Rod-structure of Stationary and Axisymmetric Metrics

Troels Harmark

Niels Bohr Institute

MG11, Berlin, July 25th, 2006

Based on:
hep-th/0408141
hep-th/0508208 (with Poul Olesen, NBI)
plus work in progress

Motivation:

In 4D, only one available solution for a given set of asymptotic charges:

The Kerr black hole is the unique black hole solution of the vacuum Einstein equations for given mass M and angular momentum J (pure gravity).

In higher-dimensional General Relativity, the situation is very different:

- On Kaluza-Klein spaces $\mathcal{M}^{D-1} \times S^1$ (pure gravity):
  Phase structure very rich, there are interesting phase transitions between phases, and in some cases there are an uncountable infinite number of available static solutions (for given mass and tension).

For reviews: See Kol (2004) and TH & Obers (2005)

- In 5D flat space $\mathcal{M}^5$ (pure gravity):
  Two different type of solutions: The Myers-Perry rotating black hole (topology S^3) and the Emparan-Reall rotating black ring (topology $S^2 \times S^1$). For given mass M and angular momenta J_1, $J_2 = 0$ there are up to 3 available solutions.

To understand the phase structure of higher-dimensional General Relativity, it is important to search for new black hole solutions

But it is very difficult to find exact solutions due to the non-linearity of GR

However, for ***stationary and axisymmetric metrics***, which corresponds to D-dimensional space-times with D-2 commuting Killing vector fields, the vacuum Einstein equations simplify significantly

In this class of metrics: Kerr black hole (4D), Myers-Perry black hole (5D), the black ring of Emparan & Reall (5D).

Known results in 4D:

- Weyl (1917) for static metrics (2 orthogonal commuting Killing vector fields).
- Papapetrou (1953, 1966) for stationary metrics (2 commuting Killing vector fields).

Known results in D>4:

- Emparan & Reall (2001) for static metrics, i.e. for D-2 orthogonal commuting Killing vector fields

Goal I: To find canonical form of metric and the vacuum Einstein equations for D-dimensional stationary & axisymmetric metrics, i.e. metrics with D-2 commuting Killing vector fields.

Important since it can provide new insights into the solutions that we already know (e.g. rotating black holes and black rings in 5D).

Important to develop new method for finding exact solutions since we expect more solutions to exist than the ones we know (e.g. 5D black ring with two angular momenta)

Important to get a handle on how many different solutions one can have: How unique/non-unique is a black hole solution.

Goal II: To understand the structure of stationary & axisymmetric metrics, i.e. their sources, and to find constructive approach to find solutions

Conjecture: A stationary & axisymmetric solution is unique, given its *rod-structure*

Recent work on stationary & axisymmetric black holes:

Hollands, Ishibashi & Wald (2006):
A higher-dimensional stationary rotating black hole must be axisymmetric

Solitonic solutions, inverse scattering method, integrability:

Koikawa (2005)
Pomeransky (2005) ← Generates Myers-Perry black hole
Mishima & Igushi (2005)
Tomizawa, Morisawa & Yasui (2005)
Generates S^2 rotating black ring
Figueras (2005)
Azuma & Koikawa (2005)
Iguchi & Mishima (2006)
Tomizawa & Nozawa (2006)
Generates S^1 rotating black ring
Iguchi & Mishima (2006)

Yazadjiev (2006): Solution generation in 5D-Einstein-Maxwell-dilaton gravity

Jones & Wang (2004, 2005): S-branes, Weyl card diagrams

Important limitation:

Solutions which asymptotes to D-dimensional Minkowski-space $\mathcal{M}^D$
have at most 1 + $[\frac{D-1}{2}]$ Killing vector fields

For solutions which are asymptotically flat, we can therefore treat
$\mathcal{M}^4$ and $\mathcal{M}^5$, i.e. asymptotically flat black hole solutions in 4D and 5D

We can furthermore consider solutions which
asymptote to $\mathcal{M}^4 \times T^n$ and $\mathcal{M}^5 \times T^n$

Stationary & Axisymmetric Metrics

Consider a D-dim manifold (space-time)

