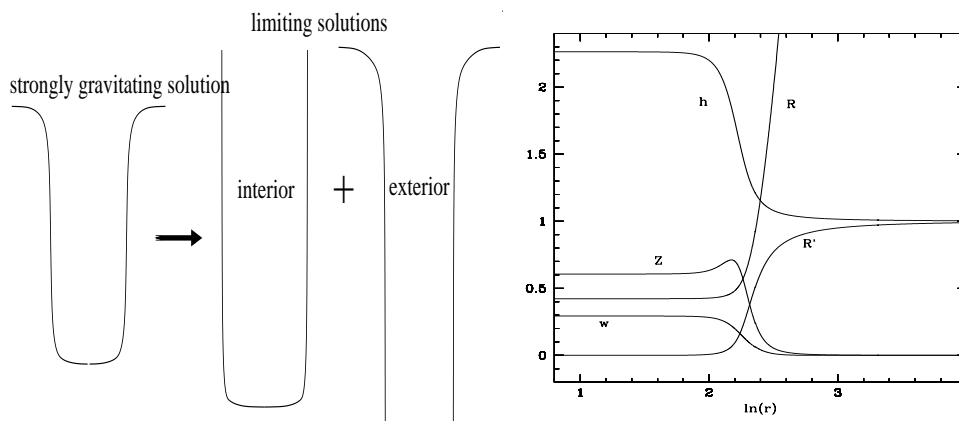


Fig. 4. Left: Schematic behavior of solutions in the strong gravity limit. Right: The profiles of the exterior part of the limiting solution.

5.2. Conclusions

Summarizing the above discussion, although the static and $SO(4)$ symmetric YM particles in $D=5$ get destroyed by gravity, in view of their scaling behavior, the static and $SO(3)$ symmetric solutions – YM vortices – do admit non-trivial curved space generalizations, since their scale invariance is broken by the asymptotic value of the ‘Higgs field’ $H^a = A_4^a$. These gravitating YM vortices comprise a one-parameter family of globally regular solutions – the fundamental branch – that interpolates between the flat space BPS monopole and the extreme non-Abelian black string. For all these solutions the YM field amplitude w is positive definite. In addition, there exist also excitations over the fundamental branch. These are solutions for which the YM field amplitude w oscillates around zero value, these solutions do not have the flat space limit.

6. Generalized YM vortices

The gravitating YM vortex solutions of the fundamental branch (but not the excited ones) have been generalized in a number of ways.

6.1. YM black strings²⁹

From the 4D viewpoint the YM vortices are regular gravitating solitons. It has been known for quite a long time²⁸ that gravitating solitons can often be generalized to include a small black hole in the center. Technically this requires replacing the boundary conditions at the regular origin (conditions of the type (10)) by the boundary conditions at the regular event horizon (conditions of the type (11)). Such a procedure has been used²⁹ to promote the regular YM vortices to black strings.

For a given κ there can be several YM vortex solutions corresponding to different parts of the spiraling curves in Fig.3. This property generalizes also to the black

string case, where one also finds several black string solutions for a given κ , provided that their even horizon radius r_h is small enough. As r_h increases, these solutions approach each other and finally merge for some maximal value $r_h^{\max}(\kappa)$. There are no black strings with $r_h > r_h^{\max}(\kappa)$, and so black strings exist only in a finite domain of the (κ, r_h) parameter plane. Black strings and regular vortex solutions with a Λ -term^{30,31} have also been considered.

