GRAVITATIONAL PERTURBATIONS OF HIGHER DIMENSIONAL ROTATING BLACK HOLES

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Assessing the stability of higher-dimensional rotating black holes requires a study of linearized gravitational perturbations around such backgrounds. We study perturbations of Myers-Perry black holes with equal angular momenta in an odd number of dimensions (greater than five), allowing for a cosmological constant. Such black exhibit enhanced symmetry: they are cohomogeneity-one solutions. This allows gravitational perturbations to be decomposed into scalar, vector and tensor types. The equations of motion for tensor perturbations reduce to a single radial equation. In the asymptotically flat case we find no evidence of any instability associated with tensor perturbations. In the asymptotically anti-de Sitter case, we demonstrate the existence of a superradiant instability that sets in precisely when the angular velocity of the black hole exceeds the speed of light from the point of view of the conformal boundary. We suggest that the endpoint of the instability may be a stationary, nonaxisymmetric black hole.

1. Introduction

Exact solutions describing higher-dimensional rotating black holes have been known for a long time[1] but the question of their classical stability is still unresolved. There are arguments suggesting that a Myers-Perry (MP) black hole will be unstable for sufficiently large angular momentum in five[2] and higher[3] dimensions. However, a convincing demonstration of this requires a study of linearized gravitational fluctuations around such backgrounds. This has only been done in the limit of vanishing angular momentum, i.e., for higher-dimensional Schwarzschild black holes.[4–6]

Another context in which stability of higher dimensional rotating black holes has been discussed is the AdS/CFT correspondence.[7] The MP solutions have been generalized to include a cosmological constant.[8,9] There is a qualitative argument that rotating, asymptotically AdS black holes might exhibit a superradiant instability.[10] The idea (inspired by the corresponding instability of a Kerr black hole in the presence of a massive scalar field[11–13]) is that superradiant perturbations are trapped

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by the AdS potential barrier at infinity and reflected towards the black hole where they get amplified and the process repeats. It can be proved that no such instability is present for black holes rotating at, or slower than, the speed of light relative to the conformal boundary[10] (essentially because superradiant modes do not fit into the AdS "box"[14]) but an instability may well be present for more rapidly rotating holes. The only way to find out is to study perturbations of such black holes. This has been done for scalar field perturbations of small four-dimensional Kerr-AdS black holes[15] but not for gravitational perturbations, large black holes, or higher dimensions.

In four dimensions, gravitational perturbations of rotating black holes can be studied analytically. The tractability of the problem arises from two miraculous properties of the Kerr metric. Firstly, the equations of motion for gravitational perturbations can be decoupled and reduced to a PDE for a single scalar quantity.[16] Secondly, this equation can be reduced to ODEs governing the radial and angular behaviour by separation of variables. It is known that this separability property of the Kerr metric does extend to some of the higher-dimensional MP metrics,[17–19] and MP metrics with a cosmological constant.[20] This makes the study of scalar field perturbations in such backgrounds tractable. However, so far no-one has succeeded in decoupling the equations of motion governing gravitational perturbations of MP black holes.

We have made progress with this problem[21] by considering the subclass of MP black holes for which the number of space-time dimensions is odd and the angular momenta are all equal, allowing for a cosmological constant. Such black holes are cohomogeneity-1: the metric depends only on a radial coordinate. In $D = 2N + 3$ dimensions, the isometry group is enhanced from $R \times U(1)^{N+1}$ to $R \times U(N + 1)$ where R denotes time translations. The horizon is a homogeneously squashed S^{2N+1} viewed as a S^1 bundle over CP^N. The symmetry enhancement allows us to classify gravitational pertubations into scalar, vector and tensor types according to how they transform under isometries of CP^N. We consider the tensor perturbations. These do not exist for $N = 1$ so we restrict ourselves to $N > 1$, i.e., spacetime dimensionality seven or greater. After separation of variables, the equations of motion for tensor perturbations reduce to a single ODE governing the radial behaviour. For completeness, we also consider massive scalar field perturbations of these black holes (for $N \geq 1$). We shall present a unified form for the radial equation that applies both to scalar fields and to tensor gravitational perturbations.

Asymptotically flat, cohomogeneity-1 Myers-Perry black holes exhibit an upper bound on their angular momentum for a given mass. Solutions saturating this bound have a regular but degenerate horizon. This means that the black holes considered here behave rather differently from ones for which some of the angular momenta vanish, which are the ones expected to exhibit a gravitational instability.[2,3] So there is no *a priori* reason to expect any instability to exist for the MP black holes considered in this paper and indeed we find no instability.

In the asymptotically anti-de Sitter case, there is also an upper bound on angular momentum for given mass and black holes saturating this bound have a regular but degenerate horizon. These extremal solutions always rotate faster than light and can be arbitrarily large compared with the AdS radius. The argument of Ref. 10 suggests that black holes near to extremality might be unstable to losing energy and angular momentum into superradiant gravitational and scalar field perturbations. We shall demonstrate that this is indeed the case. Moreover, we shall show that this instability appears as soon as the angular velocity of the hole exceeds the speed of light, i.e., as soon as the stability argument of Ref. 10 fails. The instability is a short distance instability in the sense that unstable modes exist for all wavelengths below a certain critical value determined mainly by the angular velocity of the hole. However, amongst unstable modes, the shortest wavelength modes are the least unstable.

Having demonstrated the existence of an instability, it is natural to ask what the endpoint of the instability is. We propose that a black hole that suffers from this superradiant instability will evolve to a stationary, nonaxisymmetric black hole solution. The motivation behind our proposal will be explained at the end of this paper.

This paper summarizes the results that we obtained in Ref. 21. The reader is urged to consult that paper for further details. However, section 3.1 contains new material that will be explained in more detail in a future paper.[22]

2. The background solution

The higher-dimensional generalization of the Kerr solution was obtained by Myers and Perry[1] and subsequently generalized to include a cosmological constant in five[8] and higher[9] dimensions. It is parameterized by a mass parameter M and $[(D-1)/2]$ angular momentum parameters a_i. In $D = 2N+3$ dimensions with equal rotation parameters $a_i = a$ the solution is cohomogeneity 1. The metric can be written as:

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + h(r)^2 [d\psi + A_a dx^a - \Omega(r) dt]^2 + r^2 \hat{g}_{ab} dx^a dx^b \quad (1)$$

$$g(r)^2 = \left(1 + \frac{r^2}{\ell^2} - \frac{2M\Xi}{r^{2N}} + \frac{2Ma^2}{r^{2N+2}}\right)^{-1}, \qquad h(r)^2 = r^2 \left(1 + \frac{2Ma^2}{r^{2N+2}}\right), \quad (2)$$

$$f(r) = \frac{r}{g(r)h(r)}, \qquad \Omega(r) = \frac{2Ma}{r^{2N} h^2}, \qquad \Xi = 1 - \frac{a^2}{\ell^2}, \quad (3)$$

where $\hat{g}_{ab}$ is the Fubini-Study metric on CP^N with Ricci tensor $\hat{R}_{ab} = 2(N+1)\hat{g}_{ab}$, and $A = A_a dx^a$ is a 1-form such that $J = \frac{1}{2} dA$ is the Kähler form on CP^N. This way of writing the metric arises from the fact that S^{2N+1} can be written as an S^1 fibre over CP^N. The fibre is parameterized by the coordinate ψ, which has period 2π.