Assume D-2 commuting Killing vector fields (linearly independent) $V_{(i)}$ with i=1,2…,D-2

We consider solutions to the vacuum Einstein equations

$$R_{\mu\nu} = 0$$

This defines our class of metrics

For this class of solutions we can write the metric in the canonical form:

$$ds^2 = G_{ij} dx^i dx^j + e^{2\nu}(dr^2 + dz^2) \ , \ r = \sqrt{|\det G_{ij}|}$$

(given a few extra technical assumptions…)

Canonical form of metric:

$$ds^2 = G_{ij}dx^i dx^j + e^{2\nu}(dr^2 + dz^2) \ , \ r = \sqrt{|\det G_{ij}|}$$

Canonical form for vacuum Einsteins equations:

Equation for $G_{ij}(r,z)$:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)G = \partial_r G G^{-1}\partial_r G + \partial_z G G^{-1}\partial_z G \qquad (1)$$

The two equations for $\nu(r,z)$:

$$\partial_r \nu = -\frac{1}{2r} + \frac{r}{8}\mathrm{Tr}((G^{-1}\partial_r G)^2 - (G^{-1}\partial_z G)^2)$$
$$\partial_z \nu = \frac{r}{4}\mathrm{Tr}(G^{-1}\partial_r G G^{-1}\partial_z G)$$

The two equations for $\nu(r,z)$ are integrable for a solution to Eq. (1). This means that we can find $\nu(r,z)$ for any given G(r,z) that solves Eq. (1).

$\longrightarrow$ Solving Einsteins equations reduces to the problem of finding a solution G(r,z) to Eq. (1).

Compact form of equation for $G_{ij}(r,z)$:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right) G = \partial_r G G^{-1} \partial_r G + \partial_z G G^{-1} \partial_z G \qquad (1)$$

Introduce the auxilirary angle γ so that we have the following metric for 3D Euclidean space:

$$dr^2 + r^2 d\gamma^2 + dz^2 = d\sigma_1^2 + d\sigma_2^2 + dz^2$$

$$\sigma_1 = r\cos\gamma \ , \quad \sigma_2 = r\sin\gamma$$

Then everything is axisymmetric wrt. the angle γ and we can write Eq. (1) as:

$$G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla} G)^2$$

with

$$\vec{\nabla} = (\partial_1, \partial_2, \partial_z)$$

Reduction to previously known cases:

1) 4D, solutions of vacuum Einstein equations with two commuting Killing vector fields. Gives the Papapetrou form of stationary & axisymmetric metrics in 4D:

$$ds^2 = -e^{2U}(dt + A d\phi)^2 + e^{-2U} r^2 d\phi^2 + e^{2\nu}(dr^2 + dz^2)$$

$$x^1 = t \ , \ \ x^2 = \phi$$

Reduces to the Weyl form for A = 0.

2) D > 4, solutions of vacuum Einstein equations with D-2 orthogonal commuting Killing vector fields
$\rightarrow$ Generalized Weyl metrics (Emparan & Reall, 2001)

$G_{ij}(r, z)$ diagonal

$$ds^2 = -e^{2U_1} dx_1^2 + \sum_{i=2}^{D-2} e^{2U_i} dx_i^2 + e^{2\nu}(dr^2 + dz^2)$$

$$\sum_{i=1}^{D-2} U_i = \log r$$

Behavior of solutions for r $\to$ 0:

Goal: To find regular solutions of Einstein equations

G(r,z) smooth for r > 0 $\Rightarrow$ solution regular for r > 0

Regularity for r $\to$ 0:

For r > 0: $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla} G)^2$ $\Leftarrow$ Breaks down for r = 0 since det(G(0,z)) = 0

Reason: We have sources at r=0.