6.2. ‘Twisted’ solutions^{32,33}

When performing the dimensional reduction from D=5 to D=4 in Eqs.(14) and (15) it was assumed that the Killing vector $\partial/\partial x^4$ is hypersurface orthogonal. Relaxing this condition, Eq.(14) generalizes to

$$g_{MN}dx^M dx^N = e^{-\zeta}g_{\mu\nu}dx^\mu dx^\nu + e^{2\zeta}(dx^4 + W_\mu dx^\mu)^2,$$

$$A_M^a dx^M = A_\mu^a dx^\mu + H^a(dx^4 + W_\mu dx^\mu),$$

where the twist W_μ can be viewed as a 4D vector field. Inserting this to the action (4) gives instead of (15) a four dimensional EYM-Higgs-dilaton+U(1) model. The solutions of this ‘twisted’ model carry an additional U(1) charge under the vector field W_μ . This charge can be of electric³² or magnetic³³ type. When this charge vanishes, the solutions reduce to the YM vortices/black strings.

6.3. Deformed and stationary solutions³⁴⁻³⁶

The 5D YM vortices/black strings, with or without twist, have been also generalized³⁴⁻³⁶ to the case where, after the dimensional reduction to D=4, the fields are chosen to be static and axially symmetric, rather than spherically symmetric. The ansatz for the gauge field A_μ^a and H^a contains in this case two integer winding numbers, n, m . If $n = 1, m = 0$ then the solutions are spherically symmetric monopoles. For $n > 1, m = 0$ one obtains axially symmetric solutions of the multimonopole type. Solutions with $m = 1$ describe monopole-antimonopole pairs. Both regular solutions³⁵ and black strings³⁶ have been considered within this approach. In all cases the existence of several solutions for a given value of κ has been detected.

A further generalization is achieved^{35,36} by returning back to D=5 and performing a Lorentz boost along the x^4 direction. This operation has a non-trivial effect on the 4D configurations: it produces stationary spinning solutions.

6.4. Non-Abelian braneworlds^{37,39-42}

Solutions with gravitating gauge fields can also be considered in the context of the braneworld models. As was discussed above, the non-Abelian monopole describes a one-dimensional object (vortex) in D=5. It will therefore describe a two-dimensional object (domain wall) in D=6 and a three-dimensional object (3-brane) in D=7. In

all cases the SU(2) Yang-Mills field and the triplet Higgs fields can be chosen to be static and spherically symmetric,

$$A_k^a dx^k = \epsilon_{akj} \frac{x^j}{r^2} (1 - w(r)) dx^k, \quad H^a = \frac{x^a}{r} h(r) \quad /r^2 = x^k x^k/, \quad (23)$$

with $r^2 = x^k x^k$. For the 3-brane in D=7 one chooses $x^k /k = 1, 2, 3/$ to be coordinates orthogonal to the brane, the 7D metric being

$$ds^2 = A(r) \eta_{\mu\nu} dy^\mu dy^\nu + B(r) \delta_{ik} dx^i dx^k, \quad (24)$$

where $y^\mu / \mu = 0, 1, 2, 3/$ are coordinates on the brane. Integrating the coupled EYM-Higgs- Λ equations reveals³⁷ that there are solutions satisfying the regularity condition at the brane, $A(0) = B(0) = w(0) = 1$, $h(0) = 0$, for which $A(r) \rightarrow 0$ for $r \rightarrow \infty$. Such solutions describe globally regular braneworlds confined in the monopole core with gravity localized on it. Solitonic braneworlds have also been studied in systems with a gravitating global monopole³⁸ and also for the local and global monopoles³⁹ coupled to each other and to gravity in D=7.

More general solutions of the n -brane type have been obtained⁴⁰⁻⁴² by oxidizing the $D = 5$ YM vortices to $D = 4 + n$ dimensions. The metric is then chosen to be

$$ds^2 = -e^{\xi_0(r)} (dy^0)^2 + \sum_{k=1}^n e^{\xi_k(r)} (dy^k)^2 + B(r) \delta_{ik} dx^i dx^k \quad (25)$$

with the gauge field given by

$$A_M^a dx^M = \epsilon_{akj} \frac{x^j}{r^2} (1 - w(r)) dx^k + \frac{x^a}{r} \sum_{k=1}^n h_k(r) dy^k. \quad (26)$$

What is interesting, the analog of the fixed point (19) can be obtained⁴⁰ within this approach for any n , also expressed in terms of roots of a cubic polynomial.