The spacetime metric satisfies $R_{\mu\nu} = -\ell^{-2}(D-1) g_{\mu\nu}$. Asymptotically, the solution approaches anti-de Sitter space with radius of curvature ℓ. An asymptotically flat Myers-Perry black hole can be recovered by taking $\ell \to \infty$. The event horizon located at $r = r_+$ (the largest real root of g^{-2}) is a Killing horizon of $\xi = \partial_t + \Omega_H \partial_\psi$, where the angular velocity of the horizon is:

$$\Omega_H = \frac{2Ma}{r_+^{2N+2} + 2Ma^2}. \tag{4}$$

The mass E and angular momentum J (defined with respect to ∂_ψ) are[23]

$$E = \frac{A_{2N+1}}{4\pi G} M \left(N + \frac{1}{2} + \frac{a^2}{2\ell^2} \right), \qquad J = \frac{A_{2N+1}}{4\pi G}(N+1)Ma \tag{5}$$

where A_{2N+1} is the area of a unit $2N+1$ sphere.

As written the metric is parameterised by (M, a). We shall assume $a \geq 0$, which can always be achieved by $t \to -t$ if necessary. Sometimes it will be convenient to work with more "physical" variables (Ω_H, r_+). Fortunately one can easily invert for (M, a) in terms of (Ω_H, r_+):

$$M = \frac{r_+^{2N}(1 + r_+^2 \ell^{-2})^2}{2(1 + r_+^2 \ell^{-2} - r_+^2 \Omega_H^2)}, \qquad a = \frac{r_+^2 \Omega_H}{1 + r_+^2 \ell^{-2}}. \tag{6}$$

For given r_+, existence of a regular event horizon imposes an upper bound on Ω_H:

$$\Omega_H \leq \frac{1}{\ell}\sqrt{1 + \frac{N\ell^2}{(N+1)r_+^2}}. \tag{7}$$

The extremal solution saturating this bound has a regular but degenerate horizon.

In the asymptotically AdS case, the "co-rotating" Killing vector field ξ is timelike everywhere outside the horizon if $\Omega_H \leq 1/\ell$ but becomes spacelike in a neighbourhood of infinity otherwise. With respect to the metric on the conformal boundary, ξ is timelike if $\Omega_H < 1/\ell$, null if $\Omega_H = 1/\ell$ and spacelike otherwise. For this reason, black holes with $\Omega_H > 1/\ell$ are said to be rotating faster than light. Note that the extremal black holes always rotate faster than light, and that such black holes can be arbitrarily large compared with the AdS length.

3. Perturbation equations

3.1. *Scalar, vector, tensor*

As explained in the introduction, we can make progress with the study of gravitational perturbations of these cohomogeneity-one black holes by exploiting the large symmetry group. Specifically, we can decompose a general gravitational perturbation into scalar, vector and tensor perturbations on CP^N. This is familar from perturbations of spherically symmetric black holes, where one performs an analogous decomposition on a sphere.[5] However, there are two additional complications present here.

It is natural to decompose any perturbation of our black hole into Fourier modes around the S^1 fibre parameterized by ψ, i.e., we assume that the perturbation is proportional to $\exp(im\psi)$ with m an integer. To understand the resulting equations, it is convenient to imagine performing a dimensional reduction on this S^1, i.e., reduce S^{2N+1} to CP^N. This reduction gives rise to a Kaluza-Klein magnetic field A, and the perturbation carries charge m with respect to this field. Hence we will need to understand *charged* perturbations on CP^N. This means that we will need to work with the gauge-covariant derivative $\mathcal{D}$ which acts on a field of charge m as

$$\mathcal{D}_a \equiv \nabla_a - imA_a, \tag{8}$$

with ∇ the Levi-Civita connection on CP^N.

The second complication comes from the fact that on a sphere (of dimension higher than two) there is only one second rank tensor invariant under the isometry group, namely the metric. However, on CP^N there are two such tensors: the metric and the Kähler form J. Hence in defining what we mean by scalar, vector and tensor, we have to consider contractions with both of these objects.

Consider a general metric perturbation:

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}. \tag{9}$$

Obviously, h_{tt}, h_{tr}, $h_{t\psi}$ etc. transform as scalars on CP^N. More interesting are components such as $v_a \equiv h_{ta}$, which carry a vector index a. It can be shown[22] that v_a can be decomposed as

$$v_a = w_a + \mathcal{D}_a\phi + J_a{}^b\mathcal{D}_b\psi, \tag{10}$$

where ϕ and ψ are scalars, and w_a obeys the "doubly transverse" conditions:

$$\mathcal{D}^a v_a = J^{ab}\mathcal{D}_a v_b = 0. \tag{11}$$

where we raise indices with the metric on CP^N. We can therefore decompose h_{ta} into a "vector part" w_a and a "scalar part" corresponding to ϕ, ψ. The same can be done for $h_{\psi a}$ and h_{ra}. The important thing to note is that the vector parts obey *two* transversality conditions, arising from the fact that we have two invariant tensors.

Finally, consider h_{ab}. It can be shown[22] that h_{ab} can be decomposed as

$$h_{ab} = Y_{ab} + \dots, \tag{12}$$

where the ellipsis denotes vector and scalar parts of h_{ab}, i.e., terms that can be written in terms of derivatives of vectors (analagous to w_a) and scalars, and Y_{ab} is the "tensor part" of the perturbation, defined by the (traceless, doubly transverse) conditions

$$Y_a^a = \mathcal{D}^a Y_{ab} = 0 = J^{ab}\mathcal{D}_a Y_{bc} = 0. \tag{13}$$

The reason for performing this decomposition of a general perturbation into scalar, vector and tensor types is that the linearized Einstein equations respect the decomposition, i.e., the equations of motion for scalar, vector and tensor perturbations decouple from each other. The decomposition of a general metric perturbation will give rise to many scalars, and several vectors. However, there is only a single tensor, so tensor perturbations are the simplest to study. We shall restrict attention to tensor perturbations henceforth. Tensor perturbations do not exist for $N = 1$ (recall $CP^1 = S^2$) so we shall assume $N > 1$.

