BLACK STRING SOLUTIONS WITH NEGATIVE COSMOLOGICAL CONSTANT

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- Robert B. Mann, Eugen Radu, Cristian Stelea, " **Black string solutions with negative cosmological constant**", hep-th/0604205

- Yves Brihaye and Eugen Radu, **"Rotating black string solutions with negative cosmological constant"**, to appear.

Action principle:

$$I_0 = \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\sqrt{-g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-\gamma}K$$

– solutions with a compact extradimension z:

– Black strings $\Lambda = 0$:

$$\underbrace{vacuum \quad (d-1)-dimensions} \quad + \quad dz^2$$

$$ds^2 = \frac{dr^2}{1-(\frac{r_0}{r})^{d-4}} + r^2 d\Omega_{d-3}^2 - (1-(\frac{r_0}{r})^{d-4})dt^2 + dz^2$$

– Black strings $\Lambda \neq 0$: Schwarzschild-(A)dS $\quad + \quad dz^2$ not a solution!

<u>parametrization of the d-dimensional line element</u> ($d \geq 5$)

$$ds^2 = \frac{dr^2}{f(r)} + r^2 d\Sigma_{k,d-3}^2 - b(r)dt^2 + a(r)dz^2$$

where

$$d\Sigma_{k,d-3}^2 = \begin{cases} d\Omega_{d-3}^2 & \text{for } k=+1 \\ \sum_{i=1}^{d-3} dx_i^2 & \text{for } k=0 \\ d\Xi_{d-3}^2 & \text{for } k=-1 \end{cases}$$

the equations:

$$f' = \frac{2k(d-4)}{r} + \frac{2(d-1)r}{\ell^2} - \frac{2(d-4)f}{r} - f\left(\frac{a'}{a} + \frac{b'}{b}\right) \ ,$$

$$\begin{aligned} b'' \ &= \ \frac{(d-3)(d-4)b}{r^2} - \frac{(d-3)(d-4)kb}{r^2 f} - \frac{(d-1)(d-4)b}{\ell^2 f} + \frac{(d-3)ba'}{ra} \\ &+ \frac{(d-4)b'}{r} - \frac{(d-4)kb'}{rf} - \frac{(d-1)rb'}{\ell^2 f} + \frac{a'b'}{2a} + \frac{b'^2}{b} \ , \end{aligned}$$

$$\frac{a'}{a} = 2\frac{b\big[\ell^2(d-3)(d-4)(k-f) + (d-1)(d-2)r^2\big] - (d-3)r\ell^2 f b'}{r\ell^2 f\big[rb' + 2(d-3)b\big]} \quad .$$

with $\Lambda = -(d-1)(d-2)/(2\ell^2)$

– $k = 0$: $a = r^2$, $f = 1/b = -2M/r^{d-3} + r^2/\ell^2$: AdS topological black hole

– $k = \pm 1$: no exact solution!

asymptotics $r \to \infty$

–even spacetime dimension:

$$\begin{aligned} a(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-4)/2} a_j(\frac{\ell}{r})^{2j} + c_z(\frac{\ell}{r})^{d-3} + O(1/r^{d-2}), \\ b(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-4)/2} a_j(\frac{\ell}{r})^{2j} + c_t(\frac{\ell}{r})^{d-3} + O(1/r^{d-2}), \\ f(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-4)/2} f_j(\frac{\ell}{r})^{2j} + (c_z + c_t)(\frac{\ell}{r})^{d-3} + O(1/r^{d-2}). \end{aligned}$$

$a_j,\ f_j$ are constants depending on k and d

$$a_0 = (\frac{d-4}{d-3})k \ , \quad a_1 = \frac{(d-4)^2k^2}{(d-2)(d-3)^2(d-5)},$$
$$a_2 = -\frac{(d-4)^3(3d^2-23d+26)k^3}{3(d-2)^2(d-3)^3(d-5)(d-7)} \ ,$$

$$f_0 = \frac{k(d-1)(d-4)}{(d-2)(d-3)}, \quad f_1 = 2a_1, \quad f_2 = -\frac{2(d-4)^3(d^2-8d+11)k^3}{(d-2)^2(d-3)^3(d-5)(d-7)}, \ldots .$$

