Derivation of the dipole black ring solutions

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Completely integrable sector in 5D Einstein-Maxwell gravity

$$R_{\mu\nu} = \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{6}F_{\sigma\lambda}F^{\sigma\lambda}g_{\mu\nu}\right),$$
$$\nabla_{\mu}F^{\mu\nu} = 0. \tag{1}$$

Spacetime symmetries: Three commuting Killing vectors $T = \partial/\partial t$, $K_1 = \partial/\partial X$, $K_2 = \partial/\partial Y$.

K_2-hypersurface orthogonal

Metric:

$$ds^2 = e^{2u}dY^2 + e^{-u}h_{ij}dx^i dx^j \tag{2}$$

h_{ij} - 4D Lorentz metric, $h_{ij} = h_{ij}(x^k)$

Maxwell 2-form :

$$F = dA_Y \wedge dY, \;\; A_Y = A_Y(x^k) \tag{3}$$

Dimensionally reduced equations

$$\mathcal{D}_i\mathcal{D}^i u = -\frac{1}{3}e^{-2u}h^{ij}\mathcal{D}_i A_Y \mathcal{D}_j A_Y,$$
$$\mathcal{D}_i\left(e^{-2u}\mathcal{D}^i A_Y\right) = 0, \tag{4}$$
$$R(h)_{ij} = \frac{3}{2}\partial_i u \partial_j u + \frac{1}{2}e^{-2u}\partial_i A_Y \partial_j A_Y.$$

$$M_1 = \begin{pmatrix} e^u + \frac{1}{3}e^{-u}A_Y^2 & \frac{1}{\sqrt{3}}e^{-u}A_Y \\ \frac{1}{\sqrt{3}}e^{-u}A_Y & e^{-u} \end{pmatrix} \tag{5}$$

$$\mathcal{D}_i\left[\mathcal{D}^i M_1 M_1^{-1}\right] = 0, \tag{6}$$
$$R_{ij}(h) = -\frac{3}{4}Tr\left[\partial_i M_1 \partial_j M_1^{-1}\right]. \tag{7}$$

Symmetries of the reduced equations:

$$h_{ij} \to h_{ij}, \;\; M_1 \to G M_1 G^T, G \in SL(2,R) \tag{8}$$

Further reduction along T and K_1

Metric in canonical coordinates

$$h_{ij}dx^i dx^j = -e^{2U}\left(dt + \mathcal{A}dX\right)^2 + e^{-2U}\rho^2 dX^2 + e^{-2U}e^{2\Gamma}\left(d\rho^2 + dz^2\right) \qquad (9)$$

Twist potential

$$\omega = -\frac{1}{2}\star(h)\left(T \wedge dT\right) \rightarrow \omega = df \qquad (10)$$

$$\partial_\rho f = -\frac{1}{2}\frac{e^{4U}}{\rho}\partial_z \mathcal{A}, \qquad (11)$$

$$\partial_z f = \frac{1}{2}\frac{e^{4U}}{\rho}\partial_\rho \mathcal{A}. \qquad (12)$$

$$M_2 = \begin{pmatrix} e^{2U} + 4f^2 e^{-2U} & 2fe^{-2U} \\ 2fe^{-2U} & e^{-2U} \end{pmatrix} \qquad (13)$$

2D reduced equations

$$\partial_\rho \left(\rho \partial_\rho M_1 M_1^{-1}\right) + \partial_z \left(\rho \partial_z M_1 M_1^{-1}\right) = 0,$$
$$\partial_\rho \left(\rho \partial_\rho M_2 M_2^{-1}\right) + \partial_z \left(\rho \partial_z M_2 M_2^{-1}\right) = 0,$$
$$\rho^{-1}\partial_\rho \Gamma = -\frac{1}{8}\left[Tr\left(\partial_\rho M_2 \partial_\rho M_2^{-1}\right) - Tr\left(\partial_z M_2 \partial_z M_2^{-1}\right)\right]$$
$$-\frac{3}{8}\left[Tr\left(\partial_\rho M_1 \partial_\rho M_1^{-1}\right) - Tr\left(\partial_z M_1 \partial_z M_1^{-1}\right)\right],$$
$$\rho^{-1}\partial_z \Gamma = -\frac{1}{4} Tr\left(\partial_\rho M_2 \partial_z M_2^{-1}\right)$$
$$-\frac{3}{4} Tr\left(\partial_\rho M_1 \partial_z M_1^{-1}\right).$$

The sector of Einstein-Maxwell gravity under consideration is completely integrable.