3.2. *Tensor equation of motion*

A general separable tensor perturbation takes the form

$$h_{t\mu} = h_{r\mu} = h_{\psi\mu} = 0, \qquad h_{ab} = \mathrm{Re}\left[e^{-i\omega t + im\psi} h(r)^{-1/2} r^{2-N} \Psi(r) Y_{ab}(x)\right], \quad (14)$$

where $Y_{ab}(x)$ is a charge m tensor (in the above sense, i.e., obeying equation 13) on CP^N. We shall assume that Y_{ab} can be expanded in eigenfunctions of a suitable Laplace-type operator, namely the gauge-covariant version of the Lichnerowicz operator on CP^N:

$$\Delta_L^A Y_{ab} = -2\hat{R}_{acbd} Y^{cd} - \mathcal{D}^2 Y_{ab} + 4(N+1) Y_{ab}, \quad (15)$$

where $\hat{R}_{abcd}$ is the Riemann tensor of CP^N. Note that Δ_L^A maps tensors to tensors. There is no loss of generality in assume that Y_{ab} is an eigenfunction of Δ_L^A with eigenvalue λ, i.e., a tensor harmonic.

Define a map $\mathcal{J}$ by

$$(\mathcal{J} Y)_{ab} = J_a{}^c J_b{}^d Y_{cd}. \quad (16)$$

This maps tensors to tensors, and has eigenvalues $1, -1$ whose eigenvectors we refer to as hermitian or anti-hermitian respectively. It commutes with Δ_L^A. Hence we can simultaneously diagonalize $\mathcal{J}$ and Δ_L^A. This implies that we can classify eigenfunctions of Δ_L^A into hermitian and anti-hermitian. Further, in the antihermitian case, one can distinguish between Y_{ab} of type $(2,0)$ and $(0,2)$, which we define by $J_a{}^c Y_{cb} = \epsilon i Y_{ab}$, where $\epsilon = +1, -1$ for $(2,0)$ and $(0,2)$ harmonics respectively.

Using these results, the linearized Einstein equation for a tensor perturbation reduces to the radial equation[21]

$$-\frac{f}{g}\frac{d}{dr}\left(\frac{f}{g}\frac{d\Psi}{dr}\right) + V\Psi = 0, \quad (17)$$

where the "potential" $V(r)$ is defined by

$$V(r) = V_0(r) - (\omega - m\Omega)^2 + f^2\left(\frac{m^2}{h^2} + \frac{4(1-\sigma)h^2}{r^4} + \frac{\lambda - 4(N+1) - 2\epsilon(1+\sigma)m}{r^2}\right), \quad (18)$$

with $\sigma = 1$ or -1 for anti-hermitian and hermitian Y_{ab} respectively, λ is the eigenvalue of Δ_L^A, and V_0 is defined by

$$V_0 = \frac{f^2\sqrt{h}}{r^{N+1}}\frac{d}{dr}\left[\frac{f^2 h}{r}\frac{d}{dr}(\sqrt{h} r^N)\right]. \quad (19)$$

The next step is to determine the eigenvalues λ of tensor harmonics.

3.3. *Tensor harmonics*

Recall that there are no tensor harmonics on CP^1. Uncharged tensor harmonics on CP^2 have been considered previously.[24,25] It is straightforward to generalize the results to include charge.[21] The result is that doubly transverse charged tensor harmonics must be anti-hermitian ($\sigma = 1$). The eigenvalue spectrum is

$$N = 2: \qquad \lambda = l(l+4) + 12 - m^2 + 4\epsilon m, \qquad l = 2k + |6 - \epsilon m| \tag{20}$$

where

$$k = \begin{cases} 0, 1, 2, \ldots & \text{if} \quad \epsilon m \leq 4 \\ 1, 2, 3 \ldots & \text{if} \quad \epsilon m = 5 \\ 2, 3, 4, \ldots & \text{if} \quad \epsilon m \geq 6 \end{cases} \tag{21}$$

This implies that the minimum value of l is $l_{\text{min}} = 6 - \epsilon m$ if $\epsilon m \leq 3$ and $l_{\text{min}} = \epsilon m - 2$ for $\epsilon m \geq 4$. Conversely, for given l, the allowed values of m are given by

$$\epsilon m = 6 - l,\, 6 - l + 2,\, 6 - l + 4,\, \ldots,\, l,\, l + 2, \qquad l = 2, 3, 4, \ldots \tag{22}$$

These charged tensor harmonics on CP^N can be lifted to (uncharged) eigenfunctions of the Lichnerowicz operator on S^{2N+1} where l is the usual "total angular momentum" quantum number.[a] Note $l \geq 2$, as expected for tensor harmonics.

We can obtain a general formula for the charged tensor eigenvalues for $N \geq 2$ by comparing our results with those for non-rotating black holes. If we set $a = 0$ then we have a Schwarzschild(-AdS) black hole and our tensor perturbations should form a subset of the tensor perturbations (on the sphere) considered in Refs 4–6. Demanding agreement between our results and those of Ref. 6 yields

$$\lambda = l(l + 2N) + 4N + 4\sigma - m^2 + 2\epsilon(1 + \sigma)m. \tag{23}$$

This is consistent with the above results for $N = 2$. The limitation of this approach is that it does not tell us which values of l are permitted beyond the obvious restriction $l \geq 2$.

3.4. *Unified potential*

To summarize, the equation of motion for tensor gravitational perturbations can be separated and reduced to a single radial equation (17). The same thing can be done for scalar field perturbations, and it is interesting to compare the results. Consider a scalar field obeying the Klein-Gordan equation

$$\nabla^2 \Phi - \mu^2 \Phi = 0. \tag{24}$$

Take a separable Ansatz:

$$\Phi = e^{-i\omega t + im\psi} h(r)^{-1/2} r^{-N} \Psi(r) Y(x), \tag{25}$$

where Y is a charged scalar eigenfunction of $-\mathcal{D}^2$ on CP^N. Then the Klein-Gordan equation reduces to a radial equation of exactly the same form as (17). The only difference between scalar field and tensor gravitational perturbations shows up in

[a] Note that $m = \sum_i m_i$ where $(m_1, m_2, \ldots)$ is the weight vector of SO$(2N + 2)$ corresponding to the harmonic in question.

the potential $V(r)$. This can be written in a unified form so that it applies to both types of perturbation[21]

$$V = V_0 + f^2\mu^2 - (\omega - m\Omega)^2 + \frac{f^2}{r^2}\left[l(l+2N) - m^2\left(1 - \frac{r^2}{h^2}\right) + 4(1-\sigma)\left(\frac{h^2}{r^2} - 1\right)\right], \quad (26)$$

where V_0 is defined by (19). For scalar field perturbations, $N \geq 1$, $\sigma = 1$ and $l = 2k + |m|$, $k = 0, 1, 2, \ldots$. For gravitational pertubations, $N \geq 2$, $\mu = 0$ and the permissible values of σ, l are discussed above.