–odd spacetime dimension:

$$\begin{aligned} a(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-5)/2} a_j(\frac{\ell}{r})^{2j} + \zeta\log(\frac{r}{\ell})(\frac{\ell}{r})^{d-3} + c_z(\frac{\ell}{r})^{d-3} + O(\frac{\log r}{r^{d-1}}), \\ b(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-5)/2} a_j(\frac{!}{\ell}r)^{2j} + \zeta\log(\frac{r}{\ell})(\frac{\ell}{r})^{d-3} + c_t(\frac{\ell}{r})^{d-3} + O(\frac{\log r}{r^{d-1}}), \\ f(r) &= \frac{r^2}{\ell^2} + \sum_{j=0}^{(d-5)/2} f_j(\frac{\ell}{r})^{2j} + 2\zeta\log(\frac{r}{\ell})(\frac{\ell}{r})^{d-3} + (c_z + c_t + c_0)(\frac{\ell}{r})^{d-3} + O(\frac{\log r}{r^{d-1}}), \end{aligned}$$

where we note $\zeta = a_{(d-3)/2}\sum_{k_1>0}(d-2k_1-1)\delta_{d,2k_1+1}$, while

$$c_0 = 0 \quad \text{for} \quad d=5, \quad c_0 = \frac{9k^3\ell^4}{1600} \quad \text{for} \quad d=7, \quad c_0 = -\frac{90625k^4\ell^6}{21337344} \quad \text{for} \quad d=9.$$

<u>Black hole solutions:</u> expansion near horizon $r = r_h$

$$\begin{aligned} a(r) &= a_h + \frac{2a_h(d-1)r_h}{(d-1)r_h^2 + k(d-4)\ell^2}(r-r_h) \\ &\quad + \frac{a_h(d-1)^2 r_h^2}{\big[(d-1)r_h^2 + k(d-4)\ell^2\big]^2}(r-r_h)^2 + O(r-r_h)^3, \\ b(r) &= b_1(r-r_h) - \frac{b_1(d-4)\big[(d-1)r_h^2 + (d-3)k\ell^2\big]}{2(d-1)r_h^3 + 2(d-4)kr_h\ell^2}(r-r_h)^2 + O(r-r_h)^3, \\ f(r) &= \frac{1}{r_h\ell^2}\big[(d-1)r_h^2 + k(d-4)\ell^2\big](r-r_h) \\ &\quad - \frac{(d-4)}{2r_h^2\ell^2}\big[(d-1)r_h^2 + k(d-3)\ell^2\big](r-r_h)^2 + O(r-r_h)^3. \end{aligned}$$

$k = 1$ nontrivial limit $r_h \to 0$

$$\begin{aligned} a(r) &= \bar{a}_0 + \frac{\bar{a}_0(d-1)}{(d-2)}(\frac{r}{\ell})^2 + \frac{\bar{a}_0(d-1)^2}{d(d-2)^2(d-3)}(\frac{r}{\ell})^4 + O(r^6), \\ b(r) &= \bar{b}_0 + \frac{\bar{b}_0(d!-1)}{(d-2)}(\frac{r}{\ell})^2 + \frac{\bar{b}_0(d-1)^2}{d(d-2)^2(d-3)}(\frac{r}{\ell})^4 + O(r^6), \\ f(r) &= 1 + \frac{(d-1)(d-4)}{(d-2)(d-3)}(\frac{r}{\ell})^2 + \frac{2(d-1)^2}{d(d-2)^2(d-3)}(\frac{r}{\ell})^4 + O(r^6). \end{aligned}$$

The properties of the solutions

boundary countertem action:

$$
\begin{aligned}
I_{\rm ct}^0 &= \frac{1}{8\pi G}\int d^{d-1}x\sqrt{-\gamma}\{-\frac{d-2}{\ell}-\frac{\ell\Theta\,(d-4)}{2(d-3)}\mathsf{R} \\
&-\frac{\ell^3\Theta\,(d-6)}{2(d-3)^2(d-5)}\left(\mathsf{R}_{ab}\mathsf{R}^{ab}-\frac{d-1}{4(d-2)}\mathsf{R}^2\right) \\
&+\frac{\ell^5\Theta(d-8)}{(d-3)^3(d-5)(d-7)}\big(\frac{3d-1}{4(d-2)}\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab}-\frac{d^2-1}{16(d-2)^2}\mathsf{R}^3. \\
&.-2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd}.-\frac{d-1}{4(d-2)}\nabla_a\mathsf{R}\nabla^a\mathsf{R}+\nabla^c\mathsf{R}^{ab}\nabla_c\mathsf{R}_{ab}\big)+...\},
\end{aligned}
$$

