Rotating Black Holes in Higher Dimensional Einstein-Maxwell Gravity

Alikram Aliev

TUBITAK-BU, Feza Gürsey Institute

Istanbul, TURKEY

Motivation

Black holes still remain to be one of the intriguing and puzzling objects of study in both four and higher dimensional spacetimes.

In four dimensions they were predicted in the framework of ordinary General Relativity as the endpoint of gravitational collapse of sufficiently massive stars.

Subsequently apart from their astrophysical implications, they have also played a profound role in understanding the nature of general relativity itself, resulting in the famous Singularity Theorems.

Remarkable Features:

(i) Equilibrium and Uniqueness Properties,

(ii) Quantum Properties of Evaporation of Microscopic Black Holes, etc.

Motivation

One can expect that the properties of black holes might also have played an important role in understanding the nature of Gravity in Higher Dimensions. This has triggered the study of black hole solutions in higher-dimensional gravity theories as well as in string/M-theory.

Developments have revealed New Possibilities and New Unexpected Features:

(i) For certain super-symmetric black holes in 5D it has become possible to explain the Statistical Origin of the Bekenstein-Hawking entropy,

(ii) Higher Dimensions allow Different Horizon Topologies for Black Holes,

(iii) Some basic properties change, among them are Stability and Uniqueness properties

Instabilities and Non-uniqueness

The simplest class of extended Black Holes (Black Strings) exhibits the linear perturbative instability below a certain critical mass (**Gregory-Laflamme instability**), thereby providing an example of non-uniqueness in the form of a Phase Transition between black holes and black strings in higher dimensions.

The first higher dimensional black hole solutions with the Spherical Topology of the horizon: (**Tangherlini, 1963**)

Rotating Black Hole solution: (**Myers and Perry, 1986**) This solution is not unique, unlike its four dimensional counterpart, the Kerr solution in 4D.

Rotating Black Ring solution in 5D with the horizon topology of $S^2 \times S^1$: (**Emparan and Reall, 2002**).

Rotating Charged Black Holes

Though the exact non-rotating black hole solution to the higher dimensional Einstein-Maxwell equations was found a long time ago, rotating charged black holes have been basically discussed in the framework of certain Supergravity Theories and String Theory.

The rotating black hole solution in Higher Dimensional Einstein-Maxwell Gravity, that is the counterpart of the usual Kerr-Newman solution, still remains to be found analytically !!!!

Numerical solution for some special cases: (**Jutta Kunz et. al 2005**).

I shall discuss the intermediate case of Slow Rotation, namely a new analytical solution, which describe electrically charged black holes with slow rotation in $N+1$ **Dimensions**

Weak electric charge

Let us assume that a rotating black hole in $N+1$ dimensions possesses an electric charge, which is small enough, i.e. $Q \ll M$.

In this case the space-time can still be well described by the Myers-Perry metric:

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{m}{r^{N-4}\,\Sigma}\right) dt^2 + \frac{r^{N-2}\,\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 \\
&\quad + \left(r^2 + a^2 + \frac{m a^2 \sin^2\theta}{r^{N-4}\,\Sigma}\right) \sin^2\theta\, d\phi^2 \qquad (1) \\
&\quad - \frac{2ma\sin^2\theta}{r^{N-4}\,\Sigma}\, dt\, d\phi + r^2\cos^2\theta\, d\Omega^2_{N-3}\ ,
\end{aligned}
$$

Weak electric charge

where the metric functions:

$$\Sigma = r^2 + a^2 \cos^2\theta \,, \qquad \Delta = r^{N-2}(r^2 + a^2) - m\, r^2 \,, \tag{2}$$

the parameter m is related to the mass of the black hole, while a is a parameter associated with its angular momentum and

$$d\Omega^2_{N-3} = d{\chi_1}^2 + \sin^2\chi_1\,(\,d{\chi_2}^2 + \sin^2\chi_2\,(...d{\chi_{N-3}}^2...)\,) \tag{3}$$

is the metric of a unit $(N-3)$-sphere.