Π (eigenvalues of G) = $r^2 \to 0$ for r $\to$ 0

Note: Eigenvalues real since G^T=G

$\Rightarrow$ $\mathrm{dim}(\mathrm{ker}(G(0,z))) \geq 1$

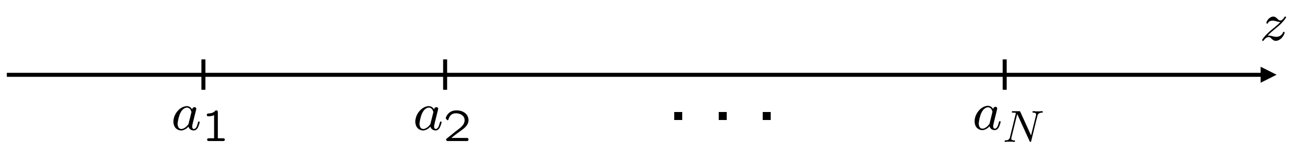
Suppose two eigenvalues $\to$ 0 for r $\to$ 0 $\Rightarrow$ $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \longrightarrow \infty$ for $r \longrightarrow 0$

Solution singular at r=0

Therefore, we require dim(ker(G(0,z))) = 1 except at isolated values of z.

$a_1, a_2, \ldots, a_N$: The isolated values of z for which dim(ker(G(0,z))) > 1

Divide z-axis into N+1 intervals $[a_{k-1}, a_k]$, k=1,…,N+1 ($a_0 = -\infty$, $a_{N+1} = \infty$)



We call an interval $[a_{k-1}, a_k]$ a ***rod*** of the solution G(r,z)

dim(ker(G(0,z))) = 1 for $a_{k-1} < z < a_k$

Consider z_* with $a_{k-1} < z_* < a_k$

Exists orthogonal matrix Λ_* such that
$$\begin{cases} (\Lambda_*^T G(0, z_*) \Lambda_*)_{11} = 0 \\ \Lambda_*^T G(0, z_*) \Lambda_* \text{ diagonal} \end{cases}$$

Define: $\tilde{G}(r, z) = \Lambda_*^T G(r, z) \Lambda_* \Longrightarrow \tilde{G}(r, z)$ a solution

Careful analysis of $\tilde{G}^{-1}\vec{\nabla}^2\tilde{G} = (\tilde{G}^{-1}\vec{\nabla}\tilde{G})^2$ for r $\to$ 0

gives that $\vec{\nabla}\tilde{G}_{1i}(0, z_*) = 0$

$\Longrightarrow$ $\tilde{G}_{1i}(0, z) = 0$ for $z \in]a_{k-1}, a_k[$

Therefore e = $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is an eigenvector for $\tilde{G}(0, z)$ with eigenvalue 0 for z $\in$]a_{k-1}, a_k[

$\Longrightarrow$ $v_{(k)} = \Lambda_* e$ is an eigenvector with eigenvalue 0 for G(0,z) for z $\in$]a_{k-1}, a_k[

Theorem:

For all N+1 rods [a_{k-1}, a_k] we can find a vector

$$v_{(k)} = v^i_{(k)} \frac{\partial}{\partial x^i}$$

vector in the (D-2)-dimensional vector space spanned by the Killing vector fields

such that

$$G(0, z) v_{(k)} = 0 \text{ for } z \in]a_{k-1}, a_k[$$

Note: $v_{(k)}$ unique in $\mathbb{R}P^{D-3}$ since dim(ker(G(0,z)))=1

We call $v_{(k)}$ the ***direction*** of the rod [a_{k-1},a_k].

The ***rod-structure*** of a solution:

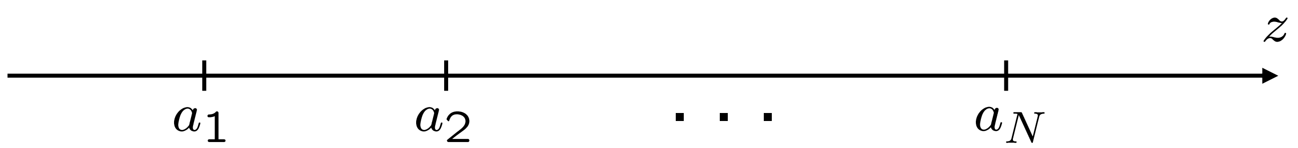
The rods (intervals) $[a_{k-1},a_k]$ and their directions $v_{(k)}$, $k=1,2,\ldots,N+1$.