Solution generating method

Let us consider two solutions $M_1 = M^{(1)}$ and $M_2 = M^{(2)}$ of the σ-model equations

$$\partial_\rho \left(\rho \partial_\rho M M^{-1} \right) + \partial_z \left(\rho \partial_z M M^{-1} \right) = 0. \quad (14)$$

In addition let us denote by $\gamma^{(i)}$ the solution of the system

$$\rho^{-1} \partial_z \gamma^{(i)} = -\frac{1}{4} Tr \left(\partial_\rho M^{(i)} \partial_z {M^{(i)}}^{-1} \right), \quad (15)$$

$$\rho^{-1} \partial_\rho \gamma^{(i)} = -\frac{1}{8} \left[Tr \left(\partial_\rho M^{(i)} \partial_\rho {M^{(i)}}^{-1} \right) - Tr \left(\partial_z M^{(i)} \partial_z {M^{(i)}}^{-1} \right) \right]. \quad (16)$$

Then we find for the metric function Γ

$$\Gamma = \gamma^{(2)} + 3\gamma^{(1)}. \quad (17)$$

From a practical point of view it is more convenient to associate the σ-model solutions $M^{(i)}$ with solutions of the vacuum Einstein equations*

$$ds^2_{E(i)} = e^{2u_E^{(i)}} dY^2 + e^{-u_E^{(i)}} \left[-e^{2U_E^{(i)}} \left(dt + \mathcal{A}_E^{(i)} dX \right)^2 \right.$$
$$\left. + e^{-2U_E^{(i)}} \rho^2 dX^2 + e^{-2U_E^{(i)}} e^{2\Gamma_E^{(i)}} (d\rho^2 + dz^2) \right],$$

which correspond to the matrixes

$$M^{(i)} = \begin{pmatrix} e^{2U_E^{(i)}} + 4\left(f_E^{(i)}\right)^2 e^{-2U_E^{(i)}} & 2f_E^{(i)} e^{-2U_E^{(i)}} \\ 2f_E^{(i)} e^{-2U_E^{(i)}} & e^{-2U_E^{(i)}} \end{pmatrix}.$$

*From now on all quantities with subscript or superscript "E" correspond to the vacuum case.

The metric function $\Gamma_E^{(i)}$ for the vacuum Einstein equations can be found from the equations of Γ by setting $A_Y = 0$ in the matrix M_1. So we obtain

$$\Gamma_E^{(i)} = \gamma^{(i)} + \Omega_E^{(i)} \tag{18}$$

where $\Omega_E^{(i)}$ is a solution to the system

$$\rho^{-1}\partial_\rho \Omega_E^{(i)} = \frac{3}{4}\left[\left(\partial_\rho u_E^{(i)}\right)^2 - \left(\partial_z u_E^{(i)}\right)^2\right], \tag{19}$$

$$\rho^{-1}\partial_z \Omega_E^{(i)} = \frac{3}{2}\partial_\rho u_E^{(i)} \partial_z u_E^{(i)}. \tag{20}$$

We then find from (17) and (18) that

$$\Gamma = \Gamma_E^{(2)} - \Omega_E^{(2)} + 3\left[\Gamma_E^{(1)} - \Omega_E^{(1)}\right]. \tag{21}$$

Comparing the matrixes M_1 and $M^{(1)}$ we obtain

$$e^{2u} = e^{4U_E^{(1)}}, \tag{22}$$
$$A_Y = 2\sqrt{3} f_E^{(1)}, \tag{23}$$

where $f_E^{(i)}$ satisfies

$$\partial_\rho f_E^{(i)} = -\frac{1}{2}\frac{e^{4U_E^{(i)}}}{\rho}\partial_z \mathcal{A}_E^{(i)}, \tag{24}$$
$$\partial_z f_E^{(i)} = \frac{1}{2}\frac{e^{4U_E^{(i)}}}{\rho}\partial_\rho \mathcal{A}_E^{(i)}. \tag{25}$$