The MP metric admits the existence of the commuting Killing vectors

$$\xi_{(0)} = \xi^\mu_{(t)}\,\frac{\partial}{\partial x^\mu}\,, \qquad \xi_{(3)} = \xi^\mu_{(\phi)}\,\frac{\partial}{\partial x^\mu}\,, \tag{4}$$

Potential one-form

In order to construct the potential one-form we shall use the well-known fact that for a Ricci-flat metric (as MP metric) a Killing 1-form field is closed and co-closed, that is, it can serve as a potential one-form for an associated test Maxwell field.

We shall take the potential one-form field as

$$A = \alpha\, \hat{\xi}_{(t)} \ , \qquad (5)$$

where the Killing one-form field: $\hat{\xi}_{(t)} = \xi_{(t)\mu}\, dx^{\mu}$ and α is an arbitrary constant parameter.

Using the integrals

$$Q = \frac{1}{A_{N-1}} \oint {}^{\star}F \ , \quad m = -\frac{1}{(N-2)\, A_{N-1}} \oint {}^{\star}d\hat{\xi}_{(t)} \qquad (6)$$

Potential one-form

With this in mind we obtain the following expression for the electromagnetic potential one-form

$$A = -\frac{Q}{(N-2)\, r^{N-4}\, \Sigma} \left(dt - a \sin^2\theta \, d\phi\right) \ . \tag{7}$$

Accordingly, the electromagnetic two-form is given by

$$F = -\frac{Q}{(N-2)\, r^{N-3}\, \Sigma^{\,2}} \left\{ H \ \left(dt - a \sin^2\theta \, d\phi\right) \wedge dr \right. \tag{8}$$

$$\left. -ra \sin 2\theta \left[a \, dt - \left(r^2 + a^2\right) \ d\phi\right] \wedge d\theta \right\} \ ,$$

where

$$H = (N-2)\, \Sigma - 2\, a^2 \cos^2\theta \ . \tag{9}$$

Arbitrary Charge

For an arbitrary amount of the electric charge of the black hole we must solve the simultaneous system of the Einstein-Maxwell equations.

Strategies resulting in the familiar Kerr-Newman metric in four dimensions

(i) The metric ansatz in the Kerr-Schild form

(ii) The complex coordinate transformation method (**Newman and Janis, 1965**).

In both cases the potential one-form (7) remains **unchanged (!)**

The metric ansatz

Using the similar approach in $N+1$ dimensions we arrive at the metric ansatz:

$$
\begin{aligned}
ds^2 \;=\; & -\left(1-\frac{m}{r^{N-4}\,\Sigma}+\frac{q^2}{r^{2(N-3)}\,\Sigma}\right)dt^2+\frac{r^{\,N-2}\,\Sigma}{\Delta}\,dr^2+\Sigma\,d\theta^2 \\
& +\left(r^2+a^2+\frac{a^2\left(mr^{N-2}-q^2\right)\sin^2\theta}{r^{2(N-3)}\,\Sigma}\right)\sin^2\theta\,d\phi^2 \\
& -\frac{2a\left(mr^{N-2}-q^2\right)\sin^2\theta}{r^{2(N-3)}\,\Sigma}\,dt\,d\phi+r^2\cos^2\theta\,d\Omega^2_{N-3}\;, \qquad (10)
\end{aligned}
$$

where q is a charge parameter and

$$
\Delta=r^{N-2}(r^2+a^2)-m\,r^2+q^2\,r^{4-N}\;. \qquad (11)
$$

The metric ansatz

This metric form is also obtained from the Myers-Perry metric by a simple re-scaling of the mass parameter

$$m \to m - q^2/r^{N-2} \ . \tag{12}$$

Straightforward calculations show that the source-free Maxwell equations

$$\partial_\nu(\sqrt{-g}\, F^{\mu\nu}) = 0 \tag{13}$$

is satisfied with potential one-form (7).