<u>Conjecture on uniqueness of solutions:</u>

Given a rod-structure, at most one solution $G(r,z)$ exists with that rod-structure

If true, this conjecture gives a complete classification of stationary & axisymmetric metrics (for pure gravity)

Consider G(r,z) diagonal ⇒ Emparans & Realls Generalized Weyl metrics:



Directions of rods are all orthogonal (or tangent): $v_{(k)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Rod-structure reduces to the rod-structure introduced by Emparan & Reall

Uniqueness-conjecture is proven true in this case

<u>Note:</u>

$$G_{ii} = \pm e^{2U_i}$$

$$\vec{\nabla}^2 U_i = 2\pi\delta^2(\sigma)\rho_i(z)$$

$$\rho_i(z) = \begin{cases} 1 & z \in \text{rod with direction } \frac{\partial}{\partial x^i} \\ 0 & \text{otherwise} \end{cases}$$

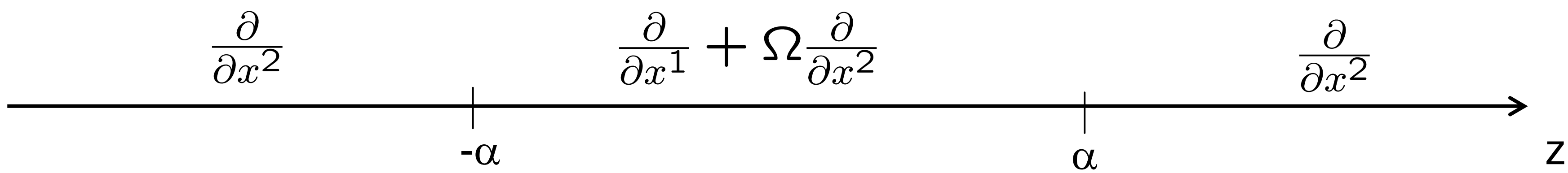
Rod $[a_{k-1}, a_k]$ is a line source for the potential U_i

⟶ The reason for calling the intervals $[a_{k-1}, a_k]$ rods

Examples of solutions:

Non-static, non-diagonal cases

4D Kerr BH

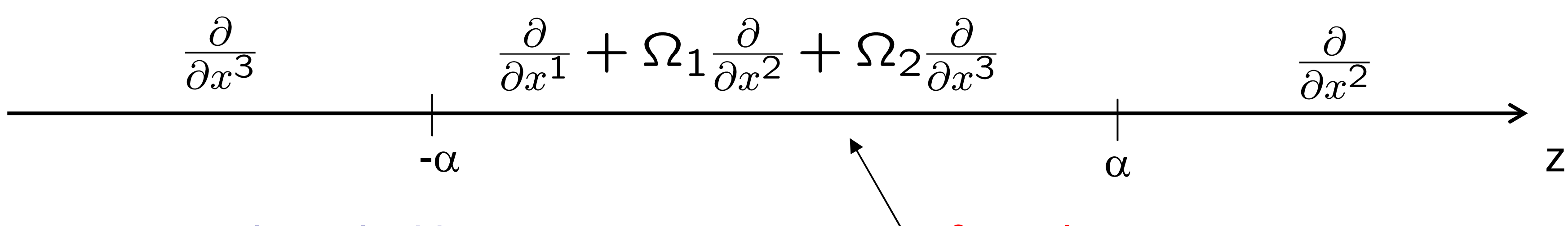


$\frac{\partial}{\partial x^1}+\Omega\frac{\partial}{\partial x^2}$ dir. of [-α,α] rod $\longrightarrow$ not orthogonal to directions of [-∞,-α] & [α,∞]

Ω :angular velocity of Kerr BH

5D Myers-Perry BH

Explicit metric in (r,z) coordinates (canonical form of metric)