Note that anti-hermitian ($\sigma = 1$) gravitational perturbations bey exactly the same equation as a massless scalar field. Therefore, one might think that results concerning the stability of (asymptotically flat) MP black holes against massless scalar field perturbations[18,26,27] would imply stability with respect to $\sigma = 1$ gravitational perturbations. However, these results concern black holes in five[18] and six[26,27] dimensions with a single non-vanishing angular momentum whereas we are interested in black holes in seven or more odd dimensions with all angular momenta equal and non-vanishing. It appears that scalar field perturbations of such black holes have not been considered previously. Furthermore, there *is* a difference between scalar field and gravitational perturbations: the lower bound on l is different for the two cases. (For $N = 2$, the lower bound for scalars can be either less than, or greater than, the lower bound for doubly transverse tensors, according to the value of m.)

Note that by introducing a "tortoise" coordinate $x(r)$ defined by

$$\frac{dx}{dr} = \frac{g}{f}, \quad (27)$$

we can convert the radial equation (17) into a 1-dimensional time-independent Schrödinger equation:

$$-\frac{d^2}{dx^2}\Psi + V\Psi = 0. \quad (28)$$

4. Stability analysis

4.1. *Boundary conditions*

The horizon is located at $r = r_+$, which corresponds to $x \to -\infty$:

$$x \sim \frac{1}{\alpha}\log\left(\frac{r - r_+}{r_+}\right), \quad (29)$$

where

$$\alpha = \frac{r_+ (g^{-2})'(r_+)}{h(r_+)}. \quad (30)$$

At the horizon, $V \to -(\omega - m\Omega_H)^2$ so the solutions behave as $\exp\left(\pm i(\omega - m\Omega_H)x\right)$. Regularity on the future horizon requires that we choose the lower sign, so we have the boundary condition

$$\Psi = \exp\left(-i(\omega - m\Omega_H)x\right)\Phi, \quad (31)$$

where $\Phi(r)$ is smooth at $r = r_+$. Substituting this into the Schrödinger equation and expanding around $r = r_+$ gives (choosing $\Phi(r_+) = 1$)

$$\Phi = 1 + \frac{V'(r_+)(r - r_+)}{\alpha^2 - 2i\alpha(\omega - m\Omega_H)} + \mathcal{O}\left((r - r_+)^2\right). \tag{32}$$

In the asymptotically flat case, we have

$$x = r + \mathcal{O}\left(r^{-2N+1}\right) \qquad \text{as} \qquad r \to \infty. \tag{33}$$

The general solution as $r \to \infty$ is a superposition of outgoing and incoming waves (proportional to e^{ikr} and e^{-ikr} respectively):

$$\Psi \sim \sqrt{kr}\left[Z_{\rm out}H^{(1)}_{l+N}(kr) + Z_{\rm in}H^{(2)}_{l+N}(kr)\right], \qquad N \geq 2 \tag{34}$$

where $H^{(i)}$ are Hankel functions and $k = \sqrt{\omega^2 - \mu^2}$. In the $N = 1$ case, the solution is as above except the order of the Hankel functions is now $[(l+1)^2 + 2M(\mu^2 - 2\omega^2)]^{1/2}$. In the asymptotically flat case, we are mainly interested in gravitational perturbations for which $\mu = 0$ and $k = \omega$.

In the asymptotically AdS case, we have $x \sim -\ell^2/r \to 0$ as $r \to \infty$. The asymptotic behaviour of the potential is

$$V \sim V_0 + \frac{r^2\mu^2}{\ell^2} \sim \frac{r^2}{\ell^4}\left((N+1)^2 - \frac{1}{4} + \mu^2\ell^2\right), \tag{35}$$

with corresponding asymptotic solutions $\Psi \sim r^{-1/2\pm\sqrt{(N+1)^2+\mu^2\ell^2}}$. Choosing the normalizable solution corresponds to the boundary condition

$$\Psi \sim r^{-1/2-\sqrt{(N+1)^2+\mu^2\ell^2}} \qquad \text{as } r \to \infty. \tag{36}$$

For stability of the AdS background we demand that a scalar field obeys the Breitenlöhner-Freedman bound[31]

$$\mu^2 \geq -\frac{(N+1)^2}{\ell^2}. \tag{37}$$

A linearized instability of the black hole would correspond to a solution of the radial equation that is regular on the future horizon and vanishing at infinity, with $\mathrm{Im}(\omega) > 0$. In the asymptotically flat case this requires $Z_{\rm in} = 0$. Note that such a solution vanishes exponentially at the horizon.

4.2. *The case* $m = 0$

It is easy to show that there can be no instability (whether asymptotically flat or asymptotically AdS) when $m = 0$ and $\mu^2 \geq 0$. If $m = 0$ then it is natural to consider the potential $\tilde{V} = V + \omega^2$, which does not depend on ω. The radial equation becomes

$$-\frac{d^2}{dx^2}\Psi(x) + \tilde{V}(x)\Psi(x) = \omega^2\Psi(x) \qquad (m = 0). \tag{38}$$

Assume that Ψ describes an unstable mode, so $\mathrm{Im}(\omega) > 0$ and Ψ vanishes at the horizon and at infinity as described above. The differential operator on the LHS of (38) is self-adjoint on such functions and hence ω^2 must be real so ω is pure imaginary and ω^2 is negative. A simple argument[21] reveals that V_0 is positive. If we assume $\mu^2 \geq 0$ then the remaining terms in $\tilde{V}$ are non-negative so $\tilde{V}$ is positive. Hence ω^2 must be positive, which is a contradiction.

For AdS black holes, it would be interesting to see if this conclusion could be extended to tachyonic scalars satisfying the Breitenlohner-Freedman bound by combining our argument with that of Ref. 31.

