Ricci flow and black holes

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Introduction

Aims

- Understand algorithms to find black holes
- Better understand off-shell gravity

Ricci flow

- Well defined for Euclidean signature metrics
- Ricci flow; $\frac{d}{d\lambda} g_{\mu\nu}(x) = -2R_{\mu\nu}(x)$
- Ricci flat metrics are fixed points of this flow
- In fact we can always do an infinitessimal diffeomorphism $x^\mu \to x^\mu + 2\xi^\mu(\lambda)$
- Then $\frac{d}{d\lambda} g_{\mu\nu} = -2R_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$
- Exactly same geometric flow
- Choosing $\xi^\mu = \triangle_S x^\mu$ gives strictly parabolic equation
- So; $\frac{d}{d\lambda} g_{\mu\nu} = -R^{(H)}_{\mu\nu} = \partial^2_\alpha g_{\mu\nu} + f(g, \partial g)$

Behaviour of Ricci flow

Fixed points of flow

- Any Ricci-flat metric is a fixed point of Ricci flow
- Consider perturbation; $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$
- Find; $\frac{d}{d\lambda} h_{\mu\nu} = -(\triangle^{(0)}_L h)_{\mu\nu} = +\partial^2_\alpha h_{\mu\nu} + 2R^{(0)}_{\mu\alpha\nu\beta} h^{\alpha\beta}$
  where we chose $\xi_\mu = -\nabla^\mu(\delta h_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} \delta h)$
- For flat space this gives diffusion:
  $\frac{d}{d\lambda} h_{\mu\nu} = \partial^2_\alpha h_{\mu\nu}$
- Hence flat is a stable fixed point of Ricci flow

Algorithm to solve Einstein equations?

- Any static black hole admits real Euclidean section
- Could use Ricci flow as an algorithm to find exotic solutions
- Note that direct elliptic approach, solving $R^{(H)}_{\mu\nu} = 0$, appears as Ricci flow on large scales

Ricci flow and black holes

Flow of Euclidean black hole

- Consider $\frac{d}{d\lambda}h_{\mu\nu} = -(\triangle_L^{(0)}h)_{\mu\nu}$
- An eigenmode, $(\triangle_L^{(0)}h_{(\mu)}) = \mu^2 h_{(\mu)}$ will go as $\sim e^{-\mu^2\lambda}$
- Hence if $(\triangle_L^{(0)})$ is not positive, the fixed point is unstable
- Schwarzschild possesses exactly one negative eigenmode; Gross-Perry-Yaffe $h_{\mu\nu} = h^{TT}_{\mu\nu}(r)$
- Static, spherically symmetric, not conformal
- Naive Ricci flow is not a good algorithm for static black holes in general!

Ricci flow as gradient flow of Euclidean action

Toy example

- For a theory we require a metric on field space
- For example: $S = \int \sqrt{g}\phi\triangle\phi$ gives $G_{AB}\delta_1\phi^A\delta_2\phi^B = \int \sqrt{g}\delta_1\phi\delta_2\phi$
- Note; abstract index A, define $G^{AB}G_{BC} = \delta^A_C$
- Given this metric one can perform a gradient flow of the action; $\frac{d}{d\lambda}\phi^A = -G^{AB}\frac{\partial}{\partial\phi^B}S$
- Property that at point on curve, transverse directions leave action stationary

Gradient descent of free energy

Ricci flow as gradient descent

- Natural metric on the space of metrics;
  $G_{AB}\delta_1 g^A \delta_2 g^B = \int \sqrt{g}\delta_1 g^{\mu\nu}(g_{\mu\alpha}g_{\nu\beta} + a\, g_{\mu\nu}g_{\alpha\beta})\delta_2 g^{\alpha\beta}$
- Then, $\frac{d}{d\lambda}g^A = -G^{AB}\frac{\partial}{\partial g^B}S$
- Gives flow; $\frac{d}{d\lambda}g_{\mu\nu}(x) = -2R_{\mu\nu}(x) + \frac{2a+1}{Da+1}Rg_{\mu\nu}$
- Well defined parablic flow for $a < -\frac{1}{D}$
- Ricci flow for $a = -\frac{1}{2}$