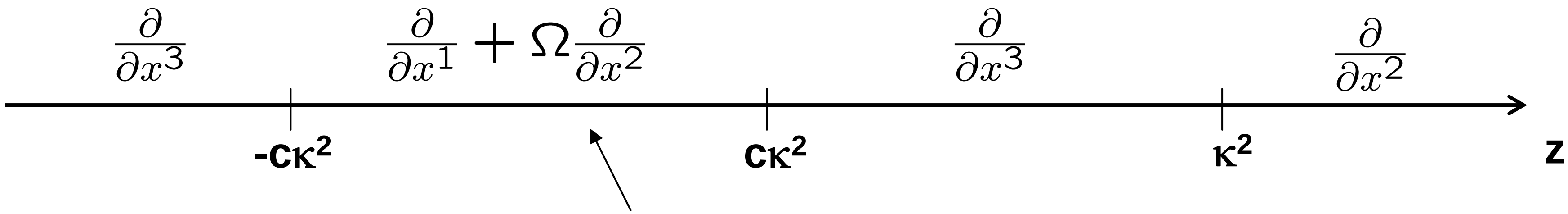
Ω_1 , Ω_2 : angular velocities

S^3 topology

5D Black ring (Emparan & Reall)

Found explicit metric in (r,z) coordinates

Four rods



horizon ⟶ S²× S¹ topology

$$x^1 = t \quad , \quad x^2 = \phi \quad , \quad x^3 = \psi$$

$$G_{11} = -\frac{(1+c)R_1 + (1-c)R_2 - 2cR_3 - 4c\kappa^2}{(1+c)R_1 + (1-c)R_2 - 2cR_3 + 4c\kappa^2}$$

$$G_{12} = -\frac{4c\kappa\sqrt{1+c}}{\sqrt{1-c}} \frac{R_3 - R_1 + (1+c)\kappa^2}{(1+c)R_1 + (1-c)R_2 - 2cR_3 + 4c\kappa^2}$$

$$G_{33} = \frac{(R_3 + z - \kappa^2)(R_2 - z + c\kappa^2)}{R_1 - z - c\kappa^2}$$

$$R_1 = \sqrt{r^2 + (z + c\kappa^2)^2} \qquad R_2 = \sqrt{r^2 + (z - c\kappa^2)^2} \qquad R_3 = \sqrt{r^2 + (z - \kappa^2)^2}$$

$$\Omega = \frac{1}{2\kappa}\sqrt{\frac{1-c}{1+c}} \qquad \Delta\phi = \Delta\psi = \frac{2\pi}{\sqrt{1+c^2}} \qquad 0 < c < 1$$

Further studies of the structure of Stationary & Axisymmetric solutions

Aim:

1. To prove that stationary & axisymmetric solutions are uniquely given by their rod-structure

$\longrightarrow$ Would give a handle on how many black holes (and other solutions) we have

2. Find constructive method to obtaining stationary & axisymmetric solutions

$\longrightarrow$ no general method exist

Problem not solved in 4D

$\longrightarrow$ Problem > 50 years

Even more interesting in D>4 since there are many more physically interesting solutions

Sources for G(r,z):

Recall: $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ **(*)**

$$d\sigma_1^2 + d\sigma_2^2 + dz^2$$
$$= dr^2 + r^2 d\gamma^2 + dz^2$$

$$\vec{\nabla} = (\partial_1, \partial_2, \partial z)$$

Reformulation:

Define $\vec{C}(r,z)$ by $\vec{C} \equiv G^{-1}\vec{\nabla}G$

i.e. $C_r = G^{-1}\partial_r G$ and $C_z = G^{-1}\partial_z G$

Then **(*)** $\Rightarrow$ $\vec{\nabla}\cdot\vec{C} = 0$ for r > 0

Including r = 0: $\boxed{\vec{\nabla}\cdot\vec{C} = 4\pi\delta^2(\sigma)\rho(z)}$

ρ(z): A source for G(r,z)
D-2 × D-2 matrix

Need also: $\boxed{\vec{\nabla}\times\vec{C} + \vec{C}\times\vec{C} = \vec{0}}$ ← from definition of $\vec{C}$

i.e. $\partial_r C_z - \partial_z C_r + [C_r, C_z] = 0$

Natural to introduce potential: $A(r,z)$ defined by $C_r = -\frac{1}{r}\partial_z A$, $C_z = \frac{1}{r}\partial_r A$