7. Non-Abelian solitons in string theory

Coming to the question of why one should study gravitating Yang-Mills fields in higher dimensions, one can say that (apart from pure curiosity) the motivation for this is provided by string theory. Gravitating Yang-Mills fields enter supersymmetry multiplets of the supergravity (SUGRA) theories to which string theory reduces at low energies. The knowledge of the basic solutions for gravitating gauge fields can therefore be useful for constructing solutions in low energy string theory. However, unlike solutions of the pure vacuum gravity, solutions of the EYM theory will not directly solve equations of SUGRA, since the latter generically contain additional fields, as for example the dilaton field. Constructing supergravity solutions with Yang-Mills fields is thus more complicated. However, imposing the supersymmetry conditions one can sometimes reduce the problem to solving first order Bogomol'nyi equations and not the second order field equations.

7.1. Heterotic solitons^{43,44}

The first example of supersymmetric solutions with gravitating Yang-Mills fields was obtained by Strominger⁴³ in the heterotic string theory. The low energy limit of the latter is a supergravity whose bosonic sector contains a Yang-Mills field already in D=10. Strominger considers a 5-brane in D=10 and makes the 6+4 split of the metric,

$$ds^2 = A(x^\mu)\eta_{MN}dy^M dy^N + B(x^\nu)\delta_{\mu\nu}dx^\mu dx^\nu,$$

where $y^M / M, M = 0, 1, 2, 3, 4, 5/$ are coordinates on the brane while the coordinates of the 4-space orthogonal to the brane are $x^\mu / \mu = 1, 2, 3, 4/$. He puts the Yang-Mills field to the orthogonal Euclidean 4-space,

$$A_M^a = 0, \quad A_\mu^a(x^\nu).$$

Since the Yang-Mills field in D=4 is conformally invariant and the relevant 4D part of the metric is conformally flat, the scale factor $B(x^\nu)$ drops out from the Yang-Mills equations. As a result, any flat space self-dual Yang-Mills instanton in D=4,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}, \quad (27)$$

will solve the Yang-Mills part of the SUGRA equations. The metric functions A, B as well as the axion and dilaton will then satisfy the Poisson equation on \mathbb{E}^4 with the source determined by $F_{\mu\nu}$, and so the solution can be expressed in quadratures. As there are many solutions of Eqs.(27), this gives a large family of heterotic 5-branes in D=10. Performing then dimensional reductions, one can construct⁴⁴ yet many more different solutions living in $D < 10$. All these are called heterotic solitons.

7.2. Non-Abelian vacua in gauged SUGRAs^{48,50,51,55,57,58}

Type I and type II string theories and also M theory reduce at low energies to supergravities in D=10 and D=11 whose multiplets do not contain Yang-Mills fields. However, the latter appear when one performs dimensional reductions on internal manifolds with non-Abelian isometries. This gives gauged SUGRAs in lower dimensions whose gauge group is related to the isometry group of the internal manifold. For example, the reduction of type II string theory on S^5 gives the gauged SUGRA in D=5 whose solutions are used in the AdS/CFT correspondence. Another example is the reduction of M theory on S^7 , which gives⁴⁵ the N=8 gauged SUGRA in D=4 with the local $SO(8) \times SU(8)$ invariance. One can then study solutions in these SUGRAs, and lifting them back to $D = 10, 11$ will give vacua of string or M theory.