Gravity in a box

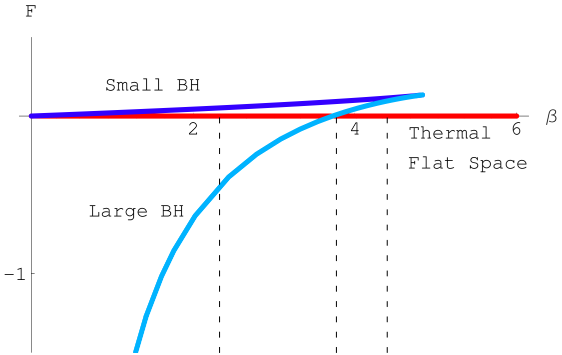
The canonical ensemble [York]

- Choose manifold with boundary $S^1 \times S^2$;
  Fix induced metric on boundary
  Circle radius β, sphere radius R
- Look for smooth Ricci flat solutions
- Fill in with topology $S^1 \times B^3$;
  Hot flat space for any β; $ds^2 = d\tau^2 + r^2 d\Omega^2$
- For sufficient temperature find black holes topology $D^2 \times S^2$
- These are Euclidean Schwarschild with 2 horizon radii, small r_- and large r_+: $0 < r_- < \frac{2}{3}R < r_+ < R$
- Only small BH's which are thermodynamically unstable have negative mode

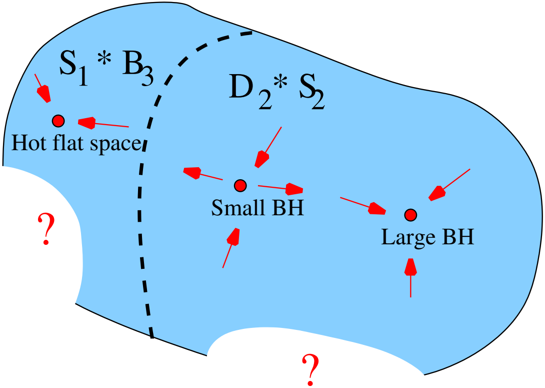
Gravity in a box, cont...

Free energy

- Find a free energy diagram:



Metric space

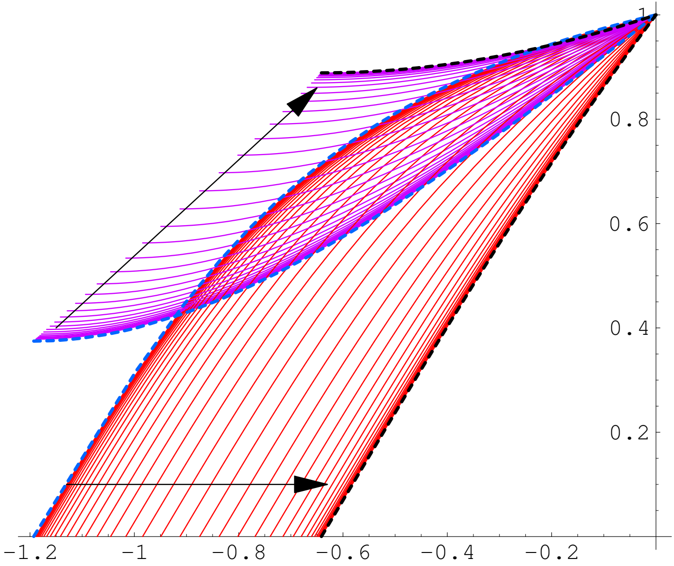


Ricci flow and black holes

The flow

- Small BH unstable fixed point – where does unstable direction flow?
- Flows preserve $U(1) \times SO(3)$ isometry
- So, $ds^2 = T(r)^2 d\tau^2 + dr^2 + S(r)^2 d\Omega^2$
- Flow 1st order so 2 directions $g^{(0)}_{\mu\nu} \pm \delta g_{\mu\nu}$
- Relatively simple to numerically compute these flows...

Flow in ‘positive’ direction

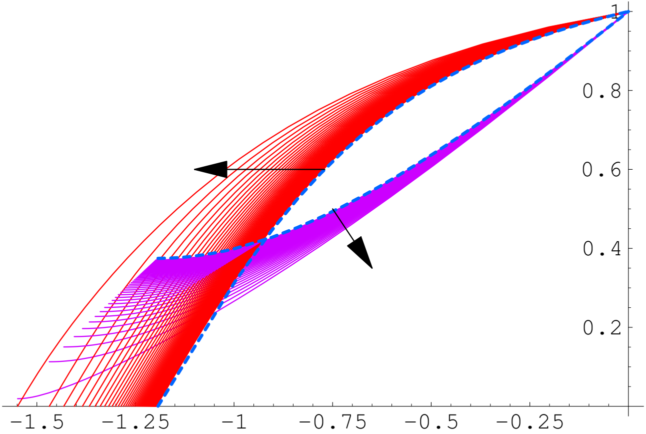


Flow in ‘negative direction

Can we flow to hot flat space?