For odd values of d, we have to add the extra terms:

$$
I_{\rm ct}^s=\tfrac{1}{8\pi G}\int d^{d-1}x\sqrt{-\gamma}\log(\tfrac{r}{\ell})\left\{\delta_{d,5}\tfrac{\ell^3}{8}(\tfrac{1}{3}\mathsf{R}^2-\mathsf{R}_{ab}\mathsf{R}^{ab})\right.
$$

$$
-\tfrac{\ell^5}{128}\left(\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab}-\tfrac{3}{25}\mathsf{R}^3-2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd}-\tfrac{1}{10}\mathsf{R}^{ab}\nabla_a\nabla_b\mathsf{R}+\mathsf{R}^{ab}\Box\mathsf{R}_{ab}-\tfrac{1}{10}\mathsf{R}\Box\mathsf{R}\right)\delta_{d,7}+\ldots
$$

boundary stress-tensor

$$
T_{ab}=\frac{2}{\sqrt{-\gamma}}\frac{\delta I}{\delta\gamma^{ab}}.
$$

Killing vector $\xi^a \Rightarrow$ conserved charge

$$
\mathfrak{Q}_\xi=\oint_\Sigma d^{d-2}S^a\;\xi^b T_{ab},
$$

Mass and Tension:

$$\begin{aligned} M &= M_0 + M_c^{(k,d)} \ , \quad M_0 = \frac{\ell^{d-4}}{16\pi G}\big[c_z - (d-2)c_t\big] L V_{k,d-3} \ , \\ \mathcal{T} &= \mathcal{T}_0 + \mathcal{T}_c^{(k,d)} \ , \quad \mathcal{T}_0 = \frac{\ell^{d-4}}{16\pi G}\big[(d-2)c_z - c_t\big] V_{k,d-3} \ , \end{aligned}$$

($V_{k,d-3}$ – the total area of the angular sector)

$$M_c^{(k,d)} = -L\mathcal{T}_c^{(k,d)} = \frac{\ell^{d-4}}{16\pi G} V_{k,d-3} L \left(\frac{1}{12}\delta_{d,5} - \frac{333}{3200}\delta_{d,7} + \dots \right) \ .$$

Hawking temperature:

$$T_H = \frac{1}{4\pi}\sqrt{\frac{b_1}{r_h \ell^2}\big[(d-1)r_h^2 + k(d-4)\ell^2\big]}.$$

Entropy:

$S = A_H/4G$, where $A_H = r_h^{d-3} V_{k,d-3} L \sqrt{a_h}$

Smarr-type formula

$$M + \mathcal{T}L = T_H S$$

the equations are solved numerically $\quad 5 \leq d \leq 12$

$d=5,\ k=1$ Copsey and Horowitz, hep-th/0602003

$k=-1$: $0 < r_{min} < r_h < \infty$

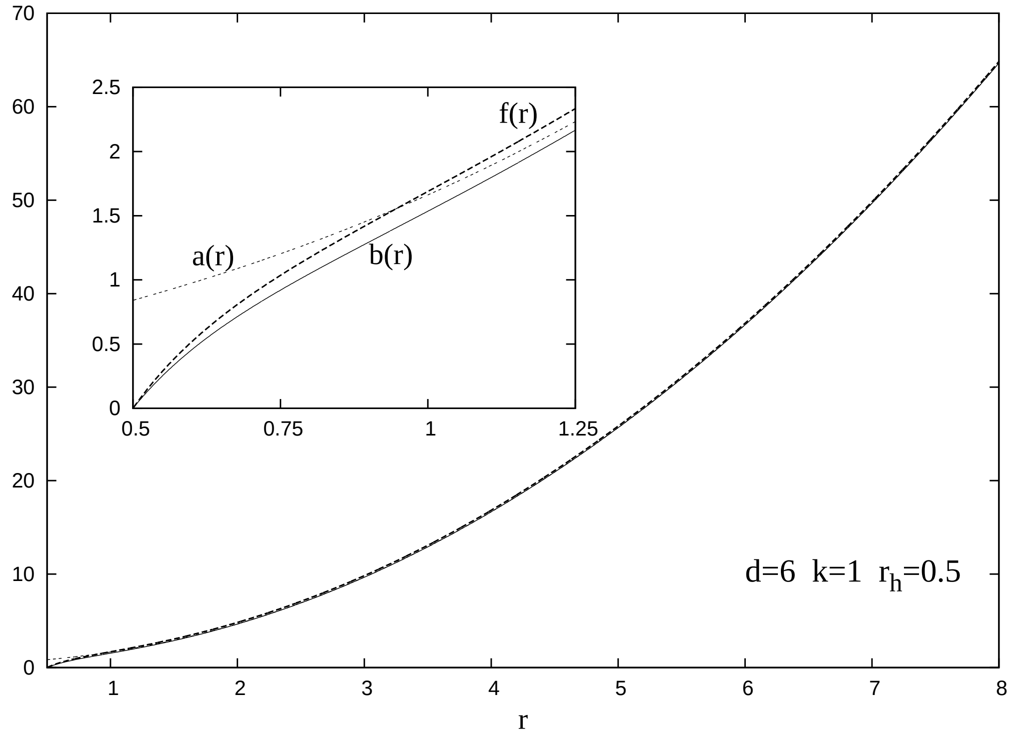
$k=-1$: $0 \leq r_h < \infty$

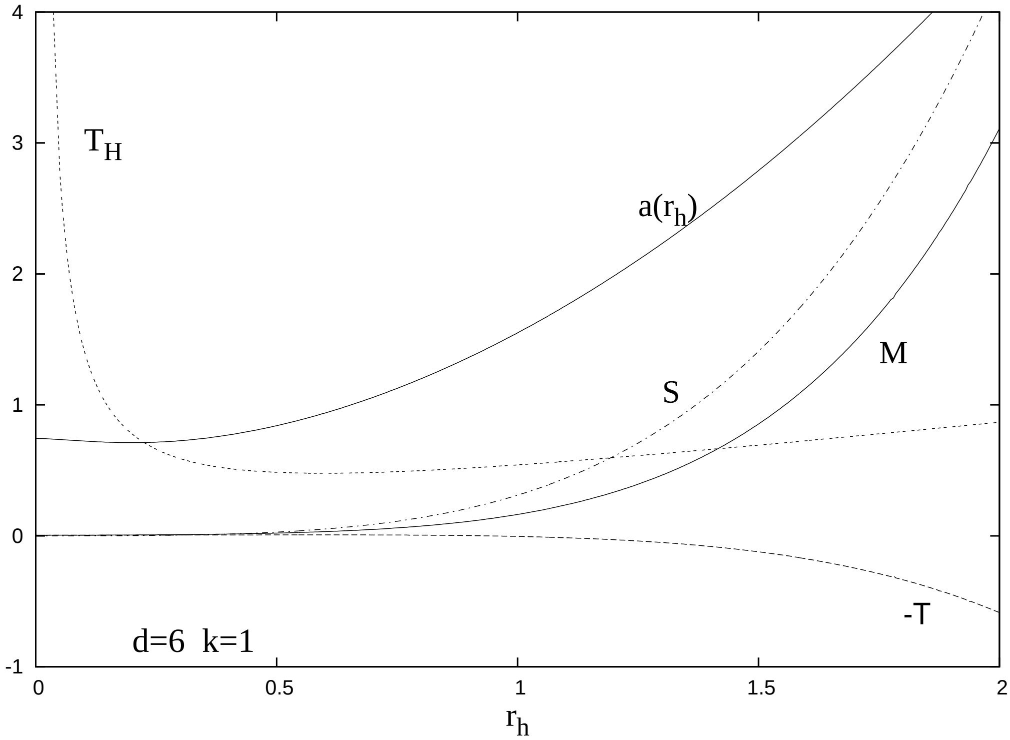
$\underline{k=1 \text{ nontrivial } r_h=0 \text{ limit}}$; nonzero mass and tension!