The field content of a gauged SUGRA model can be quite complicated, which is why the gauge fields are often set to zero to find solutions. This gives, for example, black holes with scalar hair^{46,47} in N=8 SUGRA. However, in some cases one can construct solutions also with gravitating gauge fields. Let us consider a SUGRA

model⁴⁸ in D=4 which is a consistent truncation of the maximal N=8 SUGRA,

$$\mathcal{L}_4 = \frac{1}{4}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\phi}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{8}(e^{-2\phi} + \xi^2 e^{2\phi} + 4\xi). \quad (28)$$

This theory contains a gravity-coupled Yang-Mills field with gauge group SU(2) and the dilaton, whose potential depends on a real parameter ξ . In the static and spherically symmetric case the fields are given by

$$ds_{(4)}^2 = -e^{2V(\rho)}dt^2 + e^{2\lambda(\rho)}d\rho^2 + r^2(\rho)d\Omega^2, \\ A_k^a dx^k = (w(\rho) - 1)\epsilon_{aik}n^i dn^k, \quad \phi = \phi(\rho).$$

For one-shell configurations the amplitudes V, λ, r, w, ϕ satisfy a system of second order ODEs whose solutions can be studied⁴⁹ numerically. Instead of solving these equations, one can also consider the conditions for the fields to have unbroken supersymmetries. These conditions require the existence of a non-trivial spinor ϵ satisfying the linear equations

$$0 = \frac{1}{\sqrt{2}}\gamma^\mu\partial_\mu\phi\epsilon + \frac{1}{2}e^\phi\mathcal{F}\epsilon + \frac{1}{4}(e^{-\phi} - \xi e^\phi)\epsilon, \\ 0 = \mathcal{D}_\mu\epsilon + \frac{1}{2\sqrt{2}}e^\phi\mathcal{F}\gamma_\mu\epsilon + \frac{1}{4\sqrt{2}}(e^{-\phi} + \xi e^\phi)\gamma_\mu\epsilon, \quad (29)$$

where $\mathcal{F} = \frac{i}{2}\tau^a F_{\alpha\beta}^a \gamma^\alpha \gamma^\beta$. In general these equations are inconsistent, since there are 80 equations for 16 components of ϵ , and so the only solution is $\epsilon = 0$. However, one can show⁴⁸ that if the background fields are such that the following conditions are fulfilled,

$$V' - \phi' = \xi \frac{P}{\sqrt{2}N}e^{\phi+\lambda}, \quad Q = e^{V+\phi}\frac{w}{N}, \quad \phi' = \sqrt{2}\frac{BP}{N}e^\lambda, \\ w' = -\frac{rwB}{N}e^{-\phi+\lambda}, \quad N \equiv \rho\lambda' + 1 = \sqrt{w^2 + P^2}, \quad r' = Ne^\lambda, \quad (30)$$

where Q is a constant and

$$P = e^\phi \frac{1-w^2}{\sqrt{2}r} + \frac{r}{2\sqrt{2}}(e^{-\phi} + \xi e^\phi), \quad B = -\frac{P}{\sqrt{2}r} + \frac{1}{2}e^{-\phi},$$

then there exist four independent solutions of Eqs.(29), which corresponds to the N=1 supersymmetry. Solutions⁴⁸ of the Bogomol'nyi equations (30) with the regular boundary condition at the origin, $\rho = r = 0$, comprise a continuous family labeled by ξ and show three completely different types of behavior, depending on the sign on ξ . For $\xi > 0$ the dilaton is everywhere bounded and the solutions approach the AdS metric in the asymptotic region. For $\xi < 0$ the solutions are of the 'bag of gold' type, since they have compact spatial sections with the topology of S^3 . The geometry is generically singular at one pole of the S^3 . However, for $\xi = -2$ the solution is globally regular, the dilaton is constant, the geometry is that of $\mathbb{R}^1 \times S^3$, and the gauge field potential coincides with the invariant forms on S^3 . All these solutions can be uplifted⁴⁸ to $D = 11$ to become vacua of M theory.