4.3. *Strategy*

We can look for unstable modes using the strategy adopted by Press and Teukolsky for the Kerr black hole.[28] We expect the black hole to be stable for small angular momentum because we know that the higher-dimensional Schwarzschild black hole is stable.[6] Hence, for small angular velocity, the only admissible solutions of the radial equation must have negative imaginary part, i.e., they are quasi-normal modes. If an instability is to appear as the angular velocity increases then one of these quasi-normal modes must cross the real axis in the complex ω plane.[b] Hence we expect the onset of instability to be indicated by the appearance of a real frequency mode at a critical value of the angular velocity. The strategy is to look for such a mode. By continuity it must have $Z_{\rm in} = 0$ in the asymptotically flat case, i.e., it must be purely outgoing at infinity. In the AdS case it must obey the "normalizable" boundary condition (36).

Note that the radial equation is invariant under $\omega \to -\omega$ and $m \to -m$ and, for tensors, $\epsilon \to -\epsilon$.[c] Hence we can assume $\omega \geq 0$. Following Press and Teukolsky,[30] consider the Wronskian of Ψ and $\bar{\Psi}$ for real ω. This must be constant so we obtain

$$\mathrm{Im}\left[\bar{\Psi}\partial_x\Psi\right]_{x_2}^{x_1} = 0, \tag{39}$$

for any x_1 and x_2. Taking x_1 at the horizon and x_2 at infinity and using the boundary condition at the horizon and infinity gives, for the (massless) asymptotically flat case,

$$(m\Omega_H - \omega) = \frac{2\omega}{\pi}\left(|Z_{\rm out}|^2 - |Z_{\rm in}|^2\right). \tag{40}$$

Hence a purely outgoing mode must have

$$0 \leq \omega \leq m\Omega_H \qquad \text{(asymptotically flat).} \tag{41}$$

In other words, the mode must be superradiant. In the AdS case, the LHS is unchanged but the term at infinity vanishes and we must have

$$\omega = m\Omega_H \qquad \text{(asymptotically AdS).} \tag{42}$$

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[b] Various mathematical subtleties such as modes coming in from infinity might invalidate this statement but such subtleties do not occur for Kerr[29] and we shall ignore this possibility here.

[c] This just corresponds to complex conjugation of the solution. This invariance arises from a discrete symmetry of the background which, in the coordinates of Ref. 9 is $t \to -t$, $\phi_i \to -\phi_i$. In our coordinates this amounts to $t \to -t$, $\psi \to -\psi$ and $A \to -A$.

Physically, this is simply the statement that there cannot be a constant flux of radiation through the horizon if the flux at infinity vanishes. Note that for both cases, we must have $m \geq 0$ since $\Omega_H \geq 0$.

We shall discuss the asymptotically AdS case first because the fact that we only have to consider a single value of ω makes this case simpler to analyse than the asymptotically flat case, for which we have to consider a range of values for ω.

5. Asymptotically anti-de Sitter black holes

5.1. *Form of the potential and behaviour of solutions*

Set $\omega = m\Omega_H > 0$. The potential vanishes at the horizon and is monotonically increasing just outside the horizon. For large r, the potential increases[d] proportional to r^2. What happens in between depends on the values of the parameters. For small ω, V is positive everywhere. However, for sufficiently large ω, there is a "classically allowed"[e] region in which V is negative. In more detail: V has roots at $r = r_1, r_2$. The potential is positive for $r_+ < r < r_1$, negative for $r_1 < r < r_2$ and positive for $r > r_2$, i.e., there is a potential barrier separating the classically allowed region where V is negative from the horizon.

Note that, in the AdS case considered here, the initial data $\Psi(r_+)$ and $\Psi'(r_+)$ are real and positive (since $\omega = m\Omega_H$). Hence Ψ is real everywhere. It is easy to see that Ψ will simply increase monotonically if V is positive everywhere. Hence we need V to be negative somewhere for an acceptable solution of the radial equation to exist. The solution Ψ will increase monotonically in the potential barrier, oscillate in the classically allowed region, and then match onto a sum of growing and decaying[f] terms at large r. We need to tune Ω_H until the coefficient of the growing mode vanishes, i.e., until we obtain a "bound state" solution of the radial equation.

5.2. *Small AdS black holes*

Consider the case $r_+ \ll \ell$. In this case, one can use matching techniques to solve the radial equation. One finds that a solution obeying the boundary conditions exists only for discrete values of $\omega\ell$. Since $\omega = m\Omega_H$, this translates into a condition on Ω_H:[21]

$$\Omega_H \ell = \frac{l + N + 1 + \sqrt{(N+1)^2 + \mu^2\ell^2} + 2p}{m}, \qquad p = 0, 1, 2, \ldots \tag{43}$$

The solutions with $p > 0$ correspond to "excited states" for which the solution of the radial equation oscillates (with $p+1$ extrema) before approaching zero at large

[d] If μ^2 is close to the Breitenlonher-Freedman bound then the coefficient of proportionality is negative but we shan't worry about this and our results for small black holes suggest that it doesn't change the qualitative behaviour of solutions.

[e] Of course, everything we are doing is classical but since we have written the radial equation in the form of a Schrödinger equation, we can borrow terminology such as "classically allowed", "bound state" etc. from quantum mechanics.

[f] More precisely: non-normalizable and normalizable.

r. We are interested in the onset of instability, corresponding to the smallest value of $\Omega_H \ell$ for which a solution exists, so we are mainly interested in $p = 0$.

For scalars, we have $l \geq m$ so, for given m, the smallest value of Ω_H for which we have a solution is

$$\Omega_H \ell = 1 + \frac{N + 1 + \sqrt{(N+1)^2 + \mu^2 \ell^2}}{m}. \tag{44}$$

This is the critical value of Ω_H beyond which modes with angular quantum number m become unstable. Note that it always exceeds $1/\ell$, consistent with the proof of stability for $\Omega_H \leq 1/\ell$ given in Ref. 10. However, this proof has been criticized[32] because it assumes the dominant energy condition, which is violated if $\mu^2 < 0$. Our result shows that, for small black holes, this does not matter so long as the Breitenlöhner-Freedman bound is satisfied.

Note that the critical value for Ω_H tends to $1/\ell$ from above as $m \to \infty$. This proves that, for small black holes at least, the instability sets in as soon as Ω_H exceeds $1/\ell$, with the shortest wavelength modes becoming unstable first.