Therefore, we use them to calculate the energy-momentum source on the right-hand-side of the higher dimensional Einstein equations.

Einstein Equations

$$R^{\mu}_{\nu} = 8\pi G M^{\mu}_{\nu} \ , \qquad (14)$$

where

$$M^{\mu}_{\nu} = \frac{1}{A_{N-1}} \left(F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{2(N-1)} \, \delta^{\mu}_{\nu} \, F_{\alpha\beta} F^{\alpha\beta} \right) \ . \qquad (15)$$

It is instructive to start with the case $N = 3$. Then

$$\begin{aligned} M^{0}_{0} &= -M^{3}_{3} = -\frac{Q^2}{8\pi\Sigma^3} \left(r^2 + a^2 + a^2 \sin^2\theta \right) \ , \\ M^{1}_{1} &= -M^{2}_{2} = -\frac{Q^2}{8\pi\Sigma^2} \ , \\ M^{3}_{0} &= -\frac{M^{0}_{3}}{(r^2 + a^2)\sin^2\theta} = -\frac{aQ^2}{4\pi\Sigma^3} \ , \end{aligned} \qquad (16)$$

Einstein Equations

The components of the Ricci tensor are:

$$
\begin{aligned}
R^0_0 &= -R^3_3 = -\frac{q^2}{\Sigma^3}\left(r^2 + a^2 + a^2 \sin^2\theta\right) \ , \\
R^1_1 &= -R^2_2 = -\frac{q^2}{\Sigma^2} \ , \\
R^3_0 &= -\frac{R^0_3}{(r^2+a^2)\sin^2\theta} = -\frac{2\,aq^2}{\Sigma^3} \ . \qquad (17)
\end{aligned}
$$

Inspecting equations in (14) with these expressions we find that $q^2 = GQ^2$. This is the case of ordinary Kerr-Newman black hole in 4 D.

Einstein Equations

However, for the case $N \geq 4$ equations in (14) are satisfied only with slow rotation of the black hole. To first order in the rotation parameter we obtain that

$$
\begin{aligned}
M_0^0 &= M_1^1 = -\frac{N-2}{(N-1)\,A_{N-1}}\,\frac{Q^2}{r^{2(N-1)}}\,,\\
M_2^2 &= M_3^3 = M_4^4 = \frac{1}{(N-1)\,A_{N-1}}\,\frac{Q^2}{r^{2(N-1)}}\,, \qquad (18)\\
M_3^0 &= -\,r^2\sin^2\theta\,M_0^3 = \frac{a\sin^2\theta}{A_{N-1}}\,\frac{Q^2}{r^{2(N-1)}}\,.
\end{aligned}
$$

We note that all the components M_i^i with $i \geq 4$ are equal to each other.

Einstein Equations

We also have the Ricci components

$$
\begin{aligned}
R_0^0 &= R_1^1 = -\frac{q^2}{r^{2(N-1)}}\,(N-2)^2\,, \\
R_2^2 &= R_3^3 = R_4^4 = \frac{q^2}{r^{2(N-1)}}\,(N-2)\,, \\
R_3^0 &= -r^2\sin^2\theta\,R_0^3 = \frac{q^2 a\sin^2\theta}{r^{2(N-1)}}\,(N-1)(N-2)
\end{aligned} \tag{19}
$$

along with the components R_i^i identical to each other for all $i \geq 4$. Inspecting now equations in (14) we find that

$$
q = \pm Q\left[\frac{8\pi G}{(N-2)\,(N-1)\,A_{N-1}}\right]^{1/2}. \tag{20}
$$

Solution

Finally, we have the solution:

$$
\begin{aligned}
ds^2 \quad = \quad & -\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2(N-2)}}\right)dt^2 \\
& +\left(1-\frac{m}{r^{N-2}}+\frac{q^2}{r^{2(N-2)}}\right)^{-1} dr^2 \\
& -\frac{2\,a\sin^2\theta}{r^{N-2}}\left(m-\frac{q^2}{r^{N-2}}\right)dt\,d\phi \\
& +r^2\left(d\theta^2+\sin^2\theta\,d\phi^2+\cos^2\theta\;d\Omega^2_{N-3}\right)\ .
\end{aligned}
\tag{21}
$$

while, the associated electromagnetic potential

$$
A=-\frac{Q}{(N-2)\,r^{N-2}}\left(dt-a\sin^2\theta\,d\phi\right)\ . \tag{22}
$$