Once having the metric function $e^{2u} = g_{YY}$ we can write the EM metric

$$
\begin{aligned}
ds^2 = e^{4U_E^{(1)}} dY^2 + e^{-2U_E^{(1)}} \Bigg[& -e^{2U_E^{(2)}} \left(dt + \mathcal{A}_E^{(2)} dX \right)^2 \\
& + e^{-2U_E^{(2)}} \rho^2 dX^2 \\
& + \left(\frac{e^{2\Gamma_E^{(1)}}}{e^{2\Omega_E^{(1)} + \frac{2}{3}\Omega_E^{(2)}}} \right)^3 e^{-2U_E^{(2)}} e^{2\Gamma_E^{(2)}} (d\rho^2 + dz^2) \Bigg] .
\end{aligned}
$$

Summarizing, we obtain the following result

Proposition. [hep-th/0602116] *Let us consider two solutions of the vacuum 5D Einstein equations*

$$\begin{aligned} ds^2_{E(i)} &= g^{E(i)}_{YY} dY^2 + g^{E(i)}_{00}\left(dt + \mathcal{A}^{(i)}_E dX\right)^2 \\ &\quad + \tilde{g}^{E(i)}_{XX} dX^2 + g^{E(i)}_{\rho\rho}(d\rho^2 + dz^2) \end{aligned}$$

Then the following give a solution to the 5D EM equations

$$\begin{aligned} ds^2 &= \left[|g^{E(1)}_{00}|\sqrt{g^{E(1)}_{YY}}\right]^2 dY^2 + \left[\frac{\sqrt{g^{E(2)}_{YY}}}{|g^{E(1)}_{00}|\sqrt{g^{E(1)}_{YY}}}\right] \\ &\quad \times \left[g^{E(2)}_{00}\left(dt + \mathcal{A}^{(2)}_E dX\right)^2 + \tilde{g}^{E(2)}_{XX} dX^2 \right. \\ &\quad \left. + \left(\frac{|g^{E(1)}_{00}| g^{E(1)}_{YY} g^{E(1)}_{\rho\rho}}{e^{2\Omega^{(1)}_E + \frac{2}{3}\Omega^{(2)}_E}}\right)^3 g^{E(2)}_{\rho\rho}(d\rho^2 + dz^2)\right], \\ A_Y &= \pm 2\sqrt{3} f^{(1)}_E, \end{aligned}$$

where $f_E^{(1)}$ *is a solution to the system*

$$\begin{aligned}
\partial_\rho f_E^{(1)} &= -\frac{1}{2}\frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho}\partial_z \mathcal{A}_E^{(1)},\\
\partial_z f_E^{(1)} &= \frac{1}{2}\frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho}\partial_\rho \mathcal{A}_E^{(1)},
\end{aligned}$$

and $\Omega_E^{(i)}$ *satisfy*

$$\begin{aligned}
\rho^{-1}\partial_\rho \Omega_E^{(i)} &= \frac{3}{16}\left[\left(\partial_\rho \ln\left(g_{YY}^{E(i)}\right)\right)^2 - \left(\partial_z \ln\left(g_{YY}^{E(i)}\right)\right)^2\right],\\
\rho^{-1}\partial_z \Omega_E^{(i)} &= \frac{3}{8}\partial_\rho \ln\left(g_{YY}^{E(i)}\right)\partial_z \ln\left(g_{YY}^{E(i)}\right).
\end{aligned}$$

Explicit derivation of the dipole black ring solutions

$\{neutral\ \ black\ \ ring\} + \{neutral\ \ black\ \ ring\} \rightarrow$
$\{EM\ \ dipole\ \ black\ \ ring\}$

The first solution is with parameters $\{\lambda_1, \nu, \mathcal{R}\}$ while the second is parameterized by $\{\lambda_2, \nu, \mathcal{R}\}$.

The derivation of the dipole black rings with dilaton can be found in hep-th/0604140 and hep-th/0607101.

some unresolved problems

1. $\{neutral\ black\ ring\} + \{neutral\ black\ hole\} \rightarrow$

$\{???\}$

2. $\{neutral\ black\ hole\} + \{black\ hole\} \rightarrow$
$\{???\}$

3.$\{appropriate\ \ solution\} + \{black\ hole/ring\} \rightarrow$
$\{black\ soluton\ ???\}$

4. Are there dipole black ring solutions with rotation in ϕ-direction? If YES, could they be generated via the presented solution generating method?

{black ring with rotation in ϕ-direction} + {?????}
= {dipole black ring with rotation in ϕ-direction}

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