Then $\rho(z) = -\frac{1}{2}\partial_z A(0,z)$

⟶ A(r,z) can be introduced always, and is a linear potential in diagonal case (Unlike G(r,z) which is like exp(linear) in diagonal case)

Introduce other source: $\Lambda(z) \equiv C_z(0,z)$

Equations for A(r,z): $r\partial_r(\frac{1}{r}\partial_r A) + \partial_z^2 A + \frac{1}{r}[\partial_r A, \partial_z A] = 0$

$$\partial_z A(0,z) = -2\rho(z) \qquad \lim_{r\to 0}\frac{1}{r}\partial_r A = \Lambda(z)$$

Claim : Given ρ(z) & Λ(z): A(r,z) and hence G(r,z) are uniquely determined

$\rightarrow$ Need to consider rod-structure to show this

Structure of sources ρ(z) , Λ(z) for rods:

$$\det G = \pm r^2 \longrightarrow Tr(A) = -2z \Longrightarrow Tr(\rho(z)) = 1$$

interpretation : **Total rod-density = const**

Consider a rod [z_1,z_2] with direction v . Let z$\in$]z_1,z_2[:

We can find a function λ(z) such that $\begin{cases} G(0,z)v = 0 \\ \lim_{r\to 0} v^T G(r,z)v = \pm r^2 e^{\lambda(z)} v^T v \end{cases}$

From this + further analysis:

$$\left.\begin{array}{c} \rho(z)v = v \quad , \quad \rho(z)^2 = \rho(z) \\ \exists\ w_i : \rho^T w_i = 0 \ , \quad i = 1, ..., D-3 \\ \Lambda(z)v = \lambda'(z)v \quad Tr(\rho(z)\lambda(z)) = \lambda'(z) \end{array}\right\} \quad \text{for } z \in]z_1,z_2[$$

Using all results on $\rho(z)$, $\Lambda(z)$ for rods

$\longrightarrow$ One can prove that for a rod $[z_1,z_2]$ with given $\rho(z)$, $\Lambda(z)$,

A(r,z) is uniquely determined for $z \in]z_1,z_2[$

Nontrivial, since there could be a free function coming in because of commutators in

$$r\partial_r(\tfrac{1}{r}\partial_r A) + \partial_z^2 A + \tfrac{1}{r}[\partial_r A, \partial_z A] = 0$$

Technique:

$$A(r,z) = \textstyle\sum_{n=0}^{\infty} r^{2n} \cdot A_n(z)$$

Trivial in diagonal case $\longrightarrow$ $A_{n+1} = A'_n$

Philosophy:

Question of stationary & axisymmetric solutions reduced to a one dimensional problem of finding $\rho(z)$, $\Lambda(z)$

$\longrightarrow$ A(r,z) and hence G(r,z) is given uniquely from $\rho(z)$, $\Lambda(z)$

But is the rod-structure enough to determine $\rho(z)$, $\Lambda(z)$?

No, not with the equations derived

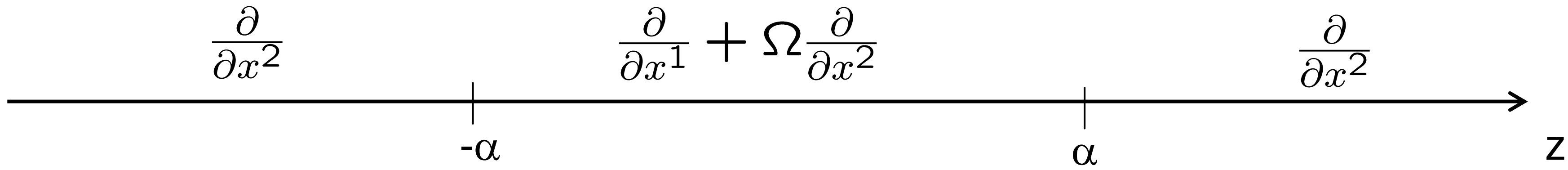
$\longrightarrow$ need more constraints

Goal:

to be able to construct $\rho(z)$, $\Lambda(z)$ from knowledge of rod-structure

How can we find more constraints ?