Now consider gravitational pertubations, for which the threshold of stability occurs at

$$\Omega_H \ell = \frac{l + 2N + 2}{m}. \tag{45}$$

For $N = 2$, taking $l = l_{\min}(m)$ and $\epsilon = 1$, this evaluates to $11, 5, 3$ for $m = 1, 2, 3$ and $1 + 4/m$ for $m \geq 4$. (Taking $\epsilon = -1$ just makes $l_{\min}$ bigger.) So the conclusion is the same as for scalar field perturbations: a superradiant gravitational instability sets in as soon as the angular velocity exceeds the speed of light, with the shortest wavelength modes becoming unstable first.

5.3. *Numerical results*

In the context of the AdS/CFT correspondence, we have to consider a higher dimensional spacetime consisting of the product of the black hole space-time with a compact internal space (e.g. a sphere). It is believed that small AdS black holes are unstable with respect to the Gregory-Laflamme instability[33] under which they are expected to localize on the internal space. This means that, although small black holes with $\Omega_H \ell \leq 1$ do not suffer from a superradiant instability, they are nevertheless unstable. In order to eliminate the GL instability we have to extend our results to large AdS black holes, i.e., $r_+ > \ell$. This can be done by solving the radial equation numerically.[21]

Our numerical results show that the critical value of $\Omega_H \ell$ is always greater than 1, and tends to 1 as $m \to \infty$. The interpretation is exactly the same as for small black holes: for any given r_+, if we start from $\Omega_H = 0$ and increase Ω_H then as soon as $\Omega_H \ell$ exceeds 1, the black hole will become unstable to all perturbations for which m exceeds some critical value. In the next section we shall demonstrate this analytically.

5.4. *WKB analysis*

In this section we consider both scalar field and gravitational perturbations governed by the effective Schrödinger equation with potential (26). As before, we are interested in modes at the threshold of instability so $\omega = m\Omega_H$. The strategy is to look at the potential for large m (and hence large ω). In this limit, WKB techniques can be used.[g] The result is that the critical value of Ω_H is given by[21]

$$\Omega_H \ell \approx 1 + \frac{1}{m}\left(l - m + N + 1 + \sqrt{(N+1)^2 - \frac{1}{4} + \mu^2\ell^2} + 2p \right). \qquad (46)$$

This formula is valid for large m with $l - m = \mathcal{O}(1)$. p is a non-negative integer. Once again, there are "excited state" solutions corresponding to positive p, $p = 0$ corresponds to the threshold of instability.

The coefficient of $1/m$ in (46) is positive (at least in all cases for which we know $l_{\min}$) so once again we see $\Omega_H \ell \to 1+$ as $m \to \infty$ so, irrespective of the size of the black hole, once its angular velocity exceeds the speed of light it becomes unstable to perturbations of arbitrarily short wavelength.

5.5. *WKB calculation of unstable modes*

So far, we have been looking for real frequency modes, whose existence indicates the onset of instability. However, the WKB approach can also be used to determine unstable modes directly. This has been used previously in a study of the superradiant instability of the Kerr black hole in the presence of a massive scalar field.[12] We allow ω to be complex: $\omega = \omega_R + i\omega_I$ and look for a suitable solution of the radial equation in the WKB approximation, which we expect to be valid for large m.

Assuming $\omega_R < m\Omega_H$ and $l - m = \mathcal{O}(1)$ we obtain the quantization condition[21]

$$\frac{\omega_R \ell}{m} \approx 1 + \frac{1}{m}\left(l - m + N + 1 + \sqrt{(N+1)^2 - \frac{1}{4} + \mu^2\ell^2} + 2p \right), \qquad p = 0, 1, 2, \ldots \qquad (47)$$

and we can bound[21]

$$0 < \omega_I \ell < \alpha \exp(-\beta m) \qquad (48)$$

for some positive constants α, β. We see that, although large m modes are the first to become unstable when $\Omega_H \ell$ exceeds 1, the growth time of the instability is exponentially large in m so these modes are the least unstable. This suggests that the most unstable modes will be those for which m is not particularly large. It would be interesting to calculate ω_I for such modes.

[g]Strictly speaking, the WKB approximation should only be valid for large μ. For gravitational perturbations we might not expect WKB to work very well. However, the same remark applies to the calculation of black hole quasi-normal modes, where WKB has been found to be accurate.[34] We shall see that the WKB results are in good agreement with our numerical results for $\mu = 0$. In any case, the WKB method is certainly reliable for sufficiently massive scalar fields.

6. Asymptotically flat black holes

6.1. *Introduction*

We know that a mode at the threshold of instability must obey $0 \leq \omega \leq m\Omega_H$, i.e., it is superradiant. The only known way that superradiant modes can actually lead to an instability is if they can be trapped by the potential at infinity, i.e., they must be bound states. This would require a local minimum in V, as in the AdS case, or for a massive scalar field in four dimensions.[h] In all cases that we have examined, the qualitative form of the potential for $\mu = 0$ is: $V \to -(\omega - m\Omega_H)^2$ as $x \to -\infty$, then V increases to a positive maximum and decreases to $-\omega^2$ as $x \to \infty$. It appears that a local minimum in V is not possible so there is no obvious sign of any gravitational instability apparent from our radial equation.

This qualitative argument is no substitute for a quantitative study. We shall analyse the radial equation in two cases: first for large m and ω using the WKB method and then numerically for $D = 7$.

6.2. *WKB approximation*

Consider large m with $r_+\omega/m$ fixed and $l \sim l_{\text{min}}$ so $l/m \to 1$. Qualitatively, the form of V is as follows.[21] It takes the value $-(\omega - m\Omega_H)^2$ at the horizon, increases to a positive maximum and then decreases to $-\omega^2$ at infinity. In other words, there is a potential barrier of height proportional to m^2 separating the classically allowed region near infinity from the classically allowed region near the horizon. The WKB method will then give $|Z_{\text{out}}/Z_{\text{in}}| \approx 1$. However, as argued above, a mode at the threshold of instability will have $|Z_{\text{out}}/Z_{\text{in}}| \to \infty$. We conclude that no such mode exists for large m and ω.

6.3. *Numerical results: asymptotically flat case*

We shall only consider gravitational perturbations in $D = 7$ so $\mu = 0$, $N = 2$.

For given l, m, our strategy (following Ref. 28) is to start with small Ω_H and search the interval $0 \leq \omega \leq m\Omega_H$ for a solution of the radial equation that is regular on the future horizon and outgoing at infinity. This is then repeated for increasing values of Ω_H up to the maximum value.

