Derivation of the dipole black ring solutions

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We present a solution generating method in 5D Einstein-Maxwell gravity. As an illustration of the method we derive explicitly the solutions describing rotating dipole black rings.

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We consider the 5D Einstein-Maxwell (EM) equations

$$R_{\mu\nu} = \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{6}F_{\sigma\lambda}F^{\sigma\lambda}g_{\mu\nu}\right), \nabla_{\mu}F^{\mu\nu} = 0, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor with respect to the spacetime metric $g_{\mu\nu}$ and $F_{\mu\nu}$ is the Maxwell tensor. We will consider spacetiems with three commuting Killing vectors: one timelike Killing vector T and two spacelike Killing vectors K_1 and K_2. We also require that the Killing vector K_2 is hypersurface orthogonal. Then, in adapted coordinates $T = \partial/\partial t$, $K_1 = \partial/\partial X$, $K_2 = \partial/\partial Y$ the 5D metric can be written in the form

$$ds^2 = g_{YY}dY^2 + g_{00}\left(dt + \mathcal{A}dX\right)^2 + \tilde{g}_{XX}dX^2 + g_{\rho\rho}(d\rho^2 + dz^2), \tag{2}$$

where all metric functions depend only on the canonical coordinates ρ and z.

We take the electromagnetic field in the form

$$F = dA_Y \wedge dY, \;\; A_Y = A_Y(\rho, z). \tag{3}$$

Let us note that this form of the electromagnetic field is compatible with the spacetime symmetries.

The following method for generating 5D EM solutions was developed:[1]

Proposition. *Let us consider two solutions of the vacuum 5D Einstein equations*

$$ds^2_{E(i)} = g^{E(i)}_{YY}dY^2 + g^{E(i)}_{00}\left(dt + \mathcal{A}^{(i)}_{E}dX\right)^2 + \tilde{g}^{E(i)}_{XX}dX^2 + g^{E(i)}_{\rho\rho}(d\rho^2 + dz^2).$$

Then the following give a solution to the 5D EM equations

$$ds^2 = \left[|g^{E(1)}_{00}|\sqrt{g^{E(1)}_{YY}}\right]^2 dY^2 + \left[\frac{\sqrt{g^{E(2)}_{YY}}}{|g^{E(1)}_{00}|\sqrt{g^{E(1)}_{YY}}}\right]\left[g^{E(2)}_{00}\left(dt + \mathcal{A}^{(2)}_{E}dX\right)^2\right.$$
$$\left. + \tilde{g}^{E(2)}_{XX}dX^2 + \left(\frac{|g^{E(1)}_{00}|g^{E(1)}_{YY}g^{E(1)}_{\rho\rho}}{e^{2\Omega^{(1)}_{E} + \frac{2}{3}\Omega^{(2)}_{E}}}\right)^3 g^{E(2)}_{\rho\rho}(d\rho^2 + dz^2)\right],$$

$$A_Y = \pm 2\sqrt{3}f^{(1)}_{E} + const,$$

where $f_E^{(1)}$ *is a solution to the system*

$$\partial_\rho f_E^{(1)} = -\frac{1}{2}\frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho}\partial_z \mathcal{A}_E^{(1)},$$
$$\partial_z f_E^{(1)} = \frac{1}{2}\frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho}\partial_\rho \mathcal{A}_E^{(1)},$$

and $\Omega_E^{(i)}$ *satisfy*

$$\rho^{-1}\partial_\rho \Omega_E^{(i)} = \frac{3}{16}\left[\left(\partial_\rho \ln\left(g_{YY}^{E(i)}\right)\right)^2 - \left(\partial_z \ln\left(g_{YY}^{E(i)}\right)\right)^2\right],$$
$$\rho^{-1}\partial_z \Omega_E^{(i)} = \frac{3}{8}\partial_\rho \ln\left(g_{YY}^{E(i)}\right)\partial_z \ln\left(g_{YY}^{E(i)}\right).$$

The presented proposition gives us a tool to generate new 5D EM solutions in a simple way from known solutions to the vacuum 5D Einstein equations.