Example: The Kerr black hole



For z ∈ [-∞,-α]:

$$\Omega = \frac{\sin\gamma\cos\gamma}{2\alpha(1+\cos\gamma)}$$

$$\rho = \begin{pmatrix} 0 & 0 \\ h & 1 \end{pmatrix}, \quad h(z) = -\frac{2\sin\gamma\cos\gamma\left(1-\frac{z}{\alpha}\cos\gamma\right)}{\alpha\left(\left(1-\frac{z}{\alpha}\cos\gamma\right)^2+\sin^2\gamma\right)^2}$$

$$\Lambda = \begin{pmatrix} -\lambda' & 0 \\ h' + 2h\lambda' & \lambda' \end{pmatrix}, \quad \lambda'(z) = \frac{2\left[\left(1-\frac{z}{\alpha}\cos\gamma\right)^2 - \sin^2\gamma\right]}{\alpha\cos\gamma\left(1-\frac{z^2}{\alpha^2}\right)\left[\left(1-\frac{z}{\alpha}\cos\gamma\right)^2+\sin^2\gamma\right]}$$

For z ∈ [α,∞]: Put z → -z

For z ∈ [-α,α]:

$$\rho = \begin{pmatrix} 1-\Omega h & h \\ \Omega - \Omega^2 h & \Omega h \end{pmatrix}, \quad h(z) = \frac{1}{\Omega} - \frac{\sin\gamma\left((1+\cos\gamma)^2 - \frac{z^2}{\alpha^2}\sin^2\gamma\right)}{\Omega^2\alpha\left((1+\cos\gamma)^2+\frac{z^2}{\alpha^2}\sin^2\gamma\right)^2},$$

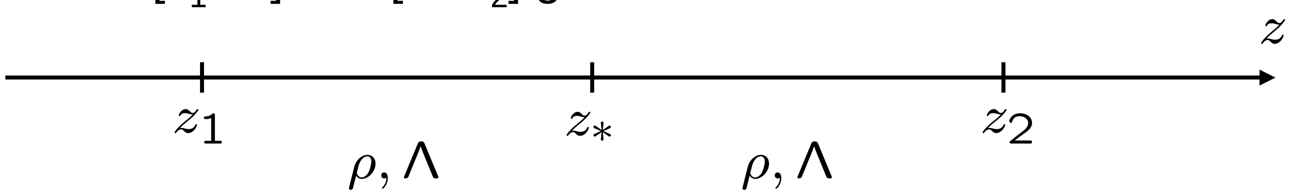
$$\Lambda = \begin{pmatrix} -\Omega h' + (1-2\Omega h)\lambda' & h' + 2h\lambda' \\ -\Omega^2 h' + 2\Omega(1-\Omega h)\lambda' & \Omega h' - (1-2\Omega h)\lambda' \end{pmatrix},$$

$$\lambda'(z) = \frac{4(1+\cos\gamma)z}{\alpha^2\left(1-\frac{z^2}{\alpha^2}\right)\left[(1+\cos\gamma)^2+\frac{z^2}{\alpha^2}\sin^2\gamma\right]}.$$

Conditions on ρ(z) and Λ(z):

Smoothness conditions at rod end points:

Two rods $[z_1, z_*]$ and $[z_*, z_2]$ given



From knowing $\rho(z)$ & $\Lambda(z)$ for $z \in]z_1, z_*[$ is it possible to get $\rho(z)$ & $\Lambda(z)$ for $z \in]z_*, z_2[$?