A convenient object to consider is the ratio

$$Z \equiv \frac{|Z_{\text{out}}|}{|Z_{\text{in}}|} = \lim_{r\to\infty} \frac{|\partial_r \Psi + i\omega\Psi|}{|\partial_r \Psi - i\omega\Psi|}. \tag{49}$$

We are looking for ω for which this ratio diverges, corresponding to a purely outgoing solution.

[h] In higher dimensions, it appears that even a mass term for a scalar is not enough to lead to a superradiant instability, at least for MP black holes with a single non-vanishing angular momentum.[35]

In all cases we have examined, the qualitative form of the potential is the same as we found in the WKB analysis above, i.e., a potential barrier with a positive maximum separates the classically allowed regions near the horizon and far from the black hole. The corresponding behaviour of the solution Ψ is: oscillation near the horizon, exponential growth in the potential barrier region and then oscillation out to infinity. If the potential barrier is large then this implies that the amplitude of oscillation far from the black hole will be large. However, we are looking for a mode with $Z_{\rm in} = 0$. From (40), such a mode obeys

$$\lim_{r\to\infty} |\Psi|^2 = \frac{m\Omega_H - \omega}{\omega}. \tag{50}$$

Hence, unless ω is very small, such a mode will *not* have a large amplitude. Hence it seems very unlikely that we will find a suitable mode when the potential barrier is large. Phrasing the argument slightly differently, if Ψ is large then $Z_{\rm in}$ and/or $Z_{\rm out}$ must be large compared with the left hand side of equation (40), which implies $Z \approx 1$. More physically, if the potential barrier is large then one expects almost perfect reflection and very little transmission, so the amplitude of Ψ is much greater far from the black hole than near the horizon.

This argument suggests that we should examine the case for which the potential barrier is smallest. The potential barrier is minimized when l is as small as possible and m as large as possible. The most favourable case (using (22)) is therefore likely to be $l = 2$, $\epsilon = 1$, $m = 4$. Our numerical results are shown in figure 1, where we plot Z against $\omega/(m\Omega_H)$ for $\Omega_H/\Omega_{\rm max} = 0.5, 0.7, 0.9, 0.99, 0.999$ where $\Omega_{\rm max} = \sqrt{N/(N+1)}/r_+$ is the upper bound on Ω_H. The curves have the same qualitative shape as for the Kerr black hole,[28] i.e., Z is very close to 1 for small ω, then increases to a maximum near $\omega = m\Omega_H$ and decreases back to 1 at $\omega = m\Omega_H$ (the latter property follows from equation (40)). The position of the maximum tends towards $\omega = m\Omega_H$ as $\Omega_H \to \Omega_{\rm max}$. The largest value for Z is $Z = 1.115$ so there is no sign of Z diverging anywhere, as would be required for an instability. Note that the amplification of energy flux in superradiant scattering is given by Z^2 so the maximum amplification apparent in our data is about 24%, and is achieved as the black hole tends to extremality and $\omega \to m\Omega_H-$. This is just as for Kerr, although for Kerr, the maximum amplification is much greater: 138%.[30]

We have repeated our analysis for other values of (l, m). The results are qualitatively similar to the case we have just discussed. For $\Omega_H/\Omega_{\rm max} = 0.99$, the largest value of Z obtained for $\epsilon = 1$ and $(l, m) = (3, 5), (3, 3)$ was $1.056, 1.000$ respectively, reflecting the fact that decreasing m tends to increase the potential barrier. In figure 2 we exhibit how Z varies with l with $m = m_{\rm max}(l) = l + 2$. The largest value of Z occurs for the $l = 2$, $m = 4$ case discussed above, and Z decreases monotonically to 1 as l increases, in agreement with our WKB analysis.

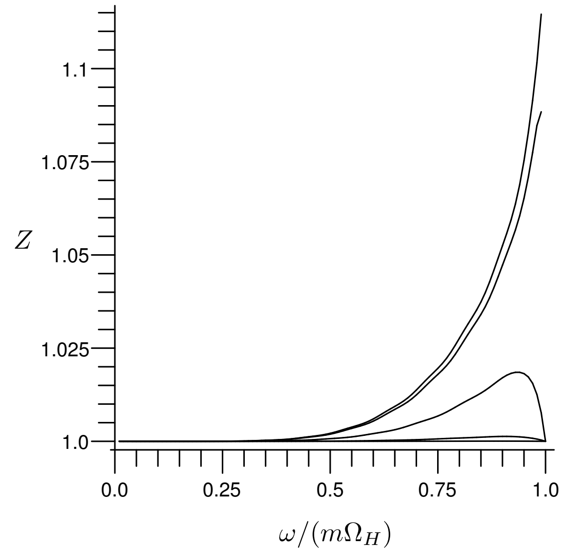

Fig. 1. Plots of Z againt $\omega/(m\Omega_H)$ for (from bottom to top) $\Omega_H/\Omega_{\max} = 0.5, 0.7, 0.9, 0.99, 0.999$ with $\epsilon = 1, l = 2, m = 4$. Note that $Z = 1$ for $\omega = m\Omega_H$ but this point has been deleted from the topmost two curves to make the figure clearer.

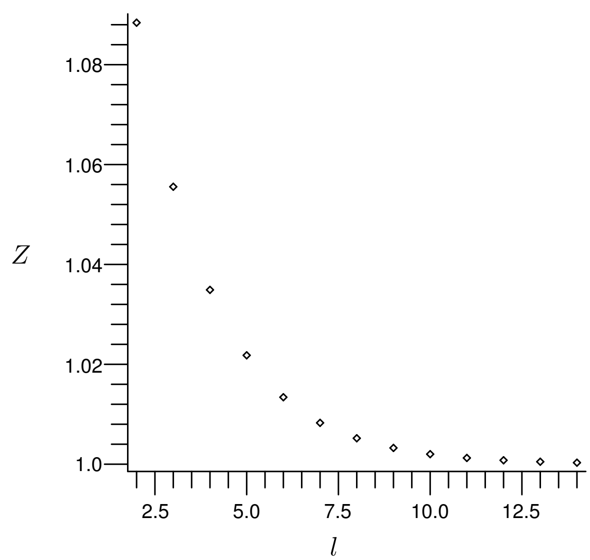

Fig. 2. Plot of Z against $l = 2, 3, \ldots$ for $m = m_{\max} = l+2$, $\epsilon = 1$, $\Omega_H/\Omega_{\max} = 0.99$, $\omega/(m\Omega_H) = 0.99$.

7. Discussion

We have shown that there exists a class of higher-dimensional rotating black hole solutions for which linearized gravitational perturbations can be studied analytically. We have concentrated on the particular case of tensor modes. The equation of motion for such modes reduces to a single ODE governing the behaviour in the radial direction. In the asymptotically flat case, our analysis of this equation shows no evidence of any instability of the black holes.