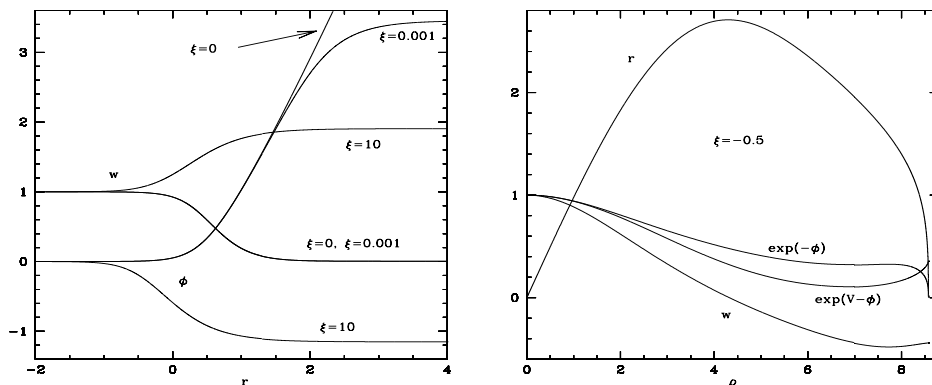


Fig. 5. The asymptotically AdS and compact solutions of the Bogomol'nyi equations (30).

For $\xi = 0$ the solution is given by⁵⁰

$$ds^2 = 2e^{2\phi} \{ -dt^2 + d\rho^2 + R^2(\rho)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \},$$

$$w = \pm \frac{\rho}{\sinh \rho}, \quad e^{2(\phi-\phi_0)} = \frac{\sinh \rho}{2R(\rho)}, \quad R(\rho) = \sqrt{2\rho \coth \rho - w^2 - 1}. \quad (31)$$

This solution can be uplifted⁵¹ to D=10, the string frame metric and the three-form in D=10 then being given by

$$ds^2 = -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 + d\rho^2 + R^2(\rho) d\Omega_2^2 + \Theta^a \Theta^a,$$

$$H = \frac{1}{2\sqrt{2}} e^{-\frac{3}{4}\phi} (F^a \wedge \Theta^a + \epsilon_{abc} \Theta^a \wedge \Theta^b \wedge \Theta^c), \quad (32)$$

where F^a is the gauge field 2-form and $\Theta^a = A^a - \theta^a$ with θ^a being the invariant forms on S^3 . This solution describes a 3-brane in D=10.

This solution was originally obtained^{50,51} by Chamseddine and Volkov, both in D=4 and D=10. A very interesting holographic interpretation for this solution was then proposed⁵² by Maldacena and Nunez, according to which this solution describes the NS-NS 5-brane wrapped on S^2 , in which case it effectively becomes a 3-brane. As a result, the solution provides a dual SUGRA description for the wrapped brane worldvolume theory, which is the (deformed) N=1 Super Yang-Mills in D=4. Since this theory is confining, one can say that the solution (31),(32) provides the dual SUGRA description for the phenomenon of confinement. For example, the value of the Wilson loop, the beta-function and other parameters of the confining gauge field theory can be obtained by simply computing some purely geometrical parameters of the solution, like areas of spheres and the values of the H -fluxes through them. As a result, this solution with self-gravitating Yang-Mills field has found quite interesting and serious applications. Generally known as solution of Maldacena and Nunez (the names of Chamseddine and Volkov seem now to be completely forgotten) it plays an important role in the analysis of string theory (see^{53,54} for recent reviews).

Non Abelian vacua have also been studied⁵⁵ in the context of a gauged SUGRA in $D=5$. The gravitational and gauge fields in this case are given by the same $SO(4)$ invariant expressions (5) as for the YM particle, and there are also the dilaton and axion. The solutions are not asymptotically flat, they have two supercharges, and they can be used⁵⁶ for a dual SUGRA description of the confining $N=1$ Super-Yang-Mills theory in $D=3$.

The discussed above supersymmetric solutions^{50,55} have also been generalized^{57,58} to the case where the spatial part of the metric is $\mathbb{R}^1 \times \Sigma$, where Σ is a maximal symmetry space, that is sphere, hyperboloid or Euclidean space. Their black hole generalizations^{59,60} have been used for a holographic description of the confinement/deconfinement phase transition in the dual gauge theory.

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