Yes: Follows from demanding smoothness (regularity) of solution at r=0 and $z=z_*$

Define for given z_* (not necessarily for rod end point):

$$p = z - z_* \quad , \quad q = \sqrt{r^2 + (z - z_*)^2}$$

We require that A(p,q) is smooth at (p,q)=(0,0)

$$\Rightarrow \quad \lim_{z \to z_*^-} \rho(z) - \lim_{z \to z_*^+} \rho(z) = \lim_{z \to z_*^+} (z - z_*)\Lambda(z) = - \lim_{z \to z_*^-} (z - z_*)\Lambda(z)$$

$$\lim_{z \to z_* +} [2\rho(z) + (z - z_*)\Lambda(z)] = \lim_{z \to z_* -} [2\rho(z) + (z - z_*)\Lambda(z)]$$

$$\lim_{z \to z_* +} \left[\rho'(z) + ((z - z_*)\Lambda(z))'\right] = \lim_{z \to z_* -} \left[\rho'(z) + ((z - z_*)\Lambda(z))'\right] \qquad \text{etc…}$$

Using these and more relations one can get $\rho(z)$ & $\Lambda(z)$ for $z \in]z_*, z_2[$ from $\rho(z)$ & $\Lambda(z)$ for $z \in]z_1, z_*[$

Smoothness conditions at rod end points
$\rightarrow$ ρ(z) and Λ(z) given from rod structure plus behavior at infinity

We need more constraints to fix ρ(z) and Λ(z)

How? $\rightarrow$ Integrability

$$\left.\begin{array}{r}\vec{\nabla}\cdot\vec{C} = 4\pi\delta^2(\sigma)\rho(z) \\ \partial_r C_z - \partial_z C_r + [C_r, C_z] = 0\end{array}\right\}$$

Equations for Sigma-model

We can generate an infinite set of currents from this:

$$\vec{\nabla}\cdot\vec{C}^{(n)} = 4\pi\delta^2(\sigma)\rho^{(n)}(z)$$

This gives an infinite set of conserved quantities for the solution

$\rightarrow$ Gives an infinite set of constraints on ρ(z) and Λ(z)

$\rightarrow$ Important ingredient: Asymptotic flatness of solution

(see my next paper for details…)

Expectation: Only one solution to the constraints (at least locally)
$\rightarrow$ Would give uniqueness of ρ(z) and Λ(z) for a given rod-structure

Conclusions:

- Found canonical form of metric for stationary & axisymmetric metrics:

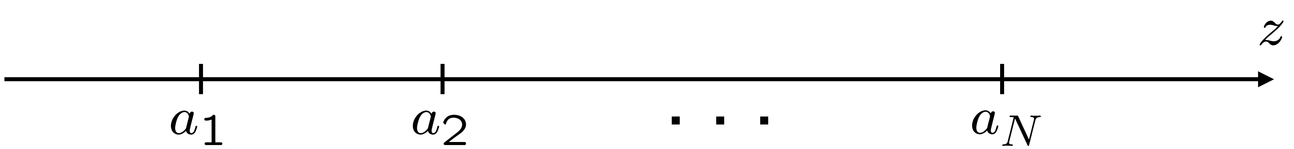
$$ds^2 = G_{ij} dx^i dx^j + e^{2\nu}(dr^2 + dz^2) \ \ , \ r = \sqrt{|\det G_{ij}|}$$

- Einstein equations reduces to solving

$$\vec{\nabla} \cdot (G^{-1} \vec{\nabla} G) = 4\pi \delta^2(\sigma) \rho(z)$$

- Found metrics for the 5D Myers-Perry black hole and the 5D black ring of Emperan & Reall in canonical form

- We identified the rod-structure of a solution:



Rods: The intervals $[a_{k-1}, a_k]$

$v_{(k)}$ the direction of rod $[a_{k-1}, a_k]$: $G(0,z)\, v_{(k)}$ for $z \in \,]\, a_{k-1}, a_k\, [$

<u>Conjecture</u>: A solution is unique given its rod-structure

If true, would provide classification of solutions

- We defined $\vec{C} = G^{-1}\vec{\nabla}G \quad , \quad \Lambda(z) = C_z(0,z)$

  Sources for G(r,z): ρ(z) & Λ(z)

  Given ρ(z) & Λ(z) $\longrightarrow$ G(r,z) determined

  ρ(z) & Λ(z) determined from rod-structure?

Future directions:

- Uniqueness theorem for rod-structure
  (at least locally, i.e. uniqueness in a neighbourhood of the solution)

- Construction of new solutions

  5D Black ring with two angular momenta

  Solutions with KK-bubbles and black holes with rotation