Gyromagnetic Ratio

The gyromagnetic ratio is the ratio of the magnetic dipole moment of a rotating charged black hole to its angular momentum.

For a black hole in four dimensions the gyromagnetic ratio $g = 2$, just like for the electron in Dirac theory, while for the usual charged matter in classical electrodynamics $g = 1$.

From the asymptotic behavior of the metric (21) we find that

$$g_{03} = -\frac{j \sin^2\theta}{r^{N-2}} + \mathcal{O}\left(\frac{1}{r^{2(N-2)}}\right) , \qquad (23)$$

which gives the specific angular momentum $j = am$.

Gyromagnetic Ratio

The magnetic dipole moment can also be determined from the far distant behavior of the magnetic field

To describe the magnetic field it is useful to introduce the magnetic $(N-2)$-form

$$\hat{B}_{N-2} = i_{\hat{\xi}_{(t)}} {}^\star F = {}^\star \left(\hat{\xi}_{(t)} \wedge F \right) , \tag{24}$$

which in the limit of slow rotation can be written as

$$\hat{B}_{N-2} = \frac{Qa}{r^2} \sqrt{\gamma} \cos^{N-3}\theta \left(\frac{2\cos\theta}{N-2} \frac{dr}{r} + \sin\theta \, d\theta \right) \wedge d\chi_1 \wedge d\chi_2 \wedge \ldots \wedge d\chi_{N-3} . \tag{25}$$

Gyromagnetic Ratio

In the asymptotic rest frame of the black hole the magnetic $(N-2)$ -form has the following orthonormal components

$$B_{\hat{r}\,\hat{\chi}_1\hat{\chi}_2\ldots\hat{\chi}_{N-3}} = \frac{2\,Qa}{N-2}\,\frac{\cos\theta}{r^N}\,,$$

$$B_{\hat{\theta}\,\hat{\chi}_1\hat{\chi}_2\ldots\hat{\chi}_{N-3}} = \frac{Qa\sin\theta}{r^N}\,. \qquad (26)$$

These expressions show that the black hole can be assigned a magnetic dipole moment given by

$$\mu = Q\,a\,. \qquad (27)$$

$$j = m\,a\,. \qquad (28)$$

Gyromagnetic Ratio

We can write

$$\mu = \frac{Q\,j}{m} = (N-1)\,\frac{Q\,J}{2\,M}\,, \tag{29}$$

where we have used the relations

$$m = \frac{16\pi G}{N-1}\,\frac{M}{A_{N-1}}\,, \qquad j = \frac{8\pi G J}{A_{N-1}}\,. \tag{30}$$

Defining now the gyromagnetic parameter g in the usual way

$$\mu = g\,\frac{Q\,J}{2\,M} \tag{31}$$

Gyromagnetic Ratio

We find that a rotating charged black hole in $N+1$ dimensions possesses the gyromagnetic ratio

$$g = N - 1 \; . \tag{32}$$

We note that this value is valid:

(i) For Black Holes with small angular momentum ($a \ll M$), but carrying an arbitrary amount of the electric charge

(ii) For Black Holes with small electric charge ($Q \ll M$), but with arbitrary rotation.

For arbitrary angular momentum and arbitrary electric charge we do not know the answer (!!!). Numerical results: (**Jutta Kunz et. al 2006**)

Details in: **A. N. Aliev, Mod. Phys. Lett. A 21, 751 (2006)**

A. N. Aliev, Phys. Rev. D 74, 024011 (2006)