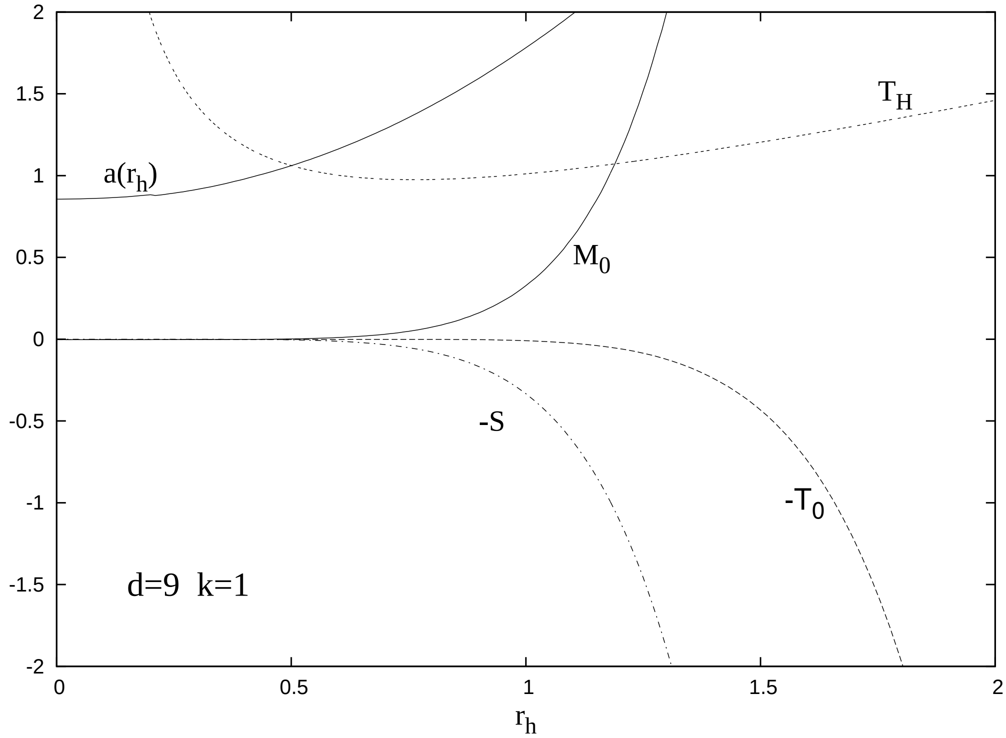
as $r_h \to 0$ we find $a(r)=b(r)$:

$c_t(d=6) \simeq -0.0801$, $c_t(d=7) \simeq -0.0439$, $c_t(d=8) \simeq 0.0403$, $c_t(d=9) \simeq 0.0229$.

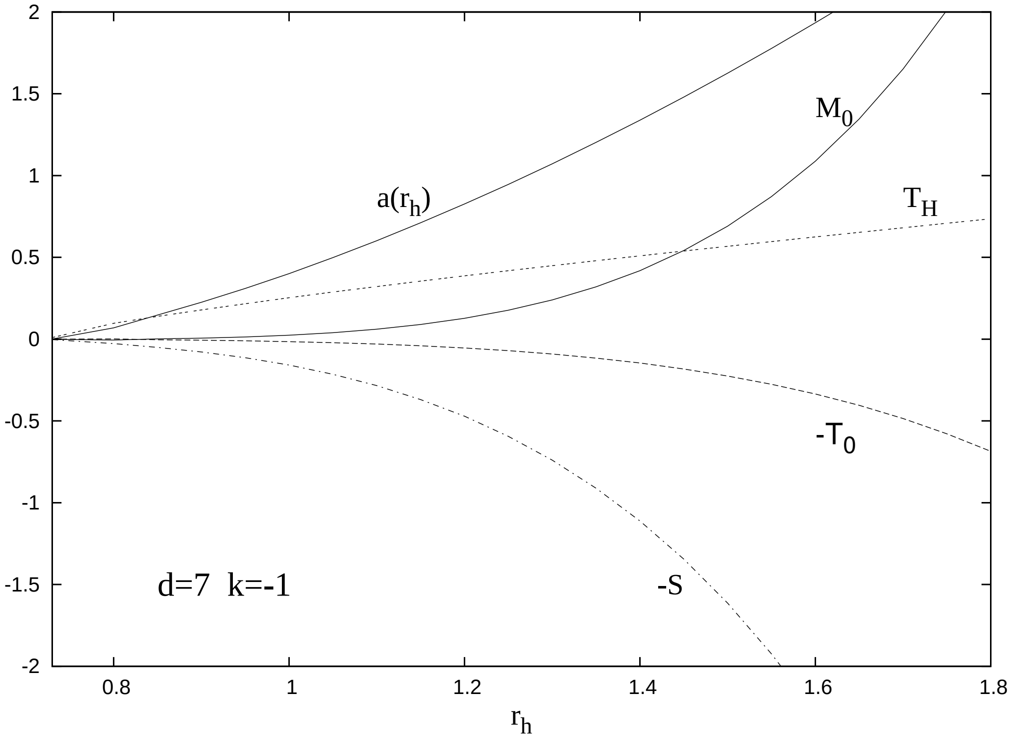
plot typical $d=6$ solution:

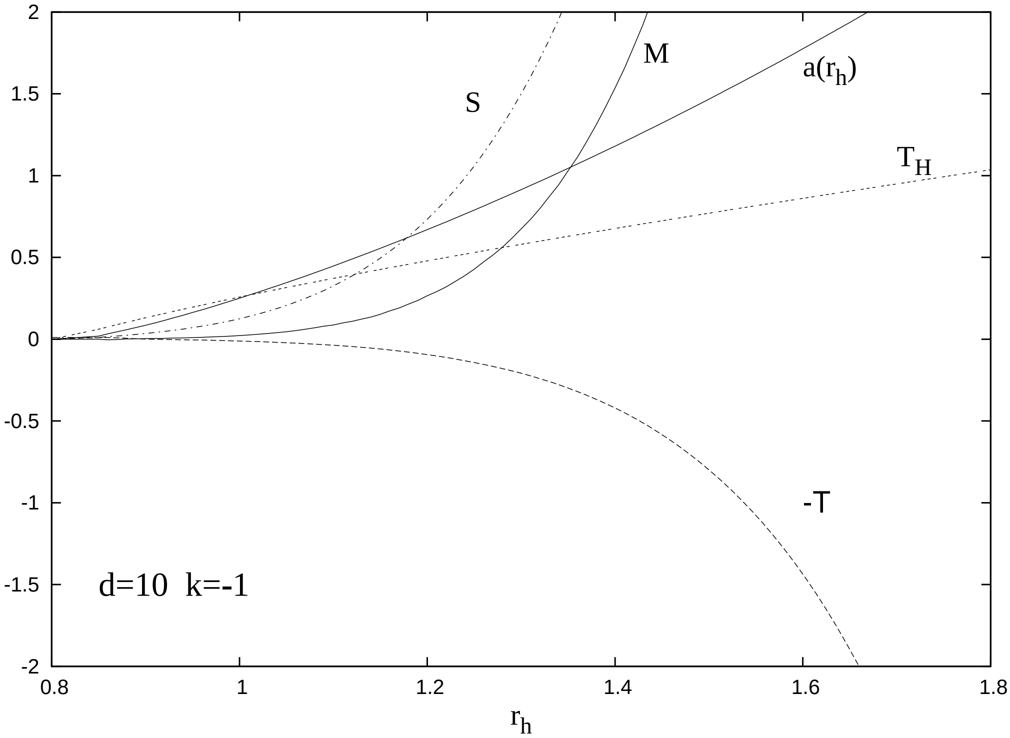




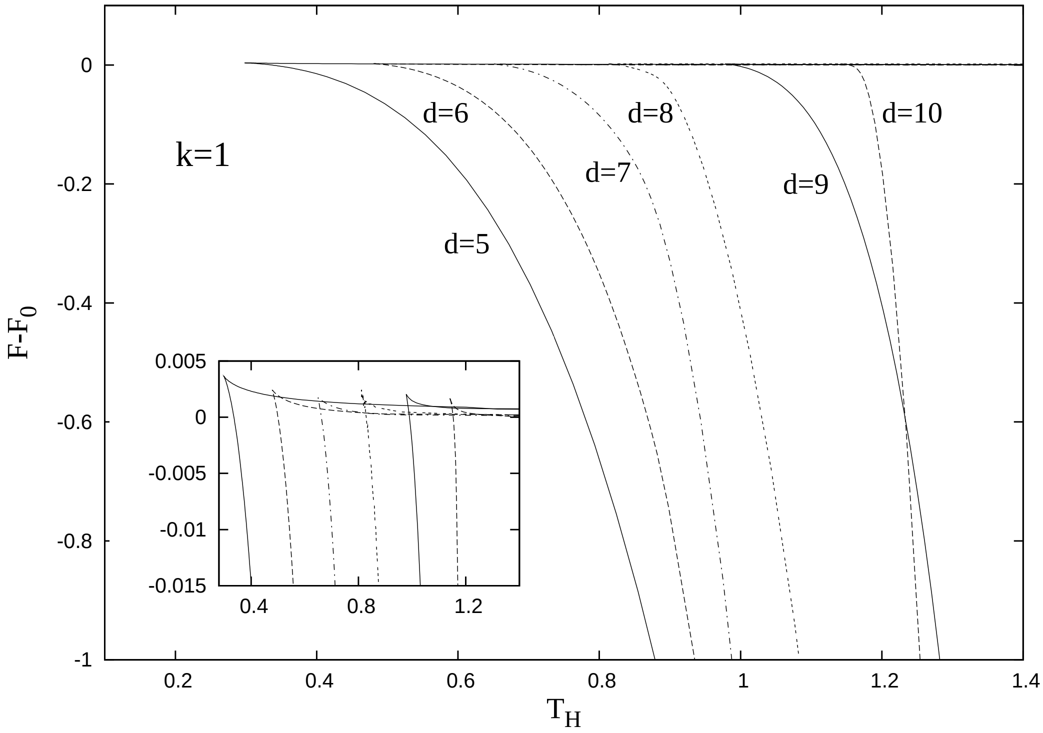


features of $k = 1$ solutions





features of $k = -1$ solutions



Boundary CFT metric: $h_{\mu\nu} = \lim_{r\to\infty} \frac{\ell^2}{r^2}\gamma_{\mu\nu}$.

$$ds^2 = h_{ab}dx^a dx^b = -dt^2 + dz^2 + \ell^2 d\Sigma_k^2,$$

CFT stress tensor:

$$\sqrt{-h}h^{ab} < \tau_{bc} >= \lim_{r\to\infty} \sqrt{-\gamma}\gamma^{ab}T_{bc},$$

$d = 5$:

$$< \tau_t^t >= \frac{36c_t - 12c_z - 1}{192\pi G\ell}, \quad < \tau_z^z >= \frac{-12c_t + 36c_z - 1}{192\pi G\ell},$$
$$< \tau_\theta^\theta >=< \tau_\phi^\phi >= \frac{-12(c_t + c_z) + 2}{192\pi G\ell}.$$

Further remarks:

new Einstein-Maxwell-dilaton solutions with Liouville potential in $d-1$-dims:

generation technique: $SL(2,R)$ symmetry

$$
\begin{aligned}
ds^2_{d-1} &= -ab(\alpha^2 a-\gamma^2 b)^{-\frac{d-4}{d-3}}dt^2+(\alpha^2 a-\gamma^2 b)^{\frac{1}{d-3}}\frac{dr^2}{f}+r^2(\alpha^2 a-\gamma^2 b)^{\frac{1}{d-3}}d\Sigma^2_{k,d-3},\\
e^{\phi} &= (\alpha^2 a-\gamma^2 b)^{-\sqrt{\frac{d-2}{2(d-3)}}}, \qquad \mathcal{A}=\frac{\alpha\beta a-\gamma\delta b}{\alpha^2 a-\gamma^2 b}dt\ .
\end{aligned}
$$

Open questions:

- $d=5$ black ring solutions with $\Lambda<0$?

- Gregory-Laflamme instability?

- $\Lambda<0$ nonuniform black hole solutions? (nontrivial $z-$ dependence)

- rotating black string solutions (Yves Brihaye and Eugen Radu - to appear)