Through the use of the proposition we can generate the "5D EM images" of all known solutions of the vacuum 5D Einstein equations with the symmetries we consider here. We shall demonstrate the application of the proposition on the case of rotating neutral black rings[2] generating in this way the EM rotating dipole black ring solutions.

The metric of the neutral black ring is given by[2,3]

$$ds_5^2 = -\frac{F(y)}{F(x)}\left(dt + C(\nu,\lambda)\mathcal{R}\frac{1+y}{F(y)}d\psi\right)^2$$
$$+\frac{\mathcal{R}^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right]$$

where

$$F(x) = 1 + \lambda x,\ \ G(x) = (1-x^2)(1+\nu x),\ \ C(\nu,\lambda) = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}. \quad (4)$$

The coordinates x, y and the parameters ν, λ satisfy

$$-1 \le x \le 1,\ \ -\infty < y \le -1,\ \ 0 < \nu \le \lambda < 1. \quad (5)$$

The dipole black ring solutions in 5D Einstein-Maxwell gravity can be generated through the use of the proposition via the scheme

$$\{neutral\ \ black\ \ ring\} + \{neutral\ \ black\ \ ring\} \rightarrow \{EM\ \ dipole\ \ black\ \ ring\}. \quad (6)$$

The first solution is with parameters $\{\lambda_1, \nu, \mathcal{R}\}$ while the second is parameterized by $\{\lambda_2, \nu, \mathcal{R}\}$. As a result we find

$$\begin{aligned} ds^2 &= -\frac{F_{\lambda_2}(y)}{F_{\lambda_2}(x)}\frac{F_{\lambda_1}(x)}{F_{\lambda_1}(y)}\left(dt + C(\nu,\lambda_2)\mathcal{R}\frac{1+y}{F_{\lambda_2}(y)}d\psi\right)^2 + \left[F_{\lambda_1}(x)F^2_{\lambda_1}(y)\right]\frac{\mathcal{R}^2 F_{\lambda_2}(x)}{(x-y)^2} \\ &\quad\times\left[-\frac{G(x)}{F^3_{\lambda_1}(y)F_{\lambda_2}(y)}d\psi^2 + \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{G(x)}{F^3_{\lambda_1}(x)F_{\lambda_2}(x)}d\phi^2\right] \\ A_\phi &= \pm\sqrt{3}C(\nu,\lambda_1)\mathcal{R}\frac{1+x}{F_{\lambda_1}(x)} + const. \end{aligned} \tag{7}$$

In order to exclude pathological behavior of the metric we must consider only negative λ_1, i.e. $\lambda_1 = -\mu,\ 0 \le \mu < 1$ and positive λ_2 and ν satisfying $0 < \nu \le \lambda_2 < 1$. One can easily see that the generated 5D EM solution is just the EM rotating dipole black ring solution.[3] Let us also recall that in order to avoid the possible conical singularities at $x = \pm 1$ and $y = -1$ we must impose

$$\Delta\phi = \Delta\psi = 2\pi\frac{(1+\mu)^{3/2}\sqrt{1-\lambda_2}}{1-\nu}, \quad \frac{1-\lambda_2}{1+\lambda_2}\left(\frac{1+\mu}{1-\mu}\right)^3 = \left(\frac{1-\nu}{1+\nu}\right)^2. \tag{8}$$

It would be interesting to find the solutions which can be generated through the use of the proposition via the schemes

$$\{neutral\ black\ ring\} + \{neutral\ black\ hole\} \rightarrow \{?\}, \tag{9}$$

$$\{neutral\ black\ hole\} + \{neutral\ black\ hole\} \rightarrow \{?\}, \tag{10}$$

$$\{appropriate\ \ solution\} + \{neutral\ black\ hole/ring\} \rightarrow \{black\ solution\ ?\}. \tag{11}$$

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