We have studied the case of asymptotically anti-de Sitter black holes in some detail. Our main result is that the superradiant instability of such black holes discussed in Ref. 10 occurs precisely when the angular velocity of the black hole exceeds the speed of light (in the sense that the co-rotating Killing field becomes space-like on the conformal boundary). In other words, the instability occurs precisely when the stability argument of Ref. 10 fails. Our results for small AdS black holes also enable us to address a loop-hole in the proof of Ref. 10, namely that it could be invalidated by the presence of a tachyonic scalar field (obeying the Breitenlohner-Freedman bound), which violates the dominant energy condition. For small black holes at least, such scalars do not behave any differently from more massive scalars, i.e., the threshold of instability is unaffected. It would be interesting to see whether the methods of Refs 10 and 31 could be combined to give a more general proof of this.

Something that has not been addressed in the literature is the end-point of the instability of Ref. 10. For the super-radiant instability of a Kerr black hole in the presence of a massive scalar, the evolution is clear: the black hole gradually loses energy and angular momentum to bound superradiant modes of the scalar field. These couple non-linearly to gravitational waves (and thereby to non-superradiant modes of the scalar field) so this energy and angular momentum is ultimately radiated to infinity. After a very long time the black hole will have lost all of its angular momentum this way.

In the AdS case, the evolution of the instability must be rather different. A black hole that is initially rotating faster than light will lose energy and angular momentum into superradiant modes of all fields of the theory under consideration. However, in AdS this cannot be radiated to infinity so instead the energy and angular momentum in fields outside the black hole must accumulate over time and backreaction will become important. If the system ultimately settles down to an equilibrium state then this must be described by a new stationary, asymptotically AdS black hole solution. Presumably the angular velocity of this new solution will not exceed the speed of light. Note that this argument does not depend on the details of the theory: it would be valid if gravity were the only field present, so there should even exist new vacuum black hole solutions.

Further evidence for the existence of new solutions comes from our analysis of modes at the threshold of instability. Since these have $\omega = m\Omega_H$, they are preserved by the co-rotating Killing vector field ξ. However they break the symmetries generated by $\partial/\partial t$ and $\partial/\partial\psi$. The existence of these modes could indicate the presence of new branches of solutions that bifurcate from the solutions of Refs 8,9 at the critical (m-dependent) value of Ω_H. The new solutions would not be invariant with respect to $\partial/\partial t$ or $\partial/\partial\psi$. In other words, they would not be stationary or axisymmetric. However, ξ would still describe a symmetry. Since the bifurcation point occurs when the original black hole is rotating faster than light, ξ would be spacelike near infinity but timelike near the horizon. So, near the bifurcation point, the new branch of solutions would correspond to black holes that are not stationary in the usual sense but nevertheless admit a Killing field that is timelike near the horizon. This Killing field becomes spacelike *outside* an "ergosphere" (this is what happens for the co-rotating Killing field of the Kerr black hole). However, if such solutions do exist, they are themselves rotating faster than light and therefore seem very likely to be unstable.

What happens as one moves further away from a bifurcation point? Obviously we can only speculate, but one possibility is that, if one moves sufficiently far along one of the new branches of solutions, one reaches solutions for which ξ is timelike everywhere outside the horizon. These would correspond to genuinely stationary black holes which are nevertheless nonaxisymmetric. There would be no violation of the theorem that a stationary black hole must be axisymmetric[37] because this theorem assumes that the stationary Killing field is *not* normal to the event horizon whereas ξ *is* normal to the horizon of all the black holes we have been discussing. If such black holes exist then it is natural to guess that these should be the new solutions describing the endpoint of the superradiant instability.

The possibility of a black hole being stationary with respect to a Killing field that does not approach the "usual" generator of global AdS time translations deserves further comment. Consider AdS_5 in global coordinates:

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2\right). \quad (51)$$

The generator of global time translations $\partial/\partial t$ is a globally timelike Killing field whose norm diverges at infinity. Now consider the Killing field

$$\frac{\partial}{\partial t} + \frac{1}{\ell}\frac{\partial}{\partial\phi_1} + \frac{1}{\ell}\frac{\partial}{\partial\phi_2}. \quad (52)$$

This is globally timelike with *constant norm*. The same construction works in any odd-dimensional AdS spacetime. Therefore there are (at least) two qualitatively different ways that an odd-dimensional asymptotically AdS space-time can be stationary: the generator of time-translations could have either unbounded norm or bounded norm. This does not appear to have been discussed before. The known AdS black hole solutions are stationary in both senses because they admit Killing fields that have the same asymptotic behaviour as $\partial/\partial t$ and $\partial/\partial\phi_i$ above. However there may well exist AdS black holes with less symmetry that are stationary only with respect to a Killing field of bounded norm.

We do not know whether the stationary nonaxisymmetric black holes discussed above must be of this form. If they are, then, since the stationary Killing field must be normal to the horizon, and since a Killing field of bounded norm is null on the conformal boundary, such black holes must be rotating at the speed of light.[i] Superficially, this makes sense because the superradiant instability "switches off" when the rotation of a black hole no longer exceeds the speed of light so one might expect the black hole to evolve to a final state rotating precisely at the speed of light. However, the evolution of the instability is a time-dependent process, during which the notion of angular velocity is not even defined, so we see no reason why the final time-independent state could not be rotating slower than light.

Finally, we note that supersymmetric black holes exist in AdS_5.[38] Supersymmetry guarantees the existence of a non-spacelike Killing vector field that is normal to the event horizon[36] and timelike with bounded norm at infinity,[39] i.e., these solutions have precisely the behaviour that we have just discussed and hence rotate at the speed of light. However, these solutions admit extra Killing fields analagous to $\partial/\partial\phi_i$ (and hence also $\partial/\partial t$) above so they are also stationary in the usual sense (i.e. with respect to a Killing field of unbounded norm). The existence of these extra Killing fields appears unrelated to supersymmetry, which raises the question of whether there exist more general supersymmetric black hole solutions without these extra symmetries. Such black holes would be nonaxisymmetric, and stationary only in the new sense that we have been discussing.

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[i] The terminology here may be a little confusing. If one defines angular velocity for such black holes in the usual way then it will vanish because the stationary Killing field is normal to the horizon. However, this is angular velocity defined with respect to a stationary bulk observer. Since the stationary Killing field is null on the conformal boundary, such an observer actually rotates at the speed of light with respect to the boundary and hence so does the black hole.

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