- Naively no! Different topology!
- Black hole topology $D^2 \times S^2$
- Hot flat space topology $S^1 \times B^3$
- But Ricci flow may lead to a singularity...

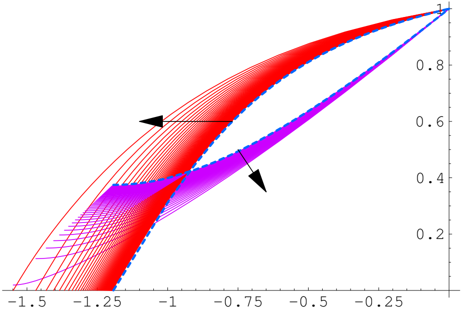
Flow in ‘negative’ direction

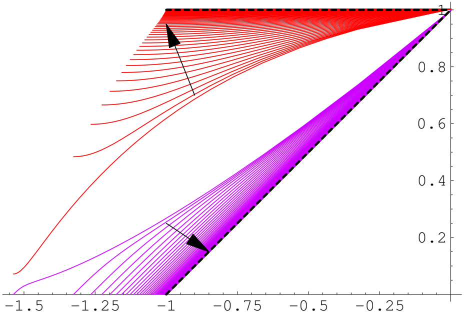


Flow in 'negative' direction

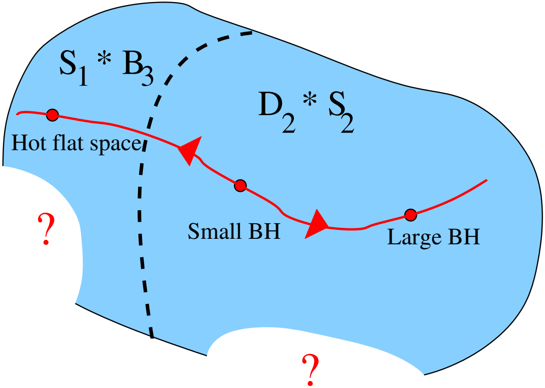
Resolve singularity by surgery

- Resolve singularity *locally* preserving $U(1) \times SO(3)$ isometries

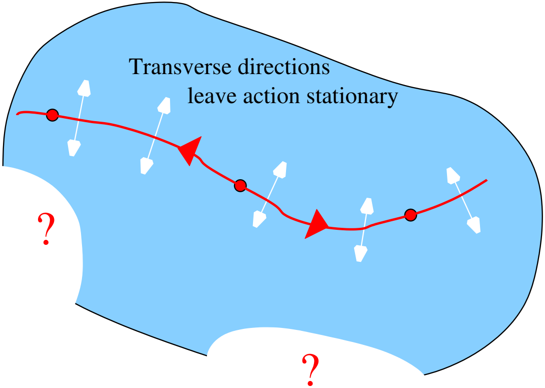




Metric space

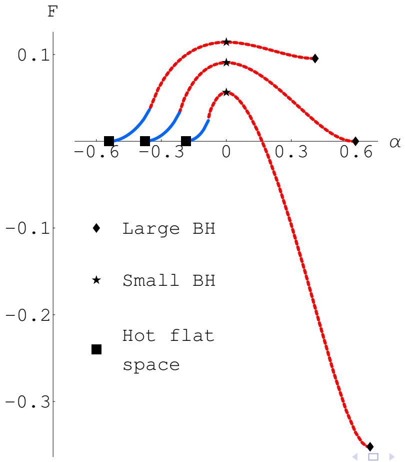


Free energy



Off shell free energy

- Free energy continuous across the singularity



Summary

- Use of Ricci flow – gradient flow – to better understand off-shell geometries
- Many conceptual things to better understand!
- Can we find an elegant algorithm to numerically find black holes?
- Relation to string theory RG flow

σ-models RG flow

Relation to string theory

- For closed string worldsheet action, the target space metric is a coupling
- Set 2-form and dilaton to be trivial
- Worldsheet beta function for this coupling gives; $\frac{d}{d\lambda} g_{\mu\nu} = -2R_{\mu\nu} + O(\alpha')$ where $\lambda = -\log \mu/\mu_0$
- Ricci flat solutions are conformal fixed points
- Our results show the black hole is unstable under RG
- Obvious question is